MESH-INDEPENDENCE OF SEMISMOOTH NEWTON METHODS FOR LAVRENTIEV-REGULARIZED STATE CONSTRAINED NONLINEAR OPTIMAL CONTROL PROBLEMS

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Abstract. A class of nonlinear elliptic optimal control problems with mixed control-state constraints arising, e.g., in Lavrentiev-type regularized state constrained optimal control is considered. Based on its first order necessary optimality conditions, a semismooth Newton method is proposed and its fast local convergence in function space as well as a mesh-independence principle for appropriate discretizations are proved. The paper ends by a numerical verification of the theoretical results including a study of the algorithm in the case of vanishing Lavrentiev-parameter. The latter process is realized numerically by a combination of a nested iteration concept and an extrapolation technique for the state with respect to the Lavrentiev-parameter.

Key words. Lavrentiev regularization, mesh independence principle, mixed pointwise control-state constraints, nested iteration, nonlinear optimal control problem, semismooth Newton method, state constraints, superlinear convergence.

AMS subject classifications. 35J60, 49K20, 49M05, 65K10.

1. Introduction.

In the last decade, the presence of pointwise state inequality constraints in optimal control problems of partial differential equations (PDEs) has opened up new challenging research directions in the development of numerical algorithms. The main difficulty when solving these problems is due to the lack of regularity of the Lagrange multipliers associated with the pointwise state constraints. As originally noticed by Casas [6] (see also [1, 7]), the Lagrange multipliers associated with the optimal solution are only (regular) Borel measures in general; see, e.g., [5, 19] for further structural properties of the multiplier. This fact complicates the numerical treatment considerably. First it raises the question of how to appropriately discretize measure-valued quantities. Secondly, well known methods such as projected Newton or primal-dual active set methods exhibit a mesh-dependent behavior.

In the recent past, there were two major efforts in devising algorithms of nonlinear programming type to overcome the difficulties due to poor multiplier regularity. On one hand, there is the Moreau-Yosida-based regularization technique of [18] which removes the state inequality constraints from the set of explicit constraints. Rather it adds an augmented Lagrangian-type penalty (respectively regularization) term to the objective functional of the control problem. As a result, from the corresponding first order necessary optimality conditions of the penalized problem one obtains a regular approximation of the measure-valued Lagrange multiplier of the original problem. The second approach [29], which we shall pursue here, consists of applying a Lavrentiev-type regularization of the pointwise state constraints. In this case, differently from the first approach, the pointwise constraints are kept as explicit constraints. Rather these are transformed into mixed control-state constraints. For the latter constraint type it is known [25, 27, 29] that the corresponding Lagrange multiplier enjoys better

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regularity properties than the original one. Compared to the Moreau-Yosida-based technique, keeping the constraints explicit preserves the overall problem structure and can be an advantage.

Throughout the present paper we, hence, focus on the following mixed control-state constrained, or equivalently Lavrentiev-regularized state constrained, model problem:

$$(P) \qquad \begin{cases} \text{minimize } J(u,y) := \frac{1}{2}\|y-y_d\|^2_{L^2} + \frac{\alpha}{2}\|u\|^2_{L^2} \\ \text{over } (u,y) \in L^2(\Omega) \times H^1_0(\Omega) \cap \mathcal{C}(\bar{\Omega}) \\ \text{subject to } Ay + d(\cdot,y) = u \quad \text{in } \Omega, \\ \qquad\qquad y_a \le \varepsilon u(x) + y(x) \le y_b \quad \text{for almost all (f.a.a.) } x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently regular boundary $\Gamma = \partial\Omega$, and A denotes a second order linear elliptic differential operator with sufficiently smooth coefficients. Further, y_d, α, the function d, $y_a < y_b \in \mathbb{R}$, and ε are fixed given data which will be specified shortly. We point out that ε is referred to as the Lavrentiev parameter. If we set $\varepsilon = 0$, then we regain the original state constrained optimal control problem. We further note that the subsequent considerations remain true for state equations involving Neumann- or Robin-type boundary conditions instead of the homogeneous Dirichlet boundary condition.

