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Second Order Optimality Conditions for a Class of Control Problems Governed by Non-Linear Integral Equations with Application to Parabolic Boundary Control

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Summary: In the paper necessary and sufficient second order optimality conditions for optimal control problems governed by weakly singular non linear HAMMERSTEIN integral equations are derived. They are applied to a semilinear parabolic boundary control problem for the one dimensional heat equation.

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1. Introduction

Within the recent years the investigation of differentiable but non-convex control problems for parabolic partial differential equations has been attracting growing interest. This is reflected by a huge number of papers on this subject, we refer only to AHMED and TEO [1], HOFFMANN and KRABS [4] or TRÖLTZSCH [10] and the references cited therein. Whereas in the convex differentiable case the first order necessary conditions are also sufficient, the presence of non-convex terms requires the consideration of higher order conditions. The main way to derive such conditions, in particular of order two, is known from many publications on mathematical programming problems in BANACH spaces. We mention, for instance, DO VAN LUU [2], GOLLAN [3], IOFFE [5], IOFFE and TIKHOMIROV [6], MAURER [8] and MAURER and ZOWE [9].

The application of the general theory to concrete problems is, however, connected with a variety of difficulties. Within the framework of optimal control of ordinary differential equations this has been adressed by MAURER and ZOWE [9].

In particular, the correct choice of function spaces and suitable norms is a specific feature for these investigations. This is known for ordinary differential equations and turns out to be even harder for partial differential equations. Here already the proof of necessary conditions is connected with difficulties, which increase in the investigation of sufficient conditions.

In this paper, we shall discuss the analysis of second order conditions for control problems governed by weakly singular HAMMERSTEIN integral equations aiming to apply them to semilinear parabolic boundary control problems.

We consider the problem to minimize

$$J(x, u) = J^1 + J^2 = \Phi\,(SN(x, u) - q) + \int_0^T f(x(t), u(t), t)\, dt \tag{1.1}$$

subject to

$$x(t) = c(t) + \int_0^t k(t, s)\, g(x(s), u(s), s)\, ds \quad \text{on} \quad [0, T] \tag{1.2}$$

$$u_1 \leqq u(t) \leqq u_2 \quad \text{a.e. on} \quad [0, T]\,. \tag{1.3}$$

Here we are given: A real HILBERT space H, $q \in H$; $\Phi : H \to \mathbb{R}$, two times continuously FRÉCHET-differentiable; real constants $u_1 < u_2$, $T > 0$; $N : L_\infty(0, T) \times L_\infty(0, T) \to L_\infty(0, T)$; $S : L_2(0, T) \to H$, linear and continuous; and continuous functions $f, g : \mathbb{R} \times \mathbb{R} \times [0, T] \to \mathbb{R}$, $c : [0, T] \to \mathbb{R}$, $k : D = [0, T] \times [0, T] \setminus \{(t, s) \mid s \geqq t\} \to \mathbb{R}$. We assume that

(A1) $\quad |k(t, s)| \leqq c\,(t-s)^{-\alpha}$ on D,

where $c \in \mathbb{R}$ and $\alpha \in (0, 1)$. Thus k is weakly singular at $t = s$.

(A2) $\quad f$ and g have all first and second order derivatives with respect to x and u. f, g, and these derivatives are continuous on $\mathbb{R} \times \mathbb{R}$ with respect to x

and u (t fixed) and measurable with respect to t (x, u fixed) (CARATHEODORY *condition*) as well as bounded on bounded subsets of $\mathbb{R} \times \mathbb{R} \times [0, T]$. N stands for the NEMYTSKI *operator*

$$N : (x(t), u(t)) \mapsto g(x(t), u(t), t)\,.$$

(A2) ensures N to be a continuous mapping from $L_\infty \times L_\infty$ to L_∞, having continuous first and second order FRÉCHET-derivatives.

Throughout the paper, we shall freely use the following notation: $\mathfrak{L}(X, Y)$: space of linear continuous operators A from X to Y; X^*: dual space to X; A^*: adjoint operator to $A \in \mathfrak{L}(X, Y)$; $\langle \cdot, \cdot \rangle$: pairing; $(\cdot, \cdot)$: inner product of H; $\|\cdot\|$: L_∞-norm, $\|\cdot\|_2$: L_2-norm. In an arbitrary BANACH-space we use the notation $B_\varepsilon(z) = \{x \mid \|z - x\| < \varepsilon\}$; $T'(x)$, $T''(x)$: first and second order FRÉCHET-derivative of $T : X \to Y$. If T is a mapping from $X \times U$ into Z, then its derivative is composed of partial derivatives. We denote them by T_x, T_u, T_{xx}, T_{xu}, and T_{uu}, respectively.

