A Domain Optimization Problem for a Nonlinear Thermoelastic System

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Abstract. A shape optimization problem for a thermoelastic system with a nonlinear boundary condition is considered. Using the material derivative method as well as the results of regularity of solutions to the state system, the sensitivity analysis of the solution to this system with respect to the variation of the domain is performed and necessary optimality conditions are derived.

1. Introduction

This paper is concerned with a shape optimization problem for an isotropic, homogeneous, elastic body occupying a two-dimensional domain and subjected to a prescribed thermal treatment. The equilibrium state of the body is described by a system of two linear weakly coupled parabolic and elliptic equations where the temperature and the displacement of the body are selected as the state variables. To model the heat exchange of the body with its surrounding medium, a nonlinear boundary condition is assumed in the parabolic equation. Existence and regularity of solutions to this state system were studied in [5, 6, 7, 10].

The shape optimization problem for the considered state system consists in finding an *initial* shape of the domain occupied by the body, such that the *final* shape of this domain after the thermal treatment will resemble a desired prescribed form as closely as possible. Note, that the solid body is subjected to a prescribed thermal treatment. Due to the temperature change, the body undergoes a thermoelastic deformation, that is, the induced thermal stresses force the body to change its shape in time. This problem was introduced by Sprekels and Tröltzsch [9].

Optimal shape design problems for systems governed by instationary equations have attracted much interest in the past years (see [2, 4, 8, 9, 11]). Among others in [9] the temperature change of the thermoelastic system was modelled by a parabolic equation with linear boundary condition. Moreover, the function describing the domain boundary was the variable subject to optimization.

In contrast to this, in this paper, we shall investigate a similar but slightly different model than introduced in [9]. We consider a nonlinear boundary condition on the part of the boundary of the domain occupied by the body. Moreover the objective functional introduced in this paper is slightly modified compared with that in [9]. Furthermore, here, the material derivative framework [8] is employed.

In the paper we shall formulate the optimal shape design problem and the underlying assumptions. Moreover the well-posedness of the state equations is investigated. Using results concerning the regularity of solutions to the thermoelastic systems [6, 7] as well as the material derivative method [8] the directional derivative of the cost functional with respect to variations of the domain is calculated. Moreover, first order necessary optimality conditions are formulated.

Throughout the paper we shall use the following notation : $H^m(\Omega)$, $m = 0, 1, 2$, will denote the Sobolev spaces of order m with norm $\| \, . \, \|_{H^m(\Omega)}$ [1], $\mathbf{n}=(n_1, n_2)$ is the unit outward vector to the boundary Γ, $\partial v/\partial n$ is the outward normal derivative of a function v on the boundary Γ of the domain Ω, $\bigtriangledown v$ is the gradient of a function v with respect to a variable x, $\triangle_x v$ denotes the Laplacian of the function v with respect to x, and $\theta_t = \partial\theta/\partial t$.

2. The thermoelastic model

Consider an isotropic, homogeneous, elastic body occupying a domain $\Omega \subset R^2$ in the plane Ox_1x_2. The domain Ω is bounded and simply connected. Its boundary Γ is supposed to be the type $C^{2,\alpha}$, with some $0 \le \alpha \le 1$. We assume that the body is loaded by a distributed force $f = f(x)$, $x = (x_1, x_2) \in D$. Hereby, $D \subset R^2$ denotes a hold-all domain containing all admissible domains Ω regarded in the shape optimization problem. We assume without loss of generality that the boundary of D is sufficiently smooth. The boundary Γ is divided into two disjoint parts Γ_1 and Γ_2. We assume $mes\Gamma_1 \ge 0$. Along the boundary Γ_2 the body is subjected to a prescribed thermal treatment, while it is thermally isolated on Γ_1. Γ_2 stands for the part of Γ which is to be shaped.

