STRONG PERSISTENCE OF A DISCRETE-TIME POPULATION WITH THREE SPECIES

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Abstract. We consider a class of discrete time prey-predator models with three interacting species defined on the two-dimensional simplex. For our choice of parameters, we show that all orbits starting from the interior of the simplex converge to the unique fixed point of the operator.

One important question in mathematical models describing the evolution of interacting populations is the long-time behavior of the absolute or relative population sizes. In particular, it is of great interest to determine whether a particular population becomes extinct in the long run. If this does not happen to a given population in the model, then one says that the population persists.

Let $f : S^2 \to [0,1]$ be continuous and let $a, b, c \in (0,1]$ be positive parameters. Here, S^2 is the 2-dimensional simplex given by

$$S^2 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_i \geq 0 \text{ for all } i \in \{1,2,3\} \text{ and } x_1 + x_2 + x_3 = 1\right\}.$$

Denote by $\operatorname{int} S^2 = \left\{(x_1, x_2, x_3) \in S^2 : x_1x_2x_3 > 0\right\}$ the interior of S^2 and by $\partial S^2 = \left\{(x_1, x_2, x_3) \in S^2 : x_1x_2x_3 = 0\right\}$ the boundary of S^2.

We consider the following evolution operator of the population which is a discrete analog of the Kolmogorov model (see [1]) of three interacting populations of the form

$$W_{f,a,b,c} : \begin{cases} x_1' = x_1(1 + (bx_3^2 - ax_1x_2)f(\mathbf{x})) \\ x_2' = x_2(1 + (ax_1^2 - cx_2x_3)f(\mathbf{x})) \\ x_3' = x_3(1 + (cx_2^2 - bx_1x_3)f(\mathbf{x})) \end{cases}$$

Note that $W_{f,a,b,c}$ maps S^2 to S^2. The same operator is analyzed in [2]. There it was shown that for negative parameters the ergodic hypothesis does not hold and that any order Cesàro mean of the trajectories diverges. In the case that two of the parameters have different sign, it was shown that trajectories converge to a vertex of the simplex. Concerning our choice of parameters, [2] shows that all orbits starting from the interior of the simplex converge to the unique fixed point in the interior of the simplex. However, the proof in [2] is restricted to a smaller class of parameters. The purpose of this paper is to provide a general proof of the statement [2, Theorem 2.12].

Theorem 1. *For any $x \in \operatorname{int} S^2$ it holds that*

$$\lim_{k\to\infty} W^k_{f,a,b,c}(x) = x^*$$

where

$$x^* := \left(\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}, \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}, \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}\right)$$

and $\lambda_1 = \sqrt[3]{bc^2}$, $\lambda_2 = \sqrt[3]{ab^2}$, $\lambda_3 = \sqrt[3]{a^2c}$.

Proof. Set $\varphi : S^2 \to \mathbb{R}$ with $\varphi(x_1, x_2, x_3) = x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3}$. Clearly the function φ is continuous on S^2 and

$$\varphi(x) = 0 \text{ iff } x \in \partial S^2, \max_{x\in S^2} \varphi(x) = \varphi(x^*) \text{ and } \varphi(x) = \varphi(x^*) \text{ iff } x = x^*.$$

Showing that φ is a Lyapunov function for $W_{f,a,b,c}$, i.e. $\varphi(W_{f,a,b,c}(x)) > \varphi(x)$ for any $x \in \operatorname{int} S^2 \setminus \{x^*\}$, the statement of the theorem follows.
Let $x \in \operatorname{int} S^2 \setminus \{x^*\}$ be arbitrary. Since $\varphi(W_{f,a,b,c}(\mathbf{x})) = \varphi(\mathbf{x})\psi(\mathbf{x})$, where

$$\psi(\mathbf{x}) = \left(1 + \left(ax_1x_2 - bx_3^2\right) f(\mathbf{x})\right)^{\lambda_1} \left(1 + \left(cx_2x_3 - ax_1^2\right) f(\mathbf{x})\right)^{\lambda_2} \left(1 + \left(bx_3x_1 - cx_2^2\right) f(\mathbf{x})\right)^{\lambda_3}$$

it is enough to show that $\psi(\mathbf{x}) > 1$. For simplicity we set

$$u = \sqrt[3]{a^2b}x_1\sqrt{f(\mathbf{x})} \qquad v = \sqrt[3]{ac^2}x_2\sqrt{f(\mathbf{x})} \qquad w = \sqrt[3]{b^2c}x_3\sqrt{f(\mathbf{x})}.$$