In (1.1–3) $u(t)$ is the *control*, we take $u \in U = L_\infty(0, T)$. Each continuous solution $x(t)$ of (1.2) is said to be a *state corresponding to* $u(t)$. The existence of g_x implies the uniqueness of the corresponding state, provided that it exists. We take $X = C[0, T]$. Then K,

$$K : (x(t), u(t)) \mapsto c(t) + \int_0^t k(t, s)\, g(x(s), u(s), s)\, ds$$

is an operator from $X \times U$ into X with continuous first and second order F-derivatives (note that the integral operator is linear and maps $L_p(0, T)$ into $C[0, T]$ for $p > (1-\alpha)^{-1}$.)

After setting $U_{ad} = \{u \in U \mid u_1 \leqq u(t) \leqq u_2\}$ the control problem admits the form

$$J(x, u) = \min! \quad x = K(x, u), \quad u \in U_{ad}\,. \tag{1.4}$$

We shall consider *locally optimal controls* $u_0(t)$, i.e. controls satisfying $J(x, u) \geqq J(x_0, u_0)$ for all (x, u) with $x = K(x, u)$, $(x, u) \in B_\varepsilon(x_0, u_0)$, $u \in U_{ad}$ and certain $\varepsilon > 0$, where x_0 is the state corresponding to u_0.

2. Necessary Optimality Conditions for (1.4)

In this section we prove optimality conditions for problems given in the abstract form (1.4). We should note that they are independent from the special background (1.1–3) and remain valid for any problem of the type (1.4) satisfying the conditions below. We begin with the first order necessary condition for a locally optimal u_0 with corresponding state x_0.

Let

$$F(x, u; y) = J(x, u) + \langle y, x - K(x, u) \rangle$$

denote the LAGRANGE function, where $\langle \cdot, \cdot \rangle$ is the scalar product in $L_2(0, T)$. In the sequel we shall denote by K_x, K_u, J_x, J_u, F_x, F_u the corresponding FRÉCHET derivatives of K, J and F at the optimal pair (x_0, u_0). It follows from the special form of K and F that these derivatives are not only linear and continuous on X and U. We can extend them continuously to $\bar{X} = \bar{U} = L_2(0, T)$.

Using the same notation for these extensions we can therefore assume $K_x \in \mathfrak{L}(\bar{X})$, $K_u \in \mathfrak{L}(\bar{X}, \bar{U})$, $F_x \in \bar{X}^*$, $F_u \in \bar{U}^*$. Moreover, the theory of VOLTERRA integral equations yields the existence of $(I - K_x)^{-1} \in \mathfrak{L}(X)$ as well as $(I - K_x)^{-1} \in \mathfrak{L}(\bar{X})$ (cf. the remarks in Section 3).

All these properties, which are fulfilled in our concrete situation, must be assumed in the next theorems, if we regard (1.4) *without* the background (1.1)–(1.3).

Theorem 1: *Let u_0 be locally optimal with corresponding optimal state x_0. Then there is a* LAGRANGE *multiplier $y \in \bar{X}^*$ such that*

$$F_x = 0, \quad \langle F_u, u - u_0 \rangle \geqq 0 \quad u \in U_{ad}\,. \tag{2.1}$$

Proof: At first we regard (1.4) in the "original" space $X \times U$. The existence of $(I - K_x)^{-1}$ in $\mathfrak{L}(X)$ is a regularity condition, which ensures that the variational inequality

$$J_x\,(x - x_0) + J_u\,(u - u_0) \geqq 0$$

holds for all $(x, u) \in X \times U$ satisfying the linearized equation

$$x - x_0 = K_x\,(x - x_0) + K_u\,(u - u_0), \quad u \in U_{ad}\,.$$

Taking advantage of the concrete integral representation of $J_x \in X^*$, $J_u \in U^*$ we recognize that they can be extended continuously to $L_2(0, T)$, i.e., we can assume $J_x \in \bar{X}^*$, $J_u \in \bar{U}^*$. Now, regarding x and u formally as elements of $\bar{X}$ and $\bar{U}$, the solution set for the linearized equation remains unchanged ($u \in U_{ad} \subset U \Rightarrow x \in X$).

