Error estimates for the finite-element approximation of a semilinear elliptic control problem

by

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Abstract: We consider the finite-element approximation of a distributed optimal control problem governed by a semilinear elliptic partial differential equation, where pointwise constraints on the control are given. We prove the existence of local approximate solutions converging to a given local reference solution. Moreover, we derive error estimates for local solutions in the maximum norm.
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1 Introduction

In this paper, we consider the finite-element discretization of the optimal control problem

$$(P) \quad \min\ J(u) = \frac{1}{2}\int_\Omega \{\, (y(x) - y_d(x))^2 + \nu u(x)^2 \,\}\, dx,$$

$$\text{subject to} \quad (y,u) \in (C(\overline{\Omega}) \cap H^1(\Omega)) \times L^\infty(\Omega),$$

$$A\,y + f(y) = u \ \text{ in } \Omega, \quad y = 0 \ \text{ on } \Gamma, \tag{1.1}$$

$$u \in U^{ad} = \{u \in L^\infty(\Omega) \mid \alpha \le u(x) \le \beta \ \text{ for a.a. } x \in \Omega\},$$

where $\Omega \subset \mathbb{R}^n$ is a convex bounded domain, Γ is the boundary of Ω, and A denotes a second-order elliptic operator of the form

$$Ay(x) = -\sum_{i,j=1}^{n} D_j(a_{ij}(x) D_i y(x)).$$

Here, D_i denotes the partial derivative with respect to x_i, u is the *control*, and $y = y(u)$ is said to be the *associated state*. The function y_d is given in $L^\infty(\Omega)$, and $\alpha < \beta$, $\nu > 0$ are real constants.

Based on a standard finite-element approximation, we set up an approximate optimal control problem (P_h). Our main aim is to estimate the error $\|\bar{u} - \bar{u}_h\|$ in the maximum norm, where $\bar{u}$ stands for a fixed locally optimal control of (P) and $\bar{u}_h$ is an associated one of (P_h). Error estimates for optimal controls certainly cannot improve those known for the solutions of elliptic equations. However, one should expect that they reflect the order of the associated estimates for equations. Due to the non-convexity of (P) and the presence of control-constraints, this is not an easy task. Optimal L^2-estimates are known since long time for linear-quadratic elliptic control problems, Falk (1973), Geveci (1979). Recently, L^∞-error estimates being optimal in that sense have been derived for the case of nonlinear equations in Arada, Casas and Tröltzsch (2001).

Moreover, we mention two further papers related to the semilinear elliptic case. Recently, Arnautu and Neittaanmäki (1998) contributed error estimates to this class of problems. Their technique, however, slightly overestimates the order of the error. We also mention the paper by Casas and Mateos (2001), who carefully study error estimates for semilinear elliptic equations. In contrast to the elliptic case, quite a number of papers was devoted to parabolic problems, although the associated theory is far from being complete. We refer to the references in Arada et al. (2001).

Our paper complements the theory presented in Arada et al. (2001), where error estimates have been derived for a subsequence $(\bar{u}_h)_h$ of *globally* optimal controls for (P_h) that converges to an optimal control $\bar{u}$ of (P) as $h \downarrow 0$. The existence of this sequence has been obtained by weak compactness arguments.

The main difference of our paper to this former one concerns the existence part. Here, we concentrate on *locally* optimal controls, since they are the natural result of numerical optimization algorithms. Suppose that a *locally* optimal control $\bar{u}$ of (P) is given. Then we expect to have a sequence $(\bar{u}_h)_h$ of *locally* optimal controls for (P_h) converging to $\bar{u}$. This should be true for *each* fixed local solution $\bar{u}$. We prove that each locally optimal control of (P) can be approximated by locally optimal controls of (P_h), while Arada et al. (2001) only guarantee that the computed global solutions contain a *subsequence* that converges to a certain globally optimal control.

Therefore, we start from a fixed reference control $\bar{u}$ being locally optimal for (P). Next we prove the existence of a sequence $(\bar{u}_h)_h$ of locally optimal controls for (P_h) converging to $\bar{u}$. We do not use compactness arguments. Finally, the order of convergence is quantified by estimating the error $\bar{u}_h - \bar{u}$. The error analysis is similar to that of our paper Arada et al. (2001).

However, our problem (P) is simplified to shorten the presentation. In our former paper, the objective functional and the nonlinearity f are more general. Following the lines of Arada et al. (2001), the results of this paper can be extended to the more general setting.

2 Assumptions and notation

The domain Ω is assumed to be a convex, bounded, and open subset in $\mathbb{R}^n$, where $n = 2$ or $n = 3$. We also assume that Ω has a boundary Γ of class $C^{1,1}$. The coefficients a_{ij} of the operator A are assumed to be in $C^{0,1}(\overline{\Omega})$, and to satisfy the ellipticity condition

$$m_0|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \qquad \forall\ (\xi, x) \in \mathbb{R}^n \times \overline{\Omega}, \qquad m_0 > 0.$$

On f, we impose the assumption

(A1) *The function $f : \mathbb{R} \to \mathbb{R}$ is of class C^2 and its first derivative f' is nonnegative. For all $M > 0$, there exists $C_M > 0$ such that*

$$|f''(y_1) - f''(y_2)| \leq C_M|y_1 - y_2|$$

for all $(y_1, y_2) \in [-M, +M]^2$.

Assumption **(A1)** permits to deal with highly nonlinear functions. For instance, $f(y) = \exp(y)$ satisfies **(A1)**.

THEOREM 2.1 *(Bonnans and Casas (1995)) Let u in $L^\infty(\Omega)$ satisfy $\|u\|_{\infty,\Omega} \leq M$. Then, for every $p > n$, equation (1.1) admits a unique solution $y = y(u) \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$. There exists a positive constant $C = C(\Omega, n, p, M)$, independent of u, such that*

$$\|y(u)\|_{W^{2,p}(\Omega)} \leq C.$$

In what follows, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the natural norms in $L^2(\Omega)$ and $L^\infty(\Omega)$, respectively, and c is a generic constant.

3 Optimality conditions for local solutions of (P)

The existence of a (global) solution to (P) can be proved by classical arguments. However, we concentrate on *local* solutions. Therefore, we just *assume* that a locally optimal reference control $\bar{u}$ is given for (P) that satisfies the standard first-order necessary and second-order sufficient optimality conditions.

A control $\bar{u} \in U^{ad}$ is said to be locally optimal or a *local solution* of (P), if there is an $r > 0$ such that

$$J(u) \geq J(\bar{u}) \qquad \forall u \in U^{ad} \text{ with } \|u - \bar{u}\|_\infty \leq r.$$

In what follows, we denote by $y(u)$ the solution y of (1.1) that is associated with u. Let $\bar y$ be the state corresponding to $\bar u$, i.e. $\bar y = y(\bar u)$.