Without loss of generality we let $\lambda_1 = \min\{\lambda_1, \lambda_2, \lambda_3\}$. Set $\lambda_4 = \min\{\lambda_2, \lambda_3\}$. Since $\lambda \mapsto (1 + \lambda^{-1}\xi)^\lambda$ is non-decreasing on $[\lambda_4, 1]$ for all $\xi \in (-\lambda_4, \infty)$, it follows that

$$\begin{aligned}\psi(\mathbf{x}) &= \left(1 + \lambda_1^{-1}\left(w^2 - uv\right)\right)^{\lambda_1} \left(1 + \lambda_2^{-1}\left(u^2 - vw\right)\right)^{\lambda_2} \left(1 + \lambda_3^{-1}\left(v^2 - uw\right)\right)^{\lambda_3} \\ &\geq \left(1 + \lambda_1^{-1}\left(w^2 - uv\right)\right)^{\lambda_1} \left(1 + \lambda_4^{-1}\left(u^2 - vw\right)\right)^{\lambda_4} \left(1 + \lambda_4^{-1}\left(v^2 - uw\right)\right)^{\lambda_4}.\end{aligned}$$

Using monotonicity and concavity of the logarithmic function, we obtain

$$\begin{aligned}&\frac{1}{\lambda_1 + 2\lambda_4} \ln \psi(\mathbf{x}) \\ &\geq -\frac{1}{\lambda_1 + 2\lambda_4}\left(\lambda_1 \ln \frac{1}{1 + \lambda_1^{-1}(w^2 - uv)} + \lambda_4 \ln \frac{1}{1 + \lambda_4^{-1}(u^2 - vw)} + \lambda_4 \ln \frac{1}{1 + \lambda_4^{-1}(v^2 - uw)}\right) \\ &\geq -\ln\left(\frac{1}{\lambda_1 + 2\lambda_4}\left(\frac{\lambda_1}{1 + \lambda_1^{-1}(w^2 - uv)} + \frac{\lambda_4}{1 + \lambda_4^{-1}(u^2 - vw)} + \frac{\lambda_4}{1 + \lambda_4^{-1}(v^2 - uw)}\right)\right) \\ &\geq -\ln\left(1 - \frac{F(\mathbf{x})}{(\lambda_1 + 2\lambda_4)\left(1 + \lambda_1^{-1}(w^2 - uv)\right)\left(1 + \lambda_4^{-1}(u^2 - vw)\right)\left(1 + \lambda_4^{-1}(v^2 - uw)\right)}\right)\end{aligned}$$

where

$$\begin{aligned}F(\mathbf{x}) = u^2 + v^2 + w^2 - uv - uw - vw &+ \frac{1}{\lambda_1}(w^2 - uv)(u^2 + v^2 - vw - uw) \\ &+ \frac{1}{\lambda_4}\left((w^2 - uv)(u^2 + v^2 - vw - uw) + 2(u^2 - vw)(v^2 - uw)\right) \\ &+ \frac{\lambda_1 + 2\lambda_4}{\lambda_1\lambda_4^2}(w^2 - uv)(u^2 - vw)(v^2 - uw).\end{aligned}$$

It remains to show that $F(\mathbf{x}) > 0$. Therefore, we write

$$F(\mathbf{x}) = F_1(\mathbf{x}) + \frac{1}{\lambda_1}F_2(\mathbf{x}) + \frac{1}{\lambda_4}F_3(\mathbf{x}) + \frac{\lambda_1 + 2\lambda_4}{\lambda_1\lambda_4^2}F_4(\mathbf{x}). \tag{1}$$