Inserting

$$x - x_0 = (I - K_x)^{-1}\, K_u\,(u - u_0)$$

into the variational inequality and introducing

$$y=-(I-K_x^*)^{-1}J_x$$

we obtain

$$y=-J_x+K_x^*y \tag{2.2}$$

$$\langle J_u-K_u^*y,\ u-u_0\rangle \geqq 0 \tag{2.3}$$

(with $K_x^*\in\mathfrak{L}(\bar{X}^*)$, $K_u^*\in\mathfrak{L}(\bar{X}^*,\bar{U}^*)$). Obviously (2.2–3) are identical with (2.1). ■

Theorem 2: *Suppose that u_0 is a locally optimal control with corresponding state x_0. Then*

$$F''(x_0,u_0;y)\,[w,w]\geqq 0 \tag{2.4}$$

for all $w=(h,v)$ satisfying

$$h=K_xh+K_uv\,, \tag{2.5}$$

$$\langle F_u, v\rangle=0\,, \tag{2.6}$$

$$v=\lambda\,(u-u_0),\quad \lambda\geqq 0,\quad u\in U_{ad}\,.$$

Proof: Let $u\in U_{ad}$, $\lambda>0$ be given arbitrarily but fixed and put $u_\varepsilon=u_0+\varepsilon\cdot\lambda\,(u-u_0)=u_0+\varepsilon\cdot v$, where $0<\varepsilon<\lambda^{-1}$. Then $u_\varepsilon\in U_{ad}$. By known arguments relying on the implicit function theorem (cf. [6] or [9]) we obtain from the existence of $(I-K_x)^{-1}$: In a neighbourhood of x_0 there is exactly one solution x_ε of $x_\varepsilon=K(x_\varepsilon,u_\varepsilon)$ and

$$x_\varepsilon=x_0+\varepsilon h+r(\varepsilon)\,,$$

where $\|r(\varepsilon)\|\cdot\varepsilon^{-1}\to 0$, $\varepsilon\to 0$ and

$$h=K_xh+K_uv\,.$$

Thus

$$\begin{aligned}0&\leqq J(x_\varepsilon,u_\varepsilon)-J(x_0,u_0)=F(x_\varepsilon,u_\varepsilon;y)-F(x_0,u_0;y)\\&=1/2F''(x_0,u_0;y)\,[(\varepsilon h+r(\varepsilon),\varepsilon v),(\varepsilon h+r(\varepsilon),\varepsilon v)]\\&\quad+o(\varepsilon^2)\end{aligned}$$

(note that $F_x=0$, $\langle F_u,v\rangle=0$)

$$\begin{aligned}&=1/2\varepsilon^2F''(x_0,u_0;y)\,[(h+r(\varepsilon)/\varepsilon,v),(h+r(\varepsilon)/\varepsilon,v)]\\&\quad+o(\varepsilon^2)\,.\end{aligned}$$

After dividing by ε^2 and $\varepsilon\to 0$ we arrive at (2.4). ■

In order to prove practicable second order sufficiency conditions we are faced with the so-called "two-norm-discrepancy", which in another sense also occured in Theorem 1: The norm used to achieve differentiability (in our case the L_∞-norm $\|\cdot\|$) is not suitable to verify sufficient second order conditions. In our problem the L_2-norm $\|\cdot\|_2$ would work. Therefore, we assume along the lines of Maurer.

(A3) Denote by $r_1(x,u)$ the first order remainder term of the Taylor expansion for $K(x,u)$ $(r_1\mid X\times U\to X)$ and analogously by r_2 the second order remainder term for $F(x,u;y)$ $(r_2\mid X\times U\to\mathbb{R})$.

Assume

(i) $|r_2(z)|\cdot\|z\|_2^{-2}\to 0$, $\|r_1(z)\|_2/\|z\|_2\to 0$, if $\|z\|\to 0$ and

(ii) $|F''(x_0,u_0;y)\,[z_1,z_2]|\leqq c\,\|z_1\|_2\,\|z_2\|_2$.

In (i), (ii) $z=(x,u)$ and the norm $\|z\|_2=\max(\|x\|_2,\|u\|_2)$ were used.

At next we can prove the second order sufficiency condition of

Theorem 3: *Assume that* (A3) *is fulfilled. If the pair (x_0,u_0) is feasible for* (1.4), *satisfies the first order necessary condition* (2.2–3) *and there exists $\delta>0$ such that*

$$F''(x_0,u_0;y)\,[w,w]\geqq\delta\,\|w\|_2^2 \tag{2.7}$$

for all $w=(h,v)$ with $v=u-u_0$, $u\in U_{ad}$ and

$$h=K_xh+K_uv\,, \tag{2.8}$$

then u_0 is a locally optimal control.

