

SUFFICIENT SECOND-ORDER OPTIMALITY CONDITIONS FOR A PARABOLIC OPTIMAL CONTROL PROBLEM WITH POINTWISE CONTROL-STATE CONSTRAINTS

A. RÖSCH AND F. TRÖLTZSCH *

Abstract. An optimal control problem for a semilinear parabolic equation is investigated, where pointwise constraints are given on the control and the state. The state constraints are of mixed (bottleneck) type, where associated Lagrange multipliers can be assumed to be bounded and measurable functions. Based on this property, a second-order sufficient optimality condition is established that considers strongly active constraints.

Keywords: Optimal control, heat equation, parabolic differential equation, sufficient second-order optimality condition, pointwise mixed control-state constraints, bottleneck constraints

AMS subject classification: 49K20, 90C48

1. Introduction. In this paper we consider the optimal control problem to minimize

$$F(y, u) = \int_{\Omega} \omega(x, y(T, x)) \, dx + \int_{\Sigma} \sigma(t, x, y(t, x), u(t, x)) \, d\Gamma dt + \int_Q q(t, x, y(t, x)) \, dx dt \quad (1.1)$$

subject to the state equations

$$\begin{aligned} (1.2) \quad & y_t + Ay = 0 && \text{in } Q = (0, T) \times \Omega \\ & \partial_n y = b(t, x, y, u) && \text{in } \Sigma = (0, T) \times \Gamma \\ & y(0, x) = y_o(x) && \text{in } \Omega, \end{aligned}$$

and subject to the mixed control-state constraints

$$(1.3) \quad 0 \leq u(t, x) \leq c(t, x) + \gamma(t, x)y(t, x) \quad \text{for } (t, x) \in \Sigma.$$

The main task of our paper is to establish second-order sufficient optimality conditions that are close to the associated necessary ones. For control-constrained problems, this issue was discussed quite completely in literature for semilinear elliptic and parabolic equations. We mention Bonnans [3], Casas/Tröltzsch/Unger [5], Goldberg/Tröltzsch [8], Heinkenschloss/Tröltzsch [9].

The main difficulty in our problem is the presence of the pointwise control-state constraint $u(t, x) \leq c(t, x) + \gamma(t, x)y(t, x)$ in (1.3). If pointwise state constraints are given, then the theory of sufficient second-order conditions is faced with specific difficulties that are still far from being solved. In particular these problems arise for pointwise state constraints of the type $y(t, x) \leq c$. Here, the difficulties are caused by the low regularity of Lagrange multipliers associated with the pointwise state constraints. The multipliers are Borel measures. We refer to Casas/Tröltzsch/Unger [6] and Raymond/Tröltzsch [12].

In our problem (1.1–1.3), the situation is slightly simpler, since the constraint (1.3) is a *mixed* control-state constraint of bottleneck type. In this case, the Lagrange multipliers are more regular, they can be assumed to be functions of $L^\infty(\Sigma)$, see Bergounioux/Tröltzsch [2] or Arada/Raymond [1].

*Technische Universität Berlin, Fakultät II – Mathematik und Naturwissenschaften, Str. des 17. Juni 136, D-10623 Berlin, Germany

Higher regularity of the multipliers is the main advantage enabling us to establish second-order conditions. Moreover, the second-order conditions should require minimum assumptions, i.e. they should be as close as possible to associated necessary conditions. Often, this task is accomplished by considering strongly active sets (see [7] for control-constrained optimal control of ordinary differential equations). Here, we extend this technique to our case of mixed constraints. Our analysis will show that this is by far not an easy problem. It indicates again that pointwise state constraints of more general type will rise even more difficult techniques. Our paper extends the results of [15], where second-order conditions were derived for a weakly singular integral state equation. This problem covered the one-dimensional parabolic case.

The paper is organized as follows: In section 2 we formulate first and second-order optimality conditions and state the main result. Section 3 contains several auxiliary results. The proof that our second-order conditions are sufficient for local optimality is presented in section 4.

In the paper we use the following notations: By $b'(t, x, y, u)$ and $b''(t, x, y, u)$ we denote the gradient and the Hessian matrix of b with respect to (y, u) :

$$b'(t, x, y, u) = \begin{pmatrix} b_y(t, x, y, u) \\ b_u(t, x, y, u) \end{pmatrix}, \quad b''(t, x, y, u) = \begin{pmatrix} b_{yy}(t, x, y, u) & b_{yu}(t, x, y, u) \\ b_{yy}(t, x, y, u) & b_{uu}(t, x, y, u) \end{pmatrix}.$$

Here, the notations $b_y(t, x, y, u) = D_y b(t, x, y, u)$ and $b_{yy}(t, x, y, u) = D_{yy} b(t, x, y, u)$ are used. The norms $|b'|$, $|b''|$ are defined by adding the absolute values of all entries of b' and b'' , respectively. By ∂_n we denote the outward normal derivative at Γ .

The following assumptions are required:

(A1) The function $b : \Sigma \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $b = b(t, x, y, u)$, satisfies the following Carathéodory condition: b is of class C^2 with respect to (y, u) . Moreover, for all $(y, u) \in \mathbb{R}^2$, it is measurable with respect to (t, x) .

For all $M > 0$, there are a constant $C_M > 0$ and a continuous, monotone increasing function $\eta \in C(\mathbb{R}^+ \cup \{0\})$ with $\eta(0) = 0$ such that:

$$\begin{aligned} & |b(t, x, 0, 0)| + |b'(t, x, 0, 0)| + |b''(t, x, 0, 0)| \leq C_M, \\ & |b''(t, x, y_1, u_1) - b''(t, x, y_2, u_2)| \leq \eta(|y_1 - y_2| + |u_1 - u_2|) \end{aligned}$$

for almost all (t, x) and all $|y|, |u|, |y_1|, |y_2|, |u_1|, |u_2| \leq M$. The same conditions are imposed for $\sigma = \sigma(t, x, y, u)$.

In addition, we suppose on $\Sigma \times \mathbb{R}^2$ that

$$b_u(t, x, y, u) \geq 0, \quad b_y(t, x, y, u) \leq 0, \quad |b(t, x, y_1, u) - b(t, x, y_2, u)| \leq L|y_1 - y_2|$$

holds for all $|y|, |u| \leq M$ and $y_1, y_2 \in \mathbb{R}$. Notice that b is supposed to be globally Lipschitz with respect to y . The constant L does not depend on M .

(A2) The function $\omega : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, $\omega = \omega(x, y)$, is of class C^2 with respect to y . Furthermore, ω is measurable with respect to x for all $y \in \mathbb{R}$. We assume $\omega(\cdot, 0) \in L^\infty(\Omega)$, $\omega_y(\cdot, 0) \in L^\infty(\Omega)$, $\omega_{yy}(\cdot, 0) \in L^\infty(\Omega)$, and

$$|\omega_{yy}(x, y_1) - \omega_{yy}(x, y_2)| \leq \eta(|y_1 - y_2|)$$

for almost all $x \in \Omega$ and all $|y_1| \leq M, |y_2| \leq M$.

The function $q = q(t, x, y)$ is assumed to satisfy the assumptions on ω , where Q is substituted for Ω and (t, x) is substituted for x .

(A3) We assume that $c, \gamma \in C(\bar{\Sigma})$, and $c(t, x) > 0$, $\gamma(t, x) \geq 0 \quad \forall (t, x) \in \bar{\Sigma}$. In addition, we require $y_0 \in C(\bar{\Omega})$, $y_0(x) \geq 0 \quad \forall x \in \Omega$.

(A4) The domain $\Omega \subset \mathbb{R}^m$ has a boundary Γ of class C^2 . The elliptic operator A is defined by

$$Ay(x) = - \sum_{i,j=1}^m D_i(a_{ij}(x)D_jy(x)),$$

where $a_{ij} \in C^{1,\nu}$ satisfy, for some positive m_0 , the condition of ellipticity

$$\sum_{i,j=1}^m a_{ij}(x)\xi_i\xi_j \geq m_0|\xi|^2.$$

Other estimates of b, ω, σ, q and their first derivatives can be derived from (A1), (A2) by the mean value theorem. For convenience, in (A1) we assume a global Lipschitz continuity for b with respect to y . This is not really a strong assumption. The maximum principle of the parabolic equation ensures a-priori-bounds on the solution of the parabolic equation. Therefore, the Lipschitz continuity with respect to y is only needed on a bounded set that is predetermined by the given data.

