An SQP Method for Optimal Control of Weakly Singular Hammerstein Integral Equations

Walter Alt*, Ralph Sontag**, Fredi Tröltzsch**

Abstract. We investigate local convergence of an SQP method for nonlinear optimal control of weakly singular Hammerstein integral equations. Sufficient conditions for local quadratic convergence of the method based are discussed.

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Keywords. Lagrange-Newton method, sequential quadratic programming, optimal control, weakly singular Hammerstein integral equations, control constraints.

1. Introduction

The Lagrange–Newton method is obtained by applying Newton's method or a generalized version of it to find a stationary point of the Lagrangian function associated to a nonlinear optimization problem. If a constraint qualification and a strong second order sufficiency condition are satisfied, the Lagrange-Newton method defines a sequential quadratic programming (SQP) algorithm. This is known for finite–dimensional spaces since several years (see e.g. FLETCHER [4] and STOER [16]).

The Lagrange-Newton method can be easily extended to infinite-dimensional optimization problems such as optimal control problems. We mention, for instance, the numerical work by MACHIELSEN [11] for systems of ordinary differential equations including state constraints. In the context of parameter identification problems we refer to KELLEY and WRIGHT [7], who consider problems with equality constraints, where the SQP method reduces to the ordinary Newton method. KUPFER and SACHS [9] discuss a reduced SQP method for a parabolic control problem with equality constraints. LEVITIN and POLYAK [10] investigated the behaviour of SQP methods for optimization problems in Hilbert spaces with convex implicit constraints.

*Mathematisches Institut, Universität Bayreuth, Postfach 10 12 51, D–8580 Bayreuth, Germany.

**Fachbereich Mathematik, Technische Universität Chemnitz, Postfach 964, D-O–9010 Chemnitz.

In Alt [1, 2] the theory of local convergence of the Lagrange-Newton method has been extended to infinite-dimensional optimization problems with more general constraints including nonlinear equality and inequality constraints. Due to specific difficulties connected with the so-called two-norm discrepancy, an extension to Banach spaces endowed with different norms was necessary. This problem was investigated recently by ALT and MALANOWSKI [3]. Their results were focused on the application to optimal control problems governed by nonlinear ODE's.

In this paper, we establish a convergence theorem for a class of optimal control problems governed by a (nonlinear) Hammerstein integral equation with weak singularity. This type of singularity is characteristic for the handling of parabolic boundary control problems by integral equations methods (cf., for instance SACHS [14], TRÖLTZSCH [17] or V. WOLFERSDORF [19]). Therefore, problems governed by weakly singular integral equations can serve a model case for PDE. The investigation of the SQP method for parabolic boundary control problems in a spatial domain of dimension one can be performed in a similar way. Details and numerical results will be published in a separate paper.

The convergence theory for the problems considered here requires a four-norm technique making use of the L_1-, L_2-, L_p- and L_∞-norm ($2 < p < \infty$). The proof of convergence can be performed extending and adapting the theory of [2] or [3]. Due to the special structure of the control problems we found it more convenient to proceed within the framework of [2].

2. The Optimal Control Problem

We shall consider the following somewhat simplified optimal control problem for a weakly singular Hammerstein integral equation:

(P) Minimize $f(x,u) = \int_0^T \varphi(t, x(t), u(t))\, dt$

subject to $x \in C[0,T]$, $u \in L_\infty(0,T)$,

$$x(t) = \int_0^t k(t,s) b(s, x(s), u(s))\, ds, \qquad (2.1)$$

$|u(t)| \le 1$.

In this setting, φ, b, and k are given realvalued functions satisfying the following assumptions:

(A1) The functions $\varphi = \varphi(t,x,u)$, $b = b(t,x,u)$ have the form

$$\varphi = \varphi_1(t,x) + \varphi_2(t,x)u + \lambda u^2\,, \quad b = b_1(t,x) + b_2(t,x)u\,,$$

where $\lambda > 0$, and $\phi_1, \phi_2, b_1, b_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$ are twice continuously differentiable with respect to x.

(A2) The functions φ_1, φ_2 and their first and second derivatives with respect to x are locally Lipschitz on $[0,T] \times \mathbb{R}$. The functions b_1, b_2 and their first and second derivatives with respect to x are uniformly bounded and Lipschitz on $[0,T] \times \mathbb{R}$ with respect to x.

For some results of our paper we shall even suppose φ_2, b_2 to be affine–linear with respect to x. Concerning the kernel k we assume

(A3) $k(t,s)$ is continuous on

$$D = \{(t,s), 0 \leq s < t \leq T\}$$

and satisfies

$$|k(t,s)| \leq c|t-s|^{-\alpha} \quad \forall (t,s) \in D \tag{2.2}$$

with a certain $\alpha \in (0,1)$.

Thus k is a weakly singular kernel being typical for the treatment of parabolic initial-boundary value problems by Green's functions.

Problem (P), although simplified, contains the main difficulties arising from the investigation of nonlinear parabolic boundary control problems. The discussion of nonlinear problems of the type (P) is quite involved, if any kind of differentiation is necessary. A two–norm technique is indispensible, cf. TRÖLTZSCH [18] for first order necessary optimality conditions and MAURER [12], GOLDBERG and TRÖLTZSCH [5, 6] for second order conditions. All what is "local" in a certain sense must be defined in the L_∞–topology. Statements invoking sufficient second order conditions must be formulated in the weaker norm of L_2. Moreover, we shall need also the spaces $L_p(0,T)$ and $L_1(0,T)$. Altogether these facts give rise to introduce the following quantities:

We shall write $L_\beta := L_\beta(0,T)$, $1 \leq \beta \leq \infty$, and $C := C[0,T]$. These spaces are equipped with their natural norm $\|\cdot\|_\beta$. By $\|\cdot\|_\infty$ we shall denote also the norm of $C[0,T]$. Moreover, we shall work with $L_{\infty,p} := C \times L_p$ endowed with the norms $\|(x,u)\|_{\infty,p} := \|x\|_\infty + \|u\|_p$ and $\|(x,u)\|_\beta := \|x\|_\beta + \|u\|_\beta$. In the case $p = \infty$ we have $\|(x,u)\|_{\infty,p} = \|(x,u)\|_\infty$. $\mathcal{L}(X,Y)$ stands for the Banach space of all continuous linear operators between Banach spaces X and Y, $\mathcal{L}(X) := \mathcal{L}(X,X)$. In the case $X = L_\beta$, $Y = L_\gamma$ the norm of $\mathcal{L}(X,Y)$ is $\|\cdot\|_{\beta\to\gamma}$. The elements of $L_{\infty,p}$ will be written as $v = (x,u)$. This notation will not cause confusion with the inner product of L_2, which is denoted by $(\cdot,\cdot)$, too.

Next, we introduce a linear operator K formally by

$$(Kx)(t) = \int_0^t k(t,s)x(s)\,ds.$$

It is a known conclusion of (2.2), that K transforms continuously $L_\beta(0,T)$ into $L_{\beta'}(0,T)$, if

$$\frac{1}{\beta'} > \frac{1}{\beta} + \alpha - 1 \tag{2.3}$$

and $1 \leq \beta \leq 1/(1-\alpha)$, cf. KRASNOSELSKIĬ a. o. [8]. Moreover, we have $K : L_p(0,T) \to C[0,T]$ (continuously) provided that

$$p > \frac{1}{1-\alpha}. \tag{2.4}$$

Now we keep once and for all one p fixed satisfying (2.4). Moreover, we define $\bar{p}$ by

$$\bar{p} = \begin{cases} p, & \text{if } \varphi_2 \text{ and } b_2 \text{ are affine–linear w. r. to } x \\ \infty, & \text{otherwise.} \end{cases} \tag{2.5}$$

The role of $\bar{p}$ will be explained later.

Although K may be regarded in different spaces we shall in any case use the same notation K. The adjoint operator $K^\star$ to $K \in \mathcal{L}(L_\beta(0,T))$, $1 \leq \beta < \infty$, has the form

$$(K^\star x)(t) = \int_t^T k(s,t)x(s)\,ds. \tag{2.6}$$

We take (2.6) as formal definition of $K^\star$. This integral operator is acting continuously between the same spaces as K.

Finally, we need the nonlinear operator $B : L_{\infty,p} \to L_p$,

$$B(x,u)(t) = b(t, x(t), u(t))$$

and the set $U^{\text{ad}} = \{u \in L_\infty : |u(t)| \leq 1 \text{ a. e. on } [0,T]\}$.

By means of these quantities our Problem (P) admits the abstract form

$$\text{(P)} \qquad \begin{aligned} & f(x,u) = \min, \\ & x - KB(x,u) = 0\,, \quad u \in U^{\text{ad}}, \end{aligned}$$

where (x,u) is taken from $L_{\infty,p}$.

The use of first and second order derivatives of $f : L_{\infty,p} \to \mathbb{R}$ and $B : L_{\infty,p} \to L_p$ requires some care, although f and B are twice continuously differentiable in the sense of Fréchet. This is a conclusion of Assumptions (A1), (A2). Derivatives will be denoted by f', f'', B', B'' etc., partial derivatives by f_x, f_u, f_{xx}, f_{xu} etc.

Owing to the very special form of f and B their derivatives can be identified with certain real functions: At $\bar{v} = (\bar{x}, \bar{u}) \in C \times U^{\text{ad}}$ in the direction $h = (x,u)$,

$$f'(\bar{v})h = \int_0^T (\varphi_x(t,\bar{v}(t))x(t) + \varphi_u(t,\bar{v}(t))u(t))\ dt.$$

By $\bar{x} \in C$ and $|\bar{u}| \leq 1$ we have φ_x, $\varphi_u \in L_\infty$. Therefore we can identify $f_x = \varphi_x$, $f_u = \varphi_u$ and write the derivative of f using the inner product of L_2,

$$f'(\bar{v})h = (f'(\bar{v}), h) = (f_x(\bar{v}), x) + (f_u(\bar{v}), u), \tag{2.7}$$

where f_x, $f_u \in L_\infty$. Analogously,

$$(B'(\bar{v})h)(t) = b_x\,(t,\bar{v}(t))\,x(t) + b_u\,(t,\bar{v}(t))\,u(t), \tag{2.8}$$

where b_x, b_u belong to L_∞. Owing to this, $B'(\bar{v})$ can also be regarded as continuous operator from $L_2 \times L_2$ to L_2 (more precisely: B' can be continuously extended in this way). We shall do so, thus

$$B'(\bar{v})h = B_x(\bar{v})x + B_u(\bar{v})u,$$

where $B_x(\bar{v}) \in \mathcal{L}(L_2)$, $B_u(\bar{v}) \in \mathcal{L}(L_2)$. Note that in this sense $B_x(\bar{v})$, $B_u(\bar{v})$ are self–adjoint. We shall see — and this is very essential — that all $\bar{v}$ occuring in our analysis belong to a bounded set S of $V = C \times L_\infty$. Therefore φ_x, φ_u, b_x, b_u are uniformly bounded and Lipschitz on S, and

$$\max(||f_x(v)||_\infty, ||f_u(v)||_\infty) \le c_f \quad \forall v \in S \tag{2.9}$$
$$|(f'(v_1) - f'(v_2), h)| \le c_f ||v_1 - v_2||_2 ||h||_2 \tag{2.10}$$
$$||B'(v)h||_2 \le c_B ||h||_2 \tag{2.11}$$
$$||(B'(v_1) - B'(v_2))h||_2 \le c_B ||v_1 - v_2||_{\infty,\bar{p}} ||h||_2 \tag{2.12}$$

$\forall v, v_i \in S$, $h \in L_\infty$ with certain positive constants c_f, c_B. In (2.11), (2.12) we need not restrict to S, as B and its derivatives are supposed to be globally Lipschitz.