Next we recall the known first-order necessary optimality conditions for (P). To this aim, we introduce the adjoint equation. Let u be in $L^\infty(\Omega)$ with state $y(u)$. The *adjoint equation* has the following form:

$$A^*\varphi + f'(y(u))\varphi = y(u) - y_d \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \Gamma. \tag{3.1}$$

Here, A^* is the formal adjoint operator of A. The solution $\varphi = \varphi(u)$ is the *adjoint state* associated with u.

THEOREM 3.1 *If $\bar u$ is a local solution of (P), then there exists an adjoint state $\bar\varphi = \varphi(\bar u) \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$ such that*

$$A^*\bar\varphi + f'(\bar y)\bar\varphi = \bar y - y_d \qquad \text{in } \Omega, \tag{3.2}$$

$$\int_\Omega (\bar\varphi + \nu\,\bar u)(u - \bar u)\,dx \geq 0 \qquad \forall\ u \in U^{ad}. \tag{3.3}$$

The classical proof is omitted. By a further discussion, the variational inequality (3.3) is seen to be equivalent to the following known relation:

$$\bar u(x) = \mathrm{Proj}_{[\alpha,\beta]}(-\frac{1}{\nu}\,\varphi(\bar u)(x)), \tag{3.4}$$

where $\mathrm{Proj}_{[\alpha,\beta]}$ denotes the projection from $\mathbb{R}$ onto $[\alpha,\beta]$. Since (P) is non-convex, the optimality conditions above are not sufficient for (local) optimality. To have this, in addition the following *second-order sufficient optimality condition* is assumed:

(SSC) *There are $\delta > 0$ and $\tau > 0$ such that*

$$J''(\bar u)v^2 \geq \delta\ \|v\|_2^2 \tag{3.5}$$

holds for all $v \in L^\infty(\Omega)$ satisfying

$$v(x) \begin{cases} \geq 0 & \text{if } \ \bar u(x) = \alpha, \\ \leq 0 & \text{if } \ \bar u(x) = \beta, \\ = 0 & \text{if } \ |\bar\varphi(x) + \nu\bar u(x)| \geq \tau > 0. \end{cases} \tag{3.6}$$

All functions v satisfying the conditions of (3.6) form a cone that we shall call the *τ-critical cone*. The set

$$A_\tau = \{x \in \Omega \mid \quad |\bar\varphi(x) + \nu\bar u(x)| \geq \tau\}$$

is the set of all points where the control constraints are *strongly active*. This notion was introduced by Dontchev, Hager, Poore and Yang (1995).

Notice that J is defined as a functional on $L^\infty(\Omega)$. It is this space, where the derivatives J' and J'' are defined. The concrete expression for the second derivative can be formulated by the Lagrange function

$$L(y,u,\varphi) = \frac{1}{2}\int_\Omega \{\, (y(x)-y_d(x))^2 + \nu u(x)^2 \,\}\, dx - \int_\Omega (-\Delta y + f(y) - u)\, \varphi \, dx,$$

which is here only formally defined (in our setting, Δy is not a function; selecting a slightly different state space for y, this can be made precise). Then, see Casas and Tröltzsch (2000),

$$\begin{aligned} J''(u)(u_1,u_2) &= D_{yy}L(y,u,\varphi)(y_1,y_2) + D_{uu}L(y,u,\varphi)(u_1,u_2) \\ &= \int_\Omega (1 - f''(y)\varphi(u))\, y_1 y_2 \, dx + \nu \int_\Omega u_1 u_2 \, dx, \end{aligned}$$

where $y_i \in H_0^1(\Omega)$ solve the linearized equation $-\Delta y_i + f'(y)y_i = u_i$. Therefore, (SSC) requires the coercivity of L'' on the cone defined by the controls u of the τ-critical cone and the associated solutions $y(u)$ of the linearized equation.

4 Finite-element approximation of (P): Basic results

4.1 The approximate problem (P_h)

Here we define a finite-element based approximation of the optimal control problem (P). To this aim, we consider a family of triangulations $(\mathcal{T}_h)_{h>0}$ of $\overline{\Omega}$. With each element $T \in \mathcal{T}_h$, we associate two parameters $\rho(T)$ and $\sigma(T)$, where $\rho(T)$ denotes the diameter of the set T and $\sigma(T)$ is the diameter of the largest ball contained in T. Define the mesh size of the grid by $h = \max_{T\in\mathcal{T}_h} \rho(T)$. We suppose that the following regularity assumptions are satisfied.

(A2) There exist two positive constants ρ and σ such that

$$\frac{\rho(T)}{\sigma(T)} \le \sigma, \qquad \frac{h}{\rho(T)} \le \rho$$

hold for all $T \in \mathcal{T}_h$ and all $h > 0$.

Let us take $\overline{\Omega}_h = \cup_{T\in\mathcal{T}_h} T$, and let Ω_h and Γ_h denote its interior and its boundary, respectively. We assume that $\overline{\Omega}_h$ is convex and that the vertices of $\mathcal{T}_h$ placed on the boundary of Γ_h are points of Γ. It is known that

$$|\Omega \setminus \Omega_h| \le Ch^2. \tag{4.1}$$

Now, to every boundary triangle T of $\mathcal{T}_h$, we associate another triangle $\hat{T} \subset \overline{\Omega}$ with curved boundary as follows: The edge between the two boundary nodes of T is substituted by the part of Γ connecting these nodes and forming a triangle with the remaining interior sides of T. We denote by $\widehat{\mathcal{T}_h}$ the union of these curved boundary triangles with the interior triangles to Ω of $\mathcal{T}_h$, so that $\overline{\Omega} = \cup_{\hat{T}\in\widehat{\mathcal{T}_h}} \hat{T}$. Let us set

$$U_h = \{u \in L^\infty(\Omega) \mid u_{|\hat{T}} \text{ is constant on all } \hat{T} \in \widehat{\mathcal{T}_h}\}, \qquad U_h^{ad} = U_h \cap U^{ad},$$

$$V_h = \{y_h \in C(\overline{\Omega}) \mid y_{h|T} \in \mathcal{P}_1, \text{ for all } T \in \mathcal{T}_h, \text{ and } y_h = 0 \text{ on } \overline{\Omega} \setminus \Omega_h\},$$

where $\mathcal{P}_1$ is the space of polynomials of degree less or equal than 1. For each $u_h \in U_h$, we denote by $y_h = y_h(u_h)$ the unique element of V_h that satisfies

$$a(y_h, \eta_h) = \int_\Omega (u_h - f(y_h))\eta_h \, dx \qquad \forall\ \eta_h \in V_h, \tag{4.2}$$

where $a : V_h \times V_h \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$a(y, \eta) = \int_\Omega (\sum_{i,j=1}^n a_{ij}(x) D_i y(x) D_j \eta(x)) \, dx.$$

