ON AN AUGMENTED LAGRANGIAN SQP METHOD FOR A CLASS OF OPTIMAL CONTROL PROBLEMS IN BANACH SPACES *

NADIR ARADA,[1] JEAN-PIERRE RAYMOND[2] AND FREDI TRÖLTZSCH[3]

Abstract. An augmented Lagrangian SQP method is discussed for a class of nonlinear optimal control problems in Banach spaces with constraints on the control. The convergence of the method is investigated by its equivalence with the generalized Newton method for the optimality system of the augmented optimal control problem. The method is shown to be quadratically convergent, if the optimality system of the standard non-augmented SQP method is strongly regular in the sense of Robinson. This result is applied to a test problem for the heat equation with Stefan-Boltzmann boundary condition. The numerical tests confirm the theoretical results.

Key words. Augmented Lagrangian SQP method in Banach spaces, optimal control, control constraints, two-norm discrepancy, generalized equation, generalized Newton method, semilinear parabolic equation.

AMS subject classifications. 49K20, 35J25

1. Introduction. We consider an Augmented Lagrangian SQP method (ALSQP method) for the following class of optimal control problems, which includes some meaningful applications to control problems for semilinear partial differential equations:

$$\text{(P)} \qquad \begin{aligned} &\text{Minimize } f(y,u),\\ &\text{subject to } \Lambda y + \phi(y) - u = 0, \quad y \in Y, \quad u \in U_{ad} \subset U. \end{aligned}$$

In this setting Y and U are real Banach spaces, $f\colon\ Y \times U \to \mathbb{R}$ and $\phi\colon\ Y \to U$ are differentiable mappings, and U_{ad} is a nonempty, closed, convex and bounded subset of U. The operator Λ is a continuous linear operator from Y to U. In general, (P) is a non-convex problem. We will refer to u as the control, and to y as the state.

In the past years, the application of ALSQP methods to optimal control or identification problems for partial differential equations has made considerable progress. The list of contributions to this field has already become rather extensive so that we shall mention only the papers by Bergounioux and Kunish [6], Ito and Kunisch [13], [14], Kauffmann [15], Kunisch and Volkwein [16], and Volkwein [25], [26].

In this paper, we extend the analysis of the ALSQP method to a Banach space setting. This generalization is needed, if, for instance, the nonlinearities of the problem cannot be well defined in Hilbert spaces. In our application, this will concern the nonlinear mapping ϕ. A natural consequence of this extension is that, in contrast to the literature about the ALSQP method, we have to deal with the well known two-norm discrepancy. Another novelty in our approach is the presence of the control constraints $u \in U_{ad}$ in (P) , which complicates the discussion of the method. To resolve the associated difficulties, we rely on known results on the convergence of the generalized Newton method for generalized equations.

*Parts of this work were done when the third author was visiting professor at the Université Paul Sabatier in Toulouse.

[1]Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany; supported by SFB 393 "Numerical simulation on massive parallel computers."

[2]UMR CNRS MIP, Université Paul Sabatier, 31062 Toulouse Cedex 4, France.

[3]Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany.

One of the main goals of this paper is to reduce the convergence analysis to one main assumption, which has to be checked for the particular applications – the strong regularity of the optimality system. In this way, we hope to have shown a general way to perform the convergence analysis of the ALSQP method.

For (P) we concentrate on a particular type of augmentation, applied only to the nonlinearity of the state equation. Splitting up the state equation into $\Lambda y + z - u = 0$ and $z - \phi(y) = 0$ we will augment only the second equation. This type of augmentation is useful for our application to parabolic boundary control problems. The convergence analysis is confirmed by numerical tests, which are compared with those performed for the (non-augmented) SQP method.

We obtain the following main results: If the optimality system of first order necessary optimality conditions for (P) is strongly regular in the sense of Robinson, then the ALSQP method will be locally quadratic convergent under natural assumptions. This result is applied to a boundary control problem for a semilinear parabolic equation. In [23], the convergence of the (non-augmented) SQP method was shown for this particular problem by verifying this strong regularity assumption. In this way, our result is immediately applicable to obtain the convergence of the augmented method in our application.

The paper is organized as follows: In Section 2 we fix the general assumptions and formulate first order necessary and second order sufficient optimality conditions. Section 3 contains our example, a semilinear parabolic control problem. The ALSQP method is presented in Section 4, where we show that its iterates are well defined in the associated Banach spaces. The convergence analysis is developed in Section 5 on the basis of the Newton method for generalized equations. The last part of our paper reports on our numerical tests with the ALSQP method.

2. General assumptions and optimality conditions. We first fix the assumptions on the spaces and mappings. The Banach spaces Y and U mentioned in the introduction stand for the ones where the following holds:

- f is a mapping of class C^2 from $Y \times U$ into $\mathbb{R}$,
- ϕ is a mapping of class C^2 from Y into U.

For several reasons, among them, the formulation of the SQP method and the sufficient second order optimality conditions, we have to introduce real Hilbert spaces Y_2 and U_2 such that Y (respectively U) is continuously and densely imbedded in Y_2 (respectively U_2). Moreover, we identify U_2 with its dual U_2^*. Therefore, denoting by U^* the dual space of U, we have the continuous imbeddings $U \subset U_2 \subset U^*$.

Let us introduce the product space $V = Y \times U$, endowed with the norm $||v||_V = ||y||_Y + ||u||_U$, and the space $V_2 = Y_2 \times U_2$, endowed with the norm $||v||_{V_2} = ||y||_{Y_2} + ||u||_{U_2}$.

Notation: We shall denote the first and second order derivatives of f and ϕ by $f'(v)$, $f''(v)$, $\phi'(y)$, $\phi''(y)$, respectively. Partial derivatives are indicated by associated subscripts such as $f_y(v)$, $f_{yu}(v)$, etc. Notice that, by their very definition, $f'(v) \in V^*$, $f''(v) \in \mathcal{L}(V, V^*)$, $\phi'(y) \in \mathcal{L}(Y, U)$ and $\phi''(y) \in \mathcal{L}(Y, \mathcal{L}(Y, U))$. The open ball in V centered at v, with radius r is denoted by $B_V(v, r)$. The same notation is used in other Banach spaces. We will denote the duality pairing between U^* and U (resp. Y^* and Y) by $\langle \cdot , \cdot \rangle_{U^* \times U}$ (resp. $\langle \cdot , \cdot \rangle_{Y^* \times Y}$), while $\langle \cdot , \cdot \rangle$ is reserved in this paper for the scalar product of U_2.

Below we list our main assumptions:

(A1) Λ is a linear, continuous, and bijective operator from Y_2 to U_2. Moreover, its restriction to Y, still denoted by Λ, is continuous and bijective from Y to U. In addition, we assume that U_{ad} is closed in U_2.

(A2) (***Extension properties***) For all $r > 0$ there is a constant $c(r) > 0$ such that, for all $v_o \in B_V(0,r)$, we have

$$|f'(v_o)v| + ||\phi'(y_o)y||_{U_2} \le c(r)||v||_{V_2} \qquad \text{for all } v \in V, \tag{2.1}$$

$$|f''(v_o)[v_1,v_2]| + ||\phi''(y_o)[y_1,y_2]||_{U^*} \le c(r)||v_1||_{V_2}||v_2||_{V_2} \tag{2.2}$$

for all $v_1, v_2 \in V$. From (2.1) it follows that $f'(v)$ can be considered as a continuous linear operator from V_2 to $\mathbb{R}$, and $\phi'(y)$ can be considered as a continuous linear operator from Y_2 to U_2.
Since $\phi''(y_o)[y_1,y_2]$ belongs to U, and $U \subset U^*$, the term $||\phi''(y_o)[y_1,y_2]||_{U^*}$ is meaningful. Moreover, $f''(v)$ (respectively $\phi''(y)$) can be considered as a continuous bilinear operator from $V_2 \times V_2$ (respectively $Y_2 \times Y_2$) into $\mathbb{R}$ (respectively U^*). In the second order derivatives we shall write $[v,v] = v^2$.

(A3) (***Lipschitz properties***) For all $v_i \in B_V(0,r)$, $i = 1,2$, there is a $c(r) > 0$ such that

$$||f'(v_1) - f'(v_2)||_{V_2^*} + ||\phi'(y_1) - \phi'(y_2)||_{\mathcal{L}(Y_2,U_2)} \le c(r)||v_1 - v_2||_V, \tag{2.3}$$

$$\begin{aligned} &|(f''(v_1) - f''(v_2))[z_1,z_2]| + ||(\phi''(y_1) - \phi''(y_2))[\eta_1,\eta_2]||_{U^*} \\ &\le c(r)\,||v_1 - v_2||_V ||z_1||_{V_2}||z_2||_{V_2} \quad \text{for all } z_i = (\eta_i, u_i) \in V,\ i = 1,2. \end{aligned} \tag{2.4}$$

(A4) (***Remainder terms***) Let $r_i^F(x_o;h)$ denote the i-th order remainder term for the Taylor expansion of a mapping F at the point x_o in the direction h. Following Ioffe [11] and Maurer [18] we assume

$$\frac{r_1^f(v_o;v)}{||v||_{V_2}} + \frac{r_2^f(v_o;v)}{||v||_{V_2}^2} \to 0 \ \text{ as } ||v||_V \to 0, \tag{2.5}$$

$$\frac{r_1^\phi(y_o;y)}{||y||_{Y_2}} + \frac{r_2^\phi(y_o;y)}{||y||_{Y_2}^2} \to 0 \ \text{ as } ||y||_Y \to 0. \tag{2.6}$$

(A5) (***Regularity***)

- For all $y \in Y$, the operator $(\Lambda + \phi'(y))$ is bijective from Y_2 to U_2. Its restriction to Y, still denoted by $\Lambda + \phi'(y)$, is bijective from Y to U.
- For all $v \in V$, $f_y(v)$ belongs to $\widehat{Y}$, where $\widehat{Y}$ is a Banach space continuously imbedded in Y^*.
- The restriction of $(\Lambda + \phi'(y))^{-*}$ to $\widehat{Y}$ is continuous from $\widehat{Y}$ to U.

The first assumption concerns the linearized state equation. The second and third assumptions are needed to get optimal regularity for the adjoint equation. Indeed, the adjoint state corresponding to $\bar{v} = (\bar{y}, \bar{u})$ is defined by $\bar{p} = (\Lambda + \phi'(\bar{y}))^{-*} f_y(\bar{v}) \in U^*$. To study the convergence of the SQP method we need that $\bar{p}$ belongs to U.

In the analysis of the Generalized Newton Method, we need the following additional regularity conditions.

(A6) For every $y \in Y$, $\phi'(y)^*$ belongs to $\mathcal{L}(U, \hat{Y})$. The mapping $y \mapsto \phi'(y)^*$ is locally of class $C^{1,1}$ from Y into $\mathcal{L}(U, \widehat{Y})$. For every $y_1, y_2 \in Y$, $[\phi''(y_1)y_2]^*$ belongs to $\mathcal{L}(U, \widehat{Y})$. The mapping $(y_1, y_2) \mapsto [\phi''(y_1)y_2]^*$ is locally of class $C^{1,1}$ from $Y \times Y$ into $\mathcal{L}(U, \widehat{Y})$.

(A7) The mapping $v \mapsto f'(v)$ is locally of class $C^{1,1}$ from V into $\hat{V}$.

3. Example - Control of a semilinear parabolic equation. Let us consider the following particular case of (P) :

$$(E) \qquad \text{Minimize } f(y,u) = \frac{1}{2}\|y(T) - y_T\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|u\|^2_{L^2(\Sigma)} + \int_\Sigma a_u\, u - \int_\Sigma a_y\, y$$

subject to

$$\begin{aligned}
&y_t - \Delta y = d && \text{in } Q = \Omega \times (0,T),\\
&y(0) = a && \text{in } \Omega,\\
&\partial_\nu y + y = b + u - \varphi(y) && \text{on } \Sigma = \Gamma \times (0,T),\\
&u_a \le u(x,t) \le u_b.
\end{aligned}$$

Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary Γ of class C^2, $T > 0, \kappa > 0$, $y_T \in L^\infty(\Omega)$, $d \in L^\infty(Q)$, $a_u \in L^\infty(\Sigma)$, $a_y \in L^\infty(\Sigma)$, $b \in L^\infty(\Sigma)$, $a \in L^\infty(\Omega)$, and $u_a < u_b$ are given fixed. The function $\varphi : \mathbb{R} \to \mathbb{R}$ is nondecreasing, and locally of class $C^{2,1}$. (The choice $\varphi = |y|^3 y$ fits into this setting.)

