ON CONVERGENCE OF A RECEDING HORIZON METHOD FOR PARABOLIC BOUNDARY CONTROL *

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Abstract. A method of receding horizon type is considered for a simplified linear-quadratic parabolic boundary control problem with bound constraints on the control. The performance of the method is examined numerically and confirmed by an associated analysis. In particular, the method is shown to converge to a unique fixed point. Moreover, a new hybrid method is suggested.

Key words. Optimal boundary control, parabolic equation, control constraints, instantaneous control, receding horizon

AMS subject classifications. 49M30, 49K20

1. Introduction.

In this paper, we discuss a suboptimal strategy of receding horizon type for the following simplified class of parabolic boundary control problems with bound constraints on the control:

$$\text{(P)} \qquad \min J(u) = \frac{1}{2}\int_0^1 (y(x,T) - y_d(x))^2 dx + \frac{\nu}{2}\int_0^T u(t)^2 dt$$

subject to

$$\begin{aligned} y_t(x,t) &= y_{xx}(x,t) \\ y(x,0) &= y_0(x) \\ y_x(0,t) &= 0 \\ y_x(1,t) &= \alpha\,(u(t) - y(1,t)), \end{aligned} \tag{1.1}$$

$x \in (0,1)$, $t \in (0,T)$, and subject to the bound constraints

$$u_a \le u(t) \le u_b \tag{1.2}$$

to be fulfilled a.e. on $[0,T]$. The control u is taken from $L^\infty(0,T)$. In this setting, T, ν, α are fixed positive constants, while $u_a < u_b$ are given real numbers. Moreover, y_d and y_0 are given in $L^2(0,1)$.

Problems of this type were frequently discussed in literature. It is easy to show that (P) admits a unique optimal control $\bar{u}$. Necessary and (by convexity) sufficient optimality conditions were derived already years ago, see the references in [21]. Thanks to the low dimension one of the domain $\Omega = (0,1), (P)$ can be easily solved numerically by various methods. It can be fully discretized and hereafter solved as a finite-dimensional quadratic programming problem, [10]. An alternative strategy is to work with different projection methods [15, 16], or to apply active set strategies such as the Bertsekas projection method [4, 10] or primal-dual active set strategies [3]. Today, it is by far not a challenge to solve (P) numerically. On the other hand, owing

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to its simplicity, the problem is a good candidate to study analysis and performance of suboptimal control methods.

Techniques of this type turned out to be efficient suboptimal strategies to solve very large scale optimal control problems. Their origin is the control of flows described by the instationary Navier-Stokes equations, where it is rather hopeless to try an accurate optimal solution in a reasonable time. We refer to [5, 7, 11]. In a number of recent papers, the advantages of this method were demonstrated again, see [11, 12, 13].

Suboptimal strategies also are interesting from another point of view. One suboptimal approach to stabilize dynamical systems leads to Model Predictive Control (MPC), [9]. Because this area has already been investigated intensively, there are many papers dealing with the properties of MPC-controlled finite-dimensional systems, see for instance [14, 19] and the references in [2, 18]. The application of MPC to partial differential equations is a more recent research topic, confer the papers [11, 13].

Encouraged by this success, we applied a method of instantaneous control type to a problem of cooling steel with linear terminal time objective functional, nonlinear parabolic equation, and constraints on control and state. We were able to drastically reduce the computing time with almost no loss of accuracy in comparison with an exact optimization [22].

However, it was reported by other scientists that the method of instantaneous control may deliver results far from optimum for other problems with terminal time functional. To study its performance in this case, we applied the following simple suboptimal strategy of instantaneous control type to (P): Split the interval $I = [0, T]$ into n small subintervals $I_j = [t_{j-1}, t_j]$ of uniform length τ, define $t_j = j\,T/n$, and take piecewise constant controls $u(t) = u_j$ on I_j.

The objective of (P) is to approximate y_d as close as possible in the L^2-norm at the final time T. Therefore, it might be natural to first choose a real control value u_1 such that $\|y(\cdot, t_1) - y_d\|$ is minimized on the first subinterval I_1, then - starting from $y(\cdot, t_1)$ - to select u_2 on I_2 such that $\|y(\cdot, t_2) - y_d\|$ is minimal etc. The idea behind seems to reflect part of our experience in daily life. Aiming to reach a target, we try to approach it in each step. However, this simple strategy exhibits a weak performance. This was experienced also by other authors.

Let us comment on the terminology "instantaneous control" at this point. The method described above consists of a sequence of optimization problems to be solved on short horizons of length τ. Therefore, it is a particular case of receding horizon techniques – a $(\tau, 1)$-receding horizon method or method of model predictive control.

Instantaneous control in its actual sense means that only one gradient step is performed in each time step to approach the optimum rather than to solve the short time problems up to their minimum, [6]. See also [7], and for the stability analysis of associated closed-loop control laws, [11, 13]. In contrast to this, the simple method explained above finds the optimum for each time step.

We present a mathematical proof that this method of receding horizon control will converge to a unique fixed point. Hence, the method can be used to stabilize the system under consideration. This stable fixed point might be far from the desired state, which explains the weak performance of the $(\tau, 1)$-receding horizon method under the cost functional J. It also serves as background to develop a better method.

This improved method is another main result of our paper. Moreover, we report on the application of the method for a domain Ω of dimension 2.

The paper is organized as follows: First, we discuss the convergence analysis for the one-dimensional case. We study the $(\tau, 1)$-receding horizon method and a more general method, where the optimization is performed over more than one time horizon, the $(l\tau, l)$-receding horizon method. Moreover, we present numerical results of the receding horizon technique described above and explain its weak performance. Next we report on other methods that improve the performance and still are very fast. Finally, the case of a two-dimensional domain is briefly sketched.

We will use the notation (t, m)-*receding horizon method* with the following meaning: t stands for the length of the time horizon, where the optimization is performed in one step of the method, while m denotes the number of time steps the horizon is shifted to obtain the next optimization period (of length t).