Proof: We shall write for short $F''[z,z]:=F''(x_0,u_0;y)\,[z,z]$. Suppose that (x,u) is feasible for (1.4) and $\max(\|x-x_0\|,\|u-u_0\|)<\varepsilon$, where ε is sufficiently small. We put $z=(x-x_0,u-u_0)$. Now

$$\begin{aligned}J(x,u)-J(x_0,u_0)&=F(x,u;y)-F(x_0,u_0;y)\\&=\langle F_x,x-x_0\rangle+\langle F_u,u-u_0\rangle+1/2F''(x_0,u_0;y)\,[z,z]+r_2(z)\\&\geqq 1/2F''(x_0,u_0;y)\,[z,z]+r_2(z)\end{aligned}$$

by (2.2–3). We have $x-x_0=K_x(x-x_0)+K_u(u-u_0)+r_1(z)$, hence

$$x-x_0=h+\tilde{r}_1(z)\,,$$

where

$$h=K_xh+K_u(u-u_0),\qquad \tilde{r}_1(z)=(I-K_x)^{-1}r_1(z)\,.$$

Therefore

$$z=(h+\tilde{r}_1,v)=w+\hat{r}_1,\qquad \hat{r}_1:=(\tilde{r}_1,0),\quad\text{and}$$

$$\begin{aligned}F''[z,z]&=F''[w,w]+2F''[w,\hat{r}_1]+F''[\hat{r}_1,\hat{r}_1]\\&\geqq\delta\,\|w\|_2^2-2c\,\|w\|_2\,\|\hat{r}_1\|_2-c\,\|\hat{r}_1\|_2^2\end{aligned}$$

(by (A3), (ii)). Further, from $\|w\|_2^2=\|z-\hat{r}_1\|_2^2\geqq(\|z\|_2-\|\hat{r}_1\|_2)^2$

$$F''[z,z]\geqq\|z\|_2^2\left\{\delta-2\delta\frac{\|\hat{r}_1\|_2}{\|z\|_2}-2c\left(1+\frac{\|\hat{r}_1\|_2}{\|z\|_2}\right)\frac{\|\hat{r}_1\|_2}{\|z\|_2}-c\,\frac{\|\hat{r}_1\|_2^2}{\|z\|_2^2}\right\}\geqq\frac{\delta}{2}\,\|z\|_2^2$$

is obtained from (A3), (i) and (2.7), provided that ε is sufficiently small. Continuing our estimation we find

$$J(x,u)-J(x_0,u_0)\geqq\|z\|_2^2\left(\frac{\delta}{4}+\frac{r_2(z)}{\|z\|_2^2}\right)\geqq\frac{\delta}{8}\,\|z\|_2^2$$

for ε sufficiently small. Therefore u_0 is locally optimal. ■

Remark: Suppose that $(I-K_x)^{-1}K_u$ is continuous from U to X in the norm $\|\cdot\|_2$. Then

$$F''(x_0,u_0;y)\,[w,w]\geqq\delta\cdot\|v\|_2^2 \tag{2.9}$$

is sufficient for (2.7). (We have $\|h\|_2 \leqq \|(I-K_x)^{-1} K_u\| \cdot \|v\|_2 \leqq c \cdot \|v\|_2$, hence $\max(\|h\|_2, \|v\|_2) \leqq \max(c\|v\|_2, \|v\|_2) = \max(1, c)\|v\|_2$, thus $\|v\|_2 \geqq \max(1, c)^{-1} \cdot \|w\|_2$ implying (2.7)). In the applications (2.9) is often easier to verify than (2.7).

3. Optimality Conditions for (1.1–3)

In this section we shall apply the results of the preceding section to our main problem (1.1–3). Suppose that $u_0(t)$ is a locally optimal control with corresponding state $x_0(t)$. In accordance with our previous notation we shall write $f_x(t) = f_x(x_0(t), u_0(t), t)$, $g_x(t) = g_x(x_0(t), u_0(t), t)$ and use a similar notation for the other derivatives. $I - K_x$ is given by