Let us verify that problem (E) satisfies all our assumptions. This problem is related to (P) as follows:

$$\begin{aligned}
&\Lambda = (y_t - \Delta y, y(0), \partial_\nu y + y),\\
&\phi(y) = (d, a, \varphi(y(\cdot))),\\
&U = L^\infty(Q) \times L^\infty(\Omega) \times L^\infty(\Sigma),\\
&Y = \{y \in W(0,T) \mid y_t - \Delta y \in L^\infty(Q),\ y(0) \in L^\infty(\Omega),\ \partial_\nu y \in L^2(\Sigma),\\
&U_{ad} = \{(0,0,u) \in U \mid u_a \le u(x,t) \le u_b \text{ a.e. on } \Sigma\},\\
&U_2 = L^2(Q) \times L^2(\Omega) \times L^2(\Sigma),\\
&Y_2 = \{y \in W(0,T) \mid y_t - \Delta y \in L^2(Q),\ y(0) \in L^2(\Omega)\},\ \partial_\nu y \in L^2(\Sigma),
\end{aligned}$$

where $W(0,T)$ is the Hilbert space defined by

$$W(0,T) = \{y \in L^2(0,T; H^1(\Omega)) \mid \frac{dy}{dt} \in L^2(0,T; (H^1(\Omega))')\}.$$

The space Y (respectively Y_2) is endowed with the norm $\|y\|_Y = \|y_t - \Delta y\|_{L^\infty(Q)} + \|y(0)\|_{L^\infty(\Omega)} + \|\partial_\nu y\|_{L^\infty(\Sigma)}$ (respectively $\|y\|_{Y_2} = \|y_t - \Delta y\|_{L^2(Q)} + \|y(0)\|_{L^2(\Omega)} + \|\partial_\nu y\|_{L^2(\Sigma)}$). Let us check the assumptions.

- The operator Λ is obviously continuous from Y_2 to U_2, and is bijective from Y_2 to U_2 (see [17]). It is also a bijection from Y to U. (see [8], [20].) Thus **(A1)** is satisfied.
- Since $Y \subset L^\infty(Q)$ with continuous imbedding ([8], [20]), we can verify that ϕ is a mapping of class C^2 from Y into U, and that f is a mapping of class C^2 from $Y \times U$

into $\mathbb{R}$. Moreover, for all $v_o = (y_o, u_o) \in Y \times U$, we have

$$f_y(y_o, u_o)y = \int_\Omega (y_o(x,T) - y_T(x))y(x,T)\,dx - \int_\Sigma a_y(x,t)y(x,t)\,dSdt$$

$$f_u(y_o, u_o)u = \int_\Sigma (\kappa u_o(x,t) + a_u(x,t))u(x,t)\,dSdt,$$

$$\phi'(y_o)y = (0, 0, \varphi'(y_o)y).$$

Thus, the derivative $f_y(v_o)$ (respectively $f_u(v_o)$) can be identified with the triplet $(0, y_o(T) - y_T, -a_y) \in L^\infty(Q) \times L^\infty(\Omega) \times L^\infty(\Sigma)$ (respectively $(0, 0, \kappa u_o + a_u) \in L^\infty(Q) \times L^\infty(\Omega) \times L^\infty(\Sigma)$). The assumptions (2.1) and (2.3) can be easily satisfied.

• To verify assumption **(A5)**, let us introduce the space $\widehat{Y} = L^\infty(Q) \times L^\infty(\Omega) \times L^\infty(\Sigma)$. This space can be identified with the subspace of Y^* of all elements having the form

$$y \mapsto \int_Q \hat{y}_Q\, y\,dxdt + \int_\Omega \hat{y}_\Omega\, y(x,T)\,dx + \int_\Sigma \hat{y}_\Sigma\, y(x,t)\,dSdt,$$

where $(\hat{y}_Q, \hat{y}_\Omega, \hat{y}_\Sigma)$ belongs to $\widehat{Y}$. ¿From the above calculations, it is clear that $f_y(v_o)$ belongs to $\widehat{Y}$. Let $y_{(d,a,u)}$ be the solution to the equation

$$(3.1)\qquad \begin{aligned} y_t - \Delta y &= d \\ y(0) &= a \\ \partial_\nu y + y + \varphi'(y_o)y &= u. \end{aligned}$$

The operator $(d, a, u) \mapsto y_{(d,a,u)}$ is continuous and bijective from U_2 into Y_2 ([17]), and from U into Y ([8], [20]). The first part of **(A5)** is satisfied. To prove the second part, let us consider the adjoint equation

$$(3.2)\qquad \begin{aligned} -\pi_t - \Delta\pi &= \hat{y}_Q \\ \pi(T) &= \hat{y}_\Omega \\ \partial_\nu \pi + \pi + \varphi'(y_o)\pi &= \hat{y}_\Sigma. \end{aligned}$$

For all $(d, a, u) \in U$, and all $\hat{y} = (\hat{y}_Q, \hat{y}_\Omega, \hat{y}_\Sigma) \in \widehat{Y}$, by using a Green formula, we obtain

$$\begin{aligned} \int_Q \pi d + \int_\Omega \pi(0)a + \int_\Sigma \pi u &= \int_Q \hat{y}_Q\, y_{(d,a,u)} + \int_\Omega \hat{y}_\Omega\, y_{(d,a,u)}(T) + \int_\Sigma \hat{y}_\Sigma y_{(d,a,u)} \\ &= \left\langle \hat{y}, (\Lambda + \phi'(y_o))^{-1}(d,a,u) \right\rangle_{Y^* \times Y} \\ &= \left\langle (\Lambda + \phi'(y_o))^{-*}\hat{y}, (d,a,u) \right\rangle_{U^* \times U}. \end{aligned}$$

Therefore $p = (\Lambda + \phi'(v_o))^{-*}(\hat{y})$ is nothing else than $(\pi, \pi(0), \pi|_\Sigma)$. With this identity, we can easily verify the second part of assumption **(A5)**.

• Let us finally discuss properties of some second order derivatives. The second derivative $\phi''(y_o)$ is given by

$$(\phi''(y_o)[y_1, y_2]) = (0, 0, \varphi''(y_o)y_1y_2).$$

For $y_i \in Y$ and $\|y_o\|_Y \le r$ we have

$$\|\phi''(y_o)y_1y_2\|_{L^1(\Sigma)} \le \|\varphi''(y_o)\|_{L^\infty(\Sigma)}\|v_1\|_{L^2(\Sigma)}\|v_2\|_{L^2(\Sigma)} \le c(r)\|v_1\|_{L^2(\Sigma)}\|v_2\|_{L^2(\Sigma)}.$$

We can interprete $\phi''(y_o)y_1y_2$ as an element of $L^1(\Sigma) \subset L^\infty(\Sigma)^*$, and (2.2) can be checked. The other assumptions on the second order derivatives, precisely (2.4) and **(A4)**, are also satisfied.

4. Optimality conditions. This section is devoted to the discussion of the first and second order optimality conditions. Let $\bar v = (\bar y, \bar u)$ be a *local solution* of (P) . This means that

$$f(\bar v) \le f(v) \tag{4.1}$$

holds for all v, which belong to a sufficiently small ball $B_V(\bar v, \varepsilon)$ and satisfy all constraints of (P) .

THEOREM 4.1. *Let $\bar v = (\bar y, \bar u)$ be a local solution of (P) and suppose that the assumptions* (**A1**), (**A2**), *and* (**A5**) *are satisfied. Then there exists a unique Lagrange multiplier $\bar p \in U$ such that*

$$f_y(\bar y, \bar u)y + \langle \bar p, \Lambda y + \phi'(\bar y)y\rangle = 0 \qquad \textit{for all } y \in Y, \tag{4.2}$$

$$\langle f_u(\bar y, \bar u) - \bar p, u - \bar u\rangle \ge 0 \qquad \textit{for all } u \in U_{ad}. \tag{4.3}$$

Proof. Since f is Fréchet-differentiable at $\bar v = (\bar y, \bar u)$, ϕ is of class C^1 from Y to U, and $\Lambda + \phi'(\bar y)$ is surjective from Y to U, there exists a unique $\bar p \in U^*$ such that (4.2) and (4.3) be satisfied (see [12], and also Theorem 2.1 in [1]). The variational equation (4.2) admits a unique solution $\bar p$ defined by $\bar p = (\Lambda + \phi'(\bar y))^{-*} f_y(\bar v)$. Due to assumptions **(A5)**, it follows that $\bar p$ belongs to U. □

We next introduce the *Lagrange function* $L : Y \times U \times U \to \mathbb{R}$,

$$L(v,p) = L(y,u,p) = f(y,u) + \langle p, \Lambda y + \phi(y) - u\rangle. \tag{4.4}$$

The system (4.2)-(4.3) is equivalent to

$$L_y(\bar v, \bar p) = 0 \qquad \text{and} \qquad L_u(\bar v, \bar p)(u - \bar u) \ge 0 \qquad \text{for all } u \in U_{ad}.$$

For shortening, we shall write the adjoint equation (4.2) in the form $f_y(\bar v) + \bar p(\Lambda + \phi'(\bar y)) = 0$. Thus the *first order optimality system* for (P) is

$$\begin{aligned} & f_y(\bar v) + \bar p(\Lambda + \phi'(\bar y)) = 0, \\ & \langle f_u(\bar v) + \bar p, u - \bar u\rangle \ge 0, \qquad \text{for all } u \in U_{ad}, \\ & \Lambda \bar y + \phi(\bar y) - \bar u = 0, \\ & \bar u \in U_{ad}. \end{aligned} \tag{4.5}$$

In what follows, the derivatives in L' and L'' refer only to the variable v, but not to the Lagrange multiplier p. Let us assume that $\bar v$ also satisfies the following:

(SSC) ***Second order sufficient optimality condition***
There is $\delta > 0$ such that

$$L''(\bar v, \bar p)v^2 \ge \delta\, \|v\|_{V_2}^2 \tag{4.6}$$

holds for all $v = (y,u) \in Y \times U$ that satisfy the linearized equation

$$\Lambda y + \phi'(\bar y)y - u = 0. \tag{4.7}$$

REMARK 4.2. *The condition (SSC) is a quite strong assumption, and does not consider active control constraints, which might occur in U_{ad}. For instance, this can be useful for constraints of the type $U_{ad} = \{u \in L^\infty(D) \mid u_a \le u(x) \le u_b$ for all $x \in D\}$. In concrete applications, the use of an associated second order assumption is possible (see for example [23]). However, we intend to shed light on the main steps, which are needed for a convergence analysis of the augmented Lagrangian SQP method, rather than to present the difficult technical details connected with weakening (SSC) . We shall adress this issue again in section 6.* Let us complete this section by some simple results, which follow from the second order sufficient condition.