2. First- and second-order optimality conditions. First, we introduce the spaces $V = H^1(\Omega)$ and $W(0, T) = \{v \in L^2(0, T; V) : v_t \in L^2(0, T; V^*)\}$. Since $W(0, T)$ is not embedded in the space $C(\bar{Q})$, which is needed to differentiate the superposition operators associated with the nonlinear functions ω, σ, q , and b , we fix $\alpha > \frac{m}{2} + 1$, $s > m + 1$ and introduce the state space

$$Y = \{y \in W(0, T) | y_t + Ay \in L^\alpha(Q), \partial_n y \in L^s(\Sigma), y(0) \in C(\bar{\Omega})\}$$

endowed with the norm

$$\|y\|_Y = \|y\|_{W(0, T)} + \|y_t + Ay\|_{L^\alpha(Q)} + \|\partial_n y\|_{L^s(\Sigma)} + \|y(0)\|_{C(\bar{\Omega})}.$$

Due to the choice of α and s , the embedding of Y into $C(\bar{Q})$ is continuous, [4], [13].

We denote by $a = a[y, v] : V \times V \rightarrow \mathbb{R}$ the bilinear form associated with A ,

$$a[y, v] = \int_{\Omega} \sum_{i,j=1}^m a_{ij}(x)D_jy(x)D_iv(x) dx.$$

A function $y \in Y$ is said to be a (weak) solution of (1.2), if y satisfies the initial value problem

$$(2.1) \quad \begin{aligned} \frac{d}{dt}(y(t), v)_{L^2(\Omega)} + a[y(t), v] &= (b(t, \cdot, y(t), u(t)), v)_{L^2(\Gamma)} \\ y(0, \cdot) &= y_0 \end{aligned}$$

for almost all t and all $v \in V$.

LEMMA 2.1. For each $u \in L^\infty(\Sigma)$, equation (1.2) admits a unique solution $y \in Y$.

For the proof we refer to [11] and [13]. In these papers, the authors use a weak solution approach. It is also possible to get a similar result by semigroup techniques, see for instance [14].

By Lemma 2.1, a solution mapping $G : L^\infty(\Sigma) \rightarrow C(\bar{Q})$ is defined that assigns to $u \in L^\infty(\Sigma)$ the solution y of (1.2). The boundary values of y are of particular

importance for us. Thus we define the mapping $S : L^\infty(\Sigma) \rightarrow C(\bar{\Sigma})$ with $S = \tau G$ that assigns to u the boundary values of y . It is known from literature that G and S are twice continuously Fréchet-differentiable. Nevertheless, for our further results it is useful to briefly sketch the proof. Let $(\bar{y}, \bar{u}) \in Y \times L^\infty(\Sigma)$ be a fixed reference pair. Later, this couple will stand for a local minimum of (1.1)–(1.3). Below we use the abbreviations $\bar{b}_u = b_u(t, x, \bar{u}(t, x), \bar{y}(t, x))$, $\bar{b}_y = b_y(t, x, \bar{u}(t, x), \bar{y}(t, x))$ with $\bar{y} = S(\bar{u})$. In the same way \bar{b}_{yy} , \bar{b}_{uu} , $\bar{\sigma}_y$ etc. are defined.

LEMMA 2.2. *The nonlinear mapping $S : L^\infty(\Sigma) \rightarrow C(\bar{Q})$ is of class C^1 . Its Fréchet derivative $S'(\bar{u})$ at \bar{u} in direction u is given by $S'(\bar{u})u = w|_\Sigma$, where w is the solution of the initial-boundary value problem*

$$(2.2) \quad \begin{aligned} w_t + Aw &= 0 && \text{in } Q \\ \partial_n w - \bar{b}_y w &= \bar{b}_u u && \text{in } \Sigma \\ w(0, x) &= 0 && \text{in } \Omega. \end{aligned}$$

Proof. Let w be the solution of (2.2) with $u = \tilde{u} - \bar{u}$ and set

$$(2.3) \quad z := \tilde{y} - \bar{y} - w = S(\tilde{u}) - S(\bar{u}) - w.$$

Next, we perform a Taylor expansion for $b(t, x, \tilde{u}(t, x), \tilde{y}(t, x))$,

$$(2.4) \quad \begin{aligned} b(t, x, \tilde{u}(t, x), \tilde{y}(t, x)) &= b(t, x, \bar{u}(t, x), \bar{y}(t, x)) + \bar{b}_u(\tilde{u}(t, x) - \bar{u}(t, x)) \\ &+ \bar{b}_y(\tilde{y}(t, x) - \bar{y}(t, x)) + r(t, x). \end{aligned}$$

The remainder term $r = r(t, x)$ depends on the point \bar{u} and on the direction h . It is known that

$$(2.5) \quad \frac{\|r(\bar{u}, h)\|_{L^\infty(\Sigma)}}{\|h\|_{L^\infty(\Sigma)}} \rightarrow 0 \quad \text{as } \|h\|_{L^\infty(\Sigma)} \rightarrow 0.$$

One can easily verify that z solves the initial-boundary value problem

$$(2.6) \quad \begin{aligned} z_t + Az &= 0 && \text{in } Q \\ \partial_n z - \bar{b}_y z &= r && \text{in } \Sigma \\ z(0, x) &= 0 && \text{in } \Omega. \end{aligned}$$

The estimate (2.5) of the remainder r implies a similar property for z ,

$$(2.7) \quad \frac{\|z(\bar{u}, h)\|_{C(\bar{Q})}}{\|h\|_{L^\infty(\Sigma)}} \rightarrow 0 \quad \text{as } \|h\|_{L^\infty(\Sigma)} \rightarrow 0,$$

and the differentiability of S is readily seen from $S(\tilde{u}) = S(\bar{u}) + w + z(\bar{u}, h)$. \square

It is possible to extend the operator $S'(\bar{u})$ to a linear continuous operator in $\mathcal{L}(L^2(\Sigma))$. From now on, we consider $S'(\bar{u})$ in this way. The known property

$$\frac{\|r(\bar{u}, h)\|_{L^2(\Sigma)}}{\|h\|_{L^2(\Sigma)}} \rightarrow 0 \quad \text{as } \|h\|_{L^\infty(\Sigma)} \rightarrow 0$$

(see Maurer [10]) implies a similar property for $z|_\Sigma$

$$\frac{\|z|_\Sigma(\bar{u}, h)\|_{L^2(\Sigma)}}{\|h\|_{L^2(\Sigma)}} \rightarrow 0 \quad \text{as } \|h\|_{L^\infty(\Sigma)} \rightarrow 0.$$

Next, we introduce the L^2 -adjoint operator $S'(\bar{u})^* \in \mathcal{L}(L^2(\Sigma))$. This operator is given by $S'(\bar{u})^* \mu = \varphi|_{\Sigma}$, where φ is the solution of the well-posed parabolic backward problem

$$(2.8) \quad \begin{aligned} -\varphi_t + A^* \varphi &= 0 && \text{in } Q \\ \partial_n \varphi - \bar{b}_y \varphi &= \bar{b}_u \mu && \text{in } \Sigma \\ \varphi(T, x) &= 0 && \text{in } \Omega, \end{aligned}$$

where A^* is the formal adjoint operator to A . In all what follows let (\bar{y}, \bar{u}) be a locally optimal reference solution of (1.1–1.3). Let us set up the associated first-order necessary optimality conditions in form of a Karush-Kuhn-Tucker type theorem.

To this aim, we introduce the Lagrange functional $L : Y \times L^\infty(\Sigma) \times Y \times L^\infty(\Sigma)^2 \rightarrow \mathbb{R}$,

$$\begin{aligned} L(y, u, p, \mu_1, \mu_2) &= F(y, u) - \int_Q (y_t + Ay)p \, dxdt - \int_{\Sigma} (\partial_n y - b)p \, d\Gamma dt \\ &\quad - \int_{\Sigma} \mu_1 u \, d\Gamma dt + \int_{\Sigma} (u - c - \gamma y)\mu_2 \, d\Gamma dt, \end{aligned}$$

where $d\Gamma$ denotes the Lebesgue surface measure induced on Γ with respect to x .