Let $\theta = \theta(x,t)$, $x \in \Omega$, $t \in [0, \mathcal{T}]$, $\mathcal{T} > 0$ given, denote the temperature of the body. Due to the temperature change the body undergoes a thermoelastic deformation. By $u = u(t,x)$, $u = (u_1, u_2)$, we denote the two-dimensional displacement vector of the body. In an equilibrium state the functions θ and u satisfy the system of state equations :

$$\begin{aligned} \theta_t(x,t) &= \Delta\theta(x,t)\,, \quad \text{in } \Omega\,, \quad 0 < t \le \mathcal{T} \\ \theta(x,0) &= \theta_0(x) \quad \text{in } \bar{\Omega} \\ \frac{\partial\theta}{\partial n}(x,t) &= g(\theta(x,t),x,t) \quad \text{on } \Gamma \quad 0 < t \le \mathcal{T} \end{aligned} \tag{1}$$

$$\begin{aligned} \mu\,\Delta u + (\lambda+\mu)\,\nabla\,(\operatorname{div} u) &= -\rho\, f + \beta\,\nabla\theta(.,\mathcal{T}) \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{on } \Gamma_1\,, \\ n_j\,\sigma_{ij} &= \beta\theta(.,\mathcal{T})n_i \quad \text{on } \Gamma_2, \quad i = 1,2,\,. \end{aligned} \tag{2}$$

Note that in (2) and throughout the paper summation convention is used. $\mu > 0$, $\lambda > 0$, $\rho > 0$, and $\beta > 0$ are fixed real constants. Moreover, real-valued functions $f \in L^2(D, R^2)$, and $g : R \times D \times [0, \mathcal{T}]$ are given. In (1) - (2) $\sigma = (\sigma_{ij}(x))$ and $\epsilon = \epsilon_{ij}(x)$ stand for the *stress* and *strain tensor*, respectively.

We assume that g satisfies the following assumptions:

(A1) g is continuously differentiable w.r. to (θ, t) on $R \times D \times [0, \mathcal{T}]$
g is continuous w.r. to x for all $(\theta, t) \in R \times [0, \mathcal{T}]$.
$| g(0, x, t) | \leq \psi_0 \; \forall (x, t) \in D \times [0, \mathcal{T}]$
$| g_\theta(\theta, x, t) | + | g_t(\theta, x, t) | \leq \psi_M \;\; \forall (\theta, x, t) \in [-M, M] \times D \times [0, \mathcal{T}]$
Here ψ_0 and ψ_M are certain real constants.

(A2) $g_\theta(\theta, x, t) \leq 0 \; \forall (\theta, x, t) \in R \times D \times [0, \mathcal{T}]$

(A3) Compatibility conditions: $\theta_0(x) = \theta_0$
is constant on D and $g(\theta_0(x), x, 0)) = 0$.

Assumption (A1) ensures in particular that $g(\theta, t, x)$ is well defined on sufficiently smooth boundaries of admissible domains Ω. The requirement that θ_0 is constant was imposed to satisfy the compatibility condition for all admissible boundaries Γ. It should be underlined that this condition is only needed to get higher regularity of the function θ.

2.1. Existence and uniqueness of solutions to the state equations

We shall work with weak solutions θ, u of (1)-(2), $\theta \in W(0, \mathcal{T})$, where the function space $W(0, \mathcal{T}) = \{v \in L^2(0, \mathcal{T}, H^1(\Omega)) \;\; | \;\; v_t \in L^2(0, \mathcal{T}, H^1(\Omega)')\}$ (see [5]). In the sequel, Q denotes the set $\Omega \times (0, \mathcal{T})$. Let H denote the space $L^2(\Omega)$ and $H^k(\Omega) = W_2^k(\Omega)$.

Definition 2.1. *$\theta \in W(0, \mathcal{T}) \cap C(\bar{Q})$ is said to be a* weak solution *of (1), if*

$$\begin{aligned} &\int_0^{\mathcal{T}} \int_\Omega [-\theta(x,t)\, \phi_t(x,t) + \nabla\theta(x,t) \nabla\phi(x,t)]\, dx\, dt \\ &= \int_0^{\mathcal{T}} \int_\Gamma g(\theta(x,t), x, t)\, \phi(x,t)\, dS\, dt + \int_\Omega \theta_o(x) \phi(x,0)\, dx \end{aligned} \tag{3}$$

for all $\phi \in W_2^{1,1}$ such that $\phi(., \mathcal{T}) = 0$.