As noted earlier, problems of the type (P) were considered in the literature recently. In [29] it was shown that the Lagrange multiplier associated with the mixed control-state constraint exists as a function in $L^2(\Omega)$. Subsequently, for the numerical solution of (P) a short-step path-following interior-point method was proposed in [26].

Since it is known from the nonlinear programming literature (see, e.g., [31]) that short-step variants of interior-point methods are usually not competitive in practice and due to our previous experience that semismooth Newton methods, or in some cases equivalently primal-dual active-set strategies, are highly efficient for control constrained optimal control of elliptic PDEs, in contrast to [26] we focus here on semismooth Newton-type methods for solving (P). The latter motivation can be substantiated by the observation that (P) can easily be transformed into an optimal control problem with pure pointwise control constraints. In fact, setting $v := \varepsilon u + y$ and, hence, $u = \frac{1}{\varepsilon}(v-y)$, we obtain the following equivalent formulation:

$$(\tilde{P^v}) \qquad \begin{cases} \text{minimize } \tilde{J}(v,y) := \frac{1}{2}\|y-y_d\|^2_{L^2} + \frac{\alpha}{2\varepsilon^2}\|v-y\|^2_{L^2} \\ \text{over } (v,y) \in L^2(\Omega) \times H^1_0(\Omega) \cap \mathcal{C}(\bar{\Omega}), \\ \text{subject to } \varepsilon Ay + \varepsilon d(\cdot,y) + y = v \quad \text{in } \Omega, \\ \qquad\qquad y_a \le v(x) \le y_b \quad \text{f.a.a. } x \in \Omega. \end{cases}$$

It turns out that the transformed problem $(\tilde{P^v})$ is useful in both, the analysis of the problem as well as the design and convergence analysis of algorithms. Indeed, we will introduce a semismooth Newton method in function space for solving $(\tilde{P^v})$ and prove its locally superlinear convergence.

Problem $(\tilde{P^v})$ is also useful when proving the mesh independent convergence of a discrete version of the semismooth Newton method introduced in this paper. This aspect is important since it guarantees a stable convergence behavior of the method as the underlying mesh is refined. For smooth nonlinear operator equations, there is an excellent body of work relying on Newton-Kantorovich [2] or Newton-Mysovskii [12] theorems for establishing mesh independence of Newton's method. We also refer to [30] for a recent affine invariant theory on asymptotic mesh independence which considerably improves, e.g., [2]. Additional results in the context of optimal control problems can be found in [3, 13]. In this context, in the presence of constraints typically a generalized equation's approach is considered with a smooth nonlinear mapping. Thus, one is back at a smooth (set-valued) operator equation. For Lagrange-Newton methods this approach analyses the outer linearization loop while the (exact) solution of the constrained subproblems is assumed and their mesh independence remains open. In sequential quadratic programming (SQP) approaches to constrained optimal control problems this would refer to a mesh independent convergence of the major (outer) iterates. However, the mesh independence of the solver of the quadratic subproblems for computing update directions for the major iterates is not considered. Our reformulation of $(\tilde{P^v})$, on the other hand, results in a nonsmooth (i.e., not Fréchet differentiable) operator equation. As, for instance, Newton-Kantorovich or Newton-Mysovskii results are not available in this case, our mesh independence analysis has to rely on different tools. Here we use the proof technique of [20]; further see [16]. Referring back to the SQP or Lagrange-Newton perspective mentioned above, our analysis can be used to study the mesh independence of semismooth Newton type solvers for the constrained subproblems. This closes the gap in the earlier analysis as in, e.g., [3] and yields a fully mesh independent method. We further mention that corresponding fast local convergence and mesh-independence results are currently not available for the short-step path-following interior-point techniques in [26].