Let us now comment on this choice for L . The heat equation (1.2) is considered in Y , while the inequality constraints (1.3) are posed in $L^\infty(\Sigma)$. Knowing the general Karush-Kuhn-Tucker theory in Banach spaces, one expects associated Lagrange multipliers $p \in Y^*$ and $\mu_i \in (L^\infty(\Sigma))^*$, together with a related quite complicated Lagrange functional. In contrast to this, special techniques for optimal control problems of bottleneck type have shown that, under natural assumptions, the Lagrange multipliers can be expressed by regular functions, i.e. $p \in W(0, T) \cap C(\bar{Q})$ and $\mu_i \in L^\infty(\Sigma)$, see Bergounioux/Tröltzsch [2] and Arada/Raymond [1]. This well known advantage of bottleneck type problems is our key idea to establish special second-order sufficient optimality conditions, which are hardly to be expected for $\mu_i \in (L^\infty(\Sigma))^*$. The existence of such regular multipliers can be shown under a Slater type condition and the assumption $\gamma(t, x) \geq 0$. Here, the nonnegativity of γ plays a crucial role.

Therefore we are justified to *assume* that an adjoint state $\bar{p} \in W(0, T) \cap C(\bar{Q})$ and Lagrange multipliers $\bar{\mu}_i \in L^\infty(\Sigma)$ exist such that $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)$ satisfies the following first-order necessary optimality system (FON),

$$(FON) \quad \left\{ \begin{array}{ll} D_y L(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2) & = 0 \\ D_u L(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2) & = 0 \\ \text{and for almost all } (t, x) \in \Sigma & \\ \bar{\mu}_1(t, x) & \geq 0 \\ \bar{\mu}_2(t, x) & \geq 0 \\ \bar{u}(t, x)\bar{\mu}_1(t, x) & = 0 \\ (\bar{u}(t, x) - c(t, x) - \gamma(t, x)\bar{y}(t, x))\bar{\mu}_2(t, x) & = 0. \end{array} \right.$$

The last two conditions of (FON) are the well-known *complementary slackness conditions*. They imply $\bar{\mu}_1(t, x) > 0 \Rightarrow \bar{u}(t, x) = 0$ and $\bar{\mu}_2(t, x) > 0 \Rightarrow \bar{u}(t, x) = c(t, x) + \gamma(t, x)\bar{y}(t, x)$. Let us express these optimality conditions also in terms of

partial differential equations. As it is well known, the first equation of (FON) is represented by the adjoint equation

$$(2.9) \quad \begin{aligned} -\bar{p}_t + A^*\bar{p} &= q_y(t, x, \bar{y}) && \text{in } Q \\ \partial_n \bar{p} - b_y(t, x, \bar{y}, \bar{u})\bar{p} &= \sigma_y(t, x, \bar{y}, \bar{u}) - \gamma\bar{\mu}_2 && \text{in } \Sigma \\ \bar{p}(T, x) &= \omega_y(x, \bar{y}(T, x)) && \text{in } \Omega. \end{aligned}$$

The second equation of (FON) is equivalent to

$$(2.10) \quad \sigma_u(t, x, \bar{y}, \bar{u}) + b_u(t, x, \bar{y}, \bar{u})\bar{p} - \bar{\mu}_1 + \bar{\mu}_2 = 0.$$

Next, we discuss a sufficient second-order optimality condition (SSC). For this purpose, we define *strongly active sets* and the associated *critical subspace*. Assume that $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)$ fulfils (FON).

DEFINITION 2.3. *Let $\delta_1, \delta_2 > 0$ be real numbers and $\bar{\mu}_1, \bar{\mu}_2 \in L^\infty(\Sigma)$ be the Lagrange multipliers introduced in (FON). The sets*

$$(2.11) \quad A_1(\delta_1) := \{(t, x) \in \Sigma : \bar{\mu}_1(t, x) \geq \delta_1\},$$

$$(2.12) \quad A_2(\delta_2) := \{(t, x) \in \Sigma : \bar{\mu}_2(t, x) - (S'(\bar{u})^*\gamma\bar{\mu}_2)(t, x) \geq \delta_2\}$$

are called *strongly active sets*. All further arguments hold true for an arbitrary choice of δ_1 and δ_2 . Later, these numbers will be chosen such that a second-order sufficient optimality condition is satisfied.

On $A_1(\delta_1)$ we have $\bar{\mu}_1 > 0$, hence the complementary slackness conditions yield $\bar{u}(t, x) = 0$. On $A_2(\delta_2)$, it holds $\bar{\mu}_2 > \delta_2 + S'(\bar{u})^*\gamma\bar{\mu}_2 \geq \delta_2 > 0$, hence the complementary slackness conditions give $\bar{u}(t, x) = c(t, x) + \gamma(t, x)\bar{y}(t, x)$. Notice that $A_1(\delta_1) \cap A_2(\delta_2) = \emptyset$, because $\bar{y} \geq 0$ and

$$c(t, x) + \gamma(t, x)\bar{y}(t, x) \geq c(t, x) > 0$$

holds true. Therefore we cannot have $\bar{u} = 0$ and $\bar{u} = c + \gamma\bar{y}$ at the same time.

DEFINITION 2.4. *We say that $(y, u) \in C(\bar{Q}) \times L^\infty(\Sigma)$ belongs to the critical subspace, if*

$$(2.13) \quad u = 0 \quad \text{on } A_1,$$

$$(2.14) \quad u = \gamma y|_\Sigma \quad \text{on } A_2,$$

and

$$(2.15) \quad \begin{aligned} y_t + Ay &= 0 && \text{in } Q \\ \partial_n y - \bar{b}_y y &= \bar{b}_u u && \text{in } \Sigma \\ y(0, x) &= 0 && \text{in } \Omega. \end{aligned}$$

Notice that (2.15) implies $y|_\Sigma = S'(\bar{u})u$. In (2.12), the expression $S'(\bar{u})^*\gamma\bar{\mu}_2$ can be evaluated by solving the backward problem

$$(2.16) \quad \begin{aligned} -\kappa_t + A^*\kappa &= 0 && \text{in } Q \\ \partial_n \kappa - \bar{b}_y \kappa &= \bar{b}_u \gamma \bar{\mu}_2 && \text{in } \Sigma \\ \kappa(T, x) &= 0 && \text{in } \Omega. \end{aligned}$$

The boundary values of κ deliver the desired expression, $\kappa|_\Sigma = S'(\bar{u})^*\gamma\bar{\mu}_2$. Knowing κ , it is easy to determine the strongly active set A_2 .

Before we formulate the second-order sufficient optimality condition, we mention for convenience the explicit expression of $L''_{(u,y)}(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)[h_y, h_u]^2$:

$$\begin{aligned}
L''_{(u,y)}(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)[h_y, h_u]^2 &= \int_{\Omega} \bar{\omega}_{yy} h_y^2 dx + \int_Q \bar{q}_{yy} h_y^2 dx dt \\
&+ \int_{\Sigma} (\bar{\sigma}_{yy} h_y^2 + 2\bar{\sigma}_{yu} h_y h_u + \bar{\sigma}_{uu} h_u^2) d\Gamma dt \\
(2.17) \qquad \qquad \qquad &+ \int_{\Sigma} (\bar{b}_{yy} h_y^2 + 2\bar{b}_{yu} h_y h_u + \bar{b}_{uu} h_u^2) \bar{p} d\Gamma dt.
\end{aligned}$$

Here, h_y, h_u denote arbitrary increments of y and u , respectively. Now we state the main result of our paper, the second-order sufficient condition.

(SSC): There exist positive numbers $\delta, \delta_1, \delta_2$ such that the coercivity condition

$$(2.18) \qquad L''_{(u,y)}(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)[h_y, h_u]^2 \geq \delta \|h_u\|_{L^2(\Sigma)}^2$$

holds true for all (h_y, h_u) belonging to the critical subspace defined upon δ_1, δ_2 .