The regularity of functions in $W(0, \mathcal{T})$ implies $\theta \in C([0, \mathcal{T}], H)$. In order to have $\nabla\theta(x, \mathcal{T})$ well defined, we need even $\theta \in C([0, \mathcal{T}], H^1(\Omega))$. This is the reason for introducing the quite strong assumptions (A1), (A3). The following result follows from [6, 7] :

Theorem 2.2. *Let the assumptions (A1)–(A3) be satisfied. Then a unique solution $\theta \in W(0, \mathcal{T}) \cap C(\bar{Q})$ of (2) exists. This solution can be repesented in the form*

$$\theta(x, t) = \theta_o(x) + \int_0^t w(x, s)\, ds, \tag{4}$$

where $w \in W(0, \mathcal{T})$.

Existence and uniqueness of $\theta \in W(0, \mathcal{T}) \cap C(\bar{Q})$ follows from Proposition 3.3 and Remark 3.4 in [6]. The representation (4) is given by Theorem 5.2 in [7].

Next we consider the weak form of the displacement equations. For convenience we introduce the space V defined by

$$V = \{u \in H^1(\Omega) \times H^1(\Omega) : u = 0 \text{ on } \;\; \Gamma_1\}. \tag{5}$$

Definition 2.3. *A function $u \in V$ is said to be a* weak solution *of (2), if*

$$\int_\Omega \sigma_{ij}(u)\epsilon_{ij}(v)dx = \int_\Omega \rho\, f\, v\, dx + \int_\Omega \beta\theta(.,\mathcal{T}) divv dx \ \ \forall v \in V \tag{6}$$

We recall that the strain tensor ϵ is defined by $\epsilon_{ij} = (\partial u^i/\partial x_j + \partial u^j/\partial x_i)/2$, $i, j = 1, 2$. The stress tensor $\sigma = \sigma_{ij}(x)$ is related to ϵ by

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & 2\mu \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{pmatrix}. \tag{7}$$

Theorem 2.4. *The assumptions (A1) – (A3) imply the existence of a unique weak solution u of (6).*

Proof. In view of Theorem 2.2 we have $\theta(.,\mathcal{T}) \in L^2(\Omega)$. Moreover, $mes\,\Gamma_1 > 0$ permits to apply Korn's first inequality. Existence and uniqueness is a known conclusion [3]. □

To achieve higher regularity of θ we impose the following stronger assumption on g:

$$(A1') \qquad g \in C^1(R \times D \times [0,\mathcal{T}])$$

Lemma 2.5. *If the stronger assumption $(A1')$ is satisfied instead of (A1), then $g(tr\theta,.,.)$ belongs to $C([0,\mathcal{T}], H^{1/2}(\Gamma))$ for the trace $tr\theta$ of the solution θ of (3).*

Proof. We know $\theta \in C(\bar{Q})$. Therefore the relation $g(tr\theta,.,.) = trg(\theta,.,.)$ is trivially true owing to the continuity assumptions on g. From Theorem 2.2 it can be deduced that $\theta \in C([0,\mathcal{T}], H^1(\Omega))$. Showing that $g(\theta,.,t)$ belongs to $C([0,\mathcal{T}], H^1(\Omega))$ we obtain the desired result. □

Corollary 2.6. *If Γ is sufficiently smooth, then θ belongs to $L^2(0,\mathcal{T}; H^2(\Omega))$.*

Proof. By (3) we know $\theta \in H^1(0,\mathcal{T}; H^1(\Omega))$, hence $\theta_t \in L^2(0,\mathcal{T}; H^1(\Omega))$ as well as $tr\theta \in H^1(0,\mathcal{T}; H^{1/2}(\Gamma))$. Lemma 2.5 yields $g(tr\theta,.,.) \in C([0,\mathcal{T}], H^{1/2}(\Gamma))$. Writing (1) in the form $\Delta\theta(x,t) = \theta_t(x,t)$ and $\partial\theta/\partial n\,(x,t)+\theta(x,t) = g(\theta(x,t),x,t)$ we obtain from the elliptic regularity theory that $\theta \in L^2(0,\mathcal{T}; H^2(\Omega))$. This follows from the fact that the mapping $(f,g) \to w$, which assigns data (f,g) the solution of $\Delta w = f,\ \partial w/\partial n + w = g$ is continuous from $L^2(\Omega) \times H^{1/2}(\Gamma)$ to $H^2(\Omega)$. □

3. The shape optimization problem

Before we formulate the shape optimization problem we shall introduce a family of admissible domains $\{\Omega_\tau\}$ depending on a parameter τ. The domain Ω_τ will be considered as the image of a mapping T_τ of the reference domain Ω. We shall employ the speed method [8] to describe the mapping T_τ. The shape optimization problem is formulated for the variational system (1)–(2) in the perturbed domain Ω_τ.