The first term is positive since

$$F_1(\mathbf{x}) = u^2 + v^2 + w^2 - uv - uw - vw = \frac{1}{2}(u - v)^2 + \frac{1}{2}(u - w)^2 + \frac{1}{2}(v - w)^2 > 0. \tag{2}$$

Motivated by the shape of first term, we rewrite the other terms. The second term can be written as

$$\begin{aligned}F_2(\mathbf{x}) &= (w^2 - uv)(u^2 + v^2 - vw - uw) \\ &= \frac{1}{2}(u - v)^2(uw + vw) + \frac{1}{2}(u - w)^2(-2uv - uw - vw) + \frac{1}{2}(v - w)^2(-2uv - uw - vw)\end{aligned} \tag{3}$$

and

$$F_2(\mathbf{x}) = \frac{1}{2}(u - v)^2(vw + w^2) + \frac{1}{2}(u - w)^2(-3vw + w^2 - 2uv) + \frac{1}{2}(v - w)^2(-2uv - w^2 - vw). \tag{4}$$

For the third term we use the formula

$$\begin{aligned} F_3(\mathbf{x}) &= (w^2-uv)(u^2+v^2-vw-uw)+2(u^2-vw)(v^2-uw) \\ &= \frac{1}{2}(u-v)^2(-2uv-3uw-3vw)+\frac{1}{2}(u-w)^2(-uw-vw)+\frac{1}{2}(v-w)^2(-uw-vw). \end{aligned} \tag{5}$$

Estimating the fourth and last term, we obtain

$$\begin{aligned} F_4(\mathbf{x}) &= (w^2-uv)(u^2-vw)(v^2-uw) \\ &= \frac{1}{2}(u-v)^2(-uw^3+u^2vw-vw^3+uv^2w)+\frac{1}{2}(u-w)^2(-uv^3+u^2vw-v^3w+uvw^2) \\ &\quad+\frac{1}{2}(v-w)^2(-u^3v+uv^2w-u^3w+uvw^2) \\ &\geq \frac{1}{2}(u-v)^2(-uw^3-vw^3)+\frac{1}{2}(u-w)^2(-uv^3-v^3w)+\frac{1}{2}(v-w)^2(-u^3v-u^3w) \end{aligned} \tag{6}$$

and

$$\begin{aligned} F_4(\mathbf{x}) &= \frac{1}{2}(u-v)^2(-2uw^3-vw^3+2uv^2w+u^2vw) \\ &\quad+\frac{1}{2}(u-w)^2(-2uv^3-v^3w+u^2vw+2uvw^2) \\ &\geq \frac{1}{2}(u-v)^2(-2uw^3-vw^3)+\frac{1}{2}(u-w)^2(-2uv^3-v^3w). \end{aligned} \tag{7}$$

We differ in the rest of the proof between two cases.
Case 1: $a^2b^2c^{-1} \leq 3/2$.
Set $\gamma = \sqrt[3]{3/2}$. Then $\sqrt[3]{a^2b^2c^{-1}} \leq \gamma$. Substituting F_1, F_2, F_3, F_4 in (1) by (2), (3), (5), (6), it follows that