LEMMA 4.3. *Suppose that the assumptions* **(A1)**-**(A5)** *are satisfied. Suppose in addition that $\bar{v}$ satisfies the second order sufficient condition (SSC). Then there exists $\rho > 0$ such that, for every $(\hat{y}, \hat{u}, \hat{p})$ given in $B_{V\times U}((\bar{y}, \bar{u}, \bar{p}), \rho)$, we have*

$$L''(\hat{y}, \hat{u}, \hat{p})v^2 \ge \frac{\delta}{2} \|v\|_{V_2}^2 \tag{4.8}$$

for all $v = (y, u) \in V$ that satisfy the perturbed linearized equation

$$\Lambda y + \phi'(\hat{y})\, y - u = 0. \tag{4.9}$$

Proof. We briefly explain the main ideas of this quite standard result, to show where the different assumptions are needed. If $(\hat{y}, \hat{u}, \hat{p})$ is sufficiently close to $(\bar{y}, \bar{u}, \bar{p})$, then the quadratic form $L''(\hat{y}, \hat{u}, \hat{p})$ is arbitrarily close to $L''(\bar{y}, \bar{u}, \bar{p})$. By (SSC), **(A2)**, and **(A3)** we derive that

$$L''(\hat{y}, \hat{u}, \hat{p})v^2 \ge \frac{7\delta}{8} \|v\|_{V_2}^2 \tag{4.10}$$

provided that $\Lambda y + \phi'(\bar{y})\, y - u = 0$. An analogous estimate has to be shown for the solutions of the perturbed equation (4.9), where ϕ' is taken at $\hat{y}$. Write for short $B := L''(\hat{y}, \hat{u}, \hat{p})$ and define z as the unique solution of $\Lambda z + \phi'(\bar{y})\, z - u = 0$ (we use the first part of **(A5)**). Then

$$\Lambda(y - z) + \phi'(\bar{y})(y - z) = -(\phi'(\hat{y}) - \phi'(\bar{y}))y. \tag{4.11}$$

The assumptions **(A1)**, **(A3)**, and **(A5)** ensure the estimate

$$\|y - z\|_{Y_2} \le c\|(\phi'(\hat{y}) - \phi'(\bar{y}))y\|_{U_2} \le c\|\hat{y} - \bar{y}\|_Y \|y\|_{Y_2} \le c\,\rho\,\|y\|_{Y_2} \tag{4.12}$$

(here and below c stands for a generic constant). Therefore,

$$\begin{aligned} B\,v^2 &= B(z + (y - z), u)^2 = B(z, u)^2 + 2B[(z, u), (y - z, 0)] + B(y - z, 0)^2 \\ &\ge 7/8\ \delta\,\|(z, u)\|_{V_2}^2 - \varepsilon\,\|(z, u)\|_{V_2}^2 - c(\varepsilon)\|y - z\|_{Y_2}^2 \\ &\ge 6/8\ \delta\,\|(z, u)\|_{V_2}^2 - c\rho^2\|y\|_{Y_2}^2 \end{aligned}$$

follows by (4.10), (4.12) and Young inequality, where $\varepsilon > 0$ can be taken arbitrarily small. Now we re-substitute z by $y + (z - y)$ and arrive by similar estimates at

$$B\,v^2 \ge \frac{5\delta}{8}\,\|(y, u)\|_{V_2}^2 - c\rho^2\|y\|_{Y_2}^2 \ge \frac{4\delta}{8}\,\|v\|_{V_2}^2,$$

provided that ρ is sufficiently small. Thus (4.8) is proven. □

Although we shall not directly apply the next result, we state it to show why the different assumptions are needed. Some of them have been assumed to deal with the well known two-norm discrepancy.

LEMMA 4.4. *Let* $(\bar{v}, \bar{p}) = (\bar{y}, \bar{u}, \bar{p})$ *satisfy the optimality system* (4.5) *of* (P) *and the second order sufficient condition* (SSC). *Suppose that the assumptions* (**A1**)-(**A5**) *are fulfilled. Then there are constants* $\varepsilon > 0$ *and* $\sigma > 0$ *such that the quadratic growth condition*

$$f(v) - f(\bar{v}) \geq \sigma \|v - \bar{v}\|_{V_2}^2 \tag{4.13}$$

holds for all admissible $v \in B_V(\bar{v}, \varepsilon)$.

Proof. The first order optimality system implies

$$f(v) - f(\bar{v}) = L(v, \bar{p}) - L(\bar{v}, \bar{p}) \geq 1/2\ L''(\bar{v}, \bar{p})(v - \bar{v})^2 + r_2^L(\bar{v}, \bar{p}; v - \bar{v}). \tag{4.14}$$

Subtracting the state equations for y and $\bar{y}$, analogously to (4.11) we find that

$$(\Lambda + \phi'(\bar{y}))(y - \bar{y}) - (u - \bar{u}) = -r_1^\phi(\bar{y}; y - \bar{y}).$$

Define h by $(\Lambda + \phi'(\bar{y}))h = r_1^\phi$. Then $v_h := (y - \bar{y} + h, u - \bar{u})$ solves the linearized equation (4.7), and the coercivity estimate of (SSC) can be applied to v_h. Moreover, **(A5)** yields

$$\|h\|_{Y_2} \leq c\, \|r_1^\phi\|_{U_2}.$$

We insert v_h in (4.14), write for short $B := L''(\bar{v}, \bar{p})$ and proceed similarly to the estimation of Bv^2 in the last proof:

$$\begin{aligned} f(v) - f(\bar{v}) &\geq 1/2\, B(v_h + v - \bar{v} - v_h)^2 + r_2^L \\ &\geq \delta/2\, \|v_h\|_{V_2}^2 - \epsilon \|v_h\|_{V_2}^2 - c\|v - \bar{v} - v_h\|_{V_2}^2 + r_2^L \\ &\geq \delta/3\, \|v - \bar{v}\|_{V_2}^2 - c\|v - \bar{v} - v_h\|_{V_2}^2 + r_2^L \\ &= \|v - \bar{v}\|_{V_2}^2 \{\delta/3 - c\frac{\|v - \bar{v} - v_h\|_{V_2}^2}{\|v - \bar{v}\|_{V_2}^2} - \frac{|r_2^L|}{\|v - \bar{v}\|_{V_2}^2}\}. \end{aligned}$$

In these estimates, the assumptions **(A2)** and **(A3)** were used. We have $\|v - \bar{v} - v_h\|_{V_2} = \|h\|_{Y_2}$, and the estimate of h by the first order remainder term r_1^ϕ can be inserted. Let $\varepsilon \to 0$. Then **(A4)** yields $r_1^\phi / \|y - \bar{y}\|_{Y_2} \to 0$ and $r_2^L / \|v - \bar{v}\|_{V_2}^2 \to 0$. Then, the quadratic growth estimate follows from classical arguments. ☐

This Lemma shows that the second order condition (SSC) is sufficient for local optimality of $(\bar{y}, \bar{u})$ in the sense of V, whenever $(\bar{y}, \bar{u})$ solves the first order optimality system. Notice that we cannot show local optimality in the sense of V_2.

5. Augmented Lagrangian method.

5.1. Augmented Lagrangian SQP method. In this section we introduce the Augmented Lagrangian SQP method (ALSQP) with some special type of augmentation. For this, we first represent (P) in the equivalent form

$$(\widetilde{\mathrm{P}}) \qquad \begin{aligned} &\text{Minimize } f(y, u), \\ &\text{subject to } z - \phi(y) = 0,\ \Lambda y + z - u = 0,\ z \in U,\ u \in U_{ad}. \end{aligned}$$

The augmentation takes into account only the nonlinear equation $z - \phi(y) = 0$. The ALSQP method is obtained by applying the classical SQP method to the problem

$$(\mathrm{P}_\alpha) \qquad \begin{aligned} &\text{Minimize } f_\alpha(y,u) = f(y,u) + \frac{\alpha}{2}\|z-\phi(y)\|^2_{U_2}, \\ &\text{subject to } z - \phi(y) = 0,\ \Lambda y + z - u = 0,\ z \in U,\ u \in U_{ad}, \end{aligned}$$

where $\alpha > 0$ is given. We define the Lagrange functional $\mathcal{L}$ for $(\tilde{P})$, and the corresponding augmented functional $\mathcal{L}_\alpha$ on $Y \times U^4$ as follows:

$$\mathcal{L}(y,u,z,p,\lambda) = f(y,u) + \langle p, \Lambda y + z - u\rangle + \langle \lambda, z - \phi(y)\rangle,$$

$$\mathcal{L}_\alpha(y,u,z,p,\lambda) = \mathcal{L}(y,u,z,p,\lambda) + \tfrac{\alpha}{2}\|z-\phi(y)\|^2_{U_2}.$$

Once again, the derivatives $\mathcal{L}'$ and $\mathcal{L}''$ will stand for derivatives with respect to (y,u,z) and do not refer to the Lagrange multipliers (p,λ). The same remark concerns $\mathcal{L}_\alpha$. Let $(y_n,u_n,z_n,p_n,\lambda_n)$ denote the current iterate of the ALSQP method, and consider the linear-quadratic problem

$$(\mathrm{QP}^\alpha_{n+1}) \qquad \begin{aligned} \text{Minimize} \quad & f'_\alpha(y_n,u_n,z_n)(y-y_n,u-u_n,z-z_n) \\ & + \tfrac{1}{2}\mathcal{L}''_\alpha(y_n,u_n,z_n,p_n,\lambda_n)(y-y_n,u-u_n,z-z_n)^2, \\ \text{subject to} \quad & z - \phi(y_n) - \phi'(y_n)(y-y_n) = 0, \\ & \Lambda y + z - u = 0,\ y \in Y,\ z \in U,\ u \in U_{ad}. \end{aligned}$$

The new iterate $(y_{n+1},u_{n+1},z_{n+1},p_{n+1},\lambda_{n+1})$ is obtained by taking the solution $(y_{n+1},u_{n+1},z_{n+1})$ of $(\mathrm{QP}^\alpha_{n+1})$ (if it exists), and the multipliers (p_{n+1},λ_{n+1}) associated with the constraints $\Lambda y + z - u = 0$, and $z - \phi(y_n) - \phi'(y_n)(y-y_n) = 0$, respectively. For $\alpha = 0$ we recover the classical SQP method.

Let us also introduce the following problem:

$$(\widehat{\mathrm{QP}}^\alpha_{n+1}) \qquad \begin{aligned} \text{Minimize} \quad & f'(v_n)(v-v_n) + \tfrac{1}{2}f''(v_n)(v-v_n)^2 \\ & -\tfrac{1}{2}\langle \lambda_n + \alpha(z_n - \phi(y_n)), \phi''(y_n)(y-y_n)^2\rangle, \\ \text{subject to} \quad & \Lambda y + \phi(y_n) + \phi'(y_n)(y-y_n) - u = 0,\ u \in U_{ad}. \end{aligned}$$

The problems $(\mathrm{QP}^\alpha_{n+1})$ and $(\widehat{\mathrm{QP}}^\alpha_{n+1})$ are equivalent in the sense precised below.

THEOREM 5.1. *Let $(y_{n+1},u_{n+1},z_{n+1})$ be a solution of (QP^α_{n+1}) with associated Lagrange multipliers $(p_{n+1},\lambda_{n+1}) \in U \times U$. Then (y_{n+1},u_{n+1}) must solve the problem $(\widehat{QP}^\alpha_{n+1})$, and the multiplier p_{n+1} is the solution to the equation*

$$\begin{aligned} p_{n+1}(\Lambda + \phi'(y_n)) = {} & f_y(v_n) + f_{yy}(v_n)(y_{n+1}-y_n) + f_{yu}(v_n)(u_{n+1}-u_n) \\ & -(\lambda_n + \alpha(z_n - \phi(y_n)))\phi''(y_n)(y_{n+1}-y_n). \end{aligned} \tag{5.1}$$

Moreover, z_{n+1} and λ_{n+1} must satisfy

$$\lambda_{n+1} = -p_{n+1} \tag{5.2}$$

$$z_{n+1} = \phi(y_n) + \phi'(y_n)(y_{n+1}-y_n). \tag{5.3}$$

Conversely, if (y_{n+1},u_{n+1}) is a solution of $(\widehat{QP}^\alpha_{n+1})$, and $(z_{n+1},p_{n+1},\lambda_{n+1})$ are defined by (5.1) – (5.3), then $(y_{n+1},u_{n+1},z_{n+1})$ is a solution to (QP^α_{n+1}) with associated Lagrange multipliers (p_{n+1},λ_{n+1}).

