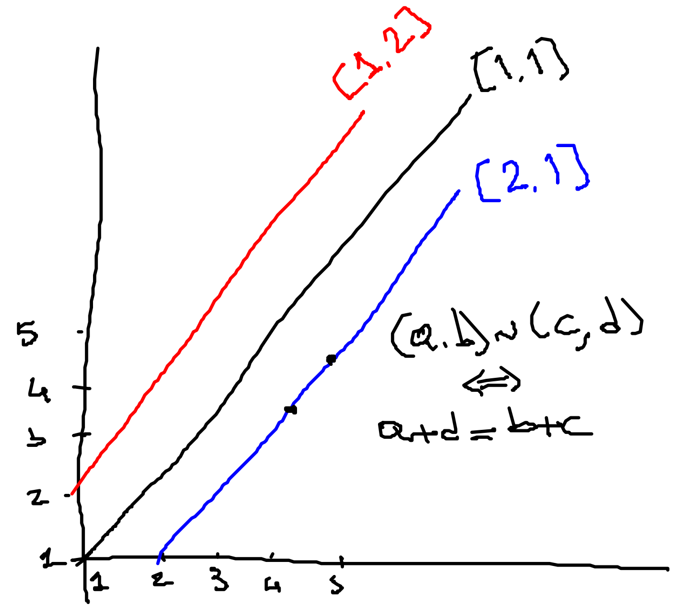
12/11

Aufgabe 12

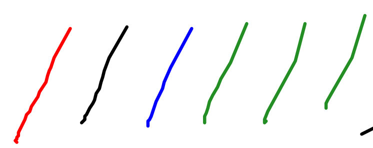
Lösbarkeit von $b+x=a$, $a,b\in\mathbb{N}$

$\mathbb{N}\times\mathbb{N}=\{(a,b),\ a,b\in\mathbb{N}\}$

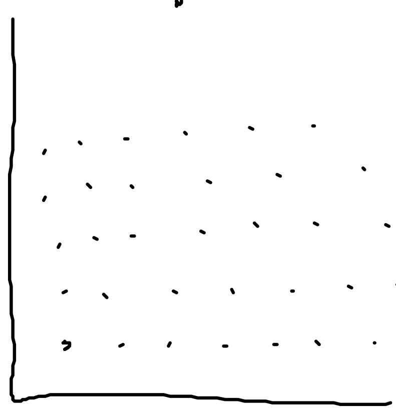
$(a,b)\sim(c,d) \Leftrightarrow a+d=b+c$



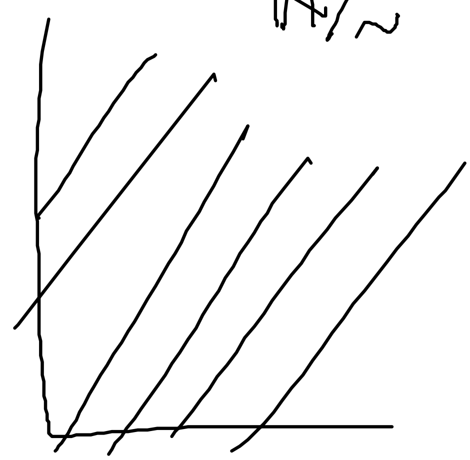
$\mathbb{N}^2/_\sim=\{[a,b],\ (a,b)\in\mathbb{N}^2\}$

$=\{$  $\}$

$\mathbb{N}^2$



$\mathbb{N}^2/_\sim$



Körper $(K,+,\cdot)$

(A5) (A1) $\forall a,b,c\in K \quad (a+b)+c=a+(b+c)$

(KO) (A2) $\forall a,b\in K \quad a+b=b+a$

(NE) $\exists\, 0\in K : \forall a\in K \quad a+0=a$

(IN) $\forall a \in K: \exists -a \in K: a + (-a) = 0$
(Inverselement)

(M1) $\forall a, b, c: (a \cdot b) \cdot c = a \cdot (b \cdot c)$

(M2) $a \cdot b = b \cdot a$

(NE) $\exists 1 \in K: \forall a \in K \; a \cdot 1 = a$

(IN) $\forall a \in K \setminus \{0\} \; \exists a^{-1} \in K: a \cdot a^{-1} = 1$

(DIS) $a \cdot (b + c) = ab + ac$

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12. c) Zeigen dass:
(A1) – (A2) – (NE) – (IN)
(M1) – (M2) – (NE) und
(NF) $\forall a, b \in \mathbb{N}^2/_\sim \; a \cdot b = 0 \Rightarrow a = 0 \vee b = 0$
gelten

(A1) $\forall [a,b], [c,d], [e,f] \in \mathbb{N}^2/_\sim$
$\Rightarrow ([a,b] + [c,d]) + [e,f] = [a,b] + ([c,d] + [e,f])$

$+: [a,b] + [c,d] = [a+c, b+d]$ Definiz

Beweis (A1)