2. Analysis of the $(\tau, 1)$-receding horizon method. Although the $(\tau, 1)$-method is a particular case of the more general (τ, l)-method, we start our presentation with this simpler case aiming at introducing the main idea and associatd basic notations. Let us first express problem (P) in a shorter setting of functional analysis. We define

$$U_{ad} = \{u \in L^2(0, T) \mid u_a \le u(t) \le u_b \text{ a.e. on } [0, T]\}.$$

For each $u \in U_{ad}$, there exists a unique weak solution y solving the equations (1.1). This is the *state* associated with u. Let $G = G(x, \xi, t)$ denote the Green's function to (1.1). It is known that y is a weak solution to (1.1) iff

$$y(x,t) = \int_0^1 G(x, \xi, t) y_0(\xi)\, d\xi + \int_0^t G(x, 1, t-s)\alpha\, u(s)\, ds. \tag{2.1}$$

As mentioned in the introduction, we split $[0, T]$ into n subintervals of uniform length $\tau = T/n$. Define on $[0, \tau]$ linear and continuous operators $D_\tau : L^2(0, 1) \to L^2(0, 1)$ and $S_\tau : L^2(0, \tau) \to L^2(0, 1)$ by

$$\begin{aligned} (D_\tau\, w)(x) &= \int_0^1 G(x, \xi, \tau)\, w(\xi) d\xi \\ (S_\tau u)(x) &= \int_0^\tau G(x, 1, \tau - s)\, \alpha\, u(s) ds. \end{aligned}$$

For the continuity of these operators we refer, for instance, to [21]. Then $(D_\tau\, w)(x) = y(x, \tau)$, where $y(x,t)$ is the unique solution of the initial-boundary value problem

$$\begin{aligned} y_t(x,t) &= y_{xx}(x,t) \\ y(x,0) &= w(x) \\ y_x(0,t) &= 0 \\ y_x(1,t) + \alpha\, y(1,t) &= 0, \end{aligned} \tag{2.2}$$

while $(S_\tau u)(x) = z(x, \tau)$, where $z(x,t)$ solves

$$\begin{aligned} z_t(x,t) &= z_{xx}(x,t) \\ z(x,0) &= 0 \\ z_x(0,t) &= 0 \\ z_x(1,t) + \alpha\, z(1,t) &= \alpha\, u(t). \end{aligned} \tag{2.3}$$

The $(\tau,1)$-receding horizon method is defined as follows. Let $e_j = e_j(t)$, $j = 1,\ldots,n$, denote the piecewise constant basis functions

$$e_j(t) = \begin{cases} 1 & \text{on } I_j \\ 0 & \text{on } [0,T]\setminus I_j. \end{cases}$$

We will apply controls u having the form

$$u(t) = \sum_{j=1}^{n} u_j\, e_j(t)$$

with unknown real numbers u_j. Define $y_j(x) = y(x,t_j)$, $j = 0,\ldots,n-1$. Starting at $j = 1$, we subsequently solve the short horizon control problems

$$\min_u \varphi(y_{j-1},u) = \frac{1}{2}\,\|\,y(\cdot,t_j) - y_d\|^2_{L^2(0,1)} + \frac{\nu}{2}\cdot\tau\cdot u^2 \qquad (P_j)$$

subject to

$$\begin{aligned} y_t(x,t) &= y_{xx}(x,t) \\ y(x,t_{j-1}) &= y_{j-1}(x) \\ y_x(0,t) &= 0 \\ y_x(1,t) + \alpha\, y(1,t) &= \alpha\, u, \end{aligned} \qquad (2.4)$$

$$u_a \le u \le u_b,$$

$t \in [t_{j-1},t_j]$, where u is a real number.

The heat equation (2.4) is autonomous in time, hence $y(x,t_j) = \tilde{y}(x,\tau)$, where $\tilde{y}(x,\tau)$ solves (2.4) in $(0,\tau)$ subject to $\tilde{y}(x,0) = y_{j-1}(x)$. We can express this fact equivalently by

$$\begin{aligned} y(\cdot,t_j) &= \int_0^1 G(\cdot,\xi,\tau)\, y_{j-1}(\xi)\, d\xi + \int_0^\tau \alpha\, G(\cdot,1,\tau-s)\, u\, ds \\ &= D_\tau\, y_{j-1} + u\, S_\tau\, e_1 \end{aligned}$$

(notice that $u \in \mathbb{R}$). In what follows, we shall indicate suboptimal controls by a bar. Therefore, the optimal solution of (P_j) is denoted by $\bar{u}_j$. Moreover, we introduce for convenience the notation $e := S_\tau\, e_1$. In our paper, $\|\cdot\|$ stands for the norm of $L^2(0,1)$, and $(\cdot,\cdot)$ denotes the associated natural inner product.

(P_j) is equivalent to a very simple one-dimensional quadratic programming problem that can be solved explicitely. In fact, define $f : L^2(0,1)\times\mathbb{R}\to\mathbb{R}$ by

$$f(y,u) := \frac{1}{2}\,\|y_d - D_\tau y - u\, e\|^2 + \frac{\nu}{2}\,\tau\, u^2.$$

Then $\varphi(y_{j-1},u) = f(y_{j-1},u)$ and

$$f(y_{j-1},u) = \frac{1}{2} a\, u^2 + b_j\, u + c_j,$$

where

$$\begin{aligned} a\ &= \|e\|^2 + \nu\,\tau \\ b_j &= (e,\, D_\tau\, y_{j-1} - y_d) \\ c_j &= \frac{1}{2}\,\|D_\tau\, y_{j-1} - y_d\|^2. \end{aligned}$$

Therefore, the solution $\bar{u}_j$ of (P_j) is obtained by minimizing $\frac{1}{2} a\, u^2 + b_j\, u$ subject to $u \in [u_a, u_b]$,

$$\bar{u}_j = \begin{cases} u_a, & \text{if } -\frac{b_j}{a} < u_a \\ -\frac{b_j}{a}, & \text{if } -\frac{b_j}{a} \in [u_a, u_b] \\ u_b, & \text{if } -\frac{b_j}{a} > u_b. \end{cases} \tag{2.5}$$

We define the suboptimal control $\bar{u}$ for (P) by $\bar{u}(t) = \bar{u}_j$, $t \in I_j$. It is obvious that $\bar{u}$ can be determined very fast. Since the minimization of (P_j) is done analytically, the only work is to set up (P_j). To do this, the most time consuming step is the computation of $D_\tau\, y_{j-1}$, i.e. the solution of the heat equation (2.2) for $w = y_{j-1}$ on $(0, \tau) \times (0, 1)$. Then a, b_j, c_j can be found by numerical integration. The function $e = S_\tau e_1$ has to be computed only once by solving (2.3). Altogether, $n+1$ PDE solves are needed.