$$x(t) \mapsto x(t) - \int_0^t k(t, s)\, g_x(s)\, x(s)\, ds .$$

K_x is an operator of potential type, thus it is a continuous mapping in $L_\infty(0, T)$ as well as in $L_2(0, T)$ (cf. KRASNOSELSKII a.o. [7, § 8]). Moreover, for n sufficiently large, $(K_x)^n$ is a contraction in $X = L_\infty(0, T)$ and $X = L_2(0, T)$. Thus the range of $(I - K_x)$ is $L_\infty(0, T)$ and $L_2(0, T)$, respectively, by a version of the BANACH fixed point theorem. Now the continuity of $(I - K_x)^{-1}$ in X and $\bar{X}$ follows from the BANACH theorem on the inverse operator. This justifies the remarks before Theorem 1, and we obtain the existence of a LAGRANGE multiplier $y(t)$ from $\bar{X}^* = L_2(0, T)$. The LAGRANGE function is

$$F(x, u; y) = J(x, u) + \int_0^T y(t) \left[x(t) - \int_0^t k(t, s)\, g(x(s), u(s), s)\, ds \right] dt$$

$$= J + \int_0^T \left(y(t)\, x(t) - g(x(t), u(t), t) \int_t^T k(s, t)\, y(s)\, ds \right) dt . \tag{3.1}$$

By means of the chain rule,

$$\langle J_u^1, h \rangle = \langle \Phi'(p_0), SN_u(x_0, u_0)\, h \rangle = \langle N_u^* S^* \Phi'(p_0), h \rangle ,$$

$$\langle J_x^1, h \rangle = \langle N_x^* S^* \Phi'(p_0), h \rangle ,$$

where $p_0 = SN(x_0, u_0) - q$. Hence

$$J_u(t) = f_u(t) + g_u(t)\, (S^* \Phi'(p_0))\, (t) ,$$

$$J_x(t) = f_x(t) + g_x(t)\, (S^* \Phi'(p_0))\, (t) ,$$

the adjoint equation (2.2) is

$$y(t) = -f_x(t) + g_x(t) \left\{ -(S^* \Phi'(p_0))\, (t) + \int_t^T k(s, t)\, y(s)\, ds \right\} , \tag{3.2}$$

and (2.3) reads

$$\int_0^T \left\{ f_u(t) + g_u(t) \left[(S^* \Phi'(p_0))\, (t) - \int_t^T k(s, t)\, y(s)\, ds \right] \right\} \cdot (u(t) - u_0(t))\, dt \geqq 0 \tag{3.3}$$

for all $u_1 \leqq u(t) \leqq u_2$. (3.2–3) are the *first order necessary conditions* for our problem. In order to establish the second order conditions we need several second order derivatives. We illustrate the computations for $J^1 = \Phi\,(SN(u, v) - q)$, the most "difficult" part of J.

It is known that for $\Psi : X \to \mathbb{R}$ the derivative $\Psi''(x_0)\,[h_1, h_2]$ can be determined as follows: Define $\varphi(x) = \langle \Psi'(x), h_1 \rangle$. Then $\Psi''(x_0)$ can be obtained just by differentiating $\varphi(x)$.

$$\Psi''(x_0)\,[h_1, h_2] = \langle \varphi'(x_0), h_2 \rangle .$$

Therefore, in order to obtain J_{uu}^1, we introduce

$$\varphi(u) = \langle J_u^1(x_0, u), h_1 \rangle = \langle \Phi'(SN(x_0, u) - q), SN_u(x_0, u)\, h_1 \rangle .$$

Now

$$\langle \varphi'(u_0), h_2 \rangle = \langle \Phi''(p_0)\, SN_u h_2, SN_u h_1 \rangle + \langle \Phi'(p_0), SN_{uu}[h_1, h_2] \rangle = J_{uu}^1[h_1, h_2] ,$$

(derivatives of N taken at (x_0, u_0)). Note that $\Phi' : H \to H$, $\Phi'' : H \to \mathfrak{L}(H)$. Hence

$$J_{uu}^1(x_0, u_0)\,[h_1, h_2] = \langle N_u^* S^* \Phi''(p_0)\, SN_u h_2, h_1 \rangle + \langle S^* \Phi'(p_0), N_{uu}[h_1, h_2] \rangle$$

$$J_{xx}^1(x_0, u_0)\,[h_1, h_2] = \langle N_x^* S^* \Phi''(p_0)\, SN_x h_2, h_1 \rangle + \langle S^* \Phi'(p_0), N_{xx}[h_1, h_2] \rangle$$

$$J_{xu}^1(x_0, u_0)\,[h_1, h_2] = \langle N_x^* S^* \Phi''(p_0)\, SN_u h_2, h_1 \rangle + \langle S^* \Phi'(p_0), N_{xu}[h_1\ h_2] \rangle = J_{ux}^1(h_1, h_2)$$

(Φ'' is self adjoint and $N_{xu} = N_{ux}$). The derivative N_{uu} is

$$(N_{uu}[h_1, h_2])\, (t) = g_{uu}(t)\, h_1(t)\, h_2(t) ,$$

and analogous expressions hold for N_{xu}, N_{xx}.