THEOREM 2.5. (Second-order sufficiency) *Assume that $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)$ fulfils the first-order optimality system (FON). If the second-order condition (SSC) is satisfied, then there exist $\delta_s > 0$ and $\varepsilon > 0$ such that the quadratic growth condition*

$$(2.19) \qquad F(y, u) - F(\bar{y}, \bar{u}) \geq \delta_s \|u - \bar{u}\|_{L^2(\Sigma)}^2$$

holds for all admissible pairs (y, u) with $\|u - \bar{u}\|_{L^\infty(\Sigma)} < \varepsilon$. Therefore, \bar{u} is a locally optimal control in the norm of $L^\infty(\Sigma)$.

The proof is contained in Section 4.

3. Auxiliary results. **LEMMA 3.1.** *Let $\beta \in L^\infty(\Sigma)$ and $f \in L^2(\Sigma)$ be given and let v be the solution of the initial boundary value problem*

$$\begin{aligned}
v_t + Av &= 0 && \text{in } Q \\
\partial_n v + \beta v &= f && \text{in } \Sigma \\
v(0, x) &= 0 && \text{in } \Omega.
\end{aligned}$$

If $f \geq 0$ holds a.e. on Σ , then also $v \geq 0$ holds true a.e. on Σ .

For a proof of this *comparison principle* we refer to Raymond/Zidani [13].

DEFINITION 3.2. *A continuous linear operator A in $\mathcal{L}(L^2(\Sigma))$ is said to be non-negative, if $u \geq 0$ a.e. on Σ implies $Au \geq 0$ a.e. on Σ . In this case, we write $A \geq 0$.*

LEMMA 3.3. (Comparison principle) *Under the assumptions (A1)-(A4), the non-negativity properties*

$$(3.1) \qquad S'(\bar{u}) \geq 0,$$

$$(3.2) \qquad (I - \gamma S'(\bar{u}))^{-1} \geq 0$$

hold true.

Proof. The operator $S'(\bar{u}) : u \mapsto w_\Sigma$, is defined upon equation (2.2). In (A1) we have assumed $b_u \geq 0$. Hence Lemma 3.1, applied with $\beta = -\bar{b}_y, f = \bar{b}_u u$, yields that $u \geq 0$ implies $w|_\Sigma \geq 0$ and (3.1) is proved.

To prove (3.2), we apply Lemma 3.1 to the system

$$(3.3) \quad \begin{aligned} v_t + Av &= 0 && \text{in } Q \\ \partial_n v - (\bar{b}_y + \gamma \bar{b}_u)v &= \bar{b}_u u && \text{in } \Sigma \\ v(0, x) &= 0 && \text{in } \Omega. \end{aligned}$$

Invoking Lemma 3.1 again, the implication $u \geq 0 \Rightarrow v|_\Sigma \geq 0$ holds. A comparison of (3.3) with (2.2) shows that

$$S'(\bar{u})(\gamma v + u) = v$$

holds. Setting $z = \gamma v + u$, we get $z = \gamma S'(\bar{u})(\gamma v + u) + u = \gamma S'(\bar{u})z + u$, hence $z = (I - \gamma S'(\bar{u}))^{-1}u$. Thanks to the implication $u \geq 0 \Rightarrow v|_\Sigma \geq 0$, (A3), and $z = \gamma v + u$, we obtain $u \geq 0 \Rightarrow z \geq 0$. This proves (3.2). \square

COROLLARY 3.4. *The property (3.1) extends to the adjoint operator $S'(\bar{u})^*$,*

$$(3.4) \quad S'(\bar{u})^* \geq 0.$$

In what follows, we repeatedly need controls u defined as follows: Let M_1, M_2 be disjoint measurable subsets of Σ such that $M_1 \cup M_2 = \Sigma$, and let $f \in L^\infty(\Sigma)$ be given. Then we would like to define u by

$$(3.5) \quad u(t, x) = \begin{cases} f(t, x) & \text{on } M_1, \\ f(t, x) + \gamma(t, x)(S'(\bar{u})u)(t, x) & \text{on } M_2. \end{cases}$$

The next lemma shows that this setting is correct:

LEMMA 3.5. *For all disjoint measurable subsets M_1, M_2 of Σ with $M_1 \cup M_2 = \Sigma$ and all $f \in L^\infty(\Sigma)$, there is exactly one function $u \in L^\infty(\Sigma)$ that satisfies condition (3.5). In addition, the implication*

$$(3.6) \quad f \geq 0 \quad \Rightarrow \quad u \geq 0$$

holds true. Moreover, the estimates

$$(3.7) \quad \|u\|_{L^\infty(\Sigma)} \leq c_1 \|f\|_{L^\infty(\Sigma)},$$

$$(3.8) \quad \|u\|_{L^2(\Sigma)} \leq c_2 \|f\|_{L^2(\Sigma)}$$

hold with certain constants c_1, c_2 that do not depend on M_1, M_2 , and f .

Proof. Suppose that $u \in L^\infty(\Sigma)$ satisfies (3.5). Put $v := S'(\bar{u})u$. Then v satisfies the heat equation with homogeneous initial data and the boundary condition

$$(3.9) \quad \partial_n v - \bar{b}_y v = \begin{cases} \bar{b}_u f & \text{on } M_1, \\ \bar{b}_u (f + \gamma v) & \text{on } M_2, \end{cases}$$

that is

$$(3.10) \quad \partial_n v - (\bar{b}_y v + \chi_{M_2} \bar{b}_u \gamma v) = \bar{b}_u f \quad \text{on } \Sigma.$$

This solution v is unique. Therefore, if u satisfies (3.5), then $v = S'(\bar{u})u$ is unique, hence u is unique, because of

$$(3.11) \quad u = \begin{cases} f & \text{on } M_1, \\ f + \gamma v & \text{on } M_2. \end{cases}$$

On the other hand, starting from M_1 , M_2 , and f , the solution v of the heat equation with homogeneous initial data and boundary condition (3.10) is defined, and the function u given by (3.11) satisfies (3.5), since, by definition of v , $u = S'(\bar{u})v$.

Finally, by Lemma 3.1 applied to (3.10) with $f := \bar{b}_u f$, the relation $f \geq 0$ implies $v \geq 0$, hence also $u \geq 0$. The estimates (3.7) and (3.8) follow immediately from those for v and (3.11). \square

To prove the main result, we later have to compare the reference pair (\bar{y}, \bar{u}) with another admissible pair (y, u) , where $y = S(u)$. Then we have to estimate the difference

$$y - \bar{y} = S(u) - S(\bar{u}) = S'(\bar{u})(u - \bar{u}) + r_1(\bar{u}, u - \bar{u}),$$

where r_1 stands for the associated first-order remainder term. In the following, we denote for short the remainder $r_1(\bar{u}, u - \bar{u})$ and the derivative $S'(\bar{u})$ by r_1 and S' , respectively, if there is no risk of notational confusion.

Before continuing our analysis of second-order sufficiency, we briefly discuss the main difficulties and our main ideas to resolve them. We start with the case of pure control constraints, which is covered by our setting for $\gamma(t, x) \equiv 0$. Then the constraints are simple box constraints,

$$0 \leq u(t, x) \leq c(t, x).$$

On A_1 , we have $\bar{u}(t, x) \equiv 0$, hence $u - \bar{u} \geq 0$ on A_1 , while $\bar{u}(t, x) = c(t, x)$ holds on A_2 , thus $u - \bar{u} \leq 0$ on A_2 . The associated terms in the Lagrange functional can be estimated as

$$\begin{aligned} \int_{A_1} \bar{\mu}_1(u - \bar{u}) \, d\Gamma dt - \int_{A_2} \bar{\mu}_2(u - \bar{u}) \, d\Gamma dt &\geq \int_{A_1} \delta_1(u - \bar{u}) \, d\Gamma dt + \int_{A_2} \delta_2(u - \bar{u}) \, d\Gamma dt \\ &= \delta_1 \|u - \bar{u}\|_{L^1(A_1)} + \delta_2 \|u - \bar{u}\|_{L^1(A_2)}. \end{aligned}$$

In the proof of the sufficiency theorem the L^1 -norms on the right-hand side will compensate for the lack of coercivity, since (2.18) does not help on $A_1 \cup A_2$.