Clearly, this suboptimal method stops after n steps. Nevertheless, let us consider infinitely many repetitions of the iteration. Each iteration assigns to an initial function $y = y(x)$ a real optimal control number $\bar{u} = \bar{u}(y(\cdot))$ for the time horizon $[0, \tau]$ and a new initial function $\bar{y}(x)$ by

$$\bar{y} = D_\tau\, y + \bar{u}\,(y(\cdot))\, e. \tag{2.6}$$

Definition 2.1. *The mapping $y \longmapsto \bar{y}$ defined by* (2.6) *in* $L^2(0,1)$ *is denoted by* Φ, $\Phi(y) := D_\tau\, y + \bar{u}(y)\, e$.

We shall prove in Section 4 that Φ has a unique fixed point

$$y^* = \bar{u}\,(y^*)\; (I - D_\tau)^{-1}\, e. \tag{2.7}$$

The function $y_f := (I - D_\tau)^{-1}\, e$ can be described as follows: We know $y_f = D_\tau y_f + e$. In other words, $y_f(x) = y(x, \tau)$, where y is the solution of an initial boundary value problem with initial value $y(x, 0) = y_f(x)$. Hence $y(x, 0)$ and $y(x, \tau)$ must coincide. We have obtained $y_f(x) = y(x, \tau)$, where y solves the following boundary value problem subject to periodic boundary conditions with respect to the time variable:

$$\begin{aligned} y_t(x,t) &= y_{xx}(x,t) \\ y(x,0) &= y(x,\tau) \\ y_x(0,t) &= 0 \\ y_x(1,t) + \alpha\, y(1,t) &= \alpha e_1(t). \end{aligned} \tag{2.8}$$

3. Performance of the $(\tau, 1)$-receding horizon method. We tested this $(\tau, 1)$-receding horizon method by a known test problem due to Schittkowski [20]. Here, the following data were given:

$T = 1.58$, $y_d(x) = 0.5\,(1 - x^2)$, $y_0(x) = 0$, $\nu = 0.001$, $u_a = -1$, $u_b = 1$, $\alpha = 1$.

To apply the method, the problem must be discretized. The intervals $(0, 1)$ and $(0, T)$ were splitted by uniform grids into $n_x = 50$ and $n_t = 100$ subintervals, and the control u was approximated by $n = 100$ basis functions e_j. The heat equation (1.1) was solved by a fully implicit finite difference method.

First, we recall the results for an exact minimization of the discretized problem, i.e. a solution of the associated finite-dimensional quadratic programming problem.

They were obtained by the Bertsekas projection method and application of a CG method to the associated unconstrained subproblems, see [1] for details.

The optimal control $u(t)$ and a comparison of the desired temperature profile y_d with the optimal final temperature $\bar{y}(x, T)$ are presented in Figure 3.1. The computed optimal control has 3 switching points separating two interior and two boundary arcs, and the computed optimal value is

$$J(u^*) = 0.000686.$$

We refer also to associated numerical tests in Eppler and Tröltzsch [8]. Moreover, we should mention here the interesting fact that, up to now, it is an open question, if this reflects the true switching structure of the optimal control for the infinite-dimensional problem. In particular, it is not yet proven that the number of switching points is finite.

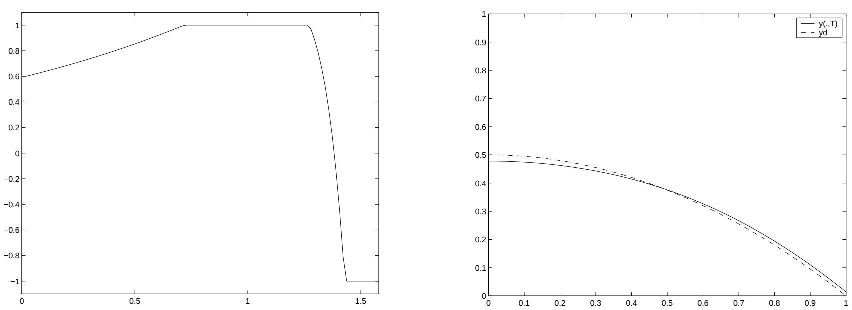

FIG. 3.1. *Optimal control u and optimal final state $y(\cdot, T)$*

Let us now report on the application of the $(\tau, 1)$-receding horizon method to this example, based on the same discretization. The computed suboptimal functions are plotted in the Figures 3.2 and the suboptimal value for the objective was

$$J(\bar{u}_{(\tau,1)}) = 0.03645.$$

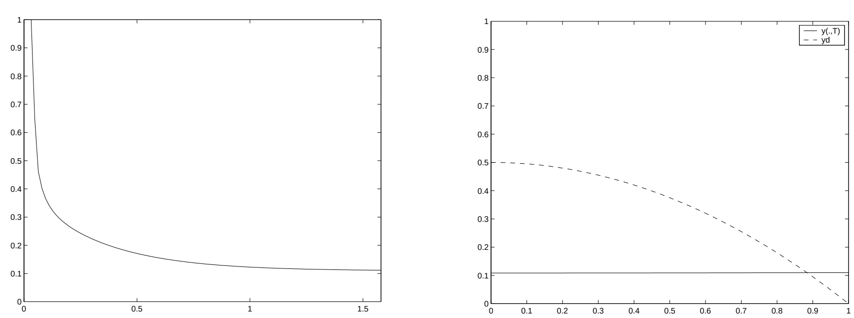

FIG. 3.2. $(\tau, 1)$*-receding horizon: Suboptimal control $\bar{u}$ and associated final state $\bar{y}(\cdot, T)$*

We observe a striking difference to the optimal data. Clearly, this is due to the fact that the time horizon τ is very short in comparison with $[0, T]$, where an exact optimization would be performed. However, in [22] it was observed that the method computed excellent values extremely close to the optimum.