The rest of the paper is organized as follows: In the next section we introduce some notation and assumptions required throughout. Further we specify the finite element discretization considered in this paper. It is based on [4], where convergence rates for control constrained optimal control problems were shown. In section 3 we discuss the first order conditions of a reduced version of $(\tilde{P^v})$ and introduce the semismooth Newton methods for the continuous as well as for the discretized case, respectively. Section 4 is devoted to our mesh-independence analysis. As the ultimate goal is to solve the state constrained problem, i.e., $\varepsilon = 0$ in (P), in section 5 we prove the strong L^2-convergence for $\varepsilon \to 0$ of the regularized optimal controls to the optimal control pertinent to (P) with $\varepsilon = 0$. Finally, in section 6 we provide a report on numerical test runs which also includes a validation of our theoretical results and a study of our algorithm in the case of vanishing Lavrentiev-parameter.

2. Basics. We start by introducing the notation used throughout the paper. Further we provide our working assumptions, and we specify the discretization concept applied to $(\tilde{P^v})$ and its first order optimality system.

2.1. Assumptions and Notation. In what follows, we assume that Ω is a convex bounded domain in $\mathbb{R}^n$, $n \in \{2, 3\}$, with a $\mathcal{C}^{1,1}$ boundary Γ. Concerning the data specified in $(\tilde{P^v})$ we suppose that the desired state y_d lies in $L^2(\Omega)$, the bounds are $y_a, y_b \in \mathbb{R}$ with $y_a < y_b$, and $\alpha > 0$ represents the cost of the control. By $\|\cdot\|_{L^2}$, as used in the objective of $(\tilde{P^v})$, and $(\cdot,\cdot)_{L^2}$ we denote the $L^2(\Omega)$-norm and the $L^2(\Omega)$-inner product, respectively. We use the analogous notation for other function space norms. The operator A denotes a second-order elliptic partial differential operator of the form

$$Ay(x) = -\sum_{i,j=1}^{n} D_i(a_{ij}(x) D_j y(x)),$$

where the coefficient functions a_{ij} belong to $C^{0,1}(\bar\Omega)$ and satisfy the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq m_0\|\xi\|_2^2 \quad \forall(\xi,x) \in \mathbb{R}^n \times \bar\Omega$$

for some constant $m_0 > 0$. The function $d(x,y)$ is a Carathéodory function from $\bar\Omega \times \mathbb{R}$ into $\mathbb{R}$, i.e., for every fixed $y \in \mathbb{R}$ the function $d(\cdot,y)$ is measurable, and d is continuous with respect to y for almost every fixed $x \in \Omega$. Furthermore, we assume that for every $x \in \Omega$ the function $d(x,\cdot)$ is of class $\mathcal{C}^2$ and $D_y d(x,\cdot)$ is nonnegative. We also suppose that for all $K > 0$ there exists a constant $C_K > 0$ such that

$$\begin{aligned} &|d(x,y)| + |D_y d(x,y)| + |D_{yy} d(x,y)| \leq C_K, \\ &|D_{yy} d(x,y_1) - D_{yy} d(x,y_2)| \leq C_K|y_1 - y_2| \end{aligned}$$

for all $(x,y,y_1,y_2) \in \Omega \times [-K,K]^3$.

From the Sobolev embedding theorem we obtain the continuous embedding

$$H_0^1(\Omega) \hookrightarrow L^p(\Omega)$$

for $p = 6$ if $n = 3$ and $p \in [6,\infty)$ if $n = 2$, i.e., there exist positive real numbers s_p such that

$$\|\cdot\|_{L^p} \leq s_p \|\cdot\|_{H^1}.$$

For later use, we also introduce the solution operator $G : L^2(\Omega) \to H_0^1(\Omega) \cap \mathcal{C}(\bar\Omega)$ that assigns to every element $v \in L^2(\Omega)$ the solution $y = y(v) \in H_0^1(\Omega) \cap \mathcal{C}(\bar\Omega)$ of the transformed state equation

$$\varepsilon Ay + \varepsilon d(\cdot,y) + y = v \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \Gamma. \tag{2.1}$$