For the second derivatives we have

$$\begin{aligned} f''(\bar{v})[h_1,h_2] = \int_0^T \{&\varphi_{xx}(t,\bar{v}(t))x_1(t)x_2(t) \\ &+\varphi_{xu}(t,\bar{v}(t))x_1(t)u_2(t) + x_2(t)u_1(t)) + 2\lambda u_1(t)u_2(t)\}dt \end{aligned} \tag{2.13}$$
$$\begin{aligned} B''(\bar{v})[h_1,h_2](t) = &b_{xx}(t,\bar{v}(t))x_1(t)x_2(t) \\ &+b_{xu}(t,\bar{v}(t))(x_1(t)u_2(t) + x_2(t)u_1(t)) \end{aligned} \tag{2.14}$$

where φ_{xx}, φ_{xu}, b_{xx}, b_{xu} are L_∞–functions. Moreover, for $1/\beta + 1/\beta' = 1$, $1 \le \beta, \beta' \le \infty$,

$$f''(v)[h_1,h_2] \le c_f ||h_1||_\beta ||h_2||_{\beta'} \quad \forall v \in S,\ \forall h_i \in L_\infty \tag{2.15}$$
$$||B''(v)[h_1,h_2]||_1 \le c_B ||h_1||_\beta ||h_2||_{\beta'} \tag{2.16}$$

(we can use w. l. o. g the same constants as in (2.9–2.12)), and

$$|(f''(v_1) - f''(v_2))[h_1,h_2]| \le \begin{cases} c_f ||v_1 - v_2||_{\infty,\beta} ||h_1||_{\infty,\beta} ||h_2||_{\beta'} \\ c_f ||v_1 - v_2||_{\infty,\bar{p}} ||h_1||_\beta ||h_2||_{\beta'} \end{cases} \tag{2.17}$$
$$||(B''(v_1) - B''(v_2))[h_1,h_2]||_1 \le \begin{cases} c_B ||v_1 - v_2||_{\infty,\beta} ||h_1||_{\infty,\beta} ||h_2||_{\beta'} \\ c_B ||v_1 - v_2||_{\infty,\bar{p}} ||h_1||_\beta ||h_2||_{\beta'} \end{cases} \tag{2.18}$$

$\forall v_i \in S,\ h_i \in L_\infty$. We shall discuss the non-trivial estimates among (2.9)–(2.18) in Lemma 6.1. As a consequence of Assumption (A1) on the form of φ and b the Problem (P) admits at least one *optimal solution* $v_0 = (x_0, u_0)$. This can be shown by standard methods.

Introducing the *Lagrange function* $\mathcal{L} = \mathcal{L}(v, y^\star) = \mathcal{L}(x, u, y^\star)$,

$$\mathcal{L}(x, u, y^\star) = f(x, u) - (y^\star, x - KB(x, u)),$$

$y^\star \in L_2$, the following *first order necessary optimality condition* can be established:

Lemma 2.1 *Let* $v_0 = (x_0, u_0)$ *be optimal for* (P). *Then a unique Lagrange multiplier* $y_0^\star \in L_\infty$ *exists such that*

$$\mathcal{L}_x(x_0, u_0, y_0^\star) = 0 \tag{2.19}$$

$$(\mathcal{L}_u(x_0, u_0, y_0^\star), u - u_0) \geq 0 \quad \forall u \in U^{\mathrm{ad}}. \tag{2.20}$$

The proof ist standard (with exception of $y_0^\star \in L_\infty$) and relies on the existence of $(I - KB_x(v_0))^{-1}$ (cf. Appendix) in all L_β–spaces, see Goldberg and Tröltzsch [5]. The multiplier $y_0^\star$ is the solution of the *adjoint equation*

$$y_0^\star = f_x(v_0) + B_x(v_0)K^\star y_0^\star \tag{2.21}$$

(being nothing else than (2.19)). Written more explicitely,

$$y_0^\star(t) = \varphi_x(t, v_0(t)) + b_x(t, v_0(t)) \int_t^T k(s,t) y_0^\star(s)\, ds. \tag{2.22}$$

Now $y_0^\star \in L_\infty$ is obvious.

The most important assumption for our theory is the *assumption* of the following *second order sufficient optimality condition*:

(SSC) There is a $\delta > 0$ such that at $v_0 = (x_0, u_0)$

$$\mathcal{L}''(v_0, y_0^\star)[h, h] \geq \delta \|h\|_2^2$$

for all $h = (x, u) \in L_{\infty,p}$ such that $x = K\,(B_x(v_0)x + B_u(v_0)u)$.

For a discussion of (SSC) we refer to Goldberg and Tröltzsch [5], [6].

3. Stability of quadratic control problems

The SQP method can be described roughly as follows: Let $v = (x, u)$ be a certain starting element with an associated Lagrange multiplier $y^\star$. Adopting the notation introduced in [2] we denote the triplet $(x, u, y^\star)$ by w and indicate the correspondence to w with a subscript, $w = (x, u, y^\star) = (x_w, u_w, y_w^\star) = (v_w, y_w^\star)$. The optimal solution corresponds to $w_0 = (x_0, u_0, y_0^\star)$.

Starting from $w = (x_w, u_w, y_w^\star)$ the next element is obtained as solution of the quadratic programming problem

$$
(\mathrm{QP})_w \qquad \begin{array}{l} F(v,w) = (f'(v_w), v - v_w) + \frac{1}{2}\mathcal{L}_{vv}(v_w, y_w^\star)[v - v_w, v - v_w] = \min! \\ \text{subject to} \\ g(v_w) + g'(v_w)(v - v_w) = 0, \quad v \in \mathcal{C}, \end{array}
$$

where

$$
\begin{aligned} g(v) = g(x,u) &= x - KB(x,u), \\ \mathcal{C} &= \left\{ v = (x,u) \,|\, u \in U^{\mathrm{ad}} \right\}. \end{aligned}
$$

By Assumption (SSC) the functional F is strictly convex. In view of the Assumptions (A1)–(A3), Lemma 6.2 shows that the convexity retains under small perturbations, i.e., if w belongs to a sufficiently small $L_{\infty,\bar{p}} \times L_p$–neighbourhood of w_0 The role of $\bar{p}$ is connected with this. $\bar{p}$ is the smallest value among p, ∞ guaranteeing this property. The feasible set of $(\mathrm{QP})_w$,

$$
\Sigma(w) = \{ v \in \mathcal{C} \,|\, g(v_w) + g'(v_w)(v - v_w) = 0 \},
$$

is always non-empty, convex, closed and bounded in $C \times L_p$. The same holds true for $\Sigma(w)$ regarded as subset of $L_2 \times L_2$. Hence $\Sigma(w)$ is weakly compact in $L_2 \times L_2$ and $(\mathrm{QP})_w$ admits a unique solution $(\bar{x}_w, \bar{u}_w) = \bar{v}_w$ with associated Lagrange multiplier $\bar{y}_w^\star$. This follows from Assumption (SSC) and Lemma 2.1. Note that $(\bar{x}_w, \bar{u}_w) \in C \times L_\infty$ follows automatically from the special form of the constraints of $(\mathrm{QP})_w$.

The next iteration is started at $w := (\bar{x}_w, \bar{u}_w, \bar{y}_w^\star)$.

Following Alt [2] we introduce

$$
G(v,w) = g(v_w) + g'(v_w)(v - v_w).
$$

The Lagrange function for $(\mathrm{QP})_w$ is

$$
\tilde{\mathcal{L}}(v, w, y^\star) = F(v,w) - (y^\star, G(v,w)).
$$

Thus the multiplier $\bar{y}_w^\star$, being the solution to $\tilde{\mathcal{L}}_x(\bar{v}_w, w, \bar{y}_w^\star) = 0$, is obtained from

$$
-\, g_x(v_w)^\star \bar{y}_w^\star + f_x(v_w) + \mathcal{L}_{xx}(v_w, y_w^\star)[h_x, \cdot] + \mathcal{L}_{xu}(v_w, y_w^\star)[h_u, \cdot] = 0 \tag{3.1}
$$

where $h_x = \bar{x}_w - x_w$, $h_u = \bar{u}_w - u_w$. More explicitely,

$$
\begin{aligned} -\bar{y}_w^\star + B_x(v_w)^\star K^\star \bar{y}_w^\star + f_x(v_w) + f_{xx}(v_w)[h_x, \cdot] + f_{xu}(v_w)[h_u, \cdot] + \\ + \left((B_{xx}(v_w)[h_x, \cdot])^\star + (B_{xu}(v_w)[h_u, \cdot])^\star \right) K^\star y_w^\star = 0. \end{aligned} \tag{3.2}
$$

Note that the adjoint operators are defined formally in the L_2–sense. (3.2) reads in explicite form

$$\bar{y}_w^\star(t) - b_x(t, v_w(t)) \int_t^T k(s,t)\bar{y}_w^\star(s)\, ds = \varphi_x(t, v_w(t)) +$$
$$+\varphi_{xx}(t, v_w(t))(\bar{x}_w(t) - x_w(t)) + \varphi_{xu}(t, v_w(t))(\bar{u}_w(t) - u_w(t)) +$$
$$+(b_{xx}(t, v_w(t))(\bar{x}_w(t) - x_w(t)) + b_{xu}(t, v_w(t))(\bar{u}_w(t) - u_w(t)))$$
$$\int_t^T k(s,t)y_w^\star(s)\, ds. \tag{3.3}$$

Lemma 3.1 *Under the Assumption* (SSC)

$$F(v, w_0) \geq \delta ||v - v_0||_2^2 = F(v_0, w_0) + \delta ||v - v_0||_2^2 \tag{3.4}$$

for all $v \in \Sigma(w_0)$*. Thus* v_0 *is the (unique) global solution of* $(\mathrm{QP})_{w_0}$*. Moreover,* $y_0^\star$ *is the Lagrange multiplier to* v_0 *regarded as solution of* $(\mathrm{QP})_{w_0}$*.*

The proof is standard (see e.g. Lemma 3.4 in ALT [2]).