Here, the optimal control has a very particular structure. In the first part of the time interval $[0, T]$, the optimal control inserts more energy than needed. Near time T the control is equal to -1, so the extra energy is dissipated. The exact optimizer recognizes that this is useful to fit y_d well at the end of the control process. In contrast, the suboptimal control strategy does not have this future perspective. It acts too cautiously and supplies only a moderate temperature.

The mathematical explanation came by the proof of the fixed point theorem 4.3, which can be found in the next section. The fixed point predicted by (2.7) is $u^* = \bar{u}(y^*) = 0.1094$. This is what Figure 3.2 shows. The right hand figure shows $y(\cdot, T) \approx u^* y_f = 0.1094 \cdot 1$. We recall that $y_f \equiv 1$ is given by (2.8).

REMARK 3.1. Considering tracking type functionals, i.e.

$$J(y,u) = \frac{1}{2}\int_0^T \int_0^1 (y(x,t) - y_Q(x,t))^2 dxdt + \frac{\gamma}{2}\int_0^T (u(t))^2 dt$$

with *time-independent* goal function, we end up with the same receding horizon method, since the optimization problems to be solved in every time step are of the same form as (P_j). Hence, the receding horizon method will converge to a fixed point that might be far from the desired state.

4. Improved strategies.

4.1. Extended time horizon. As we have seen above, a lack of future perspective can be a decisive drawback of the $(\tau, 1)$-receding horizon method. The time horizon τ is too short. Choosing a larger horizon for the optimization should improve the performance, cf. [5] and related articles for this strategy. Therefore, we next proceed as follows: Given an initial state $y(x) := y_0(x)$ we try to approach the desired state $y_d(x)$ on the extended time interval $[0, l\,\tau]$, where $l \in \mathbb{N}$ is given. The ansatz for the control is

$$u(t) = \sum_{i=1}^{l} u_i e_i(t). \tag{4.1}$$

We solve the optimal control problem

$$\min \quad J(u_1,..,u_l) = \frac{1}{2}\int_0^1 (y(x,l\,\tau) - y_d(x))^2 dx + \frac{\nu}{2}\int_0^{l\,\tau} u(t)^2 dt \tag{P$_l$}$$

subject to

$$\begin{aligned} y_t(x,t) &= y_{xx}(x,t) \\ y(x,0) &= y_0(x) \\ y_x(0,t) &= 0 \\ y_x(1,t) &= \alpha\,(u(t) - y(1,t)) \end{aligned} \tag{4.2}$$

on $[0, l\,\tau]$, subject to the ansatz (4.1) for $u(t)$ and to the constraints

$$u_a \le u_i \le u_b,$$

$i = 1,..,l$. The optimization in (P_l) is performed exactly (of course, for a discretized version of the heat equation). As a result, we obtain control coefficients $\hat{u}_1$,.., $\hat{u}_l$ and the state $y(x, l\,\tau)$ at the end of the extended horizon $[0, l\,\tau]$. Next, we put $\bar u_i := \hat u_i$, $i = 1, .., l$, and define $y_0(x) := y(x, l\,\tau)$ to be the new initial state for the shifted time horizon $[l\,\tau, 2l\,\tau]$. Then we perform the optimization on this interval. By autonomy in time, this is equivalent to solving problem (P_l) on $(0, l\,\tau)$. Let us denote its solution by $\hat u_1$,.., $\hat u_l$ again. The suboptimal controls on $[l\,\tau, 2l\,\tau]$ are now $\bar u_{l+1} := \hat u_1$, ..,$\bar u_{2l} = \hat u_l$. Next we proceed to $[2l\,\tau, 3l\,\tau]$ etc. If $k\,l\,\tau > T$, then only those controls are used that belong to subintervals contained in $[0, T]$. This approach on could denote as $(l\tau, l)$-receding horizon control method.

We will prove that a fixed point exists. For this reason, we proceed along the lines of in Section 2. First we investigate $S_{l\tau}e_i$. By definition (2.3), $S_{l\tau}e_i = y(\cdot, l\tau)$, where y solves

$$\begin{aligned} y_t(x,t) &= y_{xx}(x,t) \\ y(x,0) &= 0 \\ y_x(0,t) &= 0 \\ y_x(1,t) + \alpha\, y(1,t) &= \alpha\, e_i(t). \end{aligned} \tag{4.3}$$

Since $e_i(t) = 0$ for $t < (i-1)\tau$ the initial condition can be replaced by $y(x,(i-1)\tau) = 0$. Furthermore, we get $y(\cdot, i\tau) = S_\tau e_1 = e$ by $e_i(t) = e_1(t-(i-1)\tau)$ and (2.3). Moreover, it holds $e_i(t) = 0$ for $t > i\tau$, hence

$$S_{l\tau}e_i = y(\cdot, l\tau) = D_{(l-i)\tau}y(\cdot, i\tau) = D_{(l-i)\tau}S_\tau e_1 = D_{(l-i)\tau}e = D_\tau^{l-i}e.$$

Putting this together we find

$$S_{l\tau}u = \sum_{i=1}^{l} u_i\, D_\tau^{l-i}e.$$

Let us denote by $\bar u = (\bar u_1, \ldots, \bar u_l) =: \bar u(y)$ the solution of (P_l). Now the method assigns to an initial function y a new function $\bar y$ by

$$\bar y = D_\tau^l y + S_{l\tau}u = D_\tau^l y + \sum_{i=1}^{l} \bar u_i D_\tau^{l-i} e \tag{4.4}$$

Definition 4.1. *The mapping $y \mapsto \bar y$ given by* (4.4) *we will denote by Φ_l, i.e.*

$$\Phi_l(y) = D_\tau^l y + S_{l\tau}u = D_\tau^l y + \sum_{i=1}^{l} \bar u_i D_\tau^{l-i} e.$$

We introduce a functional $f : L^2(0,1) \times \mathbb{R}^l \to \mathbb{R}$ by

$$f(y,u) = \frac{1}{2}\|y_d - D_\tau^l y - \sum_{i=1}^{l} u_i D_\tau^{l-i} e\|^2 + \frac{\nu}{2}\tau \sum_{i=1}^{l} u_i^2 = \frac{1}{2}u^T H u + b^T u + c,$$

where

$$\begin{aligned} H &= (h_{ij}) & h_{ij} &= (D_\tau^{l-i}e,\, D_\tau^{l-j}e) + \delta_{ij}\nu\tau \\ b &= (b_i) & b_i &= (D_\tau^{l-i}e,\ D_\tau^l\, y - y_d) \\ c &= \frac{1}{2}\,\|D_\tau^l\, y - y_d\|^2. \end{aligned}$$