In other words, $y_h(u_h)$ is the approximate state associated with u_h. In the two integrals above, the test function η_h vanishes outside Ω_h so that there is no difference between integration on Ω and Ω_h. Existence and uniqueness of this solution $y_h(u_h)$ can be shown under our assumption $(A1)$, cf. Casas and Mateos (2001) and Mateos (2000). The finite-dimensional *approximate optimal control problem* (P_h) is defined by

$$(P_h) \quad \min\ J_h(u_h) = \frac{1}{2} \int_{\Omega_h} \{\, (y_h(u_h) - y_d)^2 + \nu\, u_h^2 \,\}\, dx, \qquad u_h \in U_h^{ad}.$$

The existence of at least one global solution for (P_h) follows from the continuity of J_h and the compactness of U_h^{ad}. However, this global solution need not be unique. Moreover, it can be far from the reference solution $\bar{u}$. Therefore, we do not concentrate on global solutions of (P_h). Again, we consider certain local solutions.

REMARK: *We tacitly assume that we are able to evaluate the integrals in* (4.2) *and* (P_h) *exactly. In general, numerical integration has to be used, which generates another sort of errors. We do not include them in our analysis.*

4.2 Characterization of local solutions of (P_h)

Local solutions of the approximate problem (P_h) are defined analogously to (P): A control $\bar{u}_h \in U_h^{ad}$ is a *local solution* of (P_h), if

$$J_h(u_h) \geq J_h(\bar{u}_h) \quad \forall u_h \in U_h^{ad} \text{ with } \|u_h - \bar{u}\|_\infty \leq r$$

holds for a certain $r > 0$. Associated necessary optimality conditions are similar to those for (P) in Section 3: With the solution $\bar{u}_h$ we associate the *discrete adjoint equation* for $\varphi_h \in V_h$

$$\begin{aligned}&\int_\Omega \sum_{i,j=1}^n a_{ij} D_j \varphi_h D_i \eta_h \, dx + \int_\Omega f'(y_h(\bar{u}_h)) \varphi_h \, \eta_h \, dx \\ &= \int_\Omega (y_h(\bar{u}_h) - y_d) \, \eta_h \, dx \qquad \forall \; \eta_h \in V_h.\end{aligned} \tag{4.3}$$

THEOREM 4.1 *Suppose that assumption* (**A1**) *is satisfied. If $\bar{u}_h$ is a local solution of (P_h), then there exists a unique solution $\bar{\varphi}_h = \varphi_h(\bar{u}_h) \in H_0^1(\Omega) \cap C^{0,1}(\overline{\Omega})$ of the discrete adjoint equation (4.3) such that the variational inequality*

$$\int_{\Omega_h} (\bar{\varphi}_h + \nu \, \bar{u}_h)(u - \bar{u}_h) \, dx \geq 0 \qquad \forall \; u \in U_h^{ad} \tag{4.4}$$

is satisfied.

The standard proof of this result is omitted. Throughout the sequel, for v fixed in $L^\infty(\Omega)$, we denote by $y_h(v)$ and $\varphi_h(v)$ the solutions of (4.2) and (4.3), respectively, associated with v. The next result is the discrete counterpart of (3.4). The discrete local solution $\bar{u}_h$ satisfies

$$\bar{u}_h \, |_T = \mathrm{Proj}_{[\alpha,\beta]} \big(- \frac{1}{\nu |T|} \int_T \varphi_h(\bar{u}_h)(x) \, dx \big) \qquad \forall \; T \in \mathcal{T}_h. \tag{4.5}$$

In this paper, we frequently use an interpolation operator $\Pi_h : L^2(\Omega) \to U_h$ that assigns piecewise constant functions on Ω to functions of $L^2(\Omega)$. To define Π_h, we first introduce the interpolation operator $\pi_h : L^2(\Omega) \to L^2(\Omega_h)$ by

$$(\pi_h \, v) \, |_T = \frac{1}{|T|} \int_T v(x) \, dx.$$

We extend π_h to Π_h by

$$(\Pi_h v)(x) = \begin{cases} (\pi_h \, v)(x) & \text{if } x \in T \\ (\pi_h \, v)(x_o) & \text{if } x \in \hat{T} \setminus T. \end{cases}$$

Here, x_o is the projection of x onto the boundary of the triangle T that is covered by $\hat{T}$. Let us mention an important property of Π_h : If v is a Lipschitz function, then

$$\|v - \Pi_h v\|_\infty \leq c \, h.$$

This is seen as follows: On triangles $T \in \mathcal{T}_h$ we have $\max_{x \in T} |v(x) - (\Pi_h v)(x)| = \max_{x \in T} |v(x) - (\pi_h v)(x)| \leq c\,h$ by the known properties of the interpolation operator π_h and the Lipschitz property of v. If $x \in \hat{T} \setminus T$, then

$$\begin{aligned}|v(x) - (\Pi_h v)(x)| &\leq |v(x) - v(x_o)| + |v(x_o) - (\Pi_h v)(x)| \\ &\leq c\,h + |v(x_o) - (\pi_h v)(x_o)| \leq c\,h.\end{aligned}$$

Here, we have used that $dist(x_o, T) \leq c\,h$. The same estimate follows for the L^2-norm on using (4.1). With this interpolation operator, (4.5) admits the form

$$\bar{u}_h = \mathrm{Proj}_{[\alpha,\beta]}(-\frac{1}{\nu}\Pi_h \varphi_h(\bar{u}_h)), \tag{4.6}$$

since the extension of (4.5) from boundary triangles T to $\hat{T}$ is the same on the left and right hand side of (4.6).

4.3 Error-estimates for the state and the adjoint state

Here we provide some known results on the finite element approximation of the state equation (1.1) and its adjoint equation (3.1). They are basic for the convergence analysis below and for the error estimates in the next section. Recall that $y(v)$ and $y_h(v_h)$ are the solutions of (1.1) and (4.2) corresponding to v and v_h. Analogously, $\varphi(v)$ and $\varphi_h(v_h)$ are the solutions of (3.1) and (4.3) corresponding to v and v_h.

In all what follows we tacitly assume that $(A1)$ and $(A2)$ are satisfied. Moreover, we fix once and for all a local reference solution $\bar{u}$ for (P) that satisfies (SSC). Therefore, we do not mention $(A1)$, $(A2)$, and (SSC) in the further statements.

All controls $u,\ v,\ u_h,\ v_h$ etc. used below are contained in U^{ad}. Therefore, they are uniformly bounded, and the same holds true for all associated states and adjoint states so that all $y,\ \varphi,\ y_h,\ \varphi_h$ are bounded by the same constant M.