Now we return to the given mixed control-state constraints

$$0 \leq u(t, x) \leq c(t, x) + \gamma(t, x) y(t, x).$$

To simplify our explanation, assume for a while that the control-state mapping is linear. This holds for $y_0 = 0$ functions b being linear with respect to y and u . Then $S' = S$, hence

$$(3.12) \quad 0 \leq u \leq c + \gamma S' u$$

holds for any admissible control u . On A_1 , we have again $0 = \bar{u} \leq u$, hence $u - \bar{u} \geq 0$ on A_1 . However, in contrast to the case of pure control constraints, the relation $u \leq \bar{u}$ cannot be expected on A_2 now. If $u > \bar{u}$ holds somewhere on $\Sigma \setminus A_2$, then $S' u > S' \bar{u}$ can hold on A_2 . Then the right-hand side of (3.12) is greater than $c + \gamma S' \bar{u}$ and $u > \bar{u}$ can happen.

To overcome this difficulty, we represent u in the form $u = u_1 + u_2$, such that $u_1 \leq \bar{u}$ can be shown on A_2 and u_2 stands for the additional margin of freedom that is caused by $u > \bar{u}$ outside of A_2 . Hence we split u in two parts, $u = u_1 + u_2$ on Σ , where

$$(3.13) \quad \begin{aligned} u_1 &= \bar{u}, & u_2 &= u - \bar{u} & \text{on } \Sigma \setminus A_2, \\ u_2 &= \gamma(S' u_2 + r_1), & u_1 &= u - u_2 & \text{on } A_2. \end{aligned}$$

The functions u_1 and u_2 are well defined. To see this, we apply Lemma 3.5, where $M_1 = \Sigma \setminus A_2$ and $M_2 = A_2$. On M_1 , u_2 is given by $u - \bar{u}$. On M_2 , $u_2 = \gamma r_1 + \gamma S' u_2$, hence

$$u_2 = \begin{cases} f & \text{on } M_1, \\ f + \gamma S' u_2 & \text{on } M_2. \end{cases}$$

where $f = u - \bar{u}$ on M_1 , $f = \gamma r_1$ on M_2 . Then u_2 is well defined by Lemma 3.5. Note that $S' u_2 = S'(\bar{u})(\chi_{M_1}(u - \bar{u}) + \chi_{M_2} u_2)$. From (3.10) and the properties of the remainder r_1 we get easily

$$\|u_2\|_{L^\infty(\Sigma)} \leq c_3 \|u - \bar{u}\|_{L^\infty(\Sigma)}.$$

Therefore, we find

$$(3.14) \quad \begin{aligned} \|u_1 - \bar{u}\|_{L^\infty(A_2)} &\leq \|u - \bar{u}\|_{L^\infty(A_2)} + \|u_2\|_{L^\infty(A_2)} \\ &\leq c_4 \|u - \bar{u}\|_{L^\infty(\Sigma)}. \end{aligned}$$

LEMMA 3.6. *Assume that $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)$ fulfils the first order optimality system (FON) and take any u satisfying the mixed control-state constraints. Let u be splitted by $u = u_1 + u_2$ according to formula (3.13). Then it holds*

$$(3.15) \quad \bar{u} - u_1 \geq 0 \quad \text{a.e. on } \Sigma.$$

Proof. Let A_1 and A_2 be the strongly active sets of \bar{u} defined by δ_1 and δ_2 . On A_2 , the inequality, $\bar{\mu}_2 \geq \bar{\mu}_2 - \gamma S'(\bar{u})^* \bar{\mu}_2 \geq \delta_2 > 0$ holds. Therefore, (FON) implies $\bar{u} = c + \gamma \bar{y}$ there. In addition, we know on A_2 that $u - \gamma y \leq c = \bar{u} - \gamma \bar{y}$, hence

$$u - \gamma S(u) \leq \bar{u} - \gamma S(\bar{u})$$

holds there. In view of this, we find on A_2

$$(3.16) \quad \begin{aligned} u - \gamma(S(u) - S(\bar{u})) &\leq \bar{u} \\ u - \gamma(S'(\bar{u})(u - \bar{u}) + r_1) &\leq \bar{u} \\ u_1 - \gamma S' u_1 + (u_2 - \gamma(S' u_2 + r_1)) &\leq \bar{u} - \gamma S' \bar{u} \\ u_1 - \gamma S' u_1 &\leq \bar{u} - \gamma S' \bar{u} \\ (I - \gamma S')(u_1 - \bar{u}) &\leq 0, \end{aligned}$$

where we have inserted the definition of u_2 . Outside of A_2 , it holds by definition $u_1 = \bar{u}$. We are now again in the situation that was described in Lemma 3.5. Indeed, taking $M_1 := \Sigma \setminus A_2$, $M_2 = A_2$, $f = 0$ on M_1 and $f = (I - \gamma S'(\bar{u}))(u_1 - \bar{u})$ on M_2 , we have $f \geq 0$. Applying (3.6), we obtain

$$\bar{u} - u_1 \geq 0 \quad \text{a.e. on } \Sigma,$$

which is just inequality (3.15). \square

LEMMA 3.7. *Assume that $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)$ fulfil the first-order optimality system (FON). Then the estimates*

$$(3.17) \quad \int_{\Sigma} (u - \bar{u}) \bar{\mu}_1 \, d\Gamma dt \geq \frac{\delta_1}{\varepsilon} \|u - \bar{u}\|_{L^2(A_1)}^2,$$

$$(3.18) \quad - \int_{\Sigma} (u - \bar{u} - \gamma(y - \bar{y})) \bar{\mu}_2 \, d\Gamma dt \geq \frac{\delta_2}{c_4 \varepsilon} \|u_1 - \bar{u}\|_{L^2(A_2)}^2$$

are valid for all $\varepsilon > 0$ and all admissible pairs (u, y) satisfying $\|u - \bar{u}\|_{L^\infty(\Sigma)} < \varepsilon$.

Proof. (i) Because of (FON), $\bar{\mu}_1 > 0$ can only hold if $\bar{u} = 0$. If $\bar{u} > 0$, then $\bar{\mu}_1 = 0$. Moreover, u is admissible, hence $u \geq 0$ and we have almost everywhere

$$(u - \bar{u})\bar{\mu}_1 \geq 0.$$

Therefore we get by (2.11)

$$\int_{\Sigma} (u - \bar{u})\bar{\mu}_1 \, d\Gamma dt \geq \int_{A_1} (u - \bar{u})\bar{\mu}_1 \, d\Gamma dt \geq \delta_1 \|u - \bar{u}\|_{L^1(A_1)}.$$

By our assumption, we have $\|u - \bar{u}\|_{L^\infty(\Sigma)} < \varepsilon$. In particular this inequality includes $\|u - \bar{u}\|_{L^\infty(A_1)} < \varepsilon$. Consequently,

$$\int_{\Sigma} (u - \bar{u})\bar{\mu}_1 \, d\Gamma dt \geq \delta_1 \|u - \bar{u}\|_{L^1(A_1)} \frac{\|u - \bar{u}\|_{L^\infty(A_1)}}{\varepsilon} \geq \frac{\delta_1}{\varepsilon} \|u - \bar{u}\|_{L^2(A_1)}^2,$$

and (3.17) is proved.