Proof. Let us first assume that $(y_{n+1}, u_{n+1}, z_{n+1})$ solves $(\mathrm{QP}^\alpha_{n+1})$. To show that (y_{n+1}, u_{n+1}) solves $(\widehat{\mathrm{QP}}^\alpha_{n+1})$ and that the relations (5.1)–(5.3) are satisfied, we investigate the following:

• *Explicit form of* $(\mathrm{QP}^\alpha_{n+1})$. We expand all derivatives occuring in the problem $(\mathrm{QP}^\alpha_{n+1})$. Write for short $\|\cdot\| = \|\cdot\|_{U_2}$ and introduce for convenience the functional $g(y,z) = \frac{\alpha}{2}\|z - \phi(y)\|^2$. Then

$$g'(y_n, z_n)(y,z) = \alpha\langle z_n - \phi(y_n), z - \phi'(y_n)y\rangle,$$
$$g''(y_n, z_n)(y,z)^2 = \alpha\big(\|z - \phi'(y_n)y\|^2 - \langle z_n - \phi(y_n), \phi''(y_n)y^2\rangle\big).$$

Having this, the objective to minimize in $(\mathrm{QP}^\alpha_{n+1})$ is given by

$$\begin{aligned} J(y,u,z) &= f'(y_n,u_n)(y-y_n, u-u_n) + \alpha\langle z_n - \phi(y_n), z - z_n - \phi'(y_n)(y-y_n)\rangle \\ &\quad + \frac{1}{2} f''(y_n,u_n)(y-y_n,u-u_n)^2 + \frac{\alpha}{2}\|z - z_n - \phi'(y_n)(y-y_n)\|^2 \\ &\quad - \frac{1}{2}\langle \lambda_n + \alpha(z_n - \phi(y_n)), \phi''(y_n)(y-y_n)^2\rangle. \end{aligned}$$

The minimization is subject to the constraints

$$\begin{aligned} &\Lambda y + z - u = 0, \qquad u \in U_{ad} \\ &z - \phi(y_n) - \phi'(y_n)(y - y_n) = 0. \end{aligned} \tag{5.4}$$

• *Reduction to* $(\widehat{\mathrm{QP}}^\alpha_{n+1})$. To reduce the dimension of the problem, we exploit the second one of the equations (5.4): We insert the expression $z - z_n - \phi'(y_n)(y - y_n) = \phi(y_n) - z_n$ in the functional J. Then the second and fourth items in the definition of J are constant with respect to (y,z,u). They depend only on the current iterate and can be neglected during the minimization of J. The associated functional to be minimized is

$$\begin{aligned} \tilde{J}(y,u) &= f'(y_n,u_n)(y-y_n,u-u_n) + \frac{1}{2} f''(y_n,u_n)(y-y_n,u-u_n)^2\rangle \\ &\quad - \frac{1}{2}\langle \lambda_n + \alpha(z_n - \phi(y_n)), \phi''(y_n)(y-y_n)^2\rangle. \end{aligned}$$

Moreover, we can delete the second equation of (5.4) by inserting the expression for z in the first one. This explains why (y_{n+1}, u_{n+1}) is a solution of $(\widehat{\mathrm{QP}}^\alpha_{n+1})$.

• *Necessary optimality conditions.* To derive the necessary conditions for the triplet $(y_{n+1}, u_{n+1}, z_{n+1})$, we work with the Lagrange functional

$$\tilde{\mathcal{L}} = J + \langle p, \Lambda y + z - u\rangle + \langle \lambda, z - \phi(y_n) - \phi'(y_n)(y - y_n)\rangle.$$

The conditions are $\tilde{\mathcal{L}}_y = 0$, $\tilde{\mathcal{L}}_z = 0$, $\tilde{\mathcal{L}}_u(u - u_{n+1}) \geq 0$, for all $u \in U_{ad}$. An evaluation yields

$$\begin{aligned} 0 &= f_y(v_n) + f_{yy}(v_n)(y_{n+1} - y_n) + f_{yu}(v_n)(u_{n+1} - u_n) - \phi'(y_n)^*\lambda_{n+1} \\ &\quad - (\lambda_n + \alpha(z_n - \phi(y_n)))\phi''(y_n)(y_{n+1} - y_n) + p_{n+1}\Lambda, \end{aligned} \tag{5.5}$$

$$0 = \lambda_{n+1} + p_{n+1}, \tag{5.6}$$

$$0 \leq \langle f_u(v_n) + f_{uu}(v_n)(u_{n+1} - u_n) + f_{yu}(v_n)(y_{n+1} - y_n) - p_{n+1}, u - u_{n+1}\rangle \tag{5.7}$$

for $u \in U_{ad}$. We mention for later use, that the equations (5.4) belong to the optimality system of $(\mathrm{QP}^\alpha_{n+1})$, too. The update formulas for p_{n+1} and λ_{n+1} follow from (5.5), (5.6).

We have shown one direction of the statement. The converse direction can be proved in a completely analogous manner. If (y_{n+1}, u_{n+1}) solves $(\widehat{\mathrm{QP}}^\alpha_{n+1})$, then we substitute z for $\phi(y_n) + \phi'(y_n)(y - y_n)$ and $-\lambda_{n+1}$ for p_{n+1} in the corresponding positions. Then it is easy to verify that $(y_{n+1}, u_{n+1}, z_{n+1})$ minimizes J subject to (5.4), and that λ_{n+1} is the multiplier associated to the equation $z - \phi(y_n) - \phi'(y_n)(y - y_n) = 0$. □

REMARK 5.2. *The update rules* (5.2) – (5.3) *imply that the Lagrange multiplier λ coincides with $-p$ during the iteration, while this is not necessarily true for the initial values of λ_n and p_n. Therefore, with possible exception of the first step, up to a constant, the objective functional of $(\widehat{QP}^\alpha_{n+1})$ is*

$$\tilde{J} = f'(y_n, u_n)(y - y_n, u - u_n) + \frac{1}{2} L''(y_n, u_n, p_n)(y - y_n, u - u_n)^2$$
$$- \tfrac{\alpha}{2} \langle z_n - \phi(y_n), \phi''(y_n)(y - y_n)^2 \rangle.$$

This easily follows by calculating $L''(y_n, u_n, p_n)$ from the formula (4.4). *Moreover, we are justifed to replace λ_n by $-p_n$ in the variational equation* (5.1).

Theorem 5.1 shows that the iterates of the ALSQP method can be obtained by solving the reduced problem $(\widehat{\mathrm{QP}}^\alpha_{n+1})$, provided that solutions of $(\mathrm{QP}^\alpha_{n+1})$ exist. This question of existence, can be answered by considering $(\widehat{\mathrm{QP}}^\alpha_{n+1})$ as well:

THEOREM 5.3. *Let $(\bar{y}, \bar{u}, \bar{p})$ satisfy the assumptions of Lemma* 4.3 *and let $\bar{z} = \phi(\bar{y})$. If $\|(y_n, u_n, p_n, z_n) - (\bar{y}, \bar{u}, \bar{p}, \bar{z})\|_{Y \times U^4}$ is sufficiently small, then $(\widehat{QP}^\alpha_{n+1})$ has a unique solution (y_{n+1}, u_{n+1}). Moreover, $(y_{n+1}, u_{n+1}, z_{n+1})$ (with z_{n+1} being defined by* (5.3)) *is the unique solution of (QP^α_{n+1}).*

Proof. Let us first prove the existence for $(\widehat{\mathrm{QP}}^\alpha_{n+1})$. Assume $\|(y_n, u_n, p_n, z_n) - (\bar{y}, \bar{u}, \bar{p}, \bar{z})\|_{Y \times U^4} < \rho$. In view of the remark above, the functional $\tilde{J}$ can be taken instead of J for the minimization in $(\widehat{\mathrm{QP}}^\alpha_{n+1})$. Its quadratic part is

$$\begin{aligned} & L''(v_n, p_n)(v - v_n)^2 - \alpha \langle z_n - \phi(y_n), \phi''(y_n)(y - y_n)^2 \rangle \\ &= f''(v_n)(v - v_n)^2 - \langle p_n + \alpha(z_n - \phi(y_n)), \phi''(y_n)(y - y_n)^2 \rangle \\ &= L''(v_n, \tilde{p}_n)(v - v_n)^2, \end{aligned}$$

where $\tilde{p}_n := p_n + \alpha(z_n - \phi(y_n))$. For $\rho \downarrow 0$, $\tilde{p}_n$ tends to $\bar{p}$ in U, since $z_n - \phi(y_n) \to \bar{z} - \phi(\bar{y}) = 0$. Lemma 4.3 yields that the objective functional of $(\widehat{\mathrm{QP}}^\alpha_{n+1})$ is coercive on the set $\tilde{C} = \{(y, u) \in Y_2 \times U_2 \mid \Lambda y + \phi'(y_n) y - u = 0\}$, hence it is strictly convex there. The set U_{ad} is non-empty, bounded, convex, and closed in U, and in U_2 as well. We have assumed in **(A5)** that $(\Lambda + \phi'(y))^{-1}$ is continuous from U_2 to Y_2 at all $y \in Y$, in particular at $y = y_n$. Therefore, $\tilde{C}$ is non-empty, convex, closed, and bounded in $Y_2 \times U_2$. Now existence and uniqueness of a solution $(y_{n+1}, u_{n+1}) \in Y_2 \times U_2$ to $(\widehat{\mathrm{QP}}^\alpha_{n+1})$ are standard conclusions. Moreover, $U_{ad} \subset U$, hence $u_{n+1} \in U$, and the regularity properties of $(\Lambda + \phi'(y_n))^{-1}$ guarantee that $y_{n+1} \in Y$. Further, $z_{n+1} \in U$ follows from (5.3). Existence and uniqueness for $(\mathrm{QP}^\alpha_{n+1})$ are obtained from Theorem 5.1. □

The update rules of Theorem 5.1 show that (p_{n+1}, λ_{n+1}) is uniquely determined in $U_2 \times U_2$. We get even better regularity:

COROLLARY 5.4. *If the initial element* $(y_n, u_n, z_n, p_n, \lambda_n)$ *is taken from* $Y \times U^4$, *then the iterates* $\{(y_n, u_n, z_n, p_n, \lambda_n)\}$ *generated by the ALSQP method are uniquely determined and belong to* $Y \times U^4$.

Proof. Existence and uniqueness follows from the last theorem and the update rules (5.2)–(5.3). We also know that $(y_{n+1}, u_{n+1}, z_{n+1}) \in Y \times U^2$. The only new result we have to derive is that (p_{n+1}, λ_{n+1}) remains in $U \times U$ as well. Since $\lambda_{n+1} = -p_{n+1}$, we have to verify $p_{n+1} \in U$. This, however, follows instantly from the equation (5.1): We know that $f_y(v_n)$, $f_{yy}(v_n)(y_{n+1} - y_n)$, and $f_{yu}(v_n)(u_{n+1} - u_n)$ belong to $\widehat{Y}$ (assumptions (A5), (A6), (A7)). Moreover, the same holds for $(\phi''(y_n)(y_{n+1} - y_n))^*(p_n + \alpha(z_n - \phi(y_n)))$ by (A6). Therefore, (A5) ensures the solution p_{n+1} of (5.1) to be in U. ☐

5.2. Newton method for the optimality system of (P_α). The augmented SQP method can be considered as a computational algorithm to solve the first order optimality system of (P_α) by the generalized Newton method. This equivalence will be our tool in the convergence analysis. The optimality system for (P_α) consists of the equations

$$
\begin{aligned}
&(\mathcal{L}_\alpha(w))_y = 0, \\
&(\mathcal{L}_\alpha(w))_z = 0, \\
&(\mathcal{L}_\alpha(w))_u(\tilde{u} - u) \geq 0 \qquad \text{for all } u \in U_{ad}, \\
&\Lambda y + z - u = 0, \\
&z - \phi(y) = 0,
\end{aligned}
\tag{5.8}
$$

for the unknown variable $w = (y, u, z, p, \lambda)$. The optimality system (5.8) of (P_α) is equivalent to a generalized equation. To see this, let us first introduce the following set-valued mappings:

$$
N(u) = \begin{cases} \{q \in U \mid \langle q, \tilde{u} - u \rangle \leq 0 \quad \text{for all } \tilde{u} \in U_{ad}\} & \text{if } u \in U_{ad}, \\ \emptyset & \text{if } u \notin U_{ad}. \end{cases}
$$

$$
\mathcal{N}(w) = \{0_{\widehat{Y}}\} \times \{0_U\} \times N(u) \times \{0_U\} \times \{0_U\},
$$

and consider $F : Y \times U^4 \to \widehat{Y} \times U^4$ defined by

$$
F(w) = \begin{pmatrix} f_y(y,u) - \alpha(z - \phi(y))\phi'(y) + p\Lambda - \lambda\phi'(y) \\ \alpha(z - \phi(y)) + p + \lambda \\ f_u(y,u) - p \\ \Lambda y + z - u \\ z - \phi(y) \end{pmatrix}.
\tag{5.9}
$$

Notice that $N(u)$ has a closed graph in $U \times U$. It is the restriction to U of the normal cone at U_{ad} in the point u. (For the definition of the normal cone, we refer to [5].)