THEOREM 4.2 *Let v and v_h belong to U^{ad}. Then the estimates*

$$\|y(v) - y_h(v_h)\|_{H^1(\Omega)} + \|\varphi(v) - \varphi_h(v_h)\|_{H^1(\Omega)} \leq C\,(h + \|v - v_h\|_2), \tag{4.7}$$

$$\|y(v) - y_h(v_h)\|_2 + \|\varphi(v) - \varphi_h(v_h)\|_2 \leq C\,(h^2 + \|v - v_h\|_2), \tag{4.8}$$

$$\|y(v) - y_h(v_h)\|_\infty + \|\varphi(v) - \varphi_h(v_h)\|_\infty \leq C\,(h^\lambda + \|v - v_h\|_2), \tag{4.9}$$

hold, where $C = C(\Omega, n)$ is a positive constant independent of h, and $\lambda = 2 - n/2$. Moreover, if the triangulation is of nonnegative type, then

$$\|y(v) - y_h(v_h)\|_\infty + \|\varphi(v) - \varphi_h(v_h)\|_\infty \leq C\,(h + \|v - v_h\|_2), \tag{4.10}$$

holds independently of h.

For the proof of this theorem the reader is referred to Arada et al. (2001). In all what follows, let us fix

$$\lambda = \begin{cases} 2 - n/2 & \textit{for regular triangulations} \\ 1 & \textit{for triangulations of nonnegative type.} \end{cases}$$

4.4 Convergence results

Aiming to derive error estimates, we have to find a sequence $(\bar{u}_h)_h$ of local solutions of (P_h) tending to $\bar{u}$ as $h \downarrow 0$. To solve this nontrivial problem, we proceed as follows: For $\varepsilon > 0$ we consider the auxiliary control problem

$$(P_h^\varepsilon) \quad \min \; J_h(u_h) = \frac{1}{2} \int_{\Omega_h} \{ (y_h(u_h) - y_d)^2 + \nu \, u_h^2 \} \, dx, \qquad u_h \in U_{h,\varepsilon}^{ad},$$

where

$$U_{h,\varepsilon}^{ad} = \{ u \in U_h^{ad} \,|\, (\Pi_h \bar{u})(x) - \varepsilon \le u(x) \le (\Pi_h \bar{u})(x) + \varepsilon \text{ in } \Omega \}.$$

The interpolate $\Pi_h \bar{u}$ belongs to $U_{h,\varepsilon}^{ad}$, therefore the admissible set of (P_h^ε) is not empty. This problem has a global solution u_h^ε, hence it is also a local solution for (P_h^ε). We show that this solution is even a local solution of (P_h) and tends to $\bar{u}$ as $h \downarrow 0$, provided that ε was taken sufficiently small.

It is known that the second-order condition (SSC) implies the existence of positive constants κ and r such that the quadratic growth condition

$$J(u) \ge J(\bar{u}) + \kappa \, \|u - \bar{u}\|_2^2 \tag{4.11}$$

is satisfied for all $u \in U^{ad}$ with $\|u - \bar{u}\|_\infty \le r$, cf. Casas, Tröltzsch and Unger (2000). Now take $\bar{\varepsilon} := r/2$. Then for all $\varepsilon \le \bar{\varepsilon}$ and all sufficiently small h, say $0 < h \le \bar{h}$,

$$u \in U_{h,\varepsilon}^{ad} \Rightarrow \|u - \bar{u}\|_\infty \le r, \tag{4.12}$$

because $\|u - \bar{u}\|_\infty \le \|u - \Pi_h \bar{u}\|_\infty + \|\Pi_h \bar{u} - \bar{u}\|_\infty$, the first term is not greater than $r/2$ by the definition of $U_{h,\varepsilon}^{ad}$, and the second term tends to zero as $h \downarrow 0$. Notice that (4.11) and (4.12) imply

$$J(u) \ge J(\bar{u}) + \kappa \, \|u - \bar{u}\|_2^2 \qquad \forall u \in U_{h,\bar{\varepsilon}}^{ad}. \tag{4.13}$$

LEMMA 4.1 *For all $\varepsilon \le \bar{\varepsilon}$, the objective values $J_h(u_h^\varepsilon)$ converge to $J(\bar{u})$, i.e.*

$$\lim_{h \downarrow 0} J_h(u_h^\varepsilon) = J(\bar{u}).$$

Proof. We have

$$J_h(u_h^\varepsilon) = J(u_h^\varepsilon) + (J_h(u_h^\varepsilon) - J(u_h^\varepsilon)) \ge J(\bar{u}) - c\,h,$$

since $\|u_h^\varepsilon\|_\infty$ is uniformly bounded, hence $|J_h(u_h^\varepsilon) - J(u_h^\varepsilon)| \le c\,h$. Moreover, $J(u_h^\varepsilon) \ge J(\bar{u})$ follows from (4.13). On the other hand, we know $\Pi_h \bar{u} \in U_{h,\varepsilon}^{ad}$, and the optimality of u_h^ε for (P_h^ε) gives

$$\begin{aligned} J_h(u_h^\varepsilon) &\le J_h(\Pi_h \bar{u}) = J(\bar{u}) + (J(\Pi_h \bar{u}) - J(\bar{u})) + (J_h(\Pi_h \bar{u}) - J(\Pi_h \bar{u})) \\ &\le J(\bar{u}) + c\,h, \end{aligned}$$

since $\|\Pi_h\bar{u}-\bar{u}\|_\infty \le c\,h$ and $|J_h(v)-J(v)| \le c\,h$ for all $v \in U^{ad}$. Both inequalities imply the statement of the Lemma. ∎

Lemma 4.2 *There are $0 < \varepsilon_\tau \le \bar{\varepsilon}$ and $0 < h_\tau \le \bar{h}$ such that*

$$|\varphi_h(u_h^\varepsilon)(x) + \nu u_h^\varepsilon(x)| \ge \tau/4 \tag{4.14}$$

$$u_h^\varepsilon(x) = \bar{u}(x) \tag{4.15}$$

hold for all $\varepsilon \le \varepsilon_\tau$, all $h \le h_\tau$, and all $x \in T$, if the triangle T has a non-empty intersection with $A\tau$.