Let τ be a real parameter, such that $\tau \in [0,\sigma), \sigma > 0$. We denote by $\mathbf{V}(.,.) : [0,\sigma) \times R^2 \to R^2$ a sufficiently regular vector field, i.e.,

$$\mathbf{V}(\tau,.) \in C^2(R^2,R^2)\ \forall \tau \in [0,\sigma), \quad \mathbf{V}(.,x) \in C([0,\sigma),R^2)\ \forall x \in R^2. \tag{8}$$

By $T_\tau(\mathbf{V}) : R^2 \ni X \to x(\tau,X) \in R^2$ we denote a family of mappings depending on a parameter $\tau \in [0,\sigma)$, where the vector function $x(.,X) = x(.)$ satisfies the ordinary differential equation :

$$\frac{d}{ds}x(s,X) = \mathbf{V}(s,x(s,X)) \quad s \in [0,\sigma), \quad x(0,X) = X, \ \ X \in R^2. \tag{9}$$

The family Ω_τ of domains depending on the parameter $\tau \in [0,\sigma)$ is defined as follows : $\Omega_0 = \Omega$ and

$$\Omega_\tau = T_\tau(\mathbf{V})(\Omega) = \{x \in R^2 \supset \Omega;: \ \exists X \in R^2 \text{ such that } x = x(s) \text{ , where the function } x(.) \text{ satisfies equation (9) for } \ 0 \le s \le \tau\}. \tag{10}$$

We shall assume that for any given value of the parameter τ the domain $\Omega_\tau \subset D$ is bounded, simply connected, and has $C^{2,\alpha}$, $0 \le \alpha \le 1$, continuous boundary Γ_τ. Let us denote

$$Q_\tau = (0,\mathcal{T}) \times \Omega_\tau, \quad W_\tau(0,\mathcal{T}) = \{v \in L^2(0,\mathcal{T};H^1(\Omega_\tau)); v_t \in L^2(0,\mathcal{T};H^1(\Omega_\tau)')\}.$$

The variational problem (3)–(6) in the domain Ω_τ takes the form : *for given element $g \in C^1(R \times D \times [0,\mathcal{T}])$ and $\theta_0 \in L^2(D)$ **didnŽt we assume that this function is constant?** find functions $\theta_\tau \in W_\tau(0,\mathcal{T}) \cap C(\bar{Q}_\tau)$ and $u_\tau \in V_\tau = \{v \in H^1(\Omega_\tau) : \ v = 0 \ on \ \ \Gamma_{1\tau}\}$ satisfying*

$$\begin{aligned} &\int_0^{\mathcal{T}} \int_{\Omega_\tau} [-\theta_\tau(x,t)\, \phi_t(x,t) + \nabla\theta_\tau(x,t)\nabla\phi(x,t)]\, dx\, dt \\ &= \int_0^{\mathcal{T}} \int_{\Gamma_\tau} g(\theta_\tau(x,t),x,t)\, \phi(x,t)\, dS\, dt + \int_{\Omega_\tau} \theta_{0\tau}(x)\phi(x,0)\, dx \end{aligned} \tag{11}$$

for all $\phi \in W_2^{1,1}(Q_\tau)$ such that $\phi(.,\mathcal{T}) = 0$.

$$\int_{\Omega_\tau} \sigma_{\tau ij}(u)\epsilon_{ij}(v)dx = \int_{\Omega_\tau} \rho\, f\, v\, dx + \int_{\Omega_\tau} \beta\theta_\tau(.,\mathcal{T}) divv dx \ \ \forall v \in V_\tau \tag{12}$$

For each $\tau \in [0,\sigma), \sigma > 0$ the system (11)–(12) has a unique solution $(\theta_\tau, u_\tau) \in W_\tau(0,\mathcal{T}) \cap C(\bar{Q}_\tau) \cap L^2(0,\mathcal{T};H^2(\Omega_\tau)) \cap L^\infty(Q_\tau) \times V_\tau$.