Obviously, the matrix H is symmetric, and $(H - \nu\tau I)$ is positive semidefinite, hence H is positive definite. The necessary condition for $\bar{u}$ to be solution of (P_l) is

$$(H\bar{u} + b)^T(u - \bar{u}) \geq 0 \quad \forall u \in [u_a, u_b]^l. \tag{4.5}$$

LEMMA 4.2. *The mapping Φ_l is a contraction in $L^2(0,1)$.*

Proof. Let $y_1, y_2 \in L^2(0,1)$ be arbitrary functions and $\bar{u}_i = \bar{u}(y_i)$. The variational inequality (4.5) yields

$$\begin{aligned}(H\,\bar{u}_1 + b)^T\,(\bar{u}_2 - \bar{u}_1) \geq 0\\ (H\,\bar{u}_2 + b)^T\,(\bar{u}_1 - \bar{u}_2) \geq 0.\end{aligned}$$

Adding these inequalities

$$-(\bar{u}_1 - \bar{u}_2)^T H(\bar{u}_1 - \bar{u}_2) + \sum_{i=1}^{l} \left(D_\tau^{l-i} e,\, D_\tau^l (y_1 - y_2)\right) \cdot (\bar{u}_{2i} - \bar{u}_{1i}) \geq 0$$

leads to

$$\sum_{i=1}^{l} \left(D_\tau^{l-i} e,\, D_\tau^l (y_1 - y_2)\right) \cdot (\bar{u}_{1i} - \bar{u}_{2i}) \leq -(\bar{u}_1 - \bar{u}_2)^T H(\bar{u}_1 - \bar{u}_2) \leq 0, \tag{4.6}$$

since H is positive definite. Then

$$\begin{aligned}\|\Phi_l\,(y_1) - \Phi_l\,(y_2)\|^2 &= \|D_\tau^l\,(y_1 - y_2) + \textstyle\sum_{i=1}^{l} (\bar{u}_{1i} - \bar{u}_{2i}) D_\tau^{l-i} e\|^2\\ &= \|D_\tau^l\,(y_1 - y_2)\|^2 + 2\textstyle\sum_{i=1}^{l} (\bar{u}_{1i} - \bar{u}_{2i})\left(D_\tau^l\,(y_1 - y_2), D_\tau^{l-i} e\right)\\ &\quad +(\bar{u}_1 - \bar{u}_2)^T(H - \nu\tau I)(\bar{u}_1 - \bar{u}_2)\\ &\leq \|D_\tau^l\,(y_1 - y_2)\|^2 - (\bar{u}_1 - \bar{u}_2)^T(H + \nu\tau I)(\bar{u}_1 - \bar{u}_2)\\ &\leq \|D_\tau^l\,(y_1 - y_2)\|^2 \ \leq (\|D_\tau\|^2)^l\,\|(y_1 - y_2)\|^2\end{aligned}$$

follows immediately from (4.6). This shows that Φ_l is a contraction because D_τ is a contraction as well. ∎

THEOREM 4.3. *The mapping Φ_l has unique fixed point y^* with associated control variable u^*. They satisfy*

$$y^* = \sum_{i=1}^{l} u_i^*\,(y^*)\;(I - D_\tau^l)^{-1}\,D_\tau^{l-i} e, \quad u^* = \bar{u}(y^*) \tag{4.7}$$

and

$$(H' u^* + b')^T(u - u^*) \geq 0 \quad \forall u \in [u_a, u_a]^l, \tag{4.8}$$

where

$$\begin{aligned}H' &= (h'_{ij}) \quad & h'_{ij} &= ((I - D_\tau^l)^{-1} D_\tau^{l-i} e,\, D_\tau^{l-j} e) + \delta_{ij}\nu\tau\\ b' &= (b'_j) \quad & b'_j &= -(D_\tau^{l-i} e,\, y_d).\end{aligned} \tag{4.9}$$

Proof. Existence and uniqueness of y^* follow from Lemma 4.2 and the Banach fixed point theorem. (4.7) is a consequence of (4.4) and the fact that y^* is the fixed point of Φ_l. (4.8) and (4.9) express the necessary condition (4.5) with respect to $\bar{u}$. The formulas (4.9) can be obtained as follows:

$$
\begin{aligned}
(Hu^* + b)_i &= \sum_{j=1}^{l} h_{ij} u_j^* + b_i = \sum_{j=1}^{l} h_{ij} u_j^* + (D_\tau^{l-i} e, D_\tau^l y^* - y_d) \\
&= \sum_{j=1}^{l} \{(D_\tau^{l-i} e, D_\tau^{l-j} e) + \delta_{ij} \nu \tau\} u_j^* + (D_\tau^{l-i} e, D_\tau^l y^*) + b_i'. \qquad (4.10)
\end{aligned}
$$

In view of (4.7) we find

$$
(D_\tau^{l-i} e, D_\tau^l y^*) = \sum_{j=1}^{l} (D_\tau^{l-i} e, D_\tau^l (I - D_\tau^l)^{-1} D_\tau^{l-j} e) u_j^*.
$$

Inserting this expression in (4.10),

$$
\begin{aligned}
(Hu^* + b)_i &= \sum_{j=1}^{l} \{(D_\tau^{l-i} e, [I + D_\tau^l (I - D_\tau^l)^{-1}] D_\tau^{l-j} e) + \delta_{ij} \nu \tau\} u_j^* + b_i' \\
&= \sum_{j=1}^{l} h_{ij}' u_j^* + b_i'
\end{aligned}
$$

is obtained. Here we applied

$$
I + D_\tau^l (I - D_\tau^l)^{-1} = ((I - D_\tau^l) + D_\tau^l)(I - D_\tau^l)^{-1} = (I - D_\tau^l)^{-1}.
$$

□

This method is not essentially better than the $(\tau, 1)$-receding horizon method, which is obtained for $l = 1$. For $l = 4$, the following suboptimal controls and final states were computed (all discretization parameters as above):