Proof. On A_τ we know that either $\varphi(\bar{u})(x) + \nu\bar{u}(x) \ge \tau$, where $\bar{u}(x) = \alpha$ or $\varphi(\bar{u})(x)+\nu\bar{u}(x) \le -\tau$, where $\bar{u}(x) = \beta$. Now take an arbitrary but fixed triangle T having a non-empty intersection with A_τ. If h is sufficiently small, then we can assume that one of these two cases holds for all $x \in A_\tau \cap T$, since the function $\varphi(\bar{u}) + \nu\bar{u}$ is Lipschitz continuous. We consider the case

$$\varphi(\bar{u})(x) + \nu\bar{u}(x) \ge \tau,$$

where $\bar{u}(x) \equiv \alpha$ on $A_\tau \cap T$. The arguments for $\bar{u}(x) \equiv \beta$ are analogous. Therefore, if h is sufficiently small, then

$$\varphi(\bar{u})(x) + \nu\bar{u}(x) \ge 3\tau/4 \quad \forall x \in T,$$

thus also $\bar{u}(x) \equiv \alpha$ on T. If ε is sufficiently small, say $\varepsilon \le \varepsilon_\tau$, then $\|u_h^\varepsilon - \bar{u}\|_\infty$ is so small such that

$$\begin{aligned}\varphi_h(u_h^\varepsilon) + \nu u_h^\varepsilon &= \varphi(\bar{u}) + \nu\bar{u} + (\varphi(u_h^\varepsilon) - \varphi(\bar{u})) + \nu(u_h^\varepsilon - \bar{u})\\ &\quad +(\varphi_h(u_h^\varepsilon) - \varphi(u_h^\varepsilon))\\ &\ge 2/4\tau - c\,h^\lambda \ge \tau/4\end{aligned}$$

holds on T for all sufficiently small $h \le h_\tau$. On T, the variational inequality for u_h^ε reads

$$\int_T (\varphi_h(u_h^\varepsilon) + \nu u_{h|T}^\varepsilon)(u - u_{h|T}^\varepsilon)\,dx \ge 0$$

for all $u \in \mathbb{R}$ such that $u \in [\alpha,\beta] \cap [\Pi_h\bar{u}_{|T} - \varepsilon, \Pi_h\bar{u}_{|T} + \varepsilon]$. On T, we know $\bar{u}(x) \equiv \alpha$, hence $\Pi_h\bar{u}_{|T} = \alpha$, and therefore u varies in $[\alpha, \alpha+\varepsilon]$. The positivity of $\varphi_h(u_h^\varepsilon)+\nu u_{h|T}^\varepsilon$ in the variational inequality above implies that u_h^ε must admit the left end of $[\alpha, \alpha+\varepsilon]$, i.e. $u_{h|T}^\varepsilon = \alpha = \bar{u}(x)$. ∎

By our construction, this Lemma is also true for boundary triangles $\hat{T}$.

Lemma 4.3 *If $\varepsilon \le \bar{\varepsilon}$, then $\lim_{h\downarrow 0} \|u_h^\varepsilon - \bar{u}\|_2 = 0$.*

Proof. By $u_h^\varepsilon \in U_{h,\varepsilon}^{ad}$, $\varepsilon \le \bar\varepsilon$, $h \downarrow 0$, and (4.12) we know $\|u_h^\varepsilon - \bar u\|_\infty \le r$, hence (4.11) applies,

$$J(u_h^\varepsilon) \ge J(\bar u) + \kappa\, \|u_h^\varepsilon - \bar u\|_2^2,$$

thus

$$J_h(u_h^\varepsilon) = J(u_h^\varepsilon) + (J_h(u_h^\varepsilon) - J(u_h^\varepsilon)) \ge J(\bar u) + \kappa\, \|u_h^\varepsilon - \bar u\|_2^2 - c\, h^\lambda$$

and therefore

$$J_h(u_h^\varepsilon) - J(\bar u) + c\, h^\lambda \ge \kappa\, \|u_h^\varepsilon - \bar u\|_2^2.$$

Lemma 4.1 yields $J_h(u_h^\varepsilon) \to J(\bar u)$ as $h \downarrow 0$ and the assertion of Lemma 4.3 follows immediately. ∎

THEOREM 4.3 *If $\varepsilon \le \bar\varepsilon$, then*

$$\lim_{h\downarrow 0} \|u_h^\varepsilon - \bar u\|_\infty = 0. \tag{4.16}$$

Proof. We start with the result of Lemma 4.3. From Theorem 4.2, (4.9), we deduce that $u_h^\varepsilon \to \bar u$ in $L^2(\Omega)$ implies $\|\varphi_h(u_h^\varepsilon) - \varphi(\bar u)\|_\infty \to 0$. We have the projection formulas

$$\bar u(x) = \mathrm{Proj}_{[\alpha,\beta]}(-\frac{1}{\nu}\bar\varphi(x)) \tag{4.17}$$

$$u_h^\varepsilon(x) = \mathrm{Proj}_{[\alpha_h^\varepsilon(x),\beta_h^\varepsilon(x)]}(-\frac{1}{\nu}\Pi_h\varphi_h(u_h^\varepsilon(x))), \tag{4.18}$$

where

$$\alpha_h^\varepsilon(x) = \max(\alpha, \Pi_h\bar u(x) - \varepsilon), \quad \beta_h^\varepsilon(x) = \min(\beta, \Pi_h\bar u(x) + \varepsilon).$$

Notice that α_h^ε and β_h^ε are step functions on Ω. Define analogously

$$\alpha^\varepsilon(x) = \max(\alpha, \bar u(x) - \varepsilon), \quad \beta^\varepsilon(x) = \min(\beta, \bar u(x) + \varepsilon).$$

It is quite obvious that $\bar u$ also satisfies the projection formula

$$\bar u(x) = \mathrm{Proj}_{[\alpha^\varepsilon(x),\beta^\varepsilon(x)]}(-\frac{1}{\nu}\bar\varphi(x)). \tag{4.19}$$

Indeed, $\bar u$ solves (P) with the additional restrictions $u(x) \le \bar u(x) + \varepsilon$, $u(x) \ge \bar u(x) - \varepsilon$, and both of these inequalities are not active at $\bar u$. Therefore the equations (4.17) and (4.19) are equivalent. Of course, (4.19) can also be directly derived from (4.17). We leave this to the reader. With these prerequisites, the proof can be easily completed. In view of (4.18) and (4.19)

$$\begin{aligned}
&|\bar u(x) - u_h^\varepsilon(x)| = \\
&\quad = |\mathrm{Proj}_{[\alpha^\varepsilon(x),\beta^\varepsilon(x)]}(-\tfrac{1}{\nu}\varphi(\bar u(x))) - \mathrm{Proj}_{[\alpha_h^\varepsilon(x),\beta_h^\varepsilon(x)]}(-\tfrac{1}{\nu}\Pi_h\varphi_h(u_h^\varepsilon(x)))| \\
&\quad \le |\mathrm{Proj}_{[\alpha^\varepsilon(x),\beta^\varepsilon(x)]}(-\tfrac{1}{\nu}\varphi(\bar u(x))) - \mathrm{Proj}_{[\alpha_h^\varepsilon(x),\beta_h^\varepsilon(x)]}(-\tfrac{1}{\nu}\varphi(\bar u(x)))| \\
&\quad +|\mathrm{Proj}_{[\alpha_h^\varepsilon(x),\beta_h^\varepsilon(x)]}(-\tfrac{1}{\nu}\varphi(\bar u(x))) - \mathrm{Proj}_{[\alpha_h^\varepsilon(x),\beta_h^\varepsilon(x)]}(-\tfrac{1}{\nu}\Pi_h\varphi_h(u_h^\varepsilon(x)))|.
\end{aligned}$$