We shall discuss the question of stability of the problems $(\mathrm{QP})_w$ along the lines of ALT [2]. Thereby we shall concentrate on the main points different to the presentation in [2]. In what follows, in $W = C \times L_\infty \times L_\infty$ the norms $||w||_\beta = ||x_w||_\beta + ||u_w||_\beta + ||y_w^\star||_\beta$, $1 \leq \beta \leq \infty$, $||w||_{\infty,p} = ||x_w||_\infty + ||u_w||_p + ||y_w^\star||_p$ and $||w||_W = ||x_w||_\infty + ||u_w||_{\bar{p}} + ||y_w^\star||_p$ will be used.

Lemma 3.2 *There is a* $C \times L_{\bar{p}} \times L_p$*–neigbourhood* $N_1(w_0)$ *such that for all* $w = (x_w, u_w, y_w^\star) \in N_1(w_0)$ *with* $u_w \in U^{\mathrm{ad}}$ *the Problem* $(\mathrm{QP})_w$ *has a unique solution* $\bar{v}_w = (\bar{x}_w, \bar{u}_w)$ *with (unique) Lagrange multiplier* $\bar{y}_w^\star \in L_\infty$ *and*

$$||\bar{v}_w - v_0||_2 \leq c_S ||w - w_0||_2^{1/2} \tag{3.5}$$
$$||\bar{y}_w^\star||_\infty \leq c_L, \tag{3.6}$$

where c_S *and* c_L *are independent of* w*.*

Proof. Existence and uniqueness of $\bar{v}_w \in \Sigma(w)$ have already been discussed after defining $(\mathrm{QP})_w$. It remains to show (3.5–3.6). In particular, we have for sufficiently small $N(w_0)$ in $C \times L_{\bar{p}} \times L_p$

$$\mathcal{L}_{vv}(v_w, y_w^\star)[h,h] \geq \frac{\delta}{2} ||h||_2^2 \tag{3.7}$$

for all h with $g'(v_w)h = 0$ (Lemma 6.2). By Lemma 6.4, (6.7), there is $\xi = (x_\xi, u_\xi) \in \Sigma(w_0)$ such that

$$||\bar{v}_w - \xi||_2 \leq c||v_w - v_0||_2, \tag{3.8}$$

where c is independent of w. In what follows, c will denote a generic constant. From Lemma 3.1, (3.4)

$$\delta\|\xi - v_0\|_2^2 \le F(\xi, w_0) \le F(\bar{v}_w, w) + |F(\xi, w_0) - F(\bar{v}_w, w)|. \tag{3.9}$$

Again by Lemma 6.4, (6.6), we find $\xi_w \in \Sigma(w)$ such that

$$\|\xi_w - v_0\|_2 \le c\|v_w - v_0\|_2. \tag{3.10}$$

This implies (note, that $\bar{v}_w$ solves $(\mathrm{QP})_w$)

$$F(\bar{v}_w, w) \le F(\xi_w, w) = (f'(v_w), \xi_w - v_w) + \frac{1}{2}\mathcal{L}_{vv}(v_w, y_w^\star)[\xi_w - v_w, \xi_w - v_w]. \tag{3.11}$$

On $N(w_0)$ the norms $\|x_w\|_\infty$, $\|u_w\|_\infty$, $\|y_w^\star\|_\infty$ are uniformly bounded. The same refers to $\|\xi_w\|_\infty$, as $\xi_w \in \Sigma(w)$ (apply Lemma 6.3). Thus $\|f'(v_w)\|_2$, $\|\mathcal{L}_{vv}(v_w, y_w^\star)[\xi_w - v_w, \cdot]\|_2$ are bounded independently of w by (2.10), (2.15-2.16). We can proceed identically to [2]

$$\begin{aligned} F(\bar{v}_w, w) \le c\|\xi_w - v_w\|_2 &\le c\|\xi_w - v_0\|_2 + c\|v_0 - v_w\|_2 \\ &\le c\|v_w - v_0\|_2 \end{aligned} \tag{3.12}$$

by (3.10). From Lemma 6.5 we obtain the Lipschitz continuity of F with respect to the L_2–norm. After inserting (3.12) into (3.9),

$$\delta\|\xi - v_0\|_2^2 \le c\|v_w - v_0\|_2 + c\left(\|\xi - \bar{v}_w\|_2 + \|w - w_0\|_2\right) \le c\|w - w_0\|_2$$

by (3.8). Since

$$\|\bar{v}_w - v_0\|_2 \le \|\bar{v}_w - \xi\|_2 + \|\xi - v_0\|_2$$

the inequalities (3.10), (3.8) and $\|v_w - v_0\|_2 \le \|w - w_0\|_2$ imply (3.5).

Concerning (3.6), we regard the adjoint equation (3.1) in the form (3.3). All terms on the right hand side of (3.3) are uniformly bounded in L_∞. ($|\bar{u}_w - u_w| \le 2$, $(v_w, y_w^\star)$ is bounded as element of $N(w_0)$, $\bar{x}_w$ is bounded by Lemma 6.3 as a solution of $(\bar{x}_w - x_w) - KB_x(v_w)(\bar{x}_w - x_w) = KB_u(v_w)(\bar{u}_w - u_w)$. $K^\star y_w^\star$ is bounded in L_∞, since $K^\star : L_p \to L_\infty$ is continuous.)

According to our assumptions, the derivatives of φ and b are uniformly bounded. Hence

$$\|\bar{y}_w^\star - (B_x(v_w))^\star K^\star \bar{y}_w^\star\|_\infty \le c.$$

Again Lemma 6.3 yields $\|\bar{y}_w^\star\|_\infty \le c$. □

Remark: In all what follows let $N_1(w_0)$ be so small, such that $\|w - w_0\| \le 1$ holds for all other norms $\|\cdot\|$ used in this paper. ◇

Corollary 3.3 *There is a c_S', independent of $w \in N_1(w_0)$, such that*

$$\|\bar{v}_w - v_0\|_{\infty,p} \le c_S'\|w - w_0\|_{\infty,p}^{1/p}. \tag{3.13}$$

Proof. We obtain from (3.5)

$$\|\bar{u}_w - u_0\|_p = \left(\int_0^T |\bar{u}_w(t) - u_0(t)|^2 |\bar{u}_w(t) - u_o(t)|^{p-2}\, dt \right)^{1/p}$$
$$\leq \left(2^{p-2}\right)^{1/p} \|\bar{u}_w - u_0\|_2^{2/p} \leq c\|w - w_0\|_2^{1/p} \leq c\|w - w_0\|_{\infty,p}^{1/p}$$

as $|\bar{u}_w - u_0| \leq 2$ by $u \in U^{\mathrm{ad}}$. From $\bar{x}_w \in \Sigma(w)$,

$$\begin{aligned}
&(\bar{x}_w - x_w) - KB_x(v_w)(\bar{x}_w - x_w) \\
&\quad = KB_u(v_w)(\bar{u}_w - u_w) - (x_w - x_0) + K(B(v_w) - B(v_0)) \\
&\quad = KB_u(v_w)((\bar{u}_w - u_0) + (u_0 - u_w)) - (x_w - x_0) + K(B(v_w) - B(v_0)),
\end{aligned}$$

hence

$$\begin{aligned}
&\|(\bar{x}_w - x_w) - KB_x(v_w)(\bar{x}_w - x_w)\|_\infty \\
&\quad \leq c\|K\|_{p\to\infty}\{\|\bar{u}_w - u_0\|_p + \|u_0 - u_w\|_p\} + \|x_w - x_0\|_\infty + c\|v_w - v_0\|_{\infty,p} \\
&\quad \leq c\|w - w_0\|_{\infty,p}^{1/p} + c\|u_0 - u_w\|_p + c\|w - w_0\|_{\infty,p} \\
&\quad \leq c\left(\|w - w_0\|_{\infty,p}^{1/p} + \|w - w_0\|_{\infty,p}\right) \\
&\quad \leq c\|w - w_0\|_{\infty,p}^{1/p}
\end{aligned}$$

as $\|w - w_0\|_{\infty,p} \leq 1$. Again Lemma 6.3 yields

$$\|\bar{x}_w - x_w\|_\infty \leq c\|w - w_0\|_{\infty,p}^{1/p},$$

thus (3.13) is true. □

Corollary 3.4 *For a certain c, independent of $w \in N_1(w_0)$,*

$$\|\bar{y}_w^\star - y_0^\star\|_p \leq c\|w - w_0\|_{\infty,p}^{1/p} \tag{3.14}$$

Proof. The adjoint equations defining y_0 and $\bar{y}_w$ are (2.22) and (3.3), respectively. Thus

$$\begin{aligned}
y_0^\star(t) &= [\varphi_1'(x_0) + \varphi_2'(x_0)u_0 + (b_1'(x_0) + b_2'(x_0)u_0)K^\star y_0^\star]\,(t) \\
\bar{y}_w^\star(t) &= [\varphi_1'(x_w) + \varphi_2'(x_w)u_w + (b_1'(x_w) + b_2'(x_w)u_w)(K^\star \bar{y}_w^\star)](t) \\
&\quad + [(\varphi_1''(x_w) + \varphi_2''(x_w)u_w)(\bar{x}_w - x_w) \\
&\quad + (K^\star y_w)(b_1''(x_w) + b_2''(x_w)u_w)(\bar{x}_w - x_w)](t) \\
&\quad + [(\varphi_2'(x_w)(\bar{u}_w - u_w) + (K^\star y_w^\star)b_2'(x_w)(\bar{u}_w - u_w)](t).
\end{aligned}$$