Similarly, the other derivatives can be obtained. In this way, the second order condition (2.4) amounts to

$$Q(x, v) = Q^1(x, v) + Q^2(x, v) \geqq 0 \tag{3.4}$$

for all $x(t)$, $v(t)$ satisfying $v(t) = \lambda\,(u(t) - u_0(t))$, $\lambda \geqq 0$, $u_1 \leqq u(t) \leqq u_2$,

$$x(t) = \int_0^t k(t, s)\, (g_x(s)\, x(s) + g_u(s)\, v(s))\, ds \tag{3.5}$$

$$\int_0^T F_u(t)\, (u(t) - u_0(t))\, dt = 0 , \tag{3.6}$$

where $F_u(t)$ is the expression $\{\ldots\}$ in (3.3),

$$Q^1 = J_{xx}^1(x, x) + 2 J_{xu}^1(x, v) + J_{uu}^1(v, v)$$

$$Q^2 = \int_0^T \{[f_{xx}(t) - g_{xx}(t)\, Y(t)]\, x^2(t) + 2\,[f_{xu}(t) - g_{xu}(t)\, Y(t)]\, x(t)\, v(t) + [f_{uu}(t) - g_{uu}(t)\, Y(t)]\, v^2(t)\}\, dt ,$$

and

$$Y(t) = \int_t^T k(s, t)\, y(s)\, ds .$$

(3.4–6) combined with the first order conditions are the second order necessary conditions for $u_0(t)$. It should be mentioned that the orthogonality relation (3.6) is equivalent to

$$u(t)=\begin{cases}u_0(t), & F_u(t)\neq 0\\ \text{arbitrary}, & F_u(t)=0\end{cases} \tag{3.7}$$

as $F_u(t)\,(u(t)-u_0(t))\geqq 0$ according to (3.3). Thus the second order condition gives additional information for those t, where $F_u(t)=0$. These are exactly those points, where the first order condition (3.3) contains no information, since (3.3) yields $u_0(t)=u_1$, if $F_u(t)>0$ and $u_0(t)=u_2$, if $F_u(t)<0$.

Some additional care is needed to verify the assumptions of second order conditions. According to the coercivity condition (2.9) we require

$$Q(x, v)\geqq\delta\int_0^T v^2(t)\,dt \tag{3.8}$$

for all $(x(t), v(t))$ satisfying the linearized equation (3.5). This is a natural condition, and it remains to check Assumption (A3).

Theorem 4: *Suppose that the* LAGRANGE *multiplier y belongs to $L_\infty(0, T)$. If the feasible pair $(x_0(t), u_0(t))$ satisfies the first order necessary condition* (3.2–3) *and the sufficiency condition* (3.8) *holds for all $(x(t), v(t))$ solving* (3.5), *then $u_0(t)$ is a locally optimal control.*

Proof: This is a direct consequence of Theorem 3 and the discussion above provided that (A3) holds. Aiming to shorten our presentation we confine ourselves to the proof of (i)

$$\|r_1(z)\|_2/\|z\|_2\to 0,\quad \text{if}\quad \|z\|\to 0$$

and of (ii). At first we show (i): The remainder term r_1 is

$$\begin{aligned}(r_1(z))\,(t)=(r_1(h, v))\,(t)=&\int_0^t k(t, s)\,[(g_x(x_0(s)+\vartheta(s)\,h(s), u_0(s)+\vartheta(s)\,v(s), s)\\ &-g_x(x_0(s), u_0(s), s))\,h(s)\\ &+(g_u(x_0(s)+\vartheta(s)\,h(s)), u_0(s)+\vartheta(s)\,v(s), s)\\ &-g_u(x_0(s), u_0(s), s)\,v(s)]\,ds\\ =&\int_0^t k(t, s)\,[(g_x^\vartheta(s)-g_x(s))\,h(s)+(g_u^\vartheta(s)-g_u(s))\,v(s)]\,ds\end{aligned}$$

where $0\leqq\vartheta(t)\leqq 1$ can be assumed to be measurable (cf. KRASNOSELSKII a. o. [7, § 20]). From the continuity properties of g_x, g_u we get

$$\max(\|h\|, \|v\|)\to 0\Rightarrow\max(\|g_x^\vartheta-g_x\|, \|g_u^\vartheta-g_u\|)\to 0\,.$$