LEMMA 5.5. *The optimality system* (5.8) *of* (P_α) *is equivalent to the generalized equation*

$$0 \in F(w) + \mathcal{N}(w). \tag{5.10}$$

Proof. By calculating the derivatives of $\mathcal{L}_\alpha$ in (5.8), we easily verify that:

$$F(w) = \begin{pmatrix} (\mathcal{L}_\alpha(w))_y \\ (\mathcal{L}_\alpha(w))_z \\ (\mathcal{L}_\alpha(w))_u \\ \Lambda y + z - u \\ z - \phi(y) \end{pmatrix}.$$

Therefore, by the definition of F, (5.10) is equivalent to

$$\begin{aligned} &0 = (\mathcal{L}_\alpha(w))_y \\ &0 = (\mathcal{L}_\alpha(w))_z \\ &0 \in (\mathcal{L}_\alpha(w))_u + N(u) \\ &0 = \Lambda y + z - u \\ &0 = z - \phi(y). \end{aligned}$$

The third relation can be rewritten as:

$$u \in U_{ad} \quad \text{and} \quad (\mathcal{L}_\alpha(w))_u(\tilde{u} - u) \geq 0 \qquad \text{for all } \tilde{u} \in U_{ad}.$$

This is just the variational inequality of (5.8), and the equivalence of (5.8) and (5.10) is verified. ∎

Next we recall some facts about generalized equations and related convergence results for the Generalized Newton Method (GNM). Let $\mathcal{W}$ and $\mathcal{E}$ be Banach spaces, and let $\mathcal{O}$ be an open subset of $\mathcal{W}$. Let $\mathcal{F}$ be a differentiable mapping from $\mathcal{O}$ into $\mathcal{E}$, and $\mathcal{T}$ be a set-valued mapping from $\mathcal{O}$ into $\mathcal{P}(\mathcal{E})$ with closed graph. Consider the generalized equation

$$\omega \in \mathcal{O}, \qquad 0 \in \mathcal{F}(\omega) + \mathcal{T}(\omega). \tag{5.11}$$

The generalized Newton method for (5.11) consists in the following algorithm:

- Choose a starting point $\omega_0 \in \mathcal{O}$,
- For $k = 0, 1, \ldots$, compute ω_{k+1}, the solution to the generalized equation:

$$\omega \in \mathcal{O}, \qquad 0 \in \mathcal{F}(\omega_k) + \mathcal{F}'(\omega - \omega_k) + \mathcal{T}(\omega). \tag{5.12}$$

The generalized Newton method is locally convergent under some assumptions stated below.

(C1) Equation (5.11) admits at least one solution $\bar{\omega}$.

(C2) There exist constants $\tilde{r}(\bar{\omega})$ and $\tilde{c}(\bar{\omega})$ such that $B_{\mathcal{W}}(\bar{\omega}, \tilde{r}(\bar{\omega})) \subset \mathcal{O}$, and

$$\|\mathcal{F}'(\omega_1) - \mathcal{F}'(\omega_2)\|_{\mathcal{L}(\mathcal{W};\mathcal{E})} \leq \tilde{c}(\bar{\omega})\|\omega_1 - \omega_2\|_{\mathcal{W}}$$

for all $\omega_1, \omega_2 \in B_{\mathcal{W}}(\bar{\omega}, \tilde{r}(\bar{\omega}))$.

DEFINITION 5.6. *The generalized equation is said to be strongly regular at $\omega^* \in \mathcal{O}$, if there exist constants $r(\omega^*)$ and $c(\omega^*)$, such that, for all $\xi \in B_{\mathcal{E}}(0, r(\omega^*))$, the perturbed generalized equation*

$$\omega \in \mathcal{O}, \qquad \xi \in \mathcal{F}(\omega^*) + \mathcal{F}'(\omega - \omega^*) + \mathcal{T}(\omega), \tag{5.13}$$

has a unique solution $S(\omega^*, \xi)$ *satisfying*

$$\|S(\omega^*, \xi_1) - S(\omega^*, \xi_2)\|_{\mathcal{W}} \le c(\omega^*)\|\xi_1 - \xi_2\|_{\mathcal{W}}$$

for all $\xi_1, \xi_2 \in B_{\mathcal{E}}(0, r(\omega^*))$. The theorem below is a variant of Robinson's implicit function theorem ([21], Theorem 2.1).

THEOREM 5.7. ([4], *Theorem* 2.5) *Assume that (5.11) is strongly regular at some* $\bar\omega \in \mathcal{O}$, *and that* $(C1)$ *and* $(C2)$ *are fulfilled. Then there exist* $\rho(\bar\omega) > 0$, $k(\bar\omega) > 0$, *and a mapping* S_0 *from* $B_{\mathcal{W}}(\bar\omega, \rho(\bar\omega)) \subset \mathcal{O}$ *into* $B_{\mathcal{W}}(\bar\omega, \rho(\bar\omega))$ *such that, for every* $\omega^* \in B_{\mathcal{W}}(\bar\omega, \rho(\bar\omega))$, $S_0(\omega^*)$ *is the unique solution to* (5.13), *and*

$$\|S_0(\omega^*) - \bar\omega\|_{\mathcal{W}} \le k(\bar\omega)\|\omega^* - \bar\omega\|^2_{\mathcal{W}}.$$

The following theorem is an extension to the generalized equation (5.11) of the well known Newton-Kantorovitch theorem. It is a direct consequence of Theorem 5.7.

THEOREM 5.8. ([4], *Theorem* 2.6) *Assume that the hypotheses of Theorem* 5.7 *are fulfilled. Then there exists* $\tilde\rho(\bar\omega) > 0$ *such that, for any starting point* $\omega_0 \in B_{\mathcal{W}}(\bar w, \tilde\rho(\bar\omega))$, *the generalized Newton method generates a unique sequence* $(\omega_k)_k$ *convergent to* $\bar\omega$, *and satisfying*

$$\|\omega_{k+1} - \bar\omega\|_{\mathcal{W}} \le k(\bar\omega)\|\omega_k - \bar\omega\|^2_{\mathcal{W}} \qquad \text{for all } k \ge 1.$$

We apply these results to set up the generalized Newton method for the generalized equation (5.10), which is the abstract formulation of the optimality system of (P_α).

LEMMA 5.9. *The generalized Newton method for solving the optimality system of* (P_α), *defined by* (5.12), *proceeds as follows: Let* $w_n = (y_n, u_n, z_n, p_n, \lambda_n) \in Y \times U^4$ *be the current iterate. Then the next iterate* $w_{n+1} = (y_{n+1}, u_{n+1}, z_{n+1}, p_{n+1}, \lambda_{n+1}) \in Y \times U^4$ *is the solution of the following generalized equation for* $w = (y, u, z, p, \lambda)$:

$$(5.14)\quad 0 = f_y(y_n, u_n) + f_{yy}(y_n, u_n)(y - y_n) + f_{yu}(y_n, u_n)(u - u_n) - \\ -(\lambda_n + \alpha(z_n - \phi(y_n))\phi''(y_n)(y - y_n) + p\Lambda - \lambda\phi'(y_n)$$

$$(5.15)\quad 0 = \lambda + p$$

$$(5.16)\quad 0 \in f_u(y_n, u_n) + f_{uu}(y_n, u_n)(u - u_n) + f_{uy}(y_n, u_n)(y - y_n) - p + N(u)$$

$$(5.17)\quad 0 = \Lambda y + z - u$$

$$(5.18)\quad 0 = z - \phi(y_n) - \phi'(y_n)(y - y_n).$$

Proof. This iteration scheme is a conclusion of the iteration rule (5.12) applied to the concrete choice of (5.9) for F. The computations are straightforward. We should only mention the following equivalent transformation, which finally leads to (5.14), (5.15): Due to the concrete expression for F given in (5.12), the first two relations in $0 \in F(w_n) + F'(w_n)(w - w_n) + \mathcal{N}(w)$ are

$$(5.19)\qquad 0 = f_y(y_n, u_n) + f_{yy}(y_n, u_n)(y - y_n) + f_{yu}(y_n, u_n)(u - u_n) - \\ -\alpha(z - \phi(y_n) - \phi'(y_n)(y - y_n))\phi'(y_n) \\ -(\lambda_n + \alpha(z_n - \phi(y_n))\phi''(y_n)(y - y_n)$$

$$(5.20)\qquad 0 = \alpha(z - \phi(y_n) - \phi'(y_n)(y - y_n)) + p + \lambda.$$

Inserting (5.18) in (5.19), (5.20) we obtain (5.14), (5.15). □

To apply Theorem 5.8 to the concrete generalized equation (5.10), we need that (5.10) be strongly regular at $\bar{w}$, and that conditions **(C1)** and **(C2)** be satisfied. The assumption of strong regularity at $\bar{w}$ must be assumed here. It has to be checked for each particular application. In general, the verification of strong regularity requires a detailed analysis. In the case of the optimal control of parabolic partial differential equations, we refer to the discussion of the SQP method in Tröltzsch [23]. The strong regularity of an associated generalized equation was proved there by means of a result on L^∞-Lipschitz stability from [22]. The associated semilinear elliptic case was studied by Unger [24].

The conditions **(C1)** and **(C2)** can be verified with assumptions **(A6)** and **(A7)**.

LEMMA 5.10. *The mapping $w \mapsto F(w)$ is of class $C^{1,1}$ from $Y \times U^4$ into $\widehat{Y} \times U^4$.*

Proof. This statement is an immediate consequence of **(A6)** and **(A7)**. ∎

THEOREM 5.11. *Let $(\bar{y}, \bar{u})$ be a local solution of (P), and let $\bar{p}$ be the associated adjoint state. Assume that the generalized equation: Find $(y, u, p) \in Y \times U^2$ such that*

$$0 = p\Lambda + p\phi'(y) + f_y(y, u),$$
$$0 \in f_u(y, u) + N(u), \tag{5.21}$$
$$0 = \Lambda y + \phi(y) - u,$$

be strongly regular at $(\bar{y}, \bar{u}, \bar{p})$. Then the generalized equation

$$\text{Find } w \in Y \times U^4 \quad \text{such that } F(w) \in \mathcal{N}(w), \tag{5.22}$$

is strongly regular at $\bar{w} = (\bar{y}, \bar{u}, \bar{z}, \bar{p}, \bar{\lambda})$, where $\bar{z} = \phi(\bar{y})$ and $\bar{\lambda} = -\bar{p}$.

Proof. Let $e = (e_p, e_\lambda, e_u, e_y, e_z)$ be a perturbation in $\widehat{Y} \times U^4$. The linearized generalized equation for (5.22) at the point $\bar{w}$, associated with the perturbation e, is

$$
\begin{aligned}
e_p &= \bar{f}_y + \bar{f}_{yy}(y - \bar{y}) + \bar{f}_{yu}(u - \bar{u}) - (\bar{\lambda} + \alpha(\bar{z} - \bar{\phi}))\bar{\phi}''(y - \bar{y}) \\
&\quad -\alpha(z - \bar{\phi} - \bar{\phi}'(y - \bar{y}))\bar{\phi}' + p\Lambda - \lambda\bar{\phi}' \\
e_\lambda &= \alpha(z - \bar{\phi} - \bar{\phi}'(y - \bar{y})) + p + \lambda, \\
e_u &\in \bar{f}_u + \bar{f}_{uu}(u - \bar{u}) + \bar{f}_{uy}(y - \bar{y}) - p + N(u), \\
e_y &= \Lambda y + z - u, \\
e_z &= z - \bar{\phi} - \bar{\phi}'(y - \bar{y}),
\end{aligned} \tag{5.23}
$$

where $\bar{f}_{yy}$ stands for $f_{yy}(\bar{y}, \bar{u})$, and the same notations is used for the other mappings. To obtain the two first equations of (5.23), we refer to the system (5.19), (5.20), where we insert $w_n = \bar{w}$ and replace the left hand side by the perturbation.