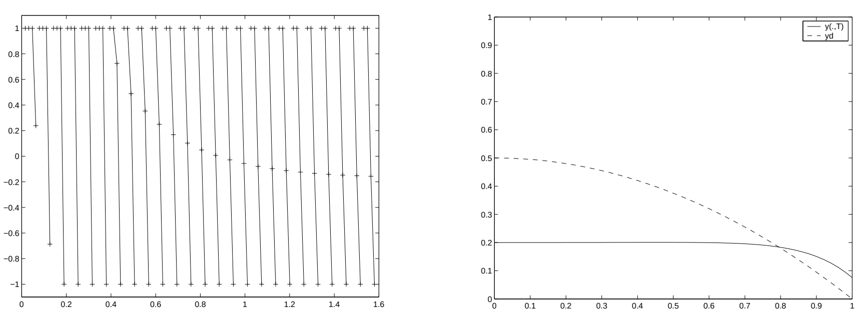

FIG. 4.1. *$(4\tau, 4)$-receding horizon: Suboptimal control $\bar{u}$ and associated final state $\bar{y}(\cdot, T)$*

The suboptimal value of the objective is

$$
J(\bar{u}_{(4\tau,4)}) = 0.01878.
$$

In this case, any computed $(4\tau, 4)$-receding horizon control has 4 components. This explains the left hand side picture of Fig. 4.1. Notice that a new horizon begins repeatedly after 4 time steps. The fixed point vector u^*, calculated by (4.7) and (4.8), is $u^* = (1.0, 1.0, -0.1483, -1.0)$. The associated y^* (dashed) and the final state $\bar{y}(\cdot, T)$ (solid line) are presented in the right hand Fig. 4.1. In a sense, on each short horizon the method reflects the long time behaviour shown in the left hand side Fig 3.1.

In (4.7), the functions $y_{f,i} = (I - D_\tau)^{-1} D_\tau^{l-i} e$ were computed by approximating the Neumann series. The expansion was truncated after $\|D_\tau^k (D_\tau^{l-i} e)\|$ was sufficiently small. The matrix H' is symmetric since D_τ is self-adjoint. However, the symmetry can be destroyed due to numerical errors in the discretized form of (4.9). To make H' symmetric, in the numerical approximation its elements were computed by

$$h'_{ij} = ((I - D_\tau^l)^{-\frac{1}{2}} D_\tau^{l-i} e, \, (I - D_\tau^l)^{-\frac{1}{2}} D_\tau^{l-j} e) + \delta_{ij} \nu \tau.$$

This is justified by $(D_\tau)^{1/2} = D_{\tau/2}$ and $\|D_{\tau/2}\| < 1$. Therefore, the power series for $(I - D_\tau^l)^{-1/2}$ converges.

4.2. $(l\tau, 1)$**-receding horizon technique.** The extension of the time horizon discussed in the last subsection improves the performance of the $(\tau, 1)$-receding horizon method. However, the larger the time horizon is, the more control values are fixed in one step, no matter what happens in the future intervals. In a well-known way, this drawback can be avoided as follows:

Again, the length of the time horizon is taken as $l\,\tau$, i.e. the horizon includes l time steps. Given an initial state $y_0 = y_0(x)$, we try to approach the desired final state y_d on $[0, l\,\tau]$ by controls $u(t)$ having the form (4.1). Again, we solve the optimal control problem (P_l) to obtain control coefficients $\hat{u}_1, \ldots, \hat{u}_l$. By (4.1), they define a control $\hat{u}(t)$ that is optimal for (P_l). In contrast to the preceding subsection, we do not use all of $\hat{u}(t)$ as part of a suboptimal control $\bar{u}(t)$ for (P). We only select the first part of $\hat{u}(t)$ that is defined on $[0, \tau]$. The remaining part, defined on $[\tau, l\,\tau]$, is ignored. Thus we set $\bar{u}_1 := \hat{u}_1$. We compute $y_1(x) = y(x, \tau)$ as the next initial state and repeat the optimization step on the shifted time interval $[\tau, (l+1)\,\tau]$. Thanks to autonomy in time, this is equivalent to solving (P_l) on $[0, l\,\tau]$ subject to $y_0(x) := y_1(x)$. Once again, we adopt only the first part of the associated optimal control, i.e. we put $\bar{u}_2 := \hat{u}_1$, while the other values $\hat{u}_2, \ldots, \hat{u}_l$ are ignored.

In (P), the interval of time is $[0, T]$ and $\tau = T/n$. After $n - l$ steps, the (τ, l)-receding horizon method has reached the subinterval $[(n - l)\,\tau, T]$. Then the whole function $\hat{u}(t)$ is taken to define the last part of the suboptimal $\bar{u}(t)$, i.e. we define $\bar{u}_{n-l+1} := \hat{u}_1, \ldots, \bar{u}_n := \hat{u}_l$.

For $l = 1$, this method recovers to the $(\tau, 1)$-receding horizon method. If $l = 2$, then it already behaves much better. We have the impression that the method converges to a fixed point as well. The gain of performance is partly connected with the optimal solution on the last interval of time $[(n - l)\,\tau, T]$, which is the most important one, due to the smoothing property of the heat equation.

We tested our example for $l = 4$, $n = n_t = 100$, $n_x = 50$. The results are presented in Fig 4.2. The suboptimal value of the objective is

$$J(\bar{u}_{(4\tau,1)}) = 0.00605.$$

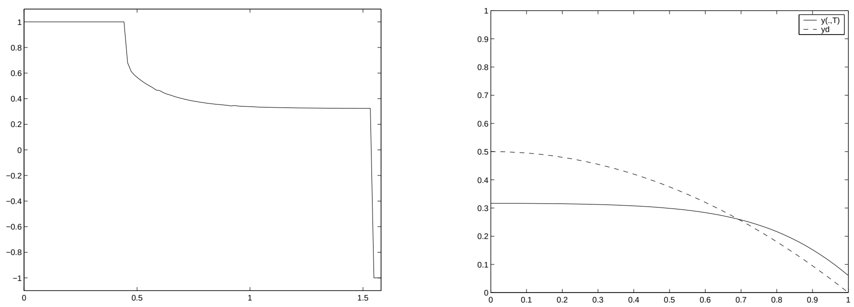

FIG. 4.2. $(4\tau, 1)$-*receding horizon: Suboptimal control* $\bar{u}$ *and associated final state* $\bar{y}(\cdot, T)$

5. Improved receding horizon method. So far, we have understood the result of Theorem 4.3 as a negative one explaining the bad performance of the $(l\tau, l)$-receding horizon method. However, we can exploit it to set up a new and – as we shall see – much better technique.