$([a,b] + [c,d]) + [e,f] \overset{\text{Def. von } +}{=} [a+c, b+d] + [e,f]$

$= [a+c+e, b+d+f] = [a+(c+e), b+(d+f)]$

$= [a,b] + ([c,d] + [e,f])$

Gilt für + in $\mathbb{N}$

(A2) $[a,b] + [c,d] = [c,d] + [a,b]$
$\Leftrightarrow (a+c, b+d] = [c+a, d+b]$

(NE) Vorgegeben: $0 = [1,1]$

Zeigen: $\forall\, [a,b] \in \mathbb{N}^2/_\sim \quad [a,b]+[1,1]=[a,b]$

$[a,b]+[1,1] = [a+1,b+1] = [a,b]$ weil $(a+1,b+1) \sim (a,b)$

(IN) Sei $x = [a,b] \in \mathbb{N}^2/_\sim$, wir definieren

$-x := [b,a]$ (vorgegeben)

zu zeigen: $[a,b]+[b,a]=[1,1]$

$[a,b]+[b,a] = [a+b,b+a] = [1,1]$ weil $\big((a+b,b+a)\sim(1,1)\big)$

Def:

$[a,b]\cdot[c,d] := [ac+bd,\, ad+bc]$ | $[1,1]\cdot[1,4]=[5,5]$ $=[1,1]$

(DIS) $\forall\, [a,b],[c,d],[e,f] \in \mathbb{N}^2/_\sim$

$[a,b]\cdot\big([c,d]+[e,f]\big) = [a,b]\cdot[c,d] + [a,b]\cdot[e,f]$

$\| \quad [ac+bd,\, ad+bc]+[ae+bf,\, af+be] =$

$= [ac+bd+ae+bf,\, ad+bc+af+be]$

$= [a(c+e)+b(d+f),\, a(d+f)+b(c+e)]$

$= [a,b]\cdot[c+e,d+f] = [a,b]\cdot\big([c,d]+[e,f]\big)$

(NF) $[a,b],\ [c,d] \in \mathbb{N}^2/\sim$

$[a,b]\cdot[c,d]=[1,1] \Rightarrow [a,b]=[1,1] \vee [c,d]=[1,1]$

$\forall a,b \in K$

$a\cdot b=0 \Rightarrow a=0 \vee b=0$

Annahme:

$[a,b]\cdot[c,d]=[1,1] \wedge [a,b]\neq[1,1]$

$[a,b]\cdot[c,d]=[1,1]$ Ich addiere $[a,b]$ links und rechts

$\Leftrightarrow [a,b]+[a,b][c,d]=[a,b]$ (Vorgegeben $1=[2,1]$)

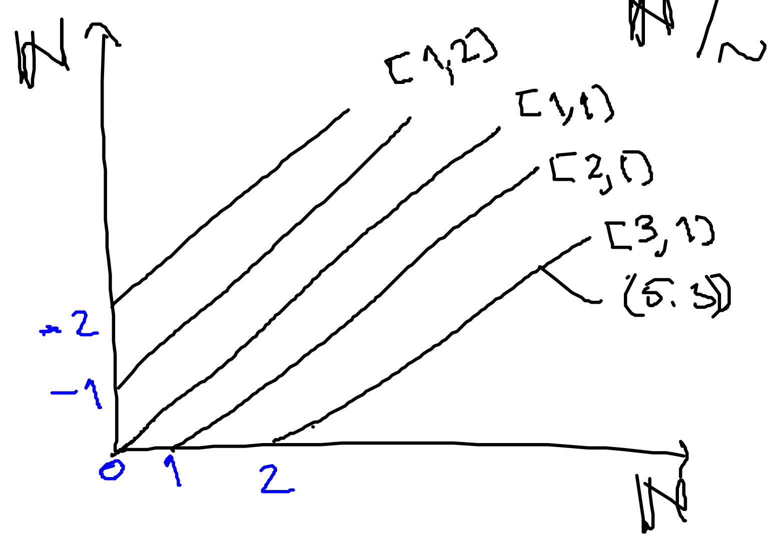
$\Leftrightarrow [a,b]\underbrace{(1+[c,d])}_{\text{Neutraleselement!}}=[a,b]$

Wenn $[a,b]\neq[1,1]$ dann muss

$1+[c,d]=1 \Leftrightarrow [c,d]=[1,1]$

d)

Donnerstag warten!



e) $b + x = a$ ist in $\mathbb{Z} = \mathbb{N}^2/\sim$ eindeutig lösbar

e.1) Lösbarkeit

$$[b+1, 1] + x = [a+1, 1]$$

$$[1, \cancel{b+1}] + [\cancel{b+1}, 1] + x = [a+1, 1] + [1, b+1]$$

$$x = [a+2, b+2] = [a, b]$$

$$3 + x = 1$$

$$(-\cancel{3}) + \cancel{3} + x = -3 + 1$$

$$x = -2$$

e.2) Eindeutig (Idee: wenn es eine 2. Lösung y gibt, dann $x = y$)

$\exists\, y = [c, d]$ die $b + y = a$ löst

$$\Leftrightarrow [b+1, 1] + [c, d] = [a+1, 1]$$

$$\Leftrightarrow [b+1+c, 1+d] = [a+1, 1]$$

$$\Leftrightarrow [b+c, d] = [a+1, 1] \qquad b+c+1 = a+d+1$$

$$\Leftrightarrow (b+c, d) \sim (a+1, 1) \Leftrightarrow$$

$$\Leftrightarrow b + c = a + d$$

$$\Leftrightarrow (a, b) \sim (c, d) \Leftrightarrow [a, b] = [c, d] \Leftrightarrow x = y$$