(ii) Next, we discuss the integral in (3.18). Because of (FON), $\bar{\mu}_2 > 0$ can only hold for $\bar{u} - c - \gamma\bar{y} = 0$. In addition, (y, u) is admissible, hence in particular $u \leq c + \gamma y$. Therefore, we obtain almost everywhere

$$-(u - \bar{u} - \gamma(y - \bar{y}))\bar{\mu}_2 \geq 0$$

and

$$-\int_{\Sigma} (u - \bar{u} - \gamma(y - \bar{y}))\bar{\mu}_2 \, d\Gamma dt \geq -\int_{A_2} (u - \bar{u} - \gamma(y - \bar{y}))\bar{\mu}_2 \, d\Gamma dt.$$

Let us discuss this integral more detailed. Expressing $y - \bar{y}$ in terms of the controls,

$$(3.19) \int_{A_2} (u - \bar{u} - \gamma(y - \bar{y}))\bar{\mu}_2 \, d\Gamma dt = \int_{A_2} (u - \bar{u} - \gamma(S'(\bar{u})(u - \bar{u}) + r_1))\bar{\mu}_2 \, d\Gamma dt$$

is found. The definition of u_1 and u_2 yields on A_2

$$u - \gamma(S'u + r_1) = u_1 + u_2 - \gamma S'u_1 - \gamma S'u_2 - \gamma r_1 = u_1 - \gamma S'u_1.$$

Inserting the last equation in (3.19), we continue by

$$\begin{aligned} \int_{A_2} (u - \bar{u} - \gamma(y - \bar{y}))\bar{\mu}_2 \, d\Gamma dt &= \int_{A_2} (u_1 - \bar{u} - \gamma(S'(\bar{u})(u_1 - \bar{u})))\bar{\mu}_2 \, d\Gamma dt \\ &= \int_{\Sigma} (u_1 - \bar{u} - \gamma(S'(\bar{u})(u_1 - \bar{u})))\chi_{A_2}\bar{\mu}_2 \, d\Gamma dt \\ &= \int_{\Sigma} (u_1 - \bar{u})(I - (\gamma S')^*)(\chi_{A_2}\bar{\mu}_2) \, d\Gamma dt \\ (3.20) \qquad \qquad \qquad &= \int_{A_2} (u_1 - \bar{u})(I - (\gamma S')^*)(\chi_{A_2}\bar{\mu}_2) \, d\Gamma dt. \end{aligned}$$

To deduce the last equation, we used $\bar{u} - u_1 = 0$ outside of A_2 . Now we discuss the expression $(I - (\gamma S')^*)(\chi_{A_2}\bar{\mu}_2)$ in (3.20). On A_2 we have

$$(I - (\gamma S')^*)(\chi_{A_2}\bar{\mu}_2) = \chi_{A_2}\bar{\mu}_2 - (\gamma S')^*(\chi_{A_2}\bar{\mu}_2) = \bar{\mu}_2 - (\gamma S')^*(\chi_{A_2}\bar{\mu}_2).$$

Using the non-negativity of S'^* following from (3.4), together with $\chi_{A_2}\bar{\mu}_2 \leq \bar{\mu}_2$, we obtain

$$(\gamma S')^*(\chi_{A_2}\bar{\mu}_2) = (S')^*(\gamma\chi_{A_2}\bar{\mu}_2) \leq (S')^*(\gamma\bar{\mu}_2) = (\gamma S')^*\bar{\mu}_2.$$

Combining these results, we continue by

$$(3.21) \quad (I - (\gamma S')^*)(\chi_{A_2}\bar{\mu}_2) = \bar{\mu}_2 - (\gamma S')^*(\chi_{A_2}\bar{\mu}_2) \geq (I - (\gamma S')^*)\bar{\mu}_2 \geq \delta_2,$$

where the last inequality follows from the definition (2.12) of A_2 . Inserting (3.15) and (3.21) in (3.20), we infer

$$\begin{aligned} - \int_{A_2} (u - \bar{u} - \gamma(y - \bar{y}))\bar{\mu}_2 \, d\Gamma dt &= - \int_{A_2} (u_1 - \bar{u})(I - (\gamma S')^*)(\chi_{A_2}\bar{\mu}_2) \, d\Gamma dt \\ &\geq \delta_2 \|\bar{u} - u_1\|_{L^1(A_2)}. \end{aligned}$$

Invoking again $\|u - \bar{u}\|_{L^\infty(\Sigma)} < \varepsilon$ and (3.14), we obtain

$$\begin{aligned} - \int_{A_2} (u - \bar{u} - \gamma(y - \bar{y}))\bar{\mu}_2 \, d\Gamma dt &\geq \delta_2 \|u_1 - \bar{u}\|_{L^1(A_2)} \cdot \frac{\|u - \bar{u}\|_{L^\infty(A_2)}}{\varepsilon} \\ &\geq \frac{\delta_2}{c_4 \varepsilon} \|u_1 - \bar{u}\|_{L^2(A_2)}^2, \end{aligned}$$

implying inequality (3.18). \square

If $A_1 \cup A_2 = \Sigma$, then the critical subspace contains only the zero-function. Then the assumptions of Theorem 2.5 are trivially fulfilled. In this case, (3.17) and (3.18) express the so-called *first-order sufficient optimality conditions*.

4. Second-order sufficient optimality condition. This section includes the proof of the sufficiency Theorem 2.5. We start from an admissible control u in a sufficiently small L^∞ -neighborhood of \bar{u} and have to show that $F(y, u) \geq F(\bar{y}, \bar{u})$. Let us introduce the increments $\delta u := u - \bar{u}$ and $\delta y := S'(\bar{u})\delta u$. We split $\delta u = u_0 + u_+$, where

$$\begin{aligned} u_0 &= 0, & u_+ &= \delta u & \text{on } A_1, \\ u_0 &= \delta u, & u_+ &= 0 & \text{on } \Sigma \setminus (A_1 \cup A_2), \\ u_0 &= \gamma S'(\bar{u})u_0, & u_+ &= \delta u - u_0 & \text{on } A_2. \end{aligned}$$

Thanks to Lemma 3.5, the definition of u_0 and hence u_+ is correct. We take $M_1 = \Sigma \setminus A_2$, $M_2 = A_2$, $f := 0$ on $A_1 \cup A_2$, $f := \delta u$ on $\Sigma \setminus (A_1 \cup A_2)$. The part u_0 belongs to the critical subspace, while u_+ is the part of δu that accounts for the effects of first-order sufficiency. Furthermore, we define $y_0 := S'u_0$ and $y_+ := S'u_+$. By the linearity of S' , we have $\delta y = y_0 + y_+$.

Below, we estimate the difference $L(y, u, \bar{p}, \bar{\mu}_1, \bar{\mu}_2) - L(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)$. Let us write for short $L(y, u) - L(\bar{y}, \bar{u})$, since $(\bar{p}, \bar{\mu}_1, \bar{\mu}_2)$ remains fixed in all the next considerations. We also do not explicitly indicate the point $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)$ where all derivatives are taken, i.e. we write $L_u u$ instead of $(D_u L)(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)u$.

LEMMA 4.1. *Under the assumptions of Theorem 2.5,*

$$(4.1) \quad L(y, u) - L(\bar{y}, \bar{u}) \geq \frac{\delta}{4} \|u_0\|_{L^2(\Sigma)}^2 - \frac{c_s}{2} \|u_+\|_{L^2(\Sigma)}^2 + r_2 + \tilde{r}_2$$

holds, where r_2, \tilde{r}_2 are second-order remainder terms with

$$\frac{|r_i|}{\|u - \bar{u}\|_{L^2(\Sigma)}^2} \rightarrow 0 \quad \text{if } \|u - \bar{u}\|_{L^\infty(\Sigma)} \rightarrow 0.$$

Proof. Using a Taylor expansion, in view of (FON) we get

$$(4.2) \quad \begin{aligned} L(y, u) - L(\bar{y}, \bar{u}) &= L_u[u - \bar{u}] + L_y[y - \bar{y}] + \frac{1}{2}(L_{uu}[u - \bar{u}]^2 \\ &\quad + 2L_{uy}[u - \bar{u}, y - \bar{y}] + L_{yy}[y - \bar{y}]^2) + r_2 \\ &= \frac{1}{2}(L_{uu}[u - \bar{u}]^2 + 2L_{uy}[u - \bar{u}, y - \bar{y}] + L_{yy}[y - \bar{y}]^2) + r_2. \end{aligned}$$

The following property of the remainder is known

$$\frac{|r_2(\bar{u}, h)|}{\|h\|_{L^2(\Sigma)}^2} \rightarrow 0 \quad \text{as } \|h\|_{L^\infty(\Sigma)} \rightarrow 0.$$

For the proof we refer to [16]. According to the notation of Lemma 3.5, we get $y - \bar{y} = \delta y + r_1$. Replacing $y - \bar{y}$ by δy in (4.2), we cause a small error of second order

$$\begin{aligned} \tilde{r}_2 &:= \frac{1}{2}(L_{uu}[u - \bar{u}]^2 + 2L_{uy}[u - \bar{u}, y - \bar{y}] + L_{yy}[y - \bar{y}]^2) \\ &\quad - \frac{1}{2}(L_{uu}[\delta u]^2 + 2L_{uy}[\delta u, \delta y] + L_{yy}[\delta y]^2). \end{aligned}$$