We have mentioned why the method $(l\tau, l)$-receding horizon control might have difficulties to approach the desired final state y_d. According to Theorem 4.3, it drives the state function y to the fixed point $y_f = c\,(I - D_\tau)^{-1}\,e$, i.e. to a multiple of $(I - D_\tau)^{-1}\,e$. This function y_f is not the optimum in the class of all possible fixed point states. So we have the freedom to choose the best multiple.

Another aspect is the cause for the better results of the $(l\tau, 1)$-receding horizon method. These are, to a large extent, connected with the optimal solution on the last time interval $[(n-l)\,\tau, T]$.

Now we will combine these two observations and introduce a two-components algorithm. The first one of this method is of $(\tau, 1)$-receding horizon type. However, the desired function y_d is changed to $\hat{y}_d = c \cdot y_f$ for a constant c which is at our disposal. Then there is a good chance to achieve this goal by the $(\tau, 1)$-receding horizon method. Proceeding in this way, we find the control u on the time interval $[0, (n-l)\,\tau]$. The remaining control variables will be the result of optimization on $[(n-l)\,\tau, T]$.

How should $\hat{y}_d$ be defined, i.e., how should the unknown constant c be selected? Suppose we are able to steer the initial distribution y_0 into $\hat{y}_d$ exactly after $n-l$ steps of the $(\tau, 1)$-receding horizon method. The last step would be the solution of

$$\min \frac{1}{2}\|y_d - c \cdot D_\tau^l y_f - \sum_{i=1}^{l} u_i D_\tau^{l-i} e\|^2 + \frac{\nu}{2}\tau \sum_{i=1}^{l} u_i^2 \tag{5.1}$$

starting from $\hat{y}_d = c \cdot y_f$ subject to $u_a \le u_i \le u_b$, $i = 1 \ldots l$. The constant c is not fixed yet. Therefore, c can be the subject of the optimization (5.1) with respect to the constraints $u_a \le c \le u_b$. Putting this together we get the following algorithm:

1. Determine y_f according to (2.8).
2. Solve (5.1) subject to $u_a \le u_i \le u_b$, $i = 1 \ldots l$, $u_a \le c \le u_b$, and get $\hat{c}, \hat{u}_1, \ldots, \hat{u}_l$.
3. Compute $u_1, \ldots, u_{n-l}$ and $y(\cdot, (n-l)\,\tau)$ by the $(\tau, 1)$-receding horizon method described in Section 2 with desired state $\hat{y}_d = \hat{c}\,y_f$.

4. Minimize the objective functional

$$\frac{1}{2}\|y_d - D_\tau^l y(\cdot,(n-l)\,\tau) - \sum_{i=1}^{l} u_i D_\tau^{l-i} e\|^2 + \frac{\nu}{2}\tau \sum_{i=1}^{l} u_i^2$$

subject to $u_a \le u_i \le u_b$, $i = 1 \ldots l$, to obtain the solution $\bar{u}_1, \ldots, \bar{u}_l$. Set $u_{n-l+1} = \bar{u}_1, \ldots, u_n = \bar{u}_l$.

This hybrid technique behaves essentially better than the former ones. With the same test parameters as above we got for $l = 4$

$$J(\bar{u}_{\text{imp}}) = 0.00265.$$

The constant $\hat{c}$ was computed as $\hat{c} = 0.41356$. This is shown by Fig. 5.1.

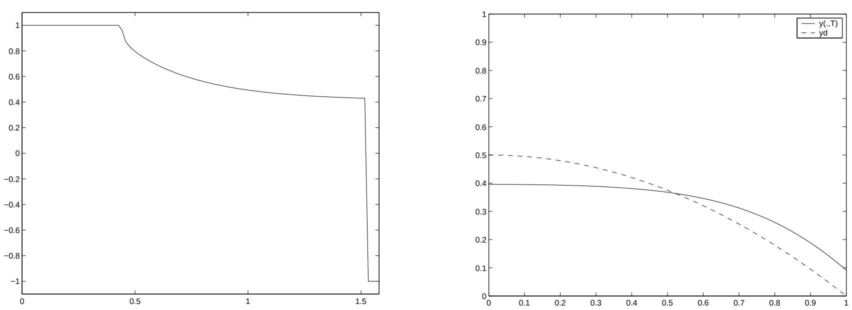

FIG. 5.1. *Improved method: Suboptimal control $\bar{u}$ and associated final state $\bar{y}(\cdot,T)$*

This strategy bears certain resemblance to the Bellman optimality principle. In the Bellman principle, starting from all possible initial states $\hat{y}_d$, the optimal control $u(\hat{y}_d)$ would be computed on $[(n-l)\,\tau, T]$. Next, the optimization is performed on $[0,(n-l)\,\tau]$ while the resulting function $y((n-l)\tau)$ is inserted as starting value $\hat{y}_d$ for $[(n-l)\,\tau, T]$. In the hybrid method, the optimization on $[(n-l)\,\tau, T]$ is restricted to the set $\{\hat{y}_d : \hat{y}_d = c y_f,\, c \in [u_a, u_b]\}$. Moreover, the optimal $\hat{y}_d$ is fixed, and the control on $[0,(n-l)\,\tau]$ is computed by the suboptimal way by the $(\tau, 1)$-receding horizon method to approach y_d.

REMARK 5.1. The result of step 2 is the state $\hat{y}_d = \hat{c}\, y_f$ that should be approximated by the $(\tau, 1)$-receding horizon method.

REMARK 5.2. Step 4 may be superfluous. If the state $\hat{y}_d$ is already well approximated by the $(\tau, 1)$-receding horizon method in step 3, then we can skip the minimization in step 4. In this case, we take the variables $\hat{u}_1, \ldots, \hat{u}_l$ to define the last control components: $u_{n-l+1} = \hat{u}_1, \ldots, u_n = \hat{u}_l$. This case occurs if the final time T is sufficiently large. Then the convergence theorem 4.3 gives the convergence of the series $\{\bar{u}_i\}$ towards $u^* \approx \hat{c}$.