Subtracting the two equations we arrive at

$$\|(\bar{y}_w^\star - y_0^\star) - ((b_1'(x_0) + b_2'(x_0)u_0)K^\star(\bar{y}_w^\star - y_0^\star)\|_p$$

$$
\begin{aligned}
&\le \|\varphi_1'(x_0) - \varphi_1'(x_w)\|_p + \|\varphi_2'(x_0)u_0 - \varphi_2'(x_w)u_w\|_p \\
&\quad + \|(b_1'(x_w) + b_2'(x_w)u_w) - (b_1'(x_0) + b_2'(x_0)u_0)\|_p \|K^\star \bar{y}_w^\star\|_\infty \\
&\quad + \|\varphi_1''(x_w) + \varphi_2''(x_w)u_w\|_\infty (\|\bar{x}_w - x_0\|_p + \|x_0 - x_w\|_p) \\
&\quad + \|K^\star y_w^\star\|_\infty \|b_1''(x_w) + b_2''(x_w)u_w\|_\infty (\|\bar{x}_w - x_0\|_p + \|x_0 - x_w\|_p) \\
&\quad + \|\varphi_2'(x_w)\|_\infty (\|\bar{u}_w - u_0\|_p + \|u_0 - u_w\|_p) \\
&\quad + \|K^\star y_w^\star\|_\infty \|b_2'(x_w)\|_\infty (\|\bar{u}_w - u_0\|_p + \|u_0 - u_w\|_p) .
\end{aligned}
$$

The quantities x_w, u_w, $y_w^\star$ belong to a $C \times L_{\bar{p}} \times L_p$–neighbourhood of $(x_0, u_0, y_0^\star)$, $\|\bar{y}_w^\star\|_\infty$ can be estimated by (3.6). Taking advantage of the Lipschitz continuity of φ_i, b_i and of the continuitiy of $K^\star : L_p \to C$,

$$
\begin{aligned}
\|(\bar{y}_w^\star - y_0^\star) - b_x(v_0)K^\star(\bar{y}_w^\star - y_0^\star)\|_p &\le c\,(\|\bar{v}_w - v_0\|_p + \|v_0 - v_w\|_p) \\
&\le c\,(\|\bar{v}_w - v_0\|_{\infty,p} + \|w - w_0\|_p) \\
&\le c\left(\|w - w_0\|_{\infty,p}^{1/p} + \|w - w_0\|_{\infty,p}\right) \\
&\le c\|w - w_0\|_{\infty,p}^{1/p} \qquad (3.15)
\end{aligned}
$$

as $\|w - w_0\|_{\infty,p} \le 1$. The proof of the corollary is finished by Lemma 6.3. □

Corollary 3.5 *There is a constant $c > 0$ such that*

$$
\|\bar{u}_w - u_0\|_\infty \le c\|w - w_0\|_{\infty,p}^{1/p} \qquad (3.16)
$$

for all $w \in N_1(w_0)$.

Proof. This result follows from the first order necessary optimality conditions for u_0 and $\bar{u}_w$. Writing down (2.20) for u_0 and

$$
(\tilde{\mathcal{L}}_u(\bar{v}_w, w, \bar{y}_w^\star), u - \bar{u}_w) \ge 0 \qquad \forall u \in U^{\mathrm{ad}}
$$

for $\bar{u}_w$ we find after some calculations

$$
\int_0^T \{\varphi_2(x_0) + b_2(x_0)K^\star y_0^\star + 2\lambda u_0\}(u - u_0)\,dt \ge 0 \quad \forall u \in U^{\mathrm{ad}} \qquad (3.17)
$$

$$
\begin{aligned}
&\int_0^T \{\varphi_2(x_w) + \varphi_2'(x_w)(\bar{x}_w - x_w) + 2\lambda\bar{u}_w + b_2'(x_w)(\bar{x}_w - x_w)K^\star y_w^\star + \\
&\quad + b_2(x_w)K^\star \bar{y}_w^\star\}(u - \bar{u}_w)\,dt \ge 0 \quad \forall u \in U^{\mathrm{ad}}. \qquad (3.18)
\end{aligned}
$$

In a standard way a discussion of (3.17), (3.18) yields

$$
\begin{aligned}
u_0(t) &= P_{[-1,1]}\{-\frac{1}{2\lambda}[\varphi_2(x_0) + b_2(x_0)K^\star y_0^\star](t)\} \\
\bar{u}_w(t) &= P_{[-1,1]}\{-\frac{1}{2\lambda}[\varphi_2(x_w) + b_2(x_w)K^\star \bar{y}_w^\star + \varphi_2'(x_w)(\bar{x}_w - x_w) \\
&\qquad + b_2'(x_w)(\bar{x}_w - x_w)K^\star y_w^\star](t)\}
\end{aligned}
$$

for almost all $t \in [0,T]$, where $P_{[-1,1]} : \mathbb{R} \to [-1,1]$ is the projection operator onto $[-1,1]$. $P_{[-1,1]}$ is Lipschitz with constant 1, hence

$$\begin{aligned}
\|u_0 - \bar{u}_w\|_\infty \le & \frac{1}{2\lambda}\{\|\varphi_2(x_0) - \varphi_2(x_w)\|_\infty + \|b_2(x_0)K^\star(y_0^\star - \bar{y}_w^\star)\|_\infty \\
& +\|(b_2(x_0) - b_2(x_w))K^\star \bar{y}_w^\star\|_\infty + \|\varphi_2'(x_w)(\bar{x}_w - x_w)\|_\infty \\
& +\|b_2'(x_w)(\bar{x}_w - x_w)K^\star y_w^\star\|_\infty\} \\
\le & c_1\|x_0 - x_w\|_\infty + c_2\|y_0^\star - \bar{y}_w^\star\|_p \\
& +c_3\|x_0 - x_w\|_\infty\|\bar{y}_w^\star\|_p + c_4(\|\bar{x}_w - x_0\|_\infty + \|x_0 - x_w\|_\infty) \\
& +c_5(\|\bar{x}_w - x_0\|_\infty + \|x_0 - x_w\|_\infty)\|y_w^\star\|_p \\
\le & c(\|w_0 - w\|_{\infty,p} + \|w_0 - w\|_{\infty,p}^{1/p}) \qquad (3.19)
\end{aligned}$$

by Lipschitz continuitiy of φ_i, b_i, (3.14), (3.13), (3.6) and the continuity of $K^\star : L_p \to C$. (3.16) follows from (3.19), as $\|w - w_0\|_{\infty,p} \le 1$. □

Remarks:

1. Repeating the proof of Corollary 3.4 with the knowledge of Corollary 3.5 we can even show

   $$\|\bar{y}_w^\star - y_0^\star\|_\infty \le c\|w - w_0\|_{\infty,p}^{1/p} \qquad (3.20)$$

   $\forall w \in N_1(w_0)$. This is not necessary for our further investigations. ◇

2. The estimates (3.13), (3.14), (3.16) and (3.20) remain true if $\|w - w_0\|_W^{1/p}$ is substituted for $\|w - w_0\|_{\infty,p}^{1/p}$, since

   $$\begin{aligned}
   \|w - w_0\|_{\infty,p} &= \|x_w - x_0\|_\infty + \|u_w - u_0\|_p + \|y_w^\star - y_0^\star\|_p \\
   &\le \|x_w - x_0\|_\infty + c\|u_w - u_0\|_{\bar{p}} + \|y_w^\star - y_0^\star\|_p \\
   &= c\|w - w_0\|_W .
   \end{aligned}$$

   ◇

4. Right hand side perturbations

Following Alt [2] we consider now the close relationship between the stability of $(QP)_w$ and certain right hand side perturbations. Let $\pi_0 = (0,0,0) \in L_\infty \times L_\infty \times C$ be the reference parameter and $\pi = (v^\star, y) = (v_x^\star, v_u^\star, y) \in L_\infty \times L_\infty \times C$ a perturbation. We consider the perturbed quadratic programming problem

$(QS)_\pi$ $\quad \tilde{F}(v,\pi) = (f'(v_0), v - v_0) + \frac{1}{2}\mathcal{L}_{vv}(v_0, y_0^\star)[v - v_0, v - v_0] - (v^\star, v - v_0) =$ min! $\qquad (4.1)$

subject to

$$\tilde{G}(v,\pi) = g(v_0) + g'(v_0)(v - v_0) - y = 0, \; v \in \mathcal{C}. \qquad (4.2)$$

Under our assumptions, $v_0 = (x_0, u_0)$ is a global solution of $(\mathrm{QS})_{\pi_0}$ with Lagrange multiplier $y_0^\star$.

$(\mathrm{QS})_\pi$ has a quadratic objective and linear constraints. As a simple consequence, the second derivative of the corresponding Lagrange function and the first derivative of the linear constraint operator do not depend on π. In view of this, the quadratic objective is coercive in the L_2–sense, uniformly with respect to π. Moreover, the feasible set is non–empty for all π. We may check this taking the admissible control u_0. Then (4.2) reads

$$x(t) - x_0(t) = \int_0^t k(s,t) b_x(s)(x(s) - x_0(s))\, ds + y(t).$$

This Volterra equation possesses a unique solution x belonging to the same L_p–space as y. Therefore, it can be shown along the lines of the preceding section that there is a ${L_\infty}^3$-neighbourhood $N(0)$ and a constant $c > 0$ such that problem $(QS)_\pi$ admits for all $\pi \in N(0)$ a unique solution $v_\pi = (x_\pi, u_\pi)$, and

$$||v_\pi - v_o||_2 \leq c||\pi||_2^{1/2}. \tag{4.3}$$

We shall improve this result without making use of (4.3) in Theorem 4.2.

Lemma 4.1 *Let $y_\pi^\star$ be the Lagrange multiplier corresponding to v_π and $1 \leq \beta \leq \infty$. Then*

$$||y_\pi^\star - y_0^\star||_\beta \leq c(||v_\pi - v_0||_\beta + ||\pi||_\beta), \tag{4.4}$$

where c is independent of π.

Proof: We have the two adjoint equations

$$\begin{aligned} y_0^\star &= f_x(v_0) + B_x(v_0) K^\star y_0^\star \\ y_\pi^\star &= f_x(v_0) - v_x^\star + (\mathcal{L}_{xx}(v_0, y_0^\star)[x_\pi - x_0, \cdot] + \mathcal{L}_{xu}(v_0, y_0^\star)[u_\pi - u_0, \cdot]) + \\ &\quad + B_x(v_0) K^\star y_\pi^\star. \end{aligned}$$

Identifying the functionals with corresponding measurable functions we arrive after subtraction at

$$\begin{aligned} (y_0^\star - y_\pi^\star - B_x(v_0)K^\star(y_0^\star - y_\pi^\star))(t) \quad &= \quad v_x^\star(t) - (\varphi_{xx}(t)(x_\pi - x_0)(t) \\ +\varphi_{xu}(t)(u_\pi - u_0)(t) - (K^\star y_0^\star)(t)&(b_{xx}(t)(x_\pi - x_0)(t) + b_{xu}(t)(u_\pi - u_0)(t)), \end{aligned}$$

where φ_{xx}, φ_{xu}, b_{xx}, b_{xu} are taken at (x_0, u_0). Thus

$$\begin{aligned} ||y_0^\star - y_\pi^\star - B_x(v_0)K^\star(y_0^\star - y_\pi^\star)||_\beta &\leq ||v_x||_\beta + c(||x_\pi - x_0||_\beta + ||u_\pi - u_0||_\beta) \\ &\leq ||\pi||_\beta + c||v_\pi - v_0||_\beta. \end{aligned}$$

Again Lemma 6.3 yields the assertion. □

Now we are able to improve the estimate (4.3) to the order 1.