It is easy to show that

$$\frac{|\tilde{r}_2|}{\|u - \bar{u}\|_{L^2(\Sigma)}^2} \rightarrow 0 \quad \text{as } \|u - \bar{u}\|_{L^\infty(\Sigma)} \rightarrow 0.$$

With these notations, we can express (4.2) in the form

$$(4.3) \quad L(y, u) - L(\bar{y}, \bar{u}) = \frac{1}{2}(L_{uu}[\delta u]^2 + 2L_{uy}[\delta u, \delta y] + L_{yy}[\delta y]^2) + r_2 + \tilde{r}_2.$$

We continue by splitting the Lagrange functional,

$$\begin{aligned} L_{uu}[\delta u]^2 + 2L_{uy}[\delta u, \delta y] + L_{yy}[\delta y]^2 &= L_{uu}[u_0]^2 + 2L_{uy}[u_0, y_0] + L_{yy}[y_0]^2 \\ &\quad + L_{uu}[u_+]^2 + 2L_{uy}[u_+, y_+] + L_{yy}[y_+]^2 \\ &\quad + 2L_{uu}[u_0, u_+] + 2L_{uy}[u_0, y_+] \\ &\quad + 2L_{uy}[u_+, y_0] + 2L_{yy}[y_0, y_+]. \end{aligned}$$

As u_0 belongs to the critical subspace, the second-order condition (SSC) yields

$$L''[u_0, y_0]^2 = L_{uu}[u_0]^2 + 2L_{uy}[u_0, y_0] + L_{yy}[y_0]^2 \geq \delta \|u_0\|_{L^2(\Sigma)}^2.$$

The other terms are easily estimated by $\|y_0\|_{L^2(\Sigma)}^2 \leq \|S'\|^2 \|u_0\|_{L^2(\Sigma)}^2$, $\|y_+\|_{L^2(\Sigma)}^2 \leq \|S'\|^2 \|u_+\|_{L^2(\Sigma)}^2$, and by means of Young's inequality,

$$\begin{aligned} &|L_{uu}[u_+]^2 + 2L_{uy}[u_+, y_+] + L_{yy}[y_+]^2 \\ &\quad + 2L_{uu}[u_0, u_+] + 2L_{uy}[u_0, y_+] \\ &\quad + 2L_{uy}[u_+, y_0] + 2L_{yy}[y_0, y_+]| \leq \frac{\delta}{2} \|u_0\|_{L^2(\Sigma)}^2 + c_s \|u_+\|_{L^2(\Sigma)}^2. \end{aligned}$$

In this setting, c_s is a certain (large) constant. Combining the last two results, we arrive at

$$L_{uu}[\delta u]^2 + 2L_{uy}[\delta u, \delta y] + L_{yy}[\delta y]^2 \geq \frac{\delta}{2}\|u_0\|_{L^2(\Sigma)}^2 - c_s\|u_+\|_{L^2(\Sigma)}^2.$$

Returning to (4.3), we end up with

$$L(y, u) - L(\bar{y}, \bar{u}) \geq \frac{\delta}{4}\|u_0\|_{L^2(\Sigma)}^2 - \frac{c_s}{2}\|u_+\|_{L^2(\Sigma)}^2 + r_2 + \tilde{r}_2,$$

which is exactly the assertion. \square

In the next Lemma, we estimate the term $\|u_+\|_{L^2(\Sigma)}^2$ in (4.1).

LEMMA 4.2. *Under the assumptions of Theorem 2.5,*

$$(4.4) \quad \left(\frac{c_s}{2} + \frac{\delta}{4}\right)\|u_+\|_{L^2(\Sigma)}^2 \leq c_5\|u_1 - \bar{u}\|_{L^2(A_2)}^2 + c_6\|r_1\|_{L^2(\Sigma)}^2 + c_7\|u - \bar{u}\|_{L^2(A_1)}^2$$

holds with certain positive constants c_5 , c_6 , and c_7 .

Proof. First, we get on A_1

$$(4.5) \quad \|u_+\|_{L^2(A_1)} = \|\delta u\|_{L^2(A_1)} = \|u - \bar{u}\|_{L^2(A_1)}.$$

On the whole set Σ we have

$$u_+ + u_0 = \delta u = u - \bar{u}.$$

We apply the operator $I - \gamma S'$ to this equation and consider the image only on the set A_2 . Using $u_0 = \gamma S' u_0$ on A_2 , we find

$$u_+ - \gamma S' u_+ = u - \gamma S' u - (\bar{u} - \gamma S' \bar{u}) \text{ on } A_2.$$

Now, u is again replaced by $u_1 + u_2$, see (3.13), to obtain on A_2

$$u_+ - \gamma S' u_+ = u_1 - \gamma S' u_1 + u_2 - \gamma S' u_2 - (\bar{u} - \gamma S' \bar{u}).$$

On A_2 , by definition, the equation $u_2 - \gamma S' u_2 = r_1$ is satisfied. Therefore, here we are able to continue by

$$u_+ - \gamma S' u_+ = u_1 - \bar{u} - (\gamma S'(\bar{u})(u_1 - \bar{u})) + r_1 \text{ on } A_2.$$

Due to our definitions, $u_+ = \delta u = u - \bar{u}$ holds on A_1 . In addition, u_+ vanishes on $\Sigma \setminus (A_1 \cup A_2)$. Therefore, we find

$$u_+ = \begin{cases} u_1 - \bar{u} + \gamma S'(\bar{u})(u_1 - \bar{u}) + r_1 & \text{on } A_2 \\ u - \bar{u} & \text{on } A_1 \\ 0 & \text{on } \Sigma \setminus (A_1 \cup A_2). \end{cases}$$

Again we have a construction that was investigated in Lemma 3.5. Setting $M_2 = A_2$, $M_1 = \Sigma \setminus A_2$ and applying (3.11), we get the inequality

$$\|u_+\|_{L^2(\Sigma)} \leq c_2\|f\|_{L^2(\Sigma)},$$

where f is defined by

$$f = \begin{cases} r_1 + (u_1 - \bar{u}) - \gamma S'(\bar{u})(u_1 - \bar{u}) & \text{on } A_2 \\ u - \bar{u} & \text{on } A_1 \\ 0 & \text{on } \Sigma \setminus (A_1 \cup A_2). \end{cases}$$

Therefore, we obtain

$$\|u_+\|_{L^2(\Sigma)} \leq c_2(\|u - \bar{u}\|_{L^2(A_1)} + c_8\|u_1 - \bar{u}\|_{L^2(\Sigma)} + \|r_1\|_{L^2(A_2)}),$$

where the positive constant c_8 is related to $\|S'\|$. Using $\|u_1 - \bar{u}\|_{L^2(\Sigma)} = \|u_1 - \bar{u}\|_{L^2(A_2)}$,

$$\|u_+\|_{L^2(\Sigma)} \leq c_9\|u_1 - \bar{u}\|_{L^2(A_2)} + c_2\|r_1\|_{L^2(A_2)} + c_2\|u - \bar{u}\|_{L^2(A_1)}$$

is found. Young's inequality yields

$$\|u_+\|_{L^2(\Sigma)}^2 \leq 3c_9\|u_1 - \bar{u}\|_{L^2(A_2)}^2 + 3c_2\|r_1\|_{L^2(\Sigma)}^2 + 3c_2\|u - \bar{u}\|_{L^2(A_1)}^2.$$

A multiplication by $(\frac{c_s}{2} + \frac{\delta}{4})$

$$\left(\frac{c_s}{2} + \frac{\delta}{4}\right) \|u_+\|_{L^2(\Sigma)}^2 \leq c_5\|u_1 - \bar{u}\|_{L^2(A_2)}^2 + c_6\|r_1\|_{L^2(\Sigma)}^2 + c_7\|u - \bar{u}\|_{L^2(A_1)}^2$$

concludes the proof of the Lemma. \square

Now we are able to prove our main result Theorem 2.5.