By U_{ad} we denote the set of admissible domains defined in the way described above. We shall consider the following shape optimization problem for the system (11)–(12) : *find a domain $\Omega_\tau \in U_{ad}$ minimizing the cost functional*

$$J(\Omega_\tau) = \lambda_1 I_1(\Omega_\tau) + \lambda_2 I_2(\Omega_\tau), \quad where \tag{13}$$

$$I_1(\Omega_\tau) = \int_{\Omega_\tau} \mid u_d - u_\tau \mid^2 dx, \ \ I_2(\Omega_\tau) = \int_{\Gamma_\tau} \mid z_d - u_\tau \mid^2 ds \tag{14}$$

and $\lambda_1 \ge 0$, $\lambda_2 \ge 0$ $(\lambda_1^2 + \lambda_2^2 \ne 0)$, are fixed real constants. Moreover, real-valued functions $u_d \in L^2(D,R^2)$, $z_d \in C(\partial D,R^2)$ are given. The pair (θ_τ, u_τ) is a solution to the system (11)–(12).

The aim of the optimization problem (13) is to find the domain Ω occupied by the body at the time $t = 0$ such that the final displacement u of the body occupying the domain Ω at the time $t = \mathcal{T}$ comes as close as possible to prescribed functions u_d in Ω and to z_d on Γ. A body occupying the domain Ω is heated according to (1). The result is a thermal elastic deformation described by (2). For the selection of the reference domain Ω_0 in numerical applications see [3].

4. Necessary optimality condition

Throughout this section we shall tacitly assume the general assumptions (A1)–(A3). We shall calculate the derivative of the cost functional (13). In order to do it, first, we calculate the shape derivative U of the solution u_τ to the system (11)–(12). Let us recall the definition [8] :

Definition 4.1. *The material derivative $\dot{u} \in V$ of the function $u_\tau \in V_\tau$ at a point $X \in \Omega$ is defined by :*

$$\lim_{\tau \to 0} \| (u_\tau \circ T_\tau - u_0)/\tau - \dot{u} \|_{H^1(\Omega)} = 0 \tag{15}$$

where $u = u_0 \in V$, the function $u^\tau = u_\tau \circ T_\tau \in V$ is an image of the function $u_\tau \in V_\tau$ in the space V.

Let us recall [8, pp. 111 - 114] that if the shape derivative $U \in V$ of the function $u_\tau \in V_\tau$ exists, then the following condition holds :

$$U = \dot{u} - \nabla u \, \mathbf{V}(0) \tag{16}$$

4.1. Sensitivity analysis of solutions to the state system

Using Definition 4.1 we can prove :

Lemma 4.2. *The material derivatives $\dot{\theta} \in W(0, \mathcal{T})$ and $\dot{u} \in V$ of the functions $\theta_\tau \in W_\tau(0, \mathcal{T})$ and $u_\tau \in V_\tau$ satisfying the system (11)–(12) are given by :*

$$\begin{aligned} &\int_0^{\mathcal{T}} \int_\Omega \{[-\dot{\theta}\frac{\partial \varphi}{\partial t} - \theta \frac{\partial \varphi}{\partial t} div \mathbf{V}(0)] + [\nabla \dot{\theta} \nabla \varphi + \\ &\nabla \theta \nabla (\nabla \varphi \mathbf{V}(0))] + [div \mathbf{V}(0) I - ({}^T D\mathbf{V}(0) + D\mathbf{V}(0)) \nabla \theta \nabla \varphi]\} dx dt = \\ &\int_0^{\mathcal{T}} \int_\Gamma \{\frac{\partial g}{\partial \theta} \dot{\theta} \varphi + g \nabla \varphi \mathbf{V}(0) + \nabla g \varphi \mathbf{V}(0) + \\ &g\varphi(div \mathbf{V}(0) - (D\mathbf{V}\mathbf{n}, \mathbf{n}))\} ds dt + \int_\Omega \theta_0 \varphi(x, 0) div \mathbf{V}(0) dx \end{aligned} \tag{17}$$

$$\int_\Omega \{a_{ijkl}[\epsilon_{ij}(\dot{u})\epsilon_{kl}(\varphi)+\bar{\epsilon}_{ij}(u)\epsilon_{kl}(\varphi)+\epsilon_{ij}(u)\bar{\epsilon}_{kl}(\varphi)]+$$