We further set $S = \imath_0 G$, where $\imath_0$ is the compact embedding operator from $H_0^1(\Omega)$ to $L^2(\Omega)$. Now we can express $(\tilde{P^v})$ as follows:

$$(P^v) \quad \begin{cases} \text{minimize } f(v) := \frac{1}{2}\|S(v) - y_d\|_{L^2}^2 + \frac{\alpha}{2\varepsilon^2}\|v - S(v)\|_{L^2}^2 \quad \text{over } v \in L^2(\Omega), \\ \text{subject to } v \in V_{ad}, \end{cases}$$

where the feasible set V_{ad} is defined by

$$V_{ad} = \{v \in L^2(\Omega) \mid y_a \leq v(x) \leq y_b \text{ f.a.a. } x \in \Omega\}.$$

2.2. Discretization. For the discretization of our model problem we employ the finite element method as outlined in [4]. In fact, let $(\mathcal{T}_h)_{h>0}$ denote a family of triangulations of $\bar\Omega$ with h, the mesh-size of the underlying mesh, being equal to the maximum diameter over all elements in $\mathcal{T}_h$. This yields $\bar\Omega_h = \bigcup_{T\in\mathcal{T}_h} T$ with Ω_h denoting the interior of $\bar\Omega_h$ and Γ_h its boundary. For the results in [4] to be applicable we require the following modification of the triangulation close to the boundary of Ω: For $n = 2$ we replace every boundary triangle T of $\mathcal{T}_h$ by another triangle $\hat T \subset \Omega$ with a curved boundary. For its precise construction we follow [9], where the edge between the two boundary nodes of T is replaced by the part of the boundary of Ω connecting these nodes and forming a triangle with the remaining interior sides of T. By $\hat{\mathcal{T}}_h$, we denote the union of these curved boundary triangles with the interior triangles in

Ω of $\mathcal{T}_h$. In this way we obtain $\bar{\Omega} = \cup_{\hat{T} \in \hat{\mathcal{T}}_h} \hat{T}$. For $n = 3$, we use tetrahedra with the analogous modification close to Γ. Based on this construction we now define the discrete control and state spaces, respectively, by

$$
\begin{aligned}
V_h &:= \{v_h \in L^\infty(\Omega) \mid v_{h|\hat{T}} \text{ is constant } \forall\, \hat{T} \in \hat{\mathcal{T}}_h\},\\
Y_h &= \{y_h \in \mathcal{C}(\bar{\Omega}) \cap H_0^1(\Omega) \;\mid y_{h|T} \in \mathcal{P}_1 \forall\, T \in \mathcal{T}_h \text{ and } y_h = 0 \text{ on } \bar{\Omega} \backslash \Omega_h\}.
\end{aligned}
$$

Above, $\mathcal{P}_1$ denotes the space of polynomials of degree less than or equal to one. The spaces V_h and Y_h are equipped with the L^2-norm respectively the H_0^1-norm. For later use, we also introduce the piecewise linear finite element basis $\{\phi_1(x), ..., \phi_{m_h}(x)\}$ of Y_h.

We are now ready to formulate the discrete approximation of (P^v):

$$
(P_h^v) \quad \begin{cases} \text{minimize } f_h(v_h) := \frac{1}{2}\|S_h(v_h) - y_d\|_{L^2}^2 + \frac{\alpha}{2\varepsilon^2}\|v_h - S_h(v_h)\|_{L^2}^2 \text{ over } v_h \in V_h \\ \text{subject to } \;\; y_a \le v_h(x) \le y_b \text{ f.a.a. } x \in \Omega, \end{cases}
$$

where $S_h = \imath_0 G_h : V_h \to L^2(\Omega)$ with G_h denoting the discrete solution operator corresponding to the discrete state equation. It assigns to every element $v_h \in V_h$ the solution $y_h = y_h(v_h) \in Y_h$ of

$$
\varepsilon a(y_h, \phi_i) + (\varepsilon d(\cdot, y_h) + y_h, \phi_i)_{L^2} = (v_h, \phi_i)_{L^2} \quad \forall i \in \{1, ..., m_h\}
$$