The first difference tends uniformly to zero, as

$$\mathrm{Proj}_{[\alpha_h^\varepsilon(x),\beta_h^\varepsilon(x)]}v(x) = \min(\beta_h^\varepsilon(x), \max(\alpha_h^\varepsilon(x), v(x)))$$

is a composition based on continuous functions, if $v \in C(\overline{\Omega})$. Therefore

$$\mathrm{Proj}_{[\alpha_h^\varepsilon(x),\beta_h^\varepsilon(x)]}v(x) \to \mathrm{Proj}_{[\alpha^\varepsilon(x),\beta^\varepsilon(x)]}v(x)$$

in $C(\overline{\Omega})$, since $\alpha_h^\varepsilon(x) \to \alpha^\varepsilon(x)$ and $\beta_h^\varepsilon(x) \to \beta^\varepsilon(x)$ in $C(\overline{\Omega})$. The second difference tends uniformly to zero, as the projection operator is Lipschitz continuous with constant 1 and $\Pi_h\varphi_h(u_h^\varepsilon(x))$ tends uniformly to $\varphi(\bar u(x))$ by Lemma 4.3 and (4.9). ∎

Finally, we show that u_h^ε is a local solution of (P_h). Intuitively, this follows from $u_h^\varepsilon \to \bar u$. Therefore u_h^ε cannot be located at the boundary of the ball $\|u_h - \Pi_h\bar u\|_\infty = \varepsilon$.

Lemma 4.4 *Suppose that $\varepsilon \le \bar\varepsilon$. Then u_h^ε is a local solution of (P_h) for all sufficiently small h.*

Proof. We have to show that

$$J_h(u_h) \ge J_h(u_h^\varepsilon) \tag{4.20}$$

holds for all $u_h \in U_h^{ad}$ such that $\|u_h - u_h^\varepsilon\|_\infty \le \varepsilon/2$. By the definition of u_h^ε we know (4.20) only for all $u_h \in U_h^{ad}$ with $\|u_h - \Pi_h\bar u\|_\infty \le \varepsilon$. Let $u_h \in U_h^{ad}$ satisfy $\|u_h - u_h^\varepsilon\|_\infty \le \varepsilon/2$. Then, if h is sufficiently small,

$$\begin{aligned}
\|u_h - \Pi_h\bar u\|_\infty &\le \|u_h - u_h^\varepsilon\|_\infty + \|u_h^\varepsilon - \bar u\|_\infty + \|\bar u - \Pi_h\bar u\|_\infty \\
&\le \varepsilon/2 + \varepsilon/4 + \varepsilon/4 = \varepsilon,
\end{aligned}$$

since u_h^ε tends to $\bar u$ by Theorem 4.3 and $\Pi_h\bar u \to \bar u$ as $h \downarrow 0$. Therefore, u_h^ε belongs to $U_{h,\varepsilon}^{ad}$, where (4.20) is satisfied. The optimality of u_h^ε is proved in the intersection of U_h^{ad} with a ball of radius $\varepsilon/2$ around u_h^ε. This is local optimality. ∎

One can also show that u_h^ε is the unique local solution of (P_h) in a certain neighborhood of $\bar u$. However, we do not discuss this here. In what follows, let us fix (P_h^ε) by $\varepsilon = \min(\bar\varepsilon, \varepsilon_\tau)$ and put $\bar u_h := u_h^\varepsilon$. In this way, a sequence of local approximate solutions $(\bar u_h)_h$ is found that tends to $\bar u$ as $h \downarrow 0$. In the next section we estimate the error $\|\bar u_h - \bar u\|$.

5 FEM-approximation of (P): Error-estimates for local solutions

In this section, we prove the error estimates for local approximate solutions in the norms of L^2 and L^∞. As outlined in the preceding subsection, we start our investigations by the sequence $(\bar{u}_h)_{h>0}$ of local solutions for (P_h), $h > 0$, converging to the fixed local reference solution $\bar{u}$ of (P) that satisfies (SSC).

To perform our analysis, we need an element u_h admissible for (P_h) so that it can serve as a test function in the variational inequality and has an optimal distance $O(h)$ to $\bar{u}$. The idea is to take $u_h = \mathrm{Proj}_{[\alpha,\beta]}(-\frac{1}{\nu}\Pi_h\varphi(\bar{u}))$. This element is admissible and close to $\bar{u}$, but we cannot expect that $\bar{u}_h - u_h$ is in the τ-critical cone where our second-order sufficient condition holds. To overcome this difficulty, we apply a splitting $\bar{u}_h - u_h = e_h + d_h$, where

$$e_h = \begin{cases} 0 & \text{on } \Omega \setminus \Omega_h \\ \bar{u}_h - u_h & \text{on } (A_\tau \cup A^i) \cap \Omega_h \\ \bar{u}_h - \bar{u} & \text{on } \Omega_h \setminus (A_\tau \cup A^i), \end{cases} \quad d_h = \begin{cases} \bar{u}_h - u_h & \text{on } \Omega \setminus \Omega_h \\ 0 & \text{on } (A_\tau \cup A^i) \cap \Omega_h \\ \bar{u} - u_h & \text{on } \Omega_h \setminus (A_\tau \cup A^i). \end{cases}$$

Here, A^i denotes the inactive set of $\bar{u}$, i.e. $A^i = \{x \in \Omega \,|\, \alpha < \bar{u}(x) < \beta\}$. We have taken $e_h = 0$ outside Ω_h to apply later Lemma 5.2.

Then e_h belongs to the τ-critical cone for all sufficiently small h:

If h is small, then in all triangles T with $T \cap A_\tau \neq \emptyset$ we know $\bar{u}(x) \equiv \alpha$ or $\bar{u}(x) \equiv \beta$, hence on T also $u_h = \Pi_h\bar{u} \equiv \bar{u}(x)$ holds. Moreover, Lemma 4.2 yields $\bar{u}_h = \bar{u}(x)$ on T. Therefore, $e_h = 0$ is true on A_τ. On A^i, the τ-critical cone does not restrict the functions. On the remaining set $\Omega_h \setminus (A_\tau \cup A^i)$, the function $\bar{u}$ is active, while $\bar{u}_h$ belongs to U^{ad}. This ensures that the difference $\bar{u}_h - \bar{u}$ has the right sign required by the τ-critical cone.