Proof. (Theorem 2.5) Inserting (4.4) in (4.1),

$$\begin{aligned} L(y, u) - L(\bar{y}, \bar{u}) &\geq \frac{\delta}{4}(\|u_0\|_{L^2(\Sigma)}^2 + \|u_+\|_{L^2(\Sigma)}^2) + r_2 + \tilde{r}_2 \\ &\quad - c_7\|u - \bar{u}\|_{L^2(A_1)}^2 - c_5\|u_1 - \bar{u}\|_{L^2(A_2)}^2 - c_6\|r_1\|_{L^2(\Sigma)}^2 \end{aligned}$$

is obtained. Next, we return to the objective F ,

$$\begin{aligned} L(y, u) - L(\bar{y}, \bar{u}) &= F(y, u) - F(\bar{y}, \bar{u}) \\ &\quad - \int_Q (y_t + Ay)\bar{p} \, dxdt - \int_\Sigma (\partial_n y - b)\bar{p} \, d\Gamma dt \\ &\quad + \int_Q (\bar{y}_t + A\bar{y})\bar{p} \, dxdt + \int_\Sigma (\partial_n \bar{y} - \bar{b})\bar{p} \, d\Gamma dt \\ &\quad - \int_\Sigma \bar{\mu}_1 u \, d\Gamma dt + \int_0^T \bar{\mu}_1 \bar{u} \, d\Gamma dt \\ &\quad + \int_\Sigma (u - c - \gamma y)\bar{\mu}_2 \, d\Gamma dt \\ &\quad - \int_\Sigma (\bar{u} - c - \gamma \bar{y})\bar{\mu}_2 \, d\Gamma dt \\ &= F(y, u) - F(\bar{y}, \bar{u}) \\ &\quad - \int_\Sigma \bar{\mu}_1 (u - \bar{u}) \, d\Gamma dt \\ &\quad + \int_\Sigma (u - \bar{u} - \gamma(y - \bar{y}))\bar{\mu}_2 \, d\Gamma dt. \end{aligned}$$

Using Lemma 3.7 we find

$$\begin{aligned} F(y, u) - F(\bar{y}, \bar{u}) &\geq \frac{\delta}{4}(\|u_0\|_{L^2(\Sigma)}^2 + \|u_+\|_{L^2(\Sigma)}^2) + r_2 + \tilde{r}_2 \\ &\quad + \left(\frac{\delta_1}{\varepsilon} - c_7\right)\|u - \bar{u}\|_{L^2(A_1)}^2 + \left(\frac{\delta_2}{c_4\varepsilon} - c_5\right)\|u_1 - \bar{u}\|_{L^2(A_2)}^2 \\ &\quad - c_6\|r_1\|_{L^2(\Sigma)}^2. \end{aligned} \tag{4.6}$$

Next, $\|\delta u\|_{L^2(\Sigma)} = \|u_0 + u_+\|_{L^2(\Sigma)}^2 \leq 2\|u_0\|_{L^2(\Sigma)}^2 + 2\|u_+\|_{L^2(\Sigma)}^2$ is applied to continue by

$$(4.7) \quad \begin{aligned} F(y, u) - F(\bar{y}, \bar{u}) &\geq \frac{\delta}{8} \|\delta u\|_{L^2(\Sigma)}^2 + r_2 + \tilde{r}_2 \\ &\quad + \left(\frac{\delta_1}{\varepsilon} - c_7\right) \|u - \bar{u}\|_{L^2(A_1)}^2 + \left(\frac{\delta_2}{c_4\varepsilon} - c_5\right) \|u_1 - \bar{u}\|_{L^2(A_2)}^2 \\ &\quad - c_6 \|r_1\|_{L^2(\Sigma)}^2. \end{aligned}$$

Take now ε sufficiently small, such that

$$\frac{\delta_1}{\varepsilon} - c_7 \geq 0 \quad \text{and} \quad \frac{\delta_2}{c_4\varepsilon} - c_5 \geq 0.$$

Then we can omit the associated terms in (4.7),

$$(4.8) \quad F(y, u) - F(\bar{y}, \bar{u}) \geq \frac{\delta}{8} \|\delta u\|_{L^2(\Sigma)}^2 + r_2 + \tilde{r}_2 - c_6 \|r_1\|_{L^2(\Sigma)}^2.$$

Due to the discussions during the proof, all terms of the right-hand side (except the first one) are small with respect to $\|u - \bar{u}\|_{L^2(\Sigma)}^2$. Therefore

$$(4.9) \quad F(y, u) - F(\bar{y}, \bar{u}) \geq \frac{\delta}{16} \|u - \bar{u}\|_{L^2(\Sigma)}^2$$

holds if $\|u - \bar{u}\|_{L^\infty(\Sigma)} < \varepsilon$ and ε is sufficiently small. The quadratic growth condition is proved. We can choose $\delta_s = \delta/16$. \square

REFERENCES

- [1] N. ARADA AND J. P. RAYMOND, *Optimal control problems with mixed control-state constraints*, SIAM J. Control Optimization, 39 (2001), pp. 1391–1407.
- [2] M. BERGOUNIOUX AND F. TRÖLTZSCH, *Optimal control of semilinear parabolic equations with state-constraints of bottleneck type*, ESAIM, Control, Optimisation and Calculus of Variations, 4 (1999), pp. 595–608.
- [3] F. BONNANS, *Second order analysis for control constrained optimal control problems of semilinear elliptic systems*, Appl. Math. Opt., 38 (1998), pp. 303–325.
- [4] E. CASAS, *Pontryagin's principle for state-constrained boundary control problems of semilinear parabolic equations*, SIAM J. Control and Optimization, 35 (1997), pp. 1297–1327.
- [5] E. CASAS, F. TRÖLTZSCH, AND A. UNGER, *Second order sufficient optimality conditions for a nonlinear elliptic control problem*, J. for Analysis and its applications (ZAA), 15 (1996), pp. 687–707.
- [6] ———, *Second order sufficient optimality conditions for some state-constrained control problems of semilinear elliptic equations*, SIAM J. Control and Optimization, 38 (2000), pp. 1369–1391.
- [7] A. DONTCHEV, W. HAGER, A. POORE, AND B. YANG, *Optimality, stability, and convergence in nonlinear control*, Appl. Math. Optim., 31 (1995), pp. 297–326.
- [8] H. GOLDBERG AND F. TRÖLTZSCH, *Second order sufficient optimality conditions for a class of non-linear parabolic boundary control problems*, SIAM J. Contr. Opt., 31 (1993), pp. 1007–1025.
- [9] M. HEINKENSCHLOSS AND F. TRÖLTZSCH, *Analysis of an SQP method for the control of the phase field equation*, tech. report, Virginia Polytechnic Institute and State University, March 1995. ICAM Report.
- [10] H. MAURER, *First and second order sufficient optimality conditions in mathematical programming and optimal control*, Math. Programming Study, 14 (1981), pp. 163–177.
- [11] J.-P. RAYMOND, *Nonlinear Boundary Control of Semilinear Parabolic Problems with Pointwise State Constraints*, Discrete and Continuous Dynamical Systems, 3 (1997), pp. 341–370.

- [12] J.-P. RAYMOND AND F. TRÖLTZSCH, *Second order sufficient optimality conditions for nonlinear parabolic control problems with state constraints*, Discrete and Continuous Dynamical Systems, 6 (2000), pp. 431–450.
- [13] J.-P. RAYMOND AND H. ZIDANI, *Hamiltonian Pontryagin's principles for control problems governed by semilinear parabolic equations*, Applied Mathematics and Optimization, 39 (1999), pp. 143–177.
- [14] A. RÖSCH AND F. TRÖLTZSCH, *An optimal control problem arising from the identification of nonlinear heat transfer laws*, Archives of Control Sciences, 1 (1992), pp. 183–195.
- [15] ———, *Sufficient second order optimality condition for a state-constrained optimal control problem of a weakly singular integral equation*, Num. Funct. Analysis and Optimization, (to appear).
- [16] F. TRÖLTZSCH, *Approximation of non-linear parabolic boundary control problem by the Fourier method – convergence of optimal controls*, Optimization, 22 (1991), pp. 83–98.