Theorem 4.2 *There is a constant c not depending on π, such that*

$$||v_\pi - v_0||_2 \le c||\pi||_2, \tag{4.5}$$

for all $\pi \in (L_2)^3$.

Proof: Let $\pi = (v_\pi^\star, y_\pi)$ be given. By definition, $g'(v_0)(v_\pi - v_0) = y_\pi$, i. e.

$$(x_\pi - x_0) - KB_x(v_0)(x_\pi - x_0) = KB_u(v_0)(u_\pi - u_0) + y_\pi.$$

Let $\xi = v_\pi - v_0 = (x_\pi - x_0, u_\pi - u_0)$. We define $\hat{\xi} = (x_{\hat{\xi}} - x_0, u_\pi - u_0)$ by

$$(x_{\hat{\xi}} - x_0) - KB_x(v_0)(x_{\hat{\xi}} - x_0) = KB_u(v_0)(u_\pi - u_0),$$

i. e. $(x_{\hat{\xi}} - x_0) = (I - KB_x(v_0))^{-1} KB_u(v_0)(u_\pi - u_0)$. By Lemma 6.3, $||x_\pi - x_{\hat{\xi}}||_2 = ||(I - KB_x(v_0))^{-1} y_\pi||_2 \le c||y_\pi||_2$ thus

$$||\xi - \hat{\xi}||_2 = ||(x_\pi - x_{\hat{\xi}}, 0)||_2 \le c||y_\pi||_2 \le c||\pi||_2, \tag{4.6}$$

and

$$\begin{aligned} ||\xi||_2^2 &\le ||\hat{\xi}||_2^2 + ||\xi - \hat{\xi}||_2^2 + 2||\hat{\xi}||_2||\xi - \hat{\xi}||_2 \\ &\le ||\hat{\xi}||_2^2 + c||\pi||_2^2 + 2c||\pi||_2(||\xi||_2 + c||\pi||_2). \end{aligned} \tag{4.7}$$

Let $Q(\xi, \xi) := \mathcal{L}_{vv}(v_0, y_0^\star)[\xi, \xi]$. By (SSC),

$$\begin{aligned} \delta||\hat{\xi}||_2^2 &\le Q(\hat{\xi}, \hat{\xi}) \\ &= Q(\xi, \xi) - 2Q(\hat{\xi}, \xi - \hat{\xi}) - Q(\xi - \hat{\xi}, \xi - \hat{\xi}) \\ &\le Q(\xi, \xi) + c||\hat{\xi}||_2||\xi - \hat{\xi}||_2 + c||y_\pi||_2^2. \end{aligned} \tag{4.8}$$

Now we can proceed completely analogous to the further proof in [2] using the L_2-norm: By means of the first order optimality conditions for v_π as a solution to $(QS)_\pi$ and for v_0 as a solution to (P) we are able to conclude

$$Q(\xi, \xi) \le (v_\pi^\star, \xi) + (y_\pi^\star - y_0^\star, g'(v_0)\xi). \tag{4.9}$$

Inserting this in (4.8) and the obtained estimate for $||\hat{\xi}||_2^2$ in (4.7) we arrive after a couple of formal manipulations at

$$\begin{aligned} ||\xi||_2^2 \le \delta^{-1}(&||v_\pi^\star||_2||\xi||_2 + ||y_\pi^\star - y_0^\star||_2||g'(v_0)||_{2\to 2}||\xi||_2 \\ &+ c||\hat{\xi}||_2||y_\pi||_2) + c(||y_\pi||_2^2 + c||\pi||_2||\xi||_2), \end{aligned}$$

provided that $||\xi||_2 \ge ||\pi||_2$ (cf. Alt[2]). From this, (4.4), $||\hat{\xi}||_2 \le ||\xi||_2 + c||\pi||_2$ (by (4.6)) and $||y_\pi||_2^2 \le ||\pi||_2||\xi||_2$ (use $||\pi||_2 \le ||\xi||_2$) the result (4.5) follows immediately with a certain constant c. In the case $||\xi||_2 \le ||\pi||_2$ (4.5) is trivially satisfied. Thus (4.5) is true with $c := \max(1, c)$. □

It is very essential for our theory to have a counterpart of (4.5) at disposal in the $L_{\bar{p}}$–norm. To this aim, we shall work with the norm

$$||\pi||_{\bar{p},\infty} = ||v_x^\star||_{\bar{p}} + ||v_u^\star||_{\bar{p}} + ||y||_\infty .$$

We recall that $\bar{p} = \infty$ in the general case and $\bar{p} = p$, if φ_2, b_2 are affine–linear with respect to x. Now take $\lambda \in (0,1)$ (sufficiently close to 1) and define p_1 by

$$\frac{1}{p_1} = \frac{1}{2} - \lambda(1-\alpha). \tag{4.10}$$

By (2.3), K maps continuously L_2 into L_{p_1}. Exploiting (4.5) with respect to u,

$$||u_\pi - u_0||_2 \le c||\pi||_2 \le c||\pi||_{p_1,\infty}. \tag{4.11}$$

The equation for $x_\pi - x_0$ is

$$x_\pi - x_0 + KB_x(x_\pi - x_0) = KB_u(u_\pi - u_0) + y,$$

thus, invoking Lemma 6.3,

$$\begin{aligned} ||x_\pi - x_0||_{p_1} &\le c_1||KB_u(u_\pi - u_0)||_{p_1} + c_2||y||_{p_1} \\ &\le c(||u_\pi - u_0||_2 + ||y||_{p_1}) \\ &\le c(c||\pi||_{p_1,\infty} + ||\pi||_{p_1,\infty}) \\ &\le c||\pi||_{p_1,\infty}. \end{aligned} \tag{4.12}$$

Next we insert (4.11 – 4.12) in (4.4), thus

$$\begin{aligned} ||y_\pi^\star - y_0^\star||_2 &\le c(||v_\pi - v_0||_2 + ||\pi||_2) \\ &\le c||\pi||_2. \end{aligned} \tag{4.13}$$

The estimate (4.11) can be improved to

$$||u_\pi - u_0||_{p_1} \le c||\pi||_{p_1,\infty} . \tag{4.14}$$

As in the proof of Corollary 3.5 we have

$$\begin{aligned} u_0 &= P_{[-1,1]}\Big[-\frac{1}{2\lambda}\left\{\varphi_2(x_0) + (K^\star y_0^\star)b_2(x_0)\right\}\Big] \\ u_\pi &= P_{[-1,1]}\Big[-\frac{1}{2\lambda}\{\varphi_2(x_0) + (K^\star y_\pi^\star)b_2(x_0) \\ &\quad +(\varphi_2'(x_0) + (K^\star y_0^\star)b_2'(x_0))(x_\pi - x_0) - v_u^\star\}\Big], \end{aligned}$$

which implies in turn

$$\begin{aligned} ||u_0 - u_\pi||_{p_1} &\le c(||x_\pi - x_0||_{p_1} + ||K^\star(y_0^\star - y_\pi^\star)||_{p_1} + ||v_u^\star||_{p_1}) \\ &\le c(||\pi||_{p_1} + ||y_0^\star - y_\pi^\star||_2) \le c(||\pi||_{p_1} + ||\pi||_2) \\ &\le c||\pi||_{p_1} \le c||\pi||_{p_1,\infty}. \end{aligned}$$

In this way, we have performed one step of a bootstrapping argument. Next, we define p_2 by

$$\frac{1}{p_2} = \frac{1}{p_1} - \lambda(1-\alpha) = \frac{1}{2} - 2\lambda(1-\alpha).$$

By the same procedure as before, (4.14) can be obtained with p_2 substituted for p_1. After finitely many steps we arrive at the case, where

$$\frac{1}{p_k} = \frac{1}{p_{k-1}} - \lambda(1-\alpha) = \frac{1}{2} - k\lambda(1-\alpha) < 1-\alpha,$$

while $p_{k-1} < 1/(1-\alpha)$. Then we obtain immediately (4.14) in the form

$$||u_\pi - u_0||_\infty \le c||\pi||_{\bar{p},\infty}\,.$$

We have just proved

Theorem 4.3 *There is a constant c not depending on π, such that*

$$||v_\pi - v_0||_\infty \le c||\pi||_{\bar{p},\infty}, \tag{4.15}$$

for all $\pi \in (L_\infty)^3$.

There is a close connection between solutions of $(\text{QP})_w$ and $(\text{QS})_\pi$. In order to link these two programs we assign to fixed $w \in C \times L_\infty \times L_\infty$, $v \in L_{\infty,p}$ and $y^\star \in L_\infty$ the elements

$$\begin{aligned} v^\star(v, y^\star, w) = f'(v_0) + \mathcal{L}_{vv}(v_0, y_0^\star)[v - v_0, \cdot] - g'(v_0)^\star y^\star \\ -f'(v_w) - \mathcal{L}_{vv}(v_w, y_w^\star)[v - v_w, \cdot] + g'(v_w)^\star y^\star \qquad (4.16) \\ y(v, w) = g(v_0) + g'(v_0)(v - v_0) - g(v_w) - g'(v_w)(v - v_w). \qquad (4.17) \end{aligned}$$

If w is sufficiently close to w_0, then it can be shown, that $\bar{v}_w$, the unique solution of $(\text{QP})_w$, also solves $(\text{QS})_{\bar{\pi}}$, where $\bar{\pi} = (v^\star(\bar{v}_w, \bar{y}_w^\star, w), y(\bar{v}_w))$. The next lemma is the main prerequisite to prove that.