The part d_h can be estimated by the optimal order $\|d_h\|_2 \leq O(h)$. Notice that $|\Omega \setminus \Omega_h| \leq c\,h$ and d_h is uniformly bounded. The part $\|e_h\|_2$ must be estimated yet.

REMARK: *In the case $A_\tau = \Omega$, the τ-critical cone consists of the zero element. Here, the second-order condition (SSC) is trivially satisfied and does not contribute to the error estimation. However, in this case, the continuity of the function $\bar{\varphi} + \nu\bar{u}$ implies that the sign is constant and then $\bar{\varphi} + \nu\bar{u} \geq \tau$ in Ω or conversely $\bar{\varphi} + \nu\bar{u} \leq -\tau$ in Ω. In the first case,* (3.2) *implies that $\bar{u} \equiv \alpha$ in Ω. In the second case, the identity $\bar{u} \equiv \beta$ in Ω holds. On the other hand, the uniform convergence $\bar{\varphi}_h + \nu\bar{u}_h \to \bar{\varphi} + \nu\bar{u}$ implies that $\bar{\varphi}_h + \nu\bar{u}_h$ has the same sign as $\bar{\varphi} + \nu\bar{u}$ for every h small enough. Then* (4.4) *leads to $\bar{u}_h \equiv u_h \equiv \bar{u}$ in Ω for every h small enough. Consequently, also $e_h = d_h = 0$ holds true for h small.*

The next auxiliary statements express important properties of J'' and J_h'', which are more or less intuitively clear. For their proofs we refer to Arada et al. (2001). First, since e_h belongs to the τ-critical cone for sufficiently small h, we obtain:

Lemma 5.1 *It holds*

$$J''(\bar{u})(e_h)^2 \geq \delta \; \|e_h\|_2^2$$

for all sufficiently small h.

The next result concerns the approximation of J'' by J_h''.

Lemma 5.2 *Suppose that* w *belongs to* U_h^{ad}. *Then*

$$|J''(w)v^2 - J_h''(w)v^2| \leq C\, h^{\lambda} \; \|v\|_2^2$$

holds for all $v \in L^2(\Omega)$ *vanishing on* $\Omega \setminus \Omega_h$, *where the constant* $C = C(\Omega, n)$ *does not depend on* v *and* h.

Lemma 5.3 *For all sufficiently small* $h > 0$,

$$J_h''(\bar{u})(e_h)^2 \geq \frac{\delta}{2} \; \|e_h\|_2^2$$

is satisfied.

Proof. This is a direct consequence of Lemma 5.1 and Lemma 5.2. ∎

Moreover, $J_h''(w)$ is in some sense Lipschitz with respect to w:

Lemma 5.4 *Let* w_1 *and* w_2 *belong to* U^{ad}. *Then*

$$\left|J_h''(w_1)v^2 - J_h''(w_2)v^2\right| \leq C\left(\|w_1 - w_2\|_\infty + h^{\lambda}\right) \; \|v\|_2^2 \qquad (5.1)$$

is satisfied for all $v \in L^2(\Omega)$ *with a constant* $C = C(\Omega, n)$ *independent of* v *and* h.

The term h^{λ} in (5.1) can be avoided, if the so-called discrete maximum principle holds for the finite-element approximation of (1.1).

By (4.4) the approximate local solution $\bar{u}_h$ satisfies

$$\int_{\Omega_h} (\varphi_h(\bar{u}_h) + \nu \bar{u}_h)(v - \bar{u}_h)(x)\, dx \geq 0 \qquad \forall\; v \in U_h^{ad}.$$

The auxiliary control u_h will not fulfill the analogous inequality

$$\int_{\Omega_h} (\varphi_h(u_h) + \nu u_h)(v - u_h)(x)\, dx \geq 0 \qquad \forall\; v \in U_h^{ad}.$$

Instead of this, we are able to show that u_h satisfies an associated perturbed variational inequality with perturbation ζ_h, namely

$$\int_{\Omega_h} (\varphi_h(u_h) + \nu u_h + \zeta_h)(v - u_h)(x)\, dx \geq 0 \qquad \forall\ v \in U_h^{ad}. \tag{5.2}$$

To this aim, we introduce $\zeta_h \in U_h$ by

$$\zeta_{h|T} = \begin{cases} \big\{ -\frac{1}{|T|} \int_T (\varphi_h(\bar{u}_h) + \nu \bar{u}_h)\, dx \big\}^+ & \text{if } \ u_{h|T} = \alpha, \\ -\big\{ \frac{1}{|T|} \int_T (\varphi_h(\bar{u}_h) + \nu \bar{u}_h)\, dx \big\}^+ & \text{if } \ u_{h|T} = \beta, \\ -\frac{1}{|T|} \int_T (\varphi_h(\bar{u}_h) + \nu \bar{u}_h)\, dx & \text{otherwise,} \end{cases}$$

for all $T \in \mathcal{T}_h$. We extend ζ_h up to the boundary of Ω analogously to the definition of the controls in U_h. As we shall verify below, the function ζ_h is constructed such that the auxiliary function u_h satisfies the first-order necessary optimality condition of the problem

$$\min \ J_h(v) + \int_{\Omega_h} \zeta_h v\, dx, \qquad v \in U_h^{ad}, \tag{5.3}$$

which is a perturbation of (P_h) by the linear functional (ζ_h, v). We have adopted the idea to work with this type of perturbation from Malanowski, Büskens and Maurer (1997). It was introduced there for the optimal control of ODEs and can be transferred to our case. Although we shall not exactly follow that method, this idea is behind our technique to show the main error estimate.

LEMMA 5.5 *The auxiliary control u_h satisfies the variational inequality (5.2).*

Proof. How can we define ζ_h to fulfill (5.2)? Select an arbitrary triangle $T \in \mathcal{T}_h$. First, observe that (5.2) can be equivalently written as

$$\Big(\int_T (\varphi_h(u_h) + \nu u_h)\, dx + |T|\ \zeta_{h|T} \Big)(v - u_{h|T}) \geq 0 \tag{5.4}$$

for all $T \in \mathcal{T}_h$ and all $v \in [\alpha, \beta]$.

(i) If $u_{h|T} = \alpha$ then $v - u_{h|T} \geq 0$ holds in (5.4) for all $v \in [\alpha, \beta]$. Therefore, ζ_h must be chosen such that $\int_T (\varphi_h(u_h) + \nu u_h)\, dx + |T|\ \zeta_{h|T} \geq 0$ holds. Obviously,

$$|T|\ \zeta_h = \Big(\int_T (\varphi_h(u_h) + \nu u_h)\, dx \Big)^- = \Big(- \int_T (\varphi_h(u_h) + \nu u_h)\, dx \Big)^+$$

meets that requirement.