$$\begin{aligned} F(\mathbf{x}) &\geq \frac{1}{2}(u-v)^2\left(1+\frac{1}{\lambda_1}(uw+vw)-\frac{1}{\lambda_4}(2uv+3uw+3vw)-\frac{3}{\lambda_1\lambda_4}(uw^3+vw^3)\right) \\ &\quad+\frac{1}{2}(u-w)^2\left(1-\frac{1}{\lambda_1}(2uv+uw+vw)-\frac{1}{\lambda_4}(uw+vw)-\frac{3}{\lambda_1\lambda_4}(uv^3+v^3w)\right) \\ &\quad+\frac{1}{2}(v-w)^2\left(1-\frac{1}{\lambda_1}(2uv+uw+vw)-\frac{1}{\lambda_4}(uw+vw)-\frac{3}{\lambda_1\lambda_4}(u^3v+u^3w)\right) \\ &\geq \frac{1}{2}(u-v)^2\left(1-\frac{1}{\lambda_4}(2uv+2uw+2vw)-\frac{3}{\lambda_1\lambda_4}(uw^3+vw^3)\right) \\ &\quad+\frac{1}{2}(u-w)^2\left(1-\frac{1}{\lambda_1}(2uv+2uw+2vw)-\frac{3}{\lambda_1\lambda_4}(uv^3+v^3w)\right) \\ &\quad+\frac{1}{2}(v-w)^2\left(1-\frac{1}{\lambda_1}(2uv+2uw+2vw)-\frac{3}{\lambda_1\lambda_4}(u^3v+u^3w)\right) \\ &\geq \frac{1}{2}(u-v)^2\left(1-2x_1x_2-2x_1x_3-2x_2x_3-3x_1x_3^3-3x_2x_3^3\right) \\ &\quad+\frac{1}{2}(u-w)^2\left(1-2x_1x_2-2\gamma x_1x_3-2x_2x_3-3x_1x_2^3-3x_2^3x_3\right) \\ &\quad+\frac{1}{2}(v-w)^2\left(1-2x_1x_2-2\gamma x_1x_3-2x_2x_3-3\gamma x_1^3x_2-3\gamma^2x_1^3x_3\right) \\ &> 0 \end{aligned}$$

where the last estimate holds since each factor is positive.
Case 2: $a^2b^2c^{-1} > 3/2$.
Note that $\lambda_2 > \lambda_3$ and hence $\lambda_4 = \lambda_3$. Moreover, $c \leq 2/3$. Substituting F_1, F_2, F_3, F_4 in (1) by

(2), (4), (5), (7), it follows that

$$
\begin{aligned}
F(\mathbf{x}) \geq\ & \frac{1}{2}(u-v)^2\left(1+\frac{1}{\lambda_1}(vw+w^2)-\frac{1}{\lambda_3}(2uv+3uw+3vw)-\frac{3}{\lambda_1\lambda_3}(2uw^3+vw^3)\right)\\
&+\frac{1}{2}(u-w)^2\left(1-\frac{1}{\lambda_1}(3vw-w^2+2uv)-\frac{1}{\lambda_3}(uw+vw)-\frac{3}{\lambda_1\lambda_3}(2uv^3+v^3w)\right)\\
&+\frac{1}{2}(v-w)^2\left(1-\frac{1}{\lambda_1}(2uv+w^2+vw)-\frac{1}{\lambda_3}(uw+vw)\right)\\
\geq\ & \frac{1}{2}(u-v)^2\left(1-\frac{1}{\lambda_3}(2uv+3uw+2vw)+\frac{1}{\lambda_1}w^2\left(1-\frac{4}{\lambda_3}uw\right)-\frac{1}{\lambda_1\lambda_3}(2uw^3+3vw^3)\right)\\
&+\frac{1}{2}(u-w)^2\left(1-\frac{1}{\lambda_1}(4vw+2uv)-\frac{1}{\lambda_3}uw-\frac{3}{\lambda_1\lambda_3}(2uv^3+v^3w)\right)\\
&+\frac{1}{2}(v-w)^2\left(1-\frac{1}{\lambda_1}(2uv+w^2+2vw)-\frac{1}{\lambda_3}uw\right)\\
\geq\ & \frac{1}{2}(u-v)^2\left(1-2x_1x_2-3x_1x_3-2x_2x_3-2x_1x_3^3-3x_2x_3^3\right)\\
&+\frac{1}{2}(u-w)^2\left(1-4\sqrt[3]{c}x_2x_3-2x_1x_2-x_1x_3-6cx_1x_2^3-3\sqrt[3]{c^4}x_2^3x_3\right)\\
&+\frac{1}{2}(v-w)^2\left(1-2x_1x_2-x_3^2-2x_2x_3-x_1x_3\right)\\
>\ & 0
\end{aligned}
$$

where the last estimate holds since each factor is positive. □

References

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