Since $\bar{z} - \bar{\phi} = 0$ and $\bar{\lambda} = -\bar{p}$, by straightforward calculations, we can easily prove that the system (5.23) is equivalent to

$$
\begin{aligned}
e_p - e_\lambda\bar{\phi}' &= \bar{f}_y + \bar{f}_{yy}(y - \bar{y}) + \bar{f}_{yu}(u - \bar{u}) + \bar{p}\bar{\phi}''(y - \bar{y}) + p(\Lambda + \bar{\phi}') \\
e_\lambda - \alpha e_z &= p + \lambda, \\
e_u &\in \bar{f}_u + \bar{f}_{uu}(u - \bar{u}) + \bar{f}_{uy}(y - \bar{y}) - p + N(u), \\
e_y - e_z &= \Lambda y + \bar{\phi} + \bar{\phi}'(y - \bar{y}) - u, \\
e_z &= z - \bar{\phi} - \bar{\phi}'(y - \bar{y}).
\end{aligned} \tag{5.24}
$$

Now we observe that the first, third, and fourth relation of (5.24) form a subsystem for (y,u,p), which does not depend on (z,λ). Once (y,u,p) is given from this subsystem, (z,λ) is uniquely determined by the remaining two equations. Let us set $\tilde e = (\tilde e_p, \tilde e_u, \tilde e_y)$, with

$$\tilde e_p = e_p - e_\lambda \bar\phi', \qquad \tilde e_u = e_u, \qquad \tilde e_y = e_y - e_z. \tag{5.25}$$

The subsystem of (5.24) can be rewritten in the form of the generalized equation

$$\begin{aligned} &\tilde e_p = \bar f_{yy}(y-\bar y) + \bar f_{yu}(u-\bar u) + p\Lambda + \bar f_y + \bar p\bar\phi''(y-\bar y),\\ &\tilde e_u \in \bar f_u + \bar f_{uu}(u-\bar u) + \bar f_{uy}(y-\bar y) - p + N(u),\\ &\tilde e_y = \Lambda y + \bar\phi + \bar\phi'(y-\bar y) - u. \end{aligned} \tag{5.26}$$

The generalized equation (5.26) is the linearization of the generalized equation (5.21) at $(\bar y,\bar u,\bar p)$, associated with the perturbation $\tilde e$. Since (5.21) was assumed to be strongly regular at $(\bar y,\bar u,\bar p)$, there exist $\tilde r \equiv r(\bar y,\bar u,\bar p) > 0$, $\tilde c \equiv c(\bar y,\bar u,\bar p) > 0$, and a mapping S from $B_{\widehat V\times U}(0,\tilde r)$ into $\widehat V \times U$, such that $S(\tilde e)$ is the unique solution to (5.26) for all $\tilde e \in B_{\widehat V\times U}(0,\tilde r)$, and $\|S(\tilde e^1) - S(\tilde e^2)\|_{\widehat V\times U} \le \tilde c\|\tilde e^1 - \tilde e^2\|_{\widehat V\times U}$. Now, we show that (5.22) is strongly regular at $\bar w$. For any e, let $\tilde e$ be given by (5.25). Then

$$\|\tilde e\|_{\widehat V\times U} \le c\,\|e\|_{\widehat Y\times U^4},$$

and there exists $\bar r > 0$ such that $\tilde e$ belongs to $B_{\widehat V\times U}(0,\tilde r)$ if $e \in B_{\widehat Y\times U^4}(0,\bar r)$. Define a mapping $\bar S$ from $B_{\widehat Y\times U^4}(0,\bar r)$ into $\widehat Y \times U^4$, as follows :

$$\bar S(e) = (S_1(\tilde e), S_2(\tilde e), z(e), S_3(\tilde e), \lambda(e)),$$

where

$$\begin{aligned} &(S_1(\tilde e), S_2(\tilde e), S_3(\tilde e)) = S(\tilde e),\\ &z(e) = e_z + \bar\phi + \bar\phi'(S_1(\tilde e) - \bar y), \qquad \lambda(e) = e_\lambda - ce_z - S_3(\tilde e). \end{aligned}$$

Then $\bar S(e)$ is clearly the unique solution to (5.23). We can easily find $\bar c > 0$ such that $\|\bar S(e^1) - \bar S(e^2)\|_{\widehat Y\times U^4} \le \bar c\|\bar e^1 - \bar e^2\|_{\widehat Y\times U^4}$. The proof is complete. ◻

Theorem 5.11 shows that once the convergence analysis for the standard non augmented Lagrange-Newton-SQP method has been done by proving strong regularity of the associated generalized equation, this analysis does not have to be repeated for analyzing convergence of the augmented method.

Up to now, we have discussed the Augmented SQP method and the Generalized Newton method separately. Now we shall show that both methods are equivalent. This equivalence is used to obtain a convergence theorem for the augmented SQP method.

THEOREM 5.12. *Let $(\bar y,\bar u)$ a local solution of (P), which satisfies together with the associated Lagrange multiplier $\bar p$ the second order sufficient optimality condition (SSC). Define $\bar z = \phi(\bar y)$, $\bar\lambda = -\bar p$, $\bar w = (\bar y,\bar u,\bar z,\bar p,\bar\lambda)$, and suppose that the generalized equation* (5.21) *is strongly regular at $\bar w$. Then there exists $r = r(\bar w) > 0$ such that, for any starting point $(y_0,u_0,z_0,p_0,\lambda_0)$ in the neighbourhood $B_W(\bar w, r)$, the ALSQP method defined according to Theorem* 5.1 *and the generalized Newton method defined in Lemma* 5.9 *generate the same sequence of iterates $(w_n)_n = (y_n,u_n,z_n,p_n,\lambda_n)_n$. Moreover, there is a constant $c_q(\bar w)$ such that the estimate*

$$\|w_{n+1} - \bar{w}\|_W \le c_q \, \|w_n - \bar{w}\|_W^2$$

is satisfied for all $n = 0, 1, 2, \ldots$

Proof. First we should mention the simple but decisive fact that $\bar{w}$ satisfies the optimality system of (P_α), since $(\bar{y}, \bar{u}, \bar{p})$ has to satisfy the optimality system for (P). Therefore, it makes sense to determine $\bar{w}$ by the generalized Newton method. Let $w_n = (y_n, u_n, z_n, p_n, \lambda_n)$ be an arbitrary current iterate, which is identical for the ALSQP method and the generalized Newton method.

In the GNM, $w_{n+1} \in W$ is found as the unique solution of (5.14)–(5.18). As concerns the ALSQP method, $(y_{n+1}, u_{n+1}) \in Y \times U$ is obtained as the unique solution of $(\widehat{\mathrm{QP}}^{\alpha}_{n+1})$, while $(z_{n+1}, p_{n+1}, \lambda_{n+1}) \in U^3$ are determined by (5.2). Therefore, $(y_{n+1}, u_{n+1}, z_{n+1}, p_{n+1}, \lambda_{n+1})$ satisfies the associated optimality system (5.4), (5.5)-(5.7) which is obviously identical with (5.14)-(5.18). Now it is clear that both the methods deliver the same new iterate $w_{n+1} \in W$. All remaining statements of the theorem follow from the convergence theorem 5.8. □

6. Numerical results.

6.1. Test example. We apply the augmented SQP method to the following one-dimensional nonlinear parabolic control problem with Stefan-Boltzmann boundary condition:

$$(E) \qquad \text{Minimize } f(y,u) = \frac{1}{2}\int_0^\ell (y(x,T) - y_T(x))^2\,dx + \frac{\kappa}{2}\int_0^T u(t)^2\,dt + \int_0^T (-a_y(t)\,y(\ell,t) + a_u(t)u(t))\,dt,$$

subject to

$$\begin{aligned}
& y_t - y_{xx} = 0 && \text{in } (0,\ell)\times(0,T)\\
& y(x,0) = a(x) && \text{in } (0,\ell)\\
& y_x(0,t) = 0 && \text{in } (0,T)\\
& y_x(\ell,t) + y(\ell,t) = b(t) + u(t) - \varphi(y(\ell,t)) && \text{in } (0,T),\\
& u_a \le u(t) \le u_b.
\end{aligned}$$

This example is a particular case of problem (E) considered in Section 3, where we take $\Omega = (0,\ell)$ and make an associated modification of the boundary condition. In an early phase of this work, we studied the numerical behaviour of the SQP method without augmentation. Here, we compare both methods. We performed our numerical tests for the following particular data:

$$\begin{aligned}
& \ell = \pi/4, \quad T = 1, \quad \kappa = \frac{\sqrt{2}}{2}(e^{2/3} - e^{1/3}),\\
& u_a = 0, \quad u_b = 1,\\
& y_T(x) = (e + e^{-1})\,cos(x),\\
& a_y(t) = e^{-2t}, \quad a_u(t) = \frac{\sqrt{2}}{2}e^{1/3},
\end{aligned}$$

$$a(x) = cos(x), \quad b(t) = \frac{1}{4}e^{-4t} - \min(u_b, \max(u_a, -(e^{1/3} - e^t))),$$
$$\varphi(y) = y\,|y|^3.$$

LEMMA 6.1. *The pair $(\bar{y}, \bar{u})$ defined by*

$$\bar{u}(t) = \min(u_b, \max(u_a, \frac{e^t - e^{1/3}}{e^{2/3} - e^{1/3}})), \qquad \bar{y}(x,t) = e^{-t}cos(x),$$

is a locally optimal solution for (6.1) *in* $C([0,\ell] \times [0,T]) \times L^\infty(0,T)$*. The associated adjoint state (Lagrange multiplier) is given by* $\bar{p}(x,t) = -e^t cos(x)$*. The triplet* $(\bar{y}, \bar{u}, \bar{p})$ *satisfies the second order sufficient optimality condition* (SSC)*.*

Proof. The proof is split into four steps.

Step 1. State equation. It is easy to see that $\bar{y}_t - \bar{y}_{xx} = 0$, $\bar{y}(x,0) = cos(x)$, and $\bar{y}_x(0,t) = 0$. Now regard the boundary condition at $x = \ell$: The left hand side is

$$\bar{y}_x(\ell,t) + \bar{y}(\ell,t) = -e^{-t}sin(\pi/4) + e^{-t}cos(\pi/4) = 0.$$

The same holds for the right hand side, since

$$b(t) + \bar{u}(t) - \varphi(\bar{y}(\ell,t)) = \frac{1}{4}e^{-4t} - \bar{u}(t) + \bar{u}(t) - (e^{-t}cos(\pi/4))^4 = 0.$$

Step 2. Adjoint equation. Again, the equations $-\bar{p}_t - \bar{p}_{xx} = 0$, $\bar{p}_x(0,t)) = 0$, and $\bar{p}(x,T) = \bar{y}(x,T) - y_T(x)$ are easy to check. It remains to verify the boundary condition at $x = \ell$:

$$p_x(\ell,t) + p(\ell,t) = -(a_y(t) + \varphi'(\bar{y}(\ell,t))p(\ell,t)).$$

It is obvious that $\bar{p}_x + \bar{p} = 0$ at $x = \ell$. The right hand side of the boundary condition has the same value, since

$$\begin{aligned} a_y(t) + \varphi'(\bar{y}(\ell,t))\bar{p}(\ell,t) &= e^{-2t} - 4\bar{y}(\ell,t)^3 e^t cos(\ell) \\ &= e^{-2t} - 4e^{-3t}cos(\pi/4)^3 e^t cos(\pi/4) \\ &= e^{-2t}(-1 + 4(\tfrac{\sqrt{2}}{2})^4) = 0. \end{aligned}$$

Step 3. Variational inequality. We must verify that $\bar{u} \in U_{ad}$ – which is trivial – and that

$$\int_0^T (\kappa\bar{u}(t) + a_u(t) + \bar{p}(\ell,t))(u(t) - \bar{u}(t))\,dt \geq 0 \qquad \text{for all } u \in U_{ad}.$$

It is well known that this holds if and only if

$$\bar{u}(t) = P_{[u_a,u_b]}\Big\{-\frac{1}{\kappa}(a_u(t) + \bar{p}(\ell,t))\Big\} = P_{[0,1]}\Big\{\frac{e^t - e^{1/3}}{e^{2/3} - e^{1/3}}\Big\},$$

where $P_{[0,1]}$ denotes projection onto $[0,1]$. This is obviously verified.