$$\bar{a}_{ijkl}\bar{\epsilon}_{ij}(u)\epsilon_{kl}(\varphi)\}dx=-\beta\int_\Omega[-{}^TD\mathbf{V}(0)\nabla\theta\varphi+\nabla\theta\varphi divV(0)+ \tag{18}$$

$$\nabla\dot{\theta}\varphi+\nabla\theta\nabla\varphi\mathbf{V}(0)-(\rho f\varphi div\mathbf{V}(0)+\rho\varphi\nabla f\mathbf{V}(0)]dx$$

$$+\beta\int_\Gamma\{\varphi\dot{\theta}\mathbf{n}+\theta\mathbf{n}\nabla\varphi\mathbf{V}(0)+\theta\varphi(div\mathbf{V}(0)\mathbf{n}-{}^TD\mathbf{V}(0)\mathbf{n})\}ds$$

where $\bar{\epsilon}_{ij}=-\frac{1}{2}\{D\varphi D\mathbf{V}(0)+{}^TD\mathbf{V}(0)^TD\varphi\}$ *and* $\bar{a}_{ijkl}=a_{ijkl}div\mathbf{V}(0)+\nabla a_{ijkl}\mathbf{V}(0)$. *Moreover,* $\mathbf{V}(0)=\mathbf{V}(0,X)$, $DV(0)$ *denotes the Jacobian matrix of the matrix* $\mathbf{V}(0)$, ${}^TD\mathbf{V}(0)$ *denotes the transposed matrix of* $D\mathbf{V}(0)$, *and* I *is an identity matrix.*

Proof. Using formulae for the transport of the function gradient and normal vector into the fixed domain [8, pp 70 - 80] we transform system (11)–(12) to the fixed cylinder $\Omega\times[0,\mathcal{T}]$:

$$\int_0^{\mathcal{T}}\int_\Omega[-\theta^\tau(x,t)\frac{\partial\varphi}{\partial t}(x,t)+A(\tau)\nabla\theta^\tau\nabla\varphi]\gamma(\tau)dxdt=$$

$$\int_0^{\mathcal{T}}\int_{\Gamma_2}g(\theta^\tau,x,t)\varphi(x,t)\omega(\tau)dsdt+\int_\Omega\theta_0^\tau(x)\varphi(x,0)\gamma(\tau)dx \tag{19}$$

$$\forall\varphi\in W_2^{1,1}(Q),\ \varphi(0,\mathcal{T})=0$$

$$\int_\Omega a_{ijkl}\epsilon_{ij}(u^\tau)\epsilon_{kl}(\eta)A(\tau)dx=$$

$$\int_\Omega[\rho f\eta-\beta^TDT_\tau^{-1}\nabla\theta^\tau(.,\mathcal{T})]\eta\gamma(\tau)dx+\int_\Gamma\beta\theta^\tau\eta\omega(\tau)ds \tag{20}$$

for all $\eta\in V$ and a.e. $t\in(0,\mathcal{T})$. DT_τ is the Jacobian matrix of the mapping T_τ, DT_τ^{-1} is the inverse of DT_τ, and ${}^TDT_\tau$ is a transpose of DT_τ. Moreover $\gamma(\tau)=detDT_\tau$, $A(\tau)={}^TDT_\tau^{-1}DT_\tau^{-1}\gamma(\tau)$, $\omega(\tau)=\parallel DT_\tau n\parallel\gamma(\tau)$.

From Proposition 2.44 and Lemma 2.49 in [8] follows the differentiability of $\gamma(\tau)$, $A(\tau)$, $\omega(\tau)$ with respect to τ in $C^1(D)$, $C^1(D,R^2)$, $C^1(D)$ respectively. Using similar arguments as in [6, 8], from (19) and (20) we obtain the convergence of θ^τ to θ in $L^\infty(Q)$ and u^τ to u in $H^1(\Omega)$ for $\tau\to 0$. Substracting equation (3) from (19) and (7) from (20) and passing to the limit with $\tau\to 0$ we obtain formulae (17) and (18). □