Hence, by the continuity of K_x in $L_2(0, T)$

$$\begin{aligned}\|r_1(w)\|_2&\leqq c\,(\|g_x^\vartheta-g_x\|\,\|h\|_2+\|g_u^\vartheta-g_u\|\,\|v\|_2)\\ &\leqq c\max(\|g_x^\vartheta-g_x\|, \|g_u^\vartheta-g_u\|)\max(\|h\|_2, \|v\|_2)\,.\end{aligned}$$

Now (3.9) and $\|w\|_2=\max(\|h\|_2, \|v\|_2)$ yields the desired result. The validity of (i) is due to the fact that the mapping

$$(h_1(t), h_2(t))\mapsto\int_0^T y(t)\int_0^t k(t, s)\,\alpha(s)\,h_1(s)\,h_2(s)\,ds\,dt$$

is continuous in $L_2(0, T)\times L_2(0, T)$, provided that $y(t)$, $\alpha(t)$ belong to $L_\infty(0, T)$. ∎

Remark: It follows from (3.2) that $y(t)$ belongs to $L_\infty(0, T)$, if the same holds for $(S^*\Phi'(p_0))\,(t)$.

4. Application to Parabolic Boundary Control Problems

In order to illustrate the application of the results of Section 3 to the control of parabolic differential equations we regard a boundary-control problem for the one-dimensional heat equation:

Minimize

$$1/2\int_0^1[z(T, x)-q(x)]^2\,dx+\frac{\nu}{2}\int_0^T u^2(t)\,dt \tag{4.1}$$

subject to

$$\begin{aligned}&z_t(t, \xi)=z_{\xi\xi}(t, \xi) &&\text{on}\quad (0, T]\times(0, 1)\\ &z(0, \xi)=0 &&\text{on}\quad [0, 1]\\ &z_\xi(t, 0)=0 &&\text{on}\quad (0, T]\\ &z_\xi(t, 1)=g(z(t, 1), u(t), t) &&\text{on}\quad (0, T],\quad |u(t)|\leqq 1\,,\end{aligned} \tag{4.2}$$

where $\nu\geqq 0$, $T>0$ and $q\in L_\infty(0, 1)$ are given, and g is defined as in Section 1. The control u is regarded in the space $L_\infty(0, T)$, but we regard u formally as element of $L_2(0, T)$.

Aiming to relate this problem to (1.1–3) we introduce the GREEN's *function*

$$G(\xi, \eta, t)=1+2\sum_{n=1}^\infty\cos(n\pi\xi)\cos(n\pi\eta)\exp(-n^2\pi^2 t)\,.$$

Then any classical solution $z(t, \xi)$ of (4.2) must satisfy the equation

$$z(t, \xi)=\int_0^t G(\xi, 1, t-s)\,g(z(s, 1), u(s), s)\,ds\,. \tag{4.3}$$

We regard (4.3) as the equation defining $z(t, \xi)$, and each continuous solution z of (4.3) is said to be a generalized solution of (4.2). Now put $x(t)=z(t, 1)$, $k(t, s)=$ $=G(1, 1, t-s)$, $H=L_2(0, 1)$, and $S: L_2(0, T)\to H$ by

$$(Su)\,(\xi)=\int_0^T G(\xi, 1, T-s)\,u(s)\,ds\,.$$

Then $z(T, \xi)=\int_0^T G(\xi, 1, T-s)\,g(x(s), u(s), s)\,ds=(SN(x, u))\,(\xi)$, and (4.1–2) admits the form

$$1/2\,\|SN(x,u)-q\|_H^2+\frac{\nu}{2}\int_0^T u^2(t)\,dt=\min!$$

$$x(t)=\int_0^t k(t,s)\,g(x(s),u(s),s)\,ds,\qquad -1\leqq u(t)\leqq 1\,, \tag{4.4}$$

being a particular case of (1.1–3), where $\Phi(x)=1/2\,\|x\|_H^2$, $f(x,u,t)=1/2\nu u^2$. It is known $|G(1,1,t)|\leqq c\cdot t^{-1/2}$, hence (A1) holds with $\alpha=1/2$.

Simple calculations yield

$$(S^*z)\,(t)=\int_0^1 G\,(\xi,1,T-t)\,z(\xi)\,d\xi\,,$$

$$\Phi'(x)=x,\qquad \Phi''(x)=I\,.$$

Let now $u_0(t)$ with corresponding state $x_0(t)$ be locally optimal for (4.4), define $z_0(t,\xi)$ by (4.3) and put $p_0(\xi)=z_0(T,\xi)-q(\xi)$.