Step 4. Second order sufficient condition. The Lagrange function is given by

$$\begin{aligned} \mathcal{L} = f &- \textstyle\int_Q (y_t - y_{xx})p\,dxdt + \int_0^l (y(x,0) - a(x))p(x,0)dx \\ &+ \textstyle\int_0^T y_x(0,t)p(0,t)\,dt + \int_0^T (y_x(\ell,t) + y(\ell,t) - b(t) - u(t))p(\ell,t)\,dt \\ &- \textstyle\int_0^T \varphi(y(\ell,t))p(\ell,t)\,dt. \end{aligned}$$

Therefore,

$$\mathcal{L}''(\bar{y},\bar{u},\bar{p})(y,u)^2 = \|y(T)\|^2_{L^2(0,\ell)} + \|u\|^2_{L^2(0,T)} - 12\int_0^T \bar{y}(\ell,t)^2\bar{p}(\ell,t)y(\ell,t)^2\,dt.$$

Since $\bar{p}$ is negative, $\mathcal{L}''(\bar{y},\bar{u},\bar{p})$ is coercive on the whole space $Y\times U$, hence (SSC) is satisfied. □

THEOREM 6.2. *The pair $(\bar{y},\bar{u})$ is a global solution of (E).*

Proof. Let (y,u) be any other admissible pair for (E). Due to the first order necessary condition, we have

$$\begin{aligned} f(y,u) &\geq f(\bar{y},\bar{u}) + \mathcal{L}(\bar{y},\bar{u},\bar{p})(y-\bar{y},u-\bar{u}) \\ &\quad -\frac{1}{2}\int_0^T \bar{p}(\ell,t)\int_0^1 \varphi''(\bar{y}(\ell,t)+\tau(y(\ell,t)-\bar{y}(\ell,t)))(y(\ell,t)-\bar{y}(\ell,t))^2\,d\tau\,dt \\ &\geq f(\bar{y},\bar{u}) \\ &\quad -\frac{1}{2}\int_0^T \bar{p}(\ell,t)\int_0^1 \varphi''(\bar{y}(\ell,t)+\tau(y(\ell,t)-\bar{y}(\ell,t)))(y(\ell,t)-\bar{y}(\ell,t))^2\,d\tau\,dt. \end{aligned}$$

¿From the positivity of $-\bar{p}$ and of $\varphi''(\bar{y}+s(y-\bar{y}))$ (independently of s and y), it follows that $f(y,u)\geq f(\bar{y},\bar{u})$. □

Next we discuss the strong regularity of the optimality system at $(\bar{y},\bar{u},\bar{p})$.

THEOREM 6.3. *The optimality system of (E) is strongly regular at $(\bar{y},\bar{u},\bar{p})$.*

Proof. The triplet $(\bar{y},\bar{u},\bar{p})$ satisfies (SSC). Moreover, (E) fits into a more general class of optimal control problems for semilinear parabolic equations, which was considered in [23]. ¿From Theorem 5.2 in [22] and Theorem 5.3 in [23], it follows that (SSC) ensures the strong regularity of the generalized equation being the abstract formulation of the associated optimality system. We only have to apply this result to problem (6.1). □

REMARK 6.4. *A study of [23] reveals that convergence of the standard SQP method can be proved for arbitrary dimension of Ω assuming a weaker form of (SSC). It requires coercivity of $\mathcal{L}''$ only on a smaller subspace that considers strongly active control constraints. This weaker assumption should be helpful for proving the convergence of the augmented SQP method as well. We shall not discuss this, since the technical effort will increase considerably.*

Now we obtain from Theorem 5.12 the following result:

COROLLARY 6.5. *The Augmented Lagrangian SQP method for (E) is locally quadratically convergent towards $(\bar{y},\bar{u},\bar{p})$.*

6.2. Algorithm. For the convenience of the reader, let us consider the problem $(\widehat{\mathrm{QP}}^{\alpha}_{n+1})$ corresponding to our test example. After simplifying we get

$$\text{(6.1)}\quad \text{Minimize}\quad \frac{1}{2}\int_0^\ell y(\cdot,T)^2\,dx + \frac{\kappa}{2}\int_0^T u^2\,dt + \frac{1}{2}\int_0^T q_n\, y(\ell,\cdot)^2\,dt - \int_0^\ell y(\cdot,T)\,y_T\,dx$$
$$+\int_0^T \big(-(a_y+q_n\,y_n(\ell,\cdot))\,y(\ell,\cdot)+a_u u\big)\,dt,$$

subject to

$$
(6.2)\qquad \begin{aligned}
& y_t - y_{xx} = 0 && \text{in } (0,\ell), \times (0,T)\\
& y(x,0) = a(x) && \text{in } (0,\ell),\\
& y_x(0,t) = 0 && \text{in } (0,T),\\
& y_x(\ell,t) + c_n y(\ell,t) = b_n(t) + u(t) && \text{in } (0,T),
\end{aligned}
$$

$$
(6.3)\qquad u_a \le u(t) \le u_b,
$$

with

$$
\begin{aligned}
& q_n = 12 y_n(\ell)^2 \,\big(- p_n(\ell) + \alpha(z_n - y_n(\ell)^4)\big), \qquad c_n = 1 + 4 y_n(\ell)^3,\\
& b_n = b + 3 y_n(\ell)^4.
\end{aligned}
$$

One specific difficulty for solving problem (6.1)-(6.3) is partially related to the control constraints. But the main difficulty appears also in the unconstrained case where a (large) linear system has to be solved. Let us consider for a moment the unconstrained case. If (u_{n+1}, y_{n+1}) is a solution of problem (6.1)-(6.2), then the optimal triplet $(u_{n+1}, y_{n+1}, p_{n+1})$ satisfies (6.2), the adjoint equation

$$
(6.4)\qquad \begin{aligned}
& p_t + p_{xx} = 0 && \text{in } (0,\ell) \times (0,T),\\
& p(x,T) = y_n(x,T) - y_T(x) && \text{in } (0,\ell),\\
& p_x(0,t) = 0 && \text{in } (0,T),\\
& p_x(\ell,t) + c_n p(\ell,t) = q_n(t) y_{n+1}(\ell,t) - q_n(t) y_n(\ell,t) - a_y && \text{in } (0,T),
\end{aligned}
$$

and

$$
(6.5)\qquad u_{n+1} = -\frac{1}{\kappa}(a_u + p_{n+1}(\ell,\cdot)).
$$

In practice, we solve $(\widehat{\mathrm{QP}}^{\alpha}_n)$ by discretization of its optimality system. The result is taken to solve $(\widehat{\mathrm{QP}}^{\alpha}_{n+1})$. The discretized version of equation (6.5) corresponds to a large-scale linear system. To solve this system, we need the solutions corresponding to the discretization of two coupled parabolic equations (the state and the adjoint equations). It is clear that the accuracy of the Augmented Lagrangian SQP-method depends on the one for solving the linear system, and consequently on the numerical methods for the partial differential equations. In our example, the state and adjoint equations are solved by using a second-order finite difference scheme (Cranck-Nicholson scheme) appropriately modified at the boundary to maintain second order approximation. The linear system is solved by using the CGM (conjugate gradient method), with a step length given by the Polak-Ribiere formula.

Let us now take into account the constraints (6.3). The optimality condition (6.5) is replaced by

$$
(6.6)\qquad \begin{aligned}
u_{n+1} &= \mathrm{Proj}_{[u_a,u_b]}\big(- \frac{1}{\kappa}(a_u + p_{n+1}(\ell,\cdot))\big)\\
&= \begin{cases}
u_a & \text{if } \kappa u_a + a_u + p_{n+1}(\ell,\cdot) > 0,\\
u_b & \text{if } \kappa u_b + a_u + p_{n+1}(\ell,\cdot) < 0,\\
-\frac{1}{\kappa}(a_u + p_{n+1}(\ell,\cdot)) & \text{if } \kappa u_{n+1} + a_u + p_{n+1}(\ell,\cdot) = 0.
\end{cases}
\end{aligned}
$$

The management of these restrictions is based on (6.6) and on an projection method by Bertsekas [7]. (See also [9] and [10] where this method is successfully applied.) More precisely, we have the following algorithm:

1 - Let $w_n = (w_n^1, \cdots, w_n^m)^T$ be the vector representing the iterate corresponding to u_n for some fixed grid. Let ε and σ be fixed positive numbers, and let $I = \{1, \cdots, m\}$ be the index set associated to w_n. (m is the dimension of the vector w_n and depends on the discretization of u_n)

2 - Solve (6.2), (6.4), (5.3), and denote by $d_n = (d_n^1, \cdots, d_n^m)^T$ the vector representing the iterate corresponding to the solution of (6.4).

3 - Define the sets of strongly active inequalities

$$I_a^\sigma = \{i \in I \mid w_n^j = u_a \text{ and } \kappa w_n^j + d_n^j + A_u^j > \sigma\},$$
$$I_b^\sigma = \{i \in I \mid w_n^j = u_b \text{ and } \kappa w_n^j + d_n^j + A_u^j < -\sigma\},$$

where $A_u = (A_u^1, \cdots, A_u^m)^T$ is the vector representing a_u.

4 - Set $\hat{u}^j = w_n^j$ for all $j \in I_a^\sigma \cup I_b^\sigma$.

5 - Solve the unconstrained problem (6.1)-(6.2) for w_n^j, $j \in I \setminus (I_a^\sigma \cup I_b^\sigma)$. (The remaining components are fixed due to 4.) Denote by v_n the vector representation of the solution.

6 - Set $w_{n+1} = P_{[u_a,u_b]} v_n$, where $P_{[u_a,u_b]}$ denotes the projection onto $[u_a, u_b]^m$.

7 - If $\|w_{n+1} - w_n\| \geq \varepsilon$ then put $w_n := w_{n+1}$, $n := n+1$ and go to 2. Otherwise stop the iteration.

6.3. Numerical tests. In the numerical tests, we focused our interest on the aspects concerning the convergence for different values of initial data and penalty parameters α, and on the rate of convergence. The programs were written in MATLAB.

Let us first summarize some general observations.

• In our example, the augmented Lagrangian algorithm performed well. In particular, the graphs of the exact solution and that of the numerical solution are (almost) identical.

• When compared with the SQP method (corresponding to $\alpha = 0$), the augmented Lagrangian SQP has the advantage of a more global behavior. Moreover, it is less sensitive to the start-up values, and is significantly faster than the SQP method for some points.

• Graphical correction of the computed controls and precisionof optimal value (up to five digits) are obtained by taking the discretization parameters with respect to the time and the space equal to 200.

• For fixed data, the number of iterations for the CGM and the Augmented SQP turned out to be independent of the mesh size.

In all the sequel, we set

$$n_u = \|\bar{u} - \hat{u}\|_2, \quad n_y = \|\bar{y} - \hat{y}\|_2, \quad n_p = \|\bar{p} - \hat{p}\|_2, \quad n_z = \|\bar{z} - \hat{z}\|_2,$$
$$e_n = \frac{\|\bar{u} - u_n\|_2 + \|\bar{y} - y_n\|_2 + \|\bar{p} - p_n\|_2 + \alpha\|\bar{z} - z_n\|_2}{\|\bar{u} - u_{n-1}\|_2^2 + \|\bar{y} - y_{n-1}\|_2^2 + \|\bar{p} - p_{n-1}\|_2^2 + \alpha\|\bar{z} - z_{n-1}\|_2^2},$$

where $(\bar{u}, \bar{y}, \bar{p}, \bar{z})$, $(\hat{u}, \hat{y}, \hat{p}, \hat{z})$ and (u_n, y_n, p_n, z_n) are the vectors respectiveely corresponding to the exact solution of (E), the numerical solution of (E), and the solution of $(\widehat{\mathrm{QP}}_n^\alpha)$. Moreover, we denote by n_t and n_x the discretization parameters with respect to the time and the space. Optimal controls were determined for the following pairs (n_x, n_t): (100,100), (200,200), (400,400).

Run 1. (SQP method.) The first test corresponds to $(u_0, y_0, p_0) = (0.5, 0.5, 0.5)$, and $\alpha = 0$. The rates for e_n, n_u, n_p, and n_z are given in Table 6.1.