From Corollary 2.6 it follows that the solution θ to the system (3)-(6) has the regularity $\theta\in L^2(0,\mathcal{T};H^2(\Omega))$. , i.e. we have ,

$$\nabla\theta\mathbf{V}(0)\in L^2(0,\mathcal{T};H^1(\Omega)) \tag{21}$$

From [7] and Corollary 2.6 we obtain the regularity $u\in H^2(\Omega)$ for the solution u to the system (3)-(6) provided either $\bar{\Gamma}=\bar{\Gamma}_1$ or $\bar{\Gamma}=\bar{\Gamma}_2$. In the case of mixed boundary conditions u may not belong to $H^2(\Omega)$ on $\bar{\Gamma}_1 \cap \bar{\Gamma}_2$. We assume that the following regularity condition is satisfied,

$$\nabla u \mathbf{V}(0) \in H^1(\Omega). \tag{22}$$

Remark that in our case (22) may be satisfied by choosing a suitable velocity field $\mathbf{V} = 0$ on $\bar{\Gamma}_1 \cap \bar{\Gamma}_2$ (see [8, p. 140]).

Integrating by parts the system (17)–(18) two times, eliminating the terms containing the derivatives of $\mathbf{V}(0)$ and taking into account (16) as well as (21)-(22) we obtain the system of equations determining the shape derivative $(\Theta, U) \in W(0, \mathcal{T}) \times V$ of the solution $(\theta_\tau, u_\tau) \in W_\tau(0, \mathcal{T}) \times V_\tau$ of the system (11)–(12) :

$$\int_0^{\mathcal{T}} \int_\Omega \{-\Theta \frac{\partial \varphi}{\partial t} + \nabla \Theta \nabla \varphi\} dxdt + \int_0^{\mathcal{T}} \int_\Gamma \{\theta \frac{\partial \varphi}{\partial t} \mathbf{V}(0) n + \nabla \theta \nabla \varphi \mathbf{V}(0)\mathbf{n}\} dsdt =$$
$$\int_0^{\mathcal{T}} \int_{\Gamma_2} \{\frac{\partial g}{\partial \theta} \Theta \varphi + \frac{\partial g}{\partial \theta} \nabla \theta \mathbf{V}(0) \varphi\} dsdt + \int_0^{\mathcal{T}} \int_{\Gamma_2} [(\frac{\partial g}{\partial n} + g)\varphi + \tag{23}$$
$$\frac{\partial \varphi}{\partial n} g + H g \varphi] \mathbf{V}(0)\mathbf{n} dsdt + \int_\Gamma \theta_0 \varphi \mathbf{V}(0)\mathbf{n} ds \quad \forall \varphi \in W(0, \mathcal{T}), \ \ \varphi(., \mathcal{T}) = 0,$$

where $H = div\, \mathbf{n}$ denotes the mean curvature of the boundary Γ.

$$\int_\Omega a_{ijkl} \epsilon_{ij}(U) \epsilon_{kl}(\eta) dx + \int_{\Gamma_1} a_{ijkl} \epsilon_{ij}(u) \epsilon_{kl}(\varphi) \mathbf{V}(0)\mathbf{n} ds = -\int_\Omega \beta \nabla \Theta \eta dx$$
$$+ \int_\Gamma \beta \{\Theta \mathbf{n} \eta + \rho f \eta \mathbf{V}(0)\mathbf{n} - \nabla \theta \eta \mathbf{V}(0)\mathbf{n} - \theta \eta \mathbf{V}(0)\mathbf{n}\} ds \quad \forall \eta \in V. \tag{24}$$

4.2. The form of the directional derivative of the cost functional

Let us recall the definition of the Euler derivative [8] :

Definition 4.3. *The Euler derivative $dJ(\Omega, \mathbf{V})$ of the cost functional $J(\Omega)$ at a point Ω in the direction of the vector field $\mathbf{V}$ is determined by*

$$dJ(\Omega, \mathbf{V}) = \limsup_{\tau \to 0} [J(\Omega_\tau) - J(\Omega)]/\tau. \tag{25}$$

Lemma 4.4. *The derivative $dJ(\Omega, \mathbf{V})$ of the cost functional (13) at a point Ω in a direction $\mathbf{V}$, defined by (25) is given by :*

$$dJ(\Omega, \mathbf{V}) = -2\lambda_1 \int_\Omega (u_d - u, U) d\Omega + \lambda_1 \int_\Gamma \mid u_d - u \mid^2 \mathbf{V}(0)\mathbf{n} d\Gamma$$
$$-2\lambda_2 \int_\Gamma (z_d - u, U) d\Gamma - 2\lambda_2 \int_\Gamma (z_d - u, \nabla u \mathbf{V}(0)\mathbf{n}) d\Gamma +$$
$$\lambda_2 \int_\Gamma \mid z_d - u \mid^2 H \mathbf{V}(0)\mathbf{n} d\Gamma, \tag{26}$$

where $U = \partial u / \partial \tau$ is the shape derivative of the function u_τ defined by (16) and $H = div\, \mathbf{n}$ is the mean curvature of the boundary Γ.