Lemma 4.4 *There are a $L_{\infty,\bar{p}}$–neighbourhood $N_2(v_0)$ of v_0 and a $C \times L_{\bar{p}} \times L_p$–neighbourhood $N_3(w_0)$ of w_0 such that*

$$||y(v, w)||_\infty \le c_1||v_w - v_0||^2_{\infty,\bar{p}} + c_2||v_w - v_0||_{\infty,\bar{p}}||v - v_0||_{\infty,\bar{p}} \tag{4.18}$$

and

$$\begin{aligned} ||v^\star(v, y^\star, w)||_{\bar{p}} \le c_3||v_w - v_0||^2_{\infty,\bar{p}} + c_4||v_w - v_0||_{\infty,\bar{p}}||v - v_0||_{\infty,\bar{p}} \\ +c_5||y^\star||_p||v_w - v_0||^2_{\infty,\bar{p}} + c_6||v_w - v_0||_{\infty,\bar{p}}||y^\star - y_0^\star||_p \\ +c_7||v_w - v_0||_{\infty,\bar{p}}||y_w^\star - y_0^\star||_p + c_8||v - v_0||_{\infty,\bar{p}}||y_w^\star - y_0^\star||_p \qquad (4.19) \end{aligned}$$

for all $v \in N_3(v_0)$, $w \in N_2(w_0)$.

Proof: For y we obtain

$$\begin{aligned}
||y(v,w)||_\infty &= ||g(v_0) - g(v_w) - g'(v_w)(v_0 - v_w)||_\infty \\
&\quad + ||(g'(v_0) - g'(v_w))(v_0 - v)||_\infty \\
&\leq ||K||_{p\to\infty} ||B(v_0) - B(v_w) - B'(v_w)(v_0 - v_w)||_p \\
&\quad + ||K||_{p\to\infty} ||(B'(v_0) - B'(v_w))(v_0 - v)||_p \\
&\leq c_1 ||v_0 - v_w||^2_{\infty,p} + c_2 ||v_0 - v_w||_{\infty,p} ||v_0 - v||_{\infty,p} \\
&\leq c_1 ||v_0 - v_w||^2_{\infty,\bar p} + c_2 ||v_0 - v_w||_{\infty,\bar p} ||v_0 - v||_{\infty,\bar p},
\end{aligned}$$

as B is twice continuously differentiable from $L_{\infty,p}$ to L_p (here Assumption (A2) of linearity with respect to u is essential) and B' is globally Lipschitz on $C \times U^{\mathrm{ad}}$.

The estimation of $||v^\star||_{\bar p}$ is more delicate. We have

$$\begin{aligned}
v^\star(v, y^\star, w) &= f'(v_0) - f'(v_w) - f''(v_w)[v_0 - v_w, \cdot] \\
&\quad + (f''(v_w) - f''(v_0))[v_0 - v, \cdot] \\
&\quad - y^\star \circ K(B'(v_0) - B'(v_w) - B''(v_w)[v_0 - v_w, \cdot]) \\
&\quad + y^\star \circ K(B''(v_0) - B''(v_w)[v_0 - v_w, \cdot] \\
&\quad + (y^\star - y_0^\star) \circ KB''(v_0)[v_w - v_0, \cdot] \\
&\quad + y_0^\star \circ K(B''(v_0) - B''(v_w))[v_0 - v, \cdot] \\
&\quad + (y_0^\star - y_w^\star) \circ KB''(v_w)[v_w - v_0, \cdot] \\
&\quad + (y_0^\star - y_w^\star) \circ KB''(v_w)[v_0 - v, \cdot] \\
&= \mathrm{I} + \mathrm{II} + \ldots + \mathrm{VIII}
\end{aligned}$$

Now we handle I — VIII separately.
I: Set $v_{\mathrm{I}}^\star = f'(v_0) - f'(v_w) - f''(v_w)[v_0 - v_w, \cdot]$.

We take $z \in L_\infty \times L_\infty$ with $||z||_q \leq 1$, $1/\bar p + 1/q = 1$, arbitrarily but fixed and apply $v_{\mathrm{I}}^\star$ to z. Differentiating the real function

$$\Psi(s) = (f'(v_w + s(v_0 - v_w)), z)$$

with respect to s we find in a standard way

$$\begin{aligned}
|(v_{\mathrm{I}}^\star, z)| &= \left| \int_0^1 (f''(v_w + s(v_0 - v_w)) - f''(v_w))[v_0 - v_w, z])\, ds \right| \\
&\leq c ||v_0 - v_w||_{\infty,\bar p} ||v_0 - v_w||_{\infty,\bar p} ||z||_q,
\end{aligned}$$

by (2.17). Thus (roughly speaking) $v_{\mathrm{I}}^\star \in L_{\bar p} \times L_{\bar p}$ and

$$||v_{\mathrm{I}}^\star||_{\bar p} \leq c_3 ||v_0 - v_w||^2_{\infty,\bar p}.$$

II: In a simpler way,

$$\begin{aligned}
|(v_{\mathrm{II}}^\star, z)| &\leq |(f''(v_w) - f''(v_0))[v_0 - v, z]| \\
&\leq c ||v_w - v_0||_{\infty,\bar p} ||v_0 - v||_{\infty,\bar p} ||z||_q
\end{aligned}$$

by (2.17) implying $||v^\star_{\rm II}||_{\bar p} \le c_4 ||v_w - v_0||_{\infty,\bar p} ||v_0 - v||_{\infty,\bar p}$.
III:

$$
\begin{aligned}
|(v^\star_{\rm III}, z)| &= |(K^\star y^\star, (B'(v_0) - B'(v_w))z - B''(v_w)[v_0 - v_w, z])| \\
&= \left| \int_0^1 (K^\star y^\star, (B''(v_w + s(v_0 - v_w)) - B''(v_w))[v_0 - v_w, z])\, ds \right| \\
&\le \int_0^1 ||K^\star y^\star||_\infty ||(B''(v_w + s(v_0 - v_w)) - B''(v_w))[v_0 - v_w, z]||_1\, ds \\
&\le ||K^\star||_{p\to\infty} ||y^\star||_p ||v_0 - v_w||^2_{\infty,\bar p} ||z||_q,
\end{aligned}
$$

by (2.18). Hence $||v^\star_{\rm III}||_{\bar p} \le c_5 ||y^\star||_p ||v_0 - v_w||^2_{\infty,\bar p}$.

In the same way, the estimations for IV — VIII can be performed. Here, as in III, the smoothing property $K^\star \in \mathcal{L}(L_p, C)$ is essential, so that $||K^\star(y^\star_w - y^\star_0)||_\infty \le c||y^\star_w - y^\star_0||_p$. □

Completely analogous to [2] Lemma 4.5 we can derive

Lemma 4.5 *Let $w = (x_w, u_w, y^\star_w) \in C \times L_\infty \times L_\infty$ be given. Suppose that $\bar v_w = (\bar x_w, \bar u_w)$ is the corresponding solution of (QP)$_w$ with Lagrange multiplier $\bar y^\star_w$. Define*

$$\bar v^\star = v^\star(\bar v_w, \bar y^\star_w, w),\ \bar y = y(\bar v_w, w),\ \bar\pi = (\bar v^\star, \bar y).$$

Then $\bar v_w$ is a global solution of (QS)$_{\bar\pi}$, and $\bar y^\star_w$ is the Lagrange multiplier for $\bar x_w$ as solution of (QS)$_{\bar\pi}$.

Now we are able to state the main result of our paper.

Theorem 4.6 *Suppose that Assumptions* (A1)–(A3) *and* (SSC) *are satisfied. Choose $\bar p$ and p according to* (2.4), (2.6). *Then there is a $C \times L_{\bar p} \times L_{\bar p}$–neighbourhood $N_4(w_0)$ of $w_0 = (x_0, u_0, y^\star_0)$ such that for all $w = (v_w, y^\star_w) \in N_4(w_0)$ the Problem* (QP)$_w$ *has a unique solution $\bar v_w$. Let $\bar y^\star_w$ be the corresponding Lagrange multiplier. Then*

$$||(\bar v_w, \bar y^\star_w) - (v_0, y^\star_0)||_{\infty,\bar p} \le \nu ||w - w_0||^2_{\infty,\bar p} \tag{4.20}$$

holds with some $\nu \in \mathbb{R}_+$.

Proof: We take $N(w_0) \subset N_1(w_0) \cap N_2(w_0)$. Let $w \in N(w_0)$ be given. According to Lemma 3.2, (QP)$_w$ admits a unique solution $\bar v$, satisfying (3.5) with multiplier $\bar y^\star_w$ satisfying (3.6). Define the perturbations $\bar v^\star$, $\bar y$, $\bar\pi$ according to Lemma 4.5. Due to the Corollaries 3.3 and 3.4, $||\bar v_w - v_0||_{\infty,\bar p}$ remains bounded on $N(w_0)$. By (3.6), the same refers to $||\bar y^\star_w - y^\star_0||_p \le c||\bar y^\star_w - y^\star_0||_\infty$. Inserting these bounds into (4.18), (4.19),

$$||\bar y||_\infty \le \ c||v_w - v_0||_{\infty,\bar p} \ \le c||w - w_0||_{\infty,\bar p} \tag{4.21}$$

$$||\bar v^\star||_{\bar p} \le \ c||v_w - v_0||_{\infty,\bar p} \ \le c||w - w_0||_{\infty,\bar p}. \tag{4.22}$$

Now, Lemma 4.5 ensures that $\bar{v}_w$ is a solution of $(\mathrm{QS})_{\bar{\pi}}$ with Lagrange multiplier $\bar{y}_w^\star$. Thus Theorem 4.3 applies

$$||\bar{v}_w - v_0||_{\infty,\bar{p}} \leq c||\bar{\pi}||_{\bar{p},\infty}. \tag{4.23}$$

By means of (4.21), (4.22) we are able to continue

$$||\bar{v}_w - v_0||_{\infty,\bar{p}} \leq c||w - w_0||_{\infty,\bar{p}}, \tag{4.24}$$

and from Lemma 4.1, (4.4)

$$\begin{aligned} ||\bar{y}_w^\star - y_0^\star||_p &\leq c||\bar{v}_w - v_0||_{\infty,\bar{p}} + ||\bar{\pi}||_{\bar{p}} \\ &\leq c||\bar{\pi}||_{\bar{p},\infty} && (4.25) \\ &\leq c||w - w_0||_{\infty,\bar{p}}. && (4.26) \end{aligned}$$

The last inequality follows from (4.21) – (4.22). Inserting (4.24), (4.26) in (4.18), (4.19),

$$\begin{aligned} ||\bar{y}||_\infty &\leq c_1||w - w_0||^2_{\infty,\bar{p}} + c_2 c||w - w_0||_{\infty,\bar{p}}||w - w_0||_{\infty,\bar{p}} \\ &\leq c||w - w_0||^2_{\infty,\bar{p}}. && (4.27) \end{aligned}$$

Similarly

$$\begin{aligned} ||\bar{v}^\star||_{\bar{p}} \leq\ & c_3||w - w_0||^2_{\infty,\bar{p}} + c_4 c||w - w_0||_{\infty,\bar{p}}||w - w_0||_{\infty,\bar{p}} \\ & + c_5 c_2||w - w_0||^2_{\infty,\bar{p}} + c_6 c||w - w_0||_{\infty,\bar{p}}||w - w_0||_{\infty,\bar{p}} \\ & + c_7||w - w_0||_{\infty,\bar{p}}||w - w_0||_{\infty,\bar{p}} + c_8||w - w_0||_{\infty,\bar{p}}||w - w_0||_{\infty,\bar{p}} \\ \leq\ & c||w - w_0||^2_{\infty,\bar{p}}. && (4.28) \end{aligned}$$

The last two inequalities yield in particular the optimal estimate

$$||\bar{\pi}||_{\bar{p},\infty} \leq c||w - w_0||^2_{\infty,\bar{p}}. \tag{4.29}$$

$\forall \varepsilon > 0$, where c_ε depends only on ε.