As z_0 is continuous and q is bounded and measurable, we have $p_0\in L_\infty(0,1)$. Therefore $(S^*p_0)\,(t)$ is bounded and measurable, too.

Then by (3.2) the adjoint state $y(t)$ is obtained as

$$y(t)=g_x(t)\left\{-\int_0^1 G\,(\xi,1,T-t)\,p_0(\xi)\,d\xi+\int_t^T G\,(1,1,s-t)\,y(s)\,ds\right\},$$

and (3.3) amounts to

$$\int_0^T [\nu u_0(t)+g_u(t)\cdot\{\ldots\}]\,(u(t)-u_0(t))\,dt\geqq 0$$

for all $|u(t)|\leqq 1$.

$S^*p_0\in L_\infty(0,T)$ implies that $y\in L_\infty(0,T)$, hence Theorem 4 can later be applied. This is the first order condition for $u_0(t)$.

Remark: This adjoint state $y(t)$ is not suitable for numerical computations. It is customary to put $\tilde y(t)=\{\ldots\}$, hence

$$\tilde y(t)=-\int_0^1 G\,(\xi,1,T-t)\,p_0(\xi)\,d\xi+\int_t^T G\,(1,1,s-t)\,g_x(s)\,\tilde y(s)\,ds\,, \tag{4.5}$$

and the *first order necessary condition* is changed to

$$\int_0^T [\nu u_0(t)+g_u(t)\,\tilde y(t)]\,(u(t)-u_0(t))\,dt\geqq 0\;\forall\,|u(t)|\leqq 1\,. \tag{4.6}$$

y is seen to be the trace at $\xi=1$ of the generalized solution of the adjoint parabolic initial-boundary value problem $-z_t=z_{\xi\xi}$, $z(T,\xi)=-p_0(\xi)$, $z_\xi(t,0)=0$, $z_\xi(t,1)=$ $=g_x(t)\,\tilde y(t)$.

As regards the second order condition, we must determine $Q(x,v)$. $\Phi''=I$ yields

$$Q(x,v)=\|S\,(N_xx+N_uv)\|_H^2+\int_0^T\left(\int_0^1 G(\xi,1,T-t)\,p_0(\xi)\,d\xi\right)$$
$$\cdot\,(g_{xx}(t)\,x^2(t)+2g_{xu}(t)\,x(t)\,v(t)+g_{uu}(t)\,v^2(t))\,dt$$

$$-\int_0^T Y(t)\,(g_{xx}(t)\,x^2(t)+2g_{xu}(t)\,x(t)\,v(t)$$
$$+g_{uu}(t)\,v^2(t))\,dt+\nu\int_0^T v^2(t)\,dt\,,$$

where $Y(t)=\int_t^T G(1,1,s-t)\,y(s)\,ds$. Hence

$$Q(x,v)=\|S\,(N_xx+N_uv)\|_H^2-\int_0^T \tilde y(t)\,g''(x_0(t),u_0(t),t)$$
$$[(x(t),v(t)),(x(t),v(t))]\,dt+\nu\int_0^T v^2(t)\,dt\,.$$

The *second order necessary condition* is

$$Q(x,v)\geqq 0$$

for all $(x(t),v(t))$ such that $x(t)=z(t,1)$, where z is the generalized solution to

$$\begin{aligned} &z_t=z_{\xi\xi}\\ &z(0,\xi)=0\\ &z_\xi(t,0)=0\\ &z_\xi(t,1)=g_x(t)\,z(t,1)+g_u(t)\,v(t)\,, \end{aligned} \tag{4.7}$$

where

$$v(t)=\begin{cases}0 & \text{if}\quad \nu u_0(t)+g_u(t)\,\tilde y(t)\neq 0\\ \lambda(u(t)-u_0(t)),\quad \lambda\geqq 0,\quad |u(t)|\leqq 1 & \text{if}\quad \nu u_0(t)+g_u(t)\,\tilde y(t)=0\,.\end{cases}$$

It should be remarked that in this sense $\|S\,(N_xx+N_uv)\|_H^2=\int_0^1 z(T,x)^2\,dx$.

The *second order sufficient condition* is: The first order necessary condition is fulfilled and

$$Q(x,v)\geqq\delta\cdot\int_0^T v^2(t)\,dt$$

for all $x(t)=z(t,1)$, $v(t)$ satisfying (4.7)

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