Proof. Follows from (13), (25) as well as Lemma 4.2. □

In order to eliminate U from (26) we introduce an adjoint state $(p,q) \in W(0,\mathcal{T}) \times V$ satisfying the following system of equations :

$$\int_0^{\mathcal{T}} \int_\Omega -\frac{\partial p}{\partial t} \varphi dx dt + \int_0^{\mathcal{T}} \int_\Omega \nabla p \nabla \varphi dx dt - \int_0^{\mathcal{T}} \int_{\Gamma_2} \frac{\partial g}{\partial \theta} p \varphi ds dt$$

$$- \int_\Omega \beta \nabla \varphi q dx + \int_\Gamma \beta q \varphi \mathbf{n} ds = 0 \quad \forall \varphi \in W(0,\mathcal{T}), p(\mathcal{T}) = 0 \tag{27}$$

$$\int_\Omega a_{ijkl} \epsilon_{ij}(q) \epsilon_{kl}(\eta) dx - 2\lambda_1 \int_\Omega (u_d - u, \eta) dx -$$

$$2\lambda_2 \int_\Gamma (z_d - u, \eta) ds = 0 \quad \forall \eta \in V. \tag{28}$$

Lemma 4.5. *The directional derivative $dJ(\Omega, \mathbf{V})$ of the cost functional (13) at a point Ω in the direction $\mathbf{V}$ is given by :*

$$dJ(\Omega, \mathbf{V}) = \int_\Gamma [\lambda_1(u_d - u, u_d - u) + \lambda_2(z_d - u, z_d - u)H$$

$$-2\lambda_2(z_d - u, \nabla u)]\mathbf{V}(0)\mathbf{n} ds - \int_\Gamma \beta\{\rho f q \mathbf{V}(0)\mathbf{n} -$$

$$\nabla\theta q \mathbf{V}(0)\mathbf{n} - \theta q \mathbf{V}(0)\mathbf{n}\} ds + \int_{\Gamma_1} a_{ijkl}\epsilon_{ij}(u)\epsilon_{kl}(q)\mathbf{V}(0)\mathbf{n} ds$$

$$- \int_0^{\mathcal{T}} \int_\Gamma \{\theta \frac{\partial p}{\partial t} \mathbf{V}(0)\mathbf{n} + \nabla\theta\nabla p \mathbf{V}(0)\mathbf{n}\} ds dt + \int_0^{\mathcal{T}} \int_{\Gamma_2} \frac{\partial g}{\partial \theta} \nabla\theta \mathbf{V}(0)\mathbf{n} p ds dt + \tag{29}$$

$$\int_0^{\mathcal{T}} \int_{\Gamma_2} [(\frac{\partial g}{\partial n} + g)p + \frac{\partial p}{\partial n} g + Hgp]\mathbf{V}(0)\mathbf{n} ds dt + \int_\Gamma \theta_0 p \mathbf{V}(0)\mathbf{n} ds$$

where $(\theta, u) \in W(0,\mathcal{T}) \times V$ and $(p,q) \in W(0,\mathcal{T}) \times V$ satisfy, respectively, the systems (3)– (6) and (27)–(28).

Proof. Follows from (23), (24), (26), (27), (28). □

The necessary optimality condition for the problem (13) has the standard form :

Lemma 4.6. *For all vector fields $\mathbf{V}$ defined by (8)–(9) an optimal solution $\hat{\Omega} \in U_{ad}$ to the problem (13) satisfies the following condition :*

$$dJ(\hat{\Omega}, \mathbf{V}) \geq 0 \tag{30}$$

where $dJ(\hat{\Omega}, \mathbf{V})$ is given by (29).

Proof. Is standard [3, 8]. □

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