(ii) If $u_{h|T} = \beta$, then $v - u_{h|T} \leq 0$, and $\int_T (\varphi_h(u_h) + \nu u_h)\, dx + |T|\ \zeta_{h|T} \leq 0$ must hold. This is accomplished by

$$|T|\ \zeta_h = (-\int_T (\varphi_h(u_h) + \nu u_h)\, dx)^+.$$

(iii) If $\alpha < u_{h|T} < \beta$, then $v - u_{h|T}$ can be positive or negative, hence ζ_h must be taken such that $\int_T (\varphi_h(u_h) + \nu u_h)\, dx + |T|\ \zeta_{h|T} = 0$. We have found the function ζ_h as defined above. ∎

Lemma 5.6 *There exists a positive constant C, independent of h, such that*

$$\|\zeta_h\|_2 \leq Ch. \tag{5.5}$$

For the proof, the reader is referred to Arada et al. (2001).

Theorem 5.1 *For all sufficiently small $h > 0$*

$$\|\bar{u} - \bar{u}_h\|_2 \leq C\, h,$$

holds with a positive constant C independent of h.

Proof. From the optimality conditions for the problem (P_h), and since u_h satisfies the optimality conditions of (5.3), we deduce that

$$J_h'(\bar{u}_h)(u_h - \bar{u}_h) \geq 0 \quad \text{and} \quad J_h'(u_h)(\bar{u}_h - u_h) + \int_{\Omega_h} \zeta_h\, (\bar{u}_h - u_h)\, dx \geq 0.$$

Therefore,

$$\begin{aligned}(J_h'(\bar{u}_h) - J_h'(u_h))(\bar{u}_h - u_h) &\leq \textstyle\int_{\Omega_h} \zeta_h\, (\bar{u}_h - u_h)\, dx \\ &\leq \|\zeta_h\|_2 \|u_h - \bar{u}_h\|_2.\end{aligned} \tag{5.6}$$

On the other hand, we have

$$\begin{aligned}&(J_h'(\bar{u}_h) - J_h'(u_h))(\bar{u}_h - u_h) = J_h''((1-\theta)\bar{u}_h + \theta u_h)(\bar{u}_h - u_h)^2 \\ &= J_h''(\bar{u})(\bar{u}_h - u_h)^2 + (J_h''((1-\theta)\bar{u}_h + \theta u_h) - J_h''(\bar{u}))(\bar{u}_h - u_h)^2 \\ &= I_1 + I_2,\end{aligned}$$

with some $\theta \in (0, 1)$. Now we estimate I_1 and I_2 separately and apply the splitting $\bar{u}_h - u_h = e_h + d_h$ introduced at the beginning of this section. In view of Lemma 5.3 and the Young inequality we obtain for sufficiently small h

$$\begin{aligned} I_1 &= J_h''(\bar u)(e_h+d_h)^2 = J_h''(\bar u)e_h^2 + 2J_h''(\bar u)(e_h,d_h) + J_h''(\bar u)d_h^2 \\ &\geq \tfrac{\delta}{2}\|e_h\|_2^2 - c\,\|e_h\|_2\|d_h\|_2 - c\,\|d_h\|_2^2 \\ &\geq \tfrac{\delta}{3}\|e_h\|_2^2 - c\,\|d_h\|_2^2 = \tfrac{\delta}{3}\|e_h+d_h-d_h\|_2^2 - c\,\|d_h\|_2^2 \\ &\geq \tfrac{\delta}{3}\|\bar u_h - u_h\|_2^2 - \tfrac{2\delta}{3}\|\bar u_h - u_h\|_2\|d_h\|_2 - c\,\|d_h\|_2^2 \\ &\geq \tfrac{\delta}{4}\|\bar u_h - u_h\|_2^2 - c\,\|d_h\|_2^2. \end{aligned}$$

For I_2 we obtain by Lemma 5.4

$$|I_2| = |J_h''((1-\theta)\bar u_h + \theta u_h) - J_h''(\bar u))(\bar u_h - u_h)^2| \leq \frac{\delta}{8}\|\bar u_h - u_h\|_2^2$$

for all sufficiently small h, since $\bar u_h \to \bar u$ and $u_h = \Pi_h \bar u \to \bar u$ as $h \downarrow 0$. Summarizing up, we have

$$I_1 + I_2 \geq \frac{\delta}{8}\|\bar u_h - u_h\|_2^2 - c\,\|d_h\|_2^2 \geq \frac{\delta}{8}\|\bar u_h - u_h\|_2^2 - c\,h^2,$$

hence (5.6) yields

$$\|\zeta_h\|_2\|u_h - \bar u_h\|_2 \geq \frac{\delta}{8}\|\bar u_h - u_h\|_2^2 - c\,h^2.$$

By the Young inequality

$$\|\zeta_h\|_2\|u_h - \bar u_h\|_2 \leq \delta/16\,\|u_h - \bar u_h\|_2^2 + c\,\|\zeta_h\|_2^2$$

is obtained. Now from the estimate (5.5),

$$c\,h^2 \geq \frac{\delta}{16}\|u_h - \bar u_h\|_2^2,$$

follows, hence $\|u_h - \bar u_h\|_2 \leq c\,h$. This, together with $\|u_h - \bar u\|_2 = \|\Pi_h \bar u - \bar u\|_2 \leq c\,h$, gives the desired estimate $\|\bar u_h - \bar u\|_2 \leq c\,h$. ∎

Now it is an easy task to improve this L^2-estimate by one in L^∞. Here, we exploit the smoothing property of the elliptic PDEs.

Theorem 5.2 *The estimate*

$$\|\bar u - \bar u_h\|_\infty \leq C\,h^\lambda$$

holds for all sufficiently small h. Here, C is a positive constant independent of h, $\lambda = 1$ if $n = 2$ or if $n = 3$ and the triangulation is of nonnegative type, and $\lambda = 1/2$ otherwise.

Proof. Invoking Theorem 4.2 and the projection formulas (3.4), (4.6) we get

$$\|\bar{u} - \bar{u}_h\|_\infty = \|\mathrm{Proj}_{[\alpha,\beta]}(-\tfrac{1}{\nu}\varphi(\bar{u})) - \mathrm{Proj}_{[\alpha,\beta]}(-\tfrac{1}{\nu}\Pi_h\varphi_h(\bar{u}_h))\|_\infty$$

$$\leq C\,(h + \|\varphi(\bar{u}) - \varphi_h(\bar{u}_h)\|_\infty) \leq C\,(h + \|\bar{u} - \bar{u}_h\|_2 + h^\lambda).$$

Therefore we obtain

$$\|\bar{u} - \bar{u}_h\|_\infty \leq C(h^\lambda + \|\bar{u} - \bar{u}_h\|_2).$$

The conclusion follows from Theorem 5.1. ∎

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