TABLE 6.1

nx	n_u	n_y	n_p	e_1	e_2	e_3	e_4
100	1.7782e-06	2.5347e-06	1.6610e-06	0.2372	0.3653	1.0575	1.3886
200	1.3725e-06	2.9337e-06	1.0724e-06	0.2472	0.3663	1.0585	0.9980
400	3.4363e-06	1.2385e-06	1.2702e-06	0.2439	0.3636	1.0583	1.1952

The SQP method shows a good convergence for this initial point. 4 iterations were needed to get the result.

Run 2. (ALSQP method.) The second test corresponds to the point $(u_0, y_0, p_0)=$ $(0.5, 0.5, 0.5)$, with $z_0 = y_0^4 + 2$, and $\alpha = 1$.

TABLE 6.2

nx	n_u	n_y	n_p	n_z	e_1	e_2	e_3
100	1.5391e-06	2.6824e-06	1.5013e-06	1.4459e-06	0.0150	1.0004	2.2378
200	1.2318e-06	3.8783e-07	5.2251e-07	5.5521e-07	0.0149	0.9256	1.0015
400	1.5223e-06	5.3737e-06	6.0455e-06	3.5205e-07	0.0149	0.8879	1.2290

The ALSQP method has a very good convergence for this choice. Convergence could always be achieved by fixing (u_0, y_0, p_0), and using other values of z_0 and α. However, the number of iterations and the speed of the method depend on these choices. As shown in Table 6.3, three iterations for the ALSQP method were needed, instead of four for the SQP method. The number of iterations for the CGM, the SQP and the ALSQP methods is independent of the mesh-size. The exact value for the cost functional is $\bar{f} = 2.7198$. In Table 6.3, we give the values of the cost functional corresponding to the different steps for $nx = nt = 200$.

TABLE 6.3

SQP method		
Iter	f_n	CGM iter
1	2.6870	2
2	2.7104	3
3	2.7195	6
4	2.7198	10

ALSQP method		
Iter	f_n	CGM iter
1	2.6878	3
2	2.7194	6
3	2.7198	11

In Figures 6.1, 6.2, and 6.3, we compare the behavior of the control, the state, and the adjoint state obtained by taking $\alpha = 0$, and $\alpha = 1$. It is clear that in the case of

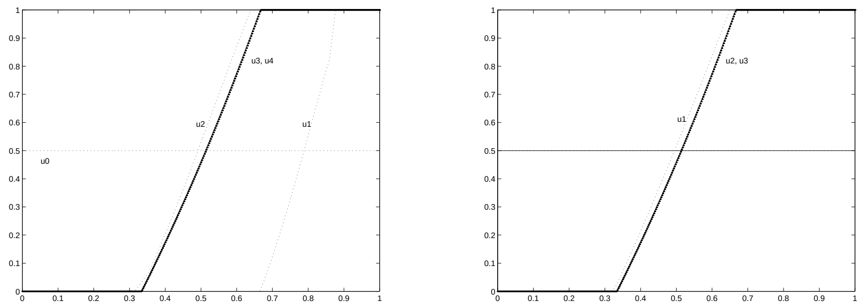

FIG. 6.1. *Controls for Run* 1 *and Run* 2

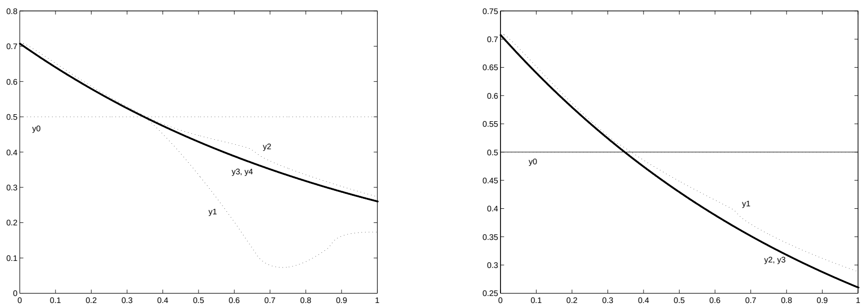

FIG. 6.2. *States* $y(\ell, t)$ *for Run* 1 *and Run* 2

the ALSQP method, the second iteration gives a good approximation to the optimal control, the optimal state, and the optimal adjoint state.

Run 3. The last test corresponds to the initial point given by $(u_0, y_0, p_0) = (0.5, 1, 2)$.

TABLE 6.4

nx	n_u	n_y	n_p	n_z
100	9.9421e-06	2.7989e-06	4.0979e-06	3.0853e-06
200	1.1864e-05	4.7523e-06	5.0999e-06	2.3842e-06
400	1.2167e-05	5.2817e-06	5.3307e-06	2.2146e-06

nx	e_1	e_2	e_3	e_4
100	0.0402	0.4017	1.3412	0.9260
200	0.0404	0.3975	1.3537	1.1471
400	0.0405	0.3956	1.3591	1.2052

For this initial point, the SQP method (corresponding to $\alpha = 0$) does not converge, while the ALSQP method converges for many choices of z_0. In our tests, the point which gives the best result is given by $z_0 = y_0^4 + 3$ with $\alpha = 1$. For this choice, 4 iterations are needed with 2, 5, 6 and 9 CG steps. The differents rates are given in Table 6.4, and the behavior of the solution is shown is Figure 6.4.

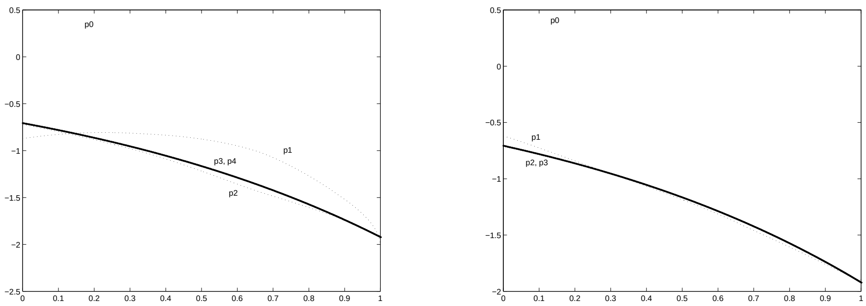

FIG. 6.3. *Adjoint states $p(\ell, t)$ for Run 1 and Run 2*

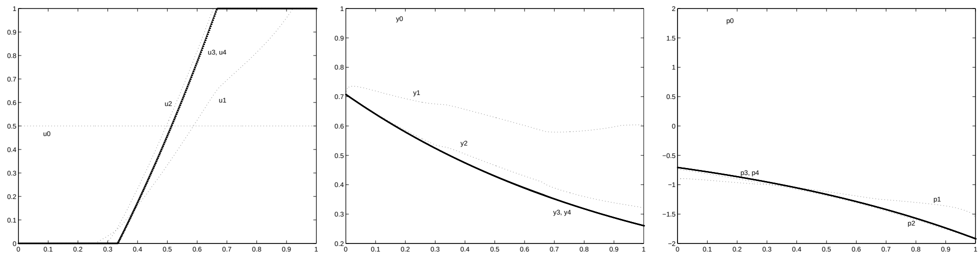

FIG. 6.4. *Controls, states, and adjoint states for Run 3*

REMARK 6.6. *The numerical results stated in Table 6.1, 6.2, and 6.3 were obtained for a fixed mesh-size (fixed grid). However, we also implemented the ALSQP method with adaptative mesh size, i.e. we started with a coarse grid and used the obtained results as startup values for the next finer grid. This method is significantly faster, and delivers essentially the same results.*

REFERENCES

[1] J. J. Alibert, J. P. Raymond, A Lagrange multiplier theorem for control problems with state constraints, Numer. Funct. Anal. Optim., Vol. 19 (1998), 697-704.

[2] W. Alt. The Lagrange–Newton method for infinite–dimensional optimization problems. *Numer. Funct. Anal. and Optimiz.*, 11:201–224, 1990.

[3] W. Alt. The Lagrange Newton method for infinite-dimensional optimization problems. *Control and Cybernetics*, 23:87–106, 1994.

[4] W. Alt. Discretization and mesh independence of Newton's method for generalized equations. In A.K. Fiacco, Ed. *Mathematical Programming with Data Perturbation.* Lecture Notes in Pure and Appl. Math., Vol. 195, Marcel Dekker, New York, 1–30.

[5] V. Barbu, Analysis and Control of Nonlinear Infinite Dimensional Systems, Academic Press, Boston/San Diego/New York, 1993.

[6] M. Bergounioux and K. Kunisch. Augmented Lagrangian techniques for elliptic state constrained optimal control problems, *SIAM J. Control Optim.*, 31:1524–1543.

[7] D.B. Bertsekas, Projected Newton methods for optimization problems with simple constraints, *SIAM J. Control Optim.*, 20:221–246.

[8] E. Casas. Pontryagin's principle for state-constrained boundary control problems of semilinear parabolic equations. *SIAM J. Control Optim.*, 35:1297–1327, 1997.

[9] H. Goldberg and F. Tröltzsch. On a SQP-Multigrid technique for nonlinear parabolic boundary control problems. In W.W Hager and P.M. Pardalos, Eds., *Optimal Control: Theory, Algorithms, and Applications*, pp. 154–177. Kluwer Academic Publishers B.V. 1998.

[10] M. Heinkenschloss and E. W. Sachs. Numerical solution of a constrained control problem for a phase field model. *Control and Estimation of Distributed Parameter Systems, Int. Ser.*

Num. Math., 118:171–188, 1994.

[11] A.D. Ioffe. Necessary and sufficient conditions for a local minimum, part 3: Second order conditions and augmented duality. *SIAM J. Control Optimization* 17 (1979), pp. 266–288.

[12] A.D. Ioffe and Tikhomirov. Theory of extremal problems. North Holland, Amsterdam, 1979.

[13] K. Ito and K. Kunisch. Augmented Lagrangian-SQP methods for nonlinear optimal control problems of tracking type. *SIAM J. Control Optim.*, 34:874–891, 1996.

[14] K. Ito and K. Kunisch. Augmented Lagrangian-SQP methods in Hilbert spaces and application to control in the coefficients problems. *SIAM J. Optim.*, 6:96–125, 1996.

[15] A. Kauffmann. Optimal control of the solid fuel ignition model. Thesis, Technical University of Berlin, October 1998.

[16] K. Kunisch, S. Volkwein. Augmented Lagrangian-SQP techniques and their approximations. *Contemporary Mathematics*, 209:147–159, 1997.

[17] O. A. Ladyzenskaja, V. A. Solonnikov, and N.N. Ural'ceva, "Linear and quasilinear equations of parabolic type", Translations of Mathematical Monographs, Vol. 23, Amer. Math. Soc., Providence, R. I., 1968.

[18] H. Maurer. First and second order sufficient optimality conditions in mathematical programming and optimal control. *Math. Programming Study* 14 (1981), pp. 163–177.

[19] J.P. Raymond and F. Tröltzsch. Second order sufficient optimality conditions for nonlinear parabolic control problems with state constraints, *Disc. Cont. Dyn. Systems* , 6:431-450, 2000.

[20] J.P. Raymond and H. Zidani. Hamiltonian Pontryagin's principles for control problems governed by semilinear parabolic equations, *Appl. Math. Optim.*, 39:143–177, 1999.

[21] S.M. Robinson. Strongly regular generalized equations. *Math. of Operations Research*, 5:43–62, 1980.

[22] F. Tröltzsch. Lipschitz stability of solutions to linear-quadratic parabolic control problems with respect to perturbations. Preprint-Series Fac. of Math., TU Chemnitz, Report 97–12, *Dynamics of Continuous, Discrete and Impulsive Systems*, accepted.

[23] F. Tröltzsch. On the Lagrange–Newton method for the optimal control of semilinear parabolic equations. *SIAM J. Optim.*, 38:294–312, 1999.

[24] A. Unger. Hinreichende Optimalitätsbedingungen 2. Ordnung und Konvergenz des SQP-Verfahrens für semilineare elliptische Randsteuerprobleme. Thesis, Technical University of Chemnitz, December 1997.

[25] S. Volkwein. Mesh-independence for an augmented Lagrangian-SQP method in Hilbert spaces. *SIAM J. Optim.*, 38:767–785, 2000.

[26] S. Volkwein. Distributed control problems for the Burgers equation. *To appear in Computational Optimization and Applications*, 2000.