Inserting (4.29) into (4.23) and (4.25) leads to the estimate (4.20). □

5. Convergence of the SQP method

In this section we introduce the SQP method and state a result on local convergence of the method.

The following sequential quadratic programming method is a straightforward extension of Wilson's method (see [15], [13]) to the infinite-dimensional Problem (P).

(SQP): *Choose a starting point* $w_1 = (v_1, y_1^\star)$. *Having* $w_k = (v_k, y_k^\star)$, *compute* $w_{k+1} = (v_{k+1}, y_{k+1}^\star)$ *to be the solution and the associated Lagrange multiplier of the quadratic optimization problem* $(\mathrm{QP})_{w_k}$. ◇

Using Theorem 4.6 it follows now by standard proof techniques that the SQP method converges quadratically to w_0 if the starting point w_1 is choosen sufficiently close to w_0 (see [2], Theorem 5.1). Let ν be defined by Theorem 4.6. Let $B_{\gamma\delta}(w_0)$ denote the ball of $L_{\infty,\bar{p}}$ around w_0 with radius $\gamma\delta$.

Theorem 5.1 *Suppose that Assumptions* (A1)–(A3) *and* (SSC) *are satisfied. Choose $\bar{p}$ and p according to* (2.4), (2.5). *Let $\gamma > 0$ be such that $\delta := \nu\gamma < 1$, and $B_{\gamma\delta}(w_0) \subset N_4(w_0)$. Then for any starting point $w_1 \in B_{\gamma\delta}(w_0)$ the SQP method computes a unique sequence w_k with*

$$||w_k - w_0||_{\infty,\bar{p}} \leq \gamma\, \delta^{2^k - 1},$$

and $w_k \in B_{\gamma\delta}(w_0)$ for $k \geq 2$. ◇

Thus we have shown local quadratic convergence of the SQP method.

6. Appendix

Lemma 6.1 *Let $a = a(x) : \mathbb{R} \to \mathbb{R}$ be a C^2-function, such that a, a', a'' are globally bounded and Lipschitz on $\mathbb{R}$. Then the nonlinear Nemytskii–operator A,*

$$(A(x,u))(t) = a(x(t)) \cdot u(t)$$

is twice continuously Fréchet differentiable from $C \times L_\beta$ into L_β for all $1 \leq \beta \leq \infty$. Moreover, let S be a bounded set of $C \times L_\infty$. Then for all v, $v_i \in S$ and all $h_i \in C \times L_\infty$, $i = 1, 2$,

$$\begin{aligned} ||A''(v)[h_1, h_2]||_1 &\leq c||h_1||||h_2||_{\beta'} \\ ||(A''(v_1) - A''(v_2))[h_1, h_2]||_1 &\leq c||v_1 - v_2||_{\infty,\beta}||h_1||_{\infty,\beta}||h_2||_{\beta'} \end{aligned}$$

($1/\beta + 1/\beta' = 1$), where c does not depend on v, v_i.

Proof. It is easy to show that the mapping $D : (x,u) \mapsto x \cdot u$ is Fréchet-differentiable from $C \times L_\beta$ to L_β. Moreover, the mapping $(x,u) \mapsto D'(x,u)$ from $C \times L_\beta$ to $\mathcal{L}(C \times L_\beta, L_\beta)$ is linear and continuous, hence Fréchet–differentiable, too. This is equivalent to the existence of the second order derivative of D. Consequently, the composition $A(x,u) = D(a(x), u)$ has this property, too. Let $v = (x,u)$, $v_i = (x_i, u_i)$, $h_i = (\xi_i, \eta_i)$, $i = 1, 2$, and l be the uniform bound and Lipschitz constant for a, a', a''. It holds

$$A''(v)[h_1, h_2](t) = a''(x(t))u(t)\xi_1(t)\xi_2(t) + a'(x(t))(\xi_1(t)\eta_2(t) + \xi_2(t)\eta_1(t)).$$

Therefore

$$\begin{aligned} ||A''(v)[h_1, h_2]||_1 &\leq l||u||_\infty||\xi_1||_\beta||\xi_2||_{\beta'} + l||\xi_1||_\beta||\eta_2||_{\beta'} + l||\xi_2||_{\beta'}||\eta_1||_\beta \\ &\leq c(||\xi_1||_\beta + ||\eta_1||_\beta)(||\xi_2||_{\beta'} + ||\eta_2||_{\beta'}) \\ &= c||h_1||_\beta||h_2||_{\beta'}. \end{aligned}$$

The Lipschitz–property of A'' is seen as follows

$$((A''(v_1) - A''(v_2))[h_1, h_2] = (a''(x_1) - a''(x_2))u_1\xi_1\xi_2 + a''(x_2)(u_1 - u_2)\xi_1\xi_2 \\ +(a'(x_1) - a'(x_2))(\xi_1\eta_2 + \xi_2\eta_1). \qquad (6.1)$$

Thus

$$\begin{aligned} &||((A''(v_1) - A''(v_2))[h_1, h_2]||_1 \\ &\leq l\,\{||x_1 - x_2||_\infty ||u_1||_\beta ||\xi_1||_\infty ||\xi_2||_{\beta'} + ||u_1 - u_2||_\beta ||\xi_1||_\infty ||\xi_2||_{\beta'} \\ &\quad + ||x_1 - x_2||_\infty (||\xi_1||_\infty ||\eta_2||_1 + ||\xi_2||_{\beta'} ||\eta_1||_\beta)\} \\ &\leq c(||x_1 - x_2||_\infty + ||u_1 - u_2||_\beta)(||\xi_1||_\infty + ||\eta_1||_\beta)(||\xi_2||_{\beta'} + ||\eta_2||_{\beta'}) \\ &= c||v_1 - v_2||_{\infty,\beta} ||h_1||_{\infty,\beta} ||h_2||_{\beta'}. \end{aligned}$$

□

Remark: Completely analogous we deduce from (6.1)

$$||(A''(v_1) - A''(v_2))[h_1, h_2]||_1 \leq c||v_1 - v_2||_\infty ||h_1||_\beta ||h_2||_{\beta'}.$$

◇ If a is in addition to the assumptions affine–linear, then $a''(x) = 0$. Then the L_1–norm in (6.1) can be estimated by

$$c||x_1 - x_2||_\infty \, (||\xi_1||_\beta ||\eta_2||_{\beta'} + ||\xi_2||_{\beta'} ||\eta_1||_\beta) \leq c||v_1 - v_2||_{\infty,p} ||h_1||_\beta ||h_2||_{\beta'}.$$

Recalling that $\bar{p} = p$, if a is affine–linear and $\bar{p} = \infty$ in the other cases, both estimates can be joined together:

$$||(A''(v_1) - A''(v_2))[h_1, h_2]||_1 \leq c||v_1 - v_2||_{\infty,\bar{p}} ||h_1||_\beta ||h_2||_{\beta'}.$$

In this way, it is now easy to derive the estimates (2.13 – 2.18).

Lemma 6.2 *There is a sufficiently small $L_\infty \times L_{\bar{p}} \times L_p$ neighbourhood $N(w_0)$ of $w_0 = (x_0, u_0, y_0^\star)$ such that for all $w = (v_w, y_w^\star) \in N(w_0)$*

$$\mathcal{L}_{vv}(v_w, y_w^\star)[h, h] \geq \frac{\delta}{2} ||h||_2^2$$

for all h such that $g'(v_w)h = 0$, $h = (x - x_w, u - u_w)$, $u \in U^{\mathrm{ad}}$.

Proof. Note that $\bar{p} = p$ iff φ_2, b_2 are affine–linear, otherwise $\bar{p} = \infty$.

$$\begin{aligned} &|\mathcal{L}_{vv}(v_w, y_w^\star)[h, h] - \mathcal{L}_{vv}(v_0, y_0^\star)[h, h]| \leq |(f''(v_w) - f''(v_0))[h, h]| + \\ &\quad + |(K^\star y_w^\star, B''(v_w)[h, h]) - (K^\star y_0^\star, B''(v_0)[h, h])| \\ &\leq c_L ||v_w - v_0||_{\infty,\bar{p}} ||h||_2^2 + |(K^\star y_w^\star, (B''(v_w) - B''(v_0))[h, h])| + \\ &\quad + |(K^\star (y_w^\star - y_0^\star), B''(v_0)[h, h])| \\ &\leq c_L ||v_w - v_0||_{\infty,\bar{p}} ||h||_2^2 + ||K^\star y_w^\star||_\infty ||B''(v_w) - B''(v_0)[h, h]||_1 + \\ &\quad + ||K^\star (y_w^\star - y_0^\star)||_\infty ||B''(v_0)[h, h]||_1 \\ &\leq c||v_w - v_0||_{\infty,\bar{p}} ||h||_2^2, \end{aligned}$$

by the L_p–boundedness of $y_w^\star$, the $L_p \to L_\infty$–continuity of $K^\star$ and (2.16 – 2.18). Moreover

$$\begin{aligned}||g'(v_w)h - g'(v_0)h||_2 &= ||K(B'(v_w) - B'(v_0))h||_2 \\ &\le ||K||_{2\to 2}||(B'(v_w) - B'(v_0))h||_2 \\ &= c \cdot c_L ||v_w - v_0||_{\infty,\bar{p}}||h||_2\end{aligned}$$

by (2.12). Now the statement follows from ALT [2], Lemma 3.5 after setting $B = \mathcal{L}_{vv}(v_w, y_w^\star)$, $\tilde{B} = \mathcal{L}_{vv}(v_0, y_0^\star)$, $A = g'(v_w)$, $\tilde{A} = g'(v_w)$. □

Lemma 6.3 *For all $\beta \in [1, \infty]$ there is a constant c_β being independent from $v = (x, u) \in C \times U^{\mathrm{ad}}$ such that*

$$\left\| (I - KB_x(v))^{-1} \right\|_{\beta\to\beta} \le c_\beta. \tag{6.2}$$

Proof. Let $y \in L_\beta$ and $v(t) = (x(t), u(t))$ with $|u(t)| \le 1$ be given. We consider the equation $(I - KB_x(v))x = y$, i. e.

$$x(t) = y(t) + \int_0^t k(t, s)b_x(s, v(s))x(s)\, ds. \tag{6.3}$$

The uniform boundedness of $b_x(v)$ implies $|b_x(t, v(t))| \le c$, independently from v. Hence

$$|x(t)| \le |y(t)| + c \int_0^t (t - s)^{-\alpha}|x(s)|\, ds. \tag{6.4}$$

All solutions of this weakly singular integral inequality are majorized by the (nonnegative) solution $z(t)$ of the corresponding integral equation, hence

$$|x(t)| \le z(t). \tag{6.5}$$

Now $||x||_\beta \le ||z||_\beta \le c||y||_\beta$ follows from standard results on Volterra integral equations. □

In the next statement $d[a, S]_\beta$ denotes the distance of the point a to the set S in the norm $|| \cdot ||_\beta$.