6. Application to a two-dimensional problem. The suboptimal control methods were applied in the former sections to the one-dimensional case. It is obvious that they can be extented to higher dimensions in a straightforward way. Our numerical tests were encouraging. The performance for optimal boundary control of the 2d heat equation was comparable to the 1d case. To show this, let us consider the optimal control problem

$$\min_{u \in U_{ad}} J(u) = \frac{1}{2}\|y(\cdot,T) - y_d\|^2_{L_2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L_2(\Gamma_c \times (0,T))}$$

with respect to

$$\begin{array}{lll} y_t(x,t) & = \Delta y(x,t) & x \in \Omega \\ y(x,0) & = y_0(x) & x \in \Omega \\ \partial_n y(x,t) & = 0 & x \in \Gamma_0 \\ \partial_n y(x,t) & = u(x,t) - y(x,t) & x \in \Gamma_c,\ t \in [0,T], \end{array} \tag{6.1}$$

where ∂_n denotes the derivative with respect to the outward normal at $\Gamma = \partial\Omega$. In the following, the domain Ω is the unit square $(0,1)^2$ with boundary $\partial\Omega = \Gamma_0 \cup \Gamma_c$, $\Sigma = \Gamma_c \times (0,T)$. The control acts on the right edge $\Gamma_c = \{(1,y) : y \in (0,1)\}$ and is required to be an element of $U_{ad} = \{w \in L_\infty(\Sigma) : u_a \leq w(x,t) \leq u_b \text{ a.e. on } \Sigma\}$.

We introduce the $(\tau,1)$-receding horizon method similar to the one-dimensional case. However, the associated analysis is more technical. In particular, the application of semigroup theory to problems with inhomogeneous boundary data requires the use of Neumann boundary operators and weakly singular Bochner integral operators. The presentation of this theory is beyond the scope of this note. We refer to the forthcoming paper [23]. Therefore, we only report on our numerical tests that were encouraging.

Similarly, one proves the contractivity of the associated mapping Φ. The essential pre-requisite is the contractivity of the mapping D_τ in $L^2(\Omega)$. This is due to the fact that the Robin-boundary Γ_c has positive measure.

Consider the test example with given data

$$T = 2,\ y_d(x) = 0.5\, x_1 x_2 + 0.25,\ y_0(x) = 0,\ \nu = 0.001,\ u_a = -1,\ u_b = 1.$$

The partial differential equation was discretized by finite differences on an equidistant grid with $n_{y_1} \times n_{y_2} = 100 \times 100$ grid points. Similarly, the time axis was divided in $n_t = 100$ subintervals. However, the control function was discretized by a slightly coarser grid with $n_{u_x} = n_{u_t} = 50$ grid points. The solution of the box-constrained optimization problems to be solved in all tested methods was performed by the primal-dual active set strategy, cf. [17], while a conjugate gradient algorithm was used to cope with the free subproblems.

Figure 6.1 shows the optimal and the suboptimal control for this problem. The suboptimal control was computed by the $(\tau,1)$-receding horizon method. Hence it converges to a fixed point for repeated time steps. The objective values are of the same order

$$J(u^*) = 0.00207, \qquad J(\bar{u}_{(\tau,1)}) = 0.00301.$$

Moreover, we applied the improved methods of Section 4 and 5. They exhibit the same performance as in the one-dimensional case. For the time horizon of length $l = 4$, we computed the following objective values:

$$J(\bar{u}_{(4\tau,1)}) = 0.00274, \qquad J(\bar{u}_{\text{imp}}) = 0.00271,$$

where 'imp' stands for the improved receding horizon method of section 5.

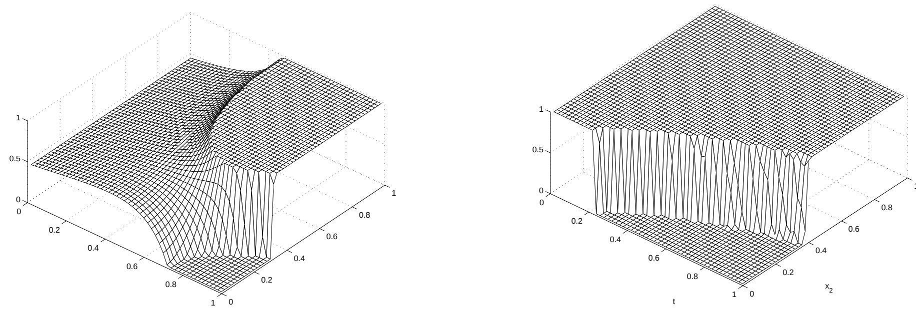

FIG. 6.1. *Optimal* u^* *and* $(\tau, 1)$*-suboptimal control* $\bar{u}$

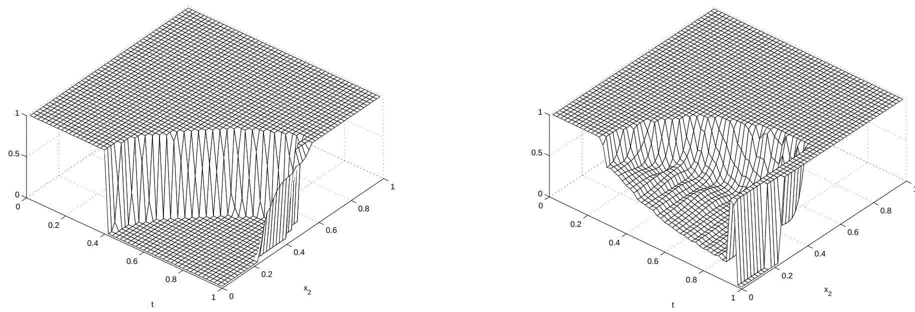

FIG. 6.2. *Suboptimal control:* $(4\tau, 1)$*-receding horizon (left) and improved receding horizon (right)*

As in the one-dimensional case, the hybrid method gives the best results. But in this example the differences between the values of the objective for the optimal control and the suboptimal approaches are smaller than for the one-dimensional example.

For the extension to nonlinear boundary conditions we refer to the forthcoming paper [23].

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