Lemma 6.4 *If $u_w \in U^{\mathrm{ad}}$ and $w = (x_w, u_w, y_w^\star)$, then $\Sigma(w) \ne \emptyset$ and*

$$d[(x_0, u_0), \Sigma(w)]_2 \le c(||x_w - x_0||_2 + ||u_w - u_0||_2) \tag{6.6}$$

Moreover, for all $(x, u) \in \Sigma(w)$

$$d[(x, u), \Sigma(w_0)]_2 \le c(||x_w - x_0||_2 + ||u_w - u_0||_2) \tag{6.7}$$

Proof. (a) Let $v_w = (x_w, u_w)$. Then $(x, u) \in \Sigma(w)$ iff

$$x_w - KB(v_w) + x - x_w - K(B_x(v_w)(x - x_w) + B_u(v_w)(u - u_w)) = 0. \quad (6.8)$$

Now we look for a special $v = (x, u) \in \Sigma(w)$ such that $||v_0 - v||_2$ is less or equal than the right hand side of (6.6). To this aim, we take $u = u_0$. Then from (6.8),

$$\begin{aligned}
&(I - KB_x(v_w))(x - x_w) = -x_w + KB(v_w) + KB_u(v_w)(u_0 - u_w) \\
&= -x_w + KB_u(v_w)(u_0 - u_w) + K(B(v_0) + B'(v_0)(v_w - v_0) + \\
&+ \frac{1}{2}B''(v_0 + \theta(v_w - v_0))[v_w - v_0, v_w - v_0],
\end{aligned}$$

where $\theta = \theta(t) \in L_\infty(0, 1)$. Now we use $x_0 = KB(v_0)$ and write B' in terms of x and u, then

$$\begin{aligned}
&x(t) - x_w(t) - \int_0^t k(t,s) b_x(v_w(s))(x(s) - x_w(s))\, ds = \\
&= -x_w(t) + x_0(t) + \int_0^t k(t,s)\{b_x(v_0(s))(x_w(s) - x_0(s)) + \\
&+ (b_u(v_w(s)) - b_u(v_0(s)))(u_0(s) - u_w(s)) + \\
&+ \frac{1}{2}b''(v_0(s) + \theta(s)(v_w(s) - v_0(s)))(v_w(s) - v_0(s))^2\}\, ds.
\end{aligned}$$

Rearranging,

$$\begin{aligned}
&(x - x_0) - KB_x(v_w)(x - x_0) = K\{(B_x(v_w) - B_x(v_0))(x_0 - x_w) + \\
&+ (B_u(v_w) - B_u(v_0))(v_0 - v_w) + \\
&+ \frac{1}{2}B''(v_0 + \theta(v_w - v_0))[v_w - v_0, v_w - v_0]\}. \quad (6.9)
\end{aligned}$$

We know, that x_w belongs to a L_∞–neighbourhood of x_0. Moreover, $u_w = u_w(t)$ and $u_0 = u_0(t)$ are uniformly bounded by 1. Denote in (6.9) the right hand side term under K by $I = I(t)$. Then

$$\begin{aligned}
|I(t)| = &\frac{1}{2}|[b_{xx}(v_0(t) + \theta(t)(v_w(t) - v_0(t)))(x_0(t) - x_w(t))]||x_0(t) - x_w(t)| \\
&+ |[b_{xu}(v_0(t) + \theta(t)(v_w(t) - v_0(t)))(u_0(t) - u_w(t))]||x_0(t) - x_w(t)| \\
&+ |[b_x(v_w(t)) - b_x(v_0(t))]||x_0(t) - x_w(t)| \\
&+ |[b_u(v_w(t)) - b_u(v_0(t))]||u_0(t) - u_w(t)| \quad (6.10)
\end{aligned}$$

All terms in the brackets are uniformly bounded, hence

$$||I||_2 \le c(||x_0 - x_w||_2 + ||u_0 - u_w||_2). \quad (6.11)$$

Thus from (6.9)

$$||(x - x_0) + KB_x(v_w)(x - x_0)||_2 \leq c||K||_{2\to 2}(||x_0 - x_w||_2 + ||u_0 - u_w||_2)$$

implying by Lemma 6.3 that $||x - x_0||_2 \leq c||v_0 - v_w||_2$.
(b) We have $v = (x, u) \in \Sigma(w)$ and look for a $\bar{v} = (\bar{x}, \bar{u}) \in \Sigma(w_0)$ close to v. Thus

$$x = K(B(v_w) + B'(v_w)(v - v_w)) \tag{6.12}$$
$$\bar{x} = x_0 + KB'(v_0)(\bar{v} - v_0) \tag{6.13}$$

(note that $x_0 = KB(v_0)$). Re-arranging (6.13),

$$\bar{x} = x_0 + K(B'(v_w)(\bar{v} - v_w) + B'(v_w)(v_w - v_0) + (B'(v_0) - B'(v_w))(\bar{v} - v_0)). \tag{6.14}$$

Now we take $\bar{u} := u$. Subtracting (6.12) from (6.14),

$$\begin{aligned}\bar{x} - x = x_0 - KB(v_w) + K(B_x(v_w)(\bar{x} - x) + B'(v_w)(v_w - v_0))\\ +K((B'(v_0) - B'(v_w))(\bar{v} - v_0)).\end{aligned}$$

Thus by $KB(v_w) = x_0 + K(B(v_w) - B(v_0))$

$$\begin{aligned}(\bar{x} - x) - KB_x(v_w)(\bar{x} - x) &= K\{B(v_w) - B(v_0) + B'(v_w)(v_w - v_0)\\ &\quad +(B'(v_0) - B'(v_w))(\bar{v} - v_0)\}\\ &= K\text{II}.\end{aligned}$$

Now we find

$$||K\text{II}||_2 \leq ||K||_{2\to 2}||v_w - v_0||_2 \tag{6.15}$$

by estimating II. Here we need that $\bar{x}$ is uniformly bounded (independently from the choice of $u \in \mathcal{C}$), thus $||\bar{v} - v_0||_\infty = ||\bar{x} - x_0||_\infty \leq c$. The last inequality follows from (6.8) with $\bar{u} = u$: $\bar{x} - x_0 = KB_x(v_0)(\bar{x} - x_0) + KB_u(v_0)(u - u_0)$. Note that $u - u_0$ is L_∞–bounded with 2.

This implies as above $||\bar{x} - x||_2 \leq c||v_w - v_0||_2$. □

Lemma 6.5 *$F(v, w)$ is Lipschitz continuous on each L_∞-bounded set of $(L_2)^5$ as a mapping from $(L_2)^5$ into $\mathbb{R}$.*

Proof: Let $v = (x, u)$, $w = (x_w, u_w, y_w^\star)$. Then

$$F(v, w) = f'(v_w)(v - v_w) + \frac{1}{2}\mathcal{L}_{vv}(v_w, y_w^\star)[v - v_w, v - v_w]$$

$$F(v_1, w_1) = f'(v_w^1)(v_1 - v_w^1) + \frac{1}{2}\mathcal{L}_{vv}(v_w^1, y_w^{\star 1})[v_1 - v_w^1, v_1 - v_w^1]$$

$$|F(v, w) - F(v_1, w_1)| = |I - I_1| + J - J_1|,$$

where I, I_1 denote the linear part and J, J_1 the nonlinear part of F.

$$\begin{aligned}|I - I_1| &\leq |f'(v_w)(v - v_1 + v_w^1 - v_w)| + |(f'(v_w^1) - f'(v_w))(v_1 - v_w^1)| \\ &\leq c_1(\|v - v_1\|_2 + \|v_w^1 - v_w\|_2) + c_2\|v_w^1 - v_w\|_2,\end{aligned}$$

as $\|f'(v_w)\|_2$ is uniformly bounded and $\|v_1 - v_w^1\|_\infty$ is bounded. Moreover

$$\begin{aligned}2|J - J_1| \leq\, &|\mathcal{L}_{vv}(v_w^1, y_w^{\star 1})[v - v_w, v - v_w] - \mathcal{L}_{vv}(v_w^1, y_w^{\star 1})[v_1 - v_w^1, v_1 - v_w^1]| \\ &+|(\mathcal{L}_{vv}(v_w, y_w^\star) - \mathcal{L}_{vv}(v_w^1, y_w^{\star 1}))[v - v_w, v - v_w]|. \qquad (6.16)\end{aligned}$$

Let $h = v - v_w$. Then

$$\begin{aligned}&|(\mathcal{L}_{vv}(v_w, y_w^\star) - \mathcal{L}_{vv}(v_w^1, y_w^{\star 1}))[h, h]| = \\ &= |(y_w^\star, KB''(v_w)[h, h]) - (y_w^{\star 1}, KB''(v_w^1)[h, h])| \\ &\leq |(y_w^\star, K(B''(v_w) - B''(v_w^1))[h, h]| + |(y_w^\star - y_w^{\star 1}, KB''(v_w^1)[h, h])| \\ &\leq c\|y_w^\star\|_2\|v_w - v_w^1\|_2\|h\|_\infty^2 + c\|y_w^\star - y_w^{\star 1}\|_2\|h\|_\infty^2 \\ &\leq c\|w - w_1\|_2\end{aligned}$$

(note that $y_w^\star$, h, v_w, v_w^1 are supposed to belong to a L_∞–bounded set). Similarly the first part in (6.16) can be estimated by $\|v - v_1\|_2 + \|v_w - v_w^1\|_2$. □ æ

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