with the bilinear form $a : Y_h \times Y_h \to \mathbb{R}$ defined by

$$
a(y, z) = \int_\Omega (\sum_{i,j=1}^n a_{ij}(x) D_i y(x) D_j z(x))\, dx.
$$

For our mesh-independence analysis in section 4 we have to work with the interpolation operator corresponding to our piecewise constant finite element discretization. Hence, we next introduce this operator and collect some of its properties. The interpolation operator $\Pi_h \in \mathcal{L}(L^2(\Omega), V_h)$ assigns to every element $v \in L^2(\Omega)$ a piecewise constant function on Ω. For its definition we follow [4]:

$$
(\Pi_h v)(x) = \begin{cases} (\pi_h v)(x) & x \in T, \\ (\pi_h v)(x_0) & x \in \hat{T} \setminus T, \end{cases}
$$

where x_0 is the projection of x onto the boundary of the triangle and the operator $\pi_h : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$
(\pi_h v)_{|T} = \frac{1}{|T|} \int_T v(x)\; dx
$$

for $T \in \mathcal{T}_h$ and $\hat{T} \in \hat{\mathcal{T}}_h$. In the following sections, we will make use of the following estimates:

$$
\|\Pi_h w\|_{L^q} \le \|w\|_{L^q} \qquad \text{for all } w \in L^q(\Omega),\; q \in [2, \infty], \tag{2.2}
$$

$$
\|w - \Pi_h w\|_{L^2} \le ch\|w\|_{H^1} \quad \text{for all } w \in H^1(\Omega). \tag{2.3}
$$

with some positive constant c. For more details we refer the reader to [20] and the references therein.

3. Optimality conditions and semismooth Newton methods. We now turn to the derivation of first- and second-order optimality conditions for (P^v). Then, a reformulation of the first order conditions is the starting point for our semismooth Newton framework.

3.1. First- and second-order optimality. The assumptions invoked in the previous section imply the existence of a solution to (P^v); see [4]. Note that due to the non-convexity of the objective functional f, we cannot expect uniqueness of the solution. Further, the methods we are about to propose typically find only local solutions. For this purpose, recall that a L^2-function v^* is said to be a local solution of (P^v) if there exists a positive real number r such that

$$f(v) \geq f(v^*) \quad \forall v \in V_{ad} \cap \bar{B}_r := \{v \in L^2(\Omega) : \|v - v^*\|_{L^2} \leq r\}.$$

The first derivative of the objective function f of (P^v) at v in an arbitrary direction $w \in L^2(\Omega)$ is given by

$$f'(v)w = (S(v) - y_d + \frac{\alpha}{\varepsilon^2}(S(v) - v), S'(v)w)_{L^2} + \frac{\alpha}{\varepsilon^2}(v - S(v), w)_{L^2}.$$

By standard arguments, this can equivalently be expressed as

$$f'(v) = p(v) - \frac{\alpha}{\varepsilon^2} y(v) + \frac{\alpha}{\varepsilon^2} v, \tag{3.1}$$

where $p = p(v) \in H_0^1(\Omega) \cap \mathcal{C}(\bar{\Omega})$ is defined as the solution of the following adjoint (state) equation

$$\varepsilon A^* p + \varepsilon d_y(\cdot, y(v))p + p = y(v) - y_d + \frac{\alpha}{\varepsilon^2}(y(v) - v) \text{ in } \Omega, \quad p = 0 \quad \text{on } \Gamma. \tag{3.2}$$

Here, A^* is the adjoint operator of A, and $y(v)$ denotes the state associated to v. The first-order necessary optimality condition for (P^v) at a (local) solution $v^* \in V_{ad}$ is then given by the variational inequality

$$(f'(v^*), v - v^*)_{L^2} \geq 0, \quad \forall v \in V_{ad}.$$

By (3.1), this is equivalent to

$$(p^* - \frac{\alpha}{\varepsilon^2} y^* + \frac{\alpha}{\varepsilon^2} v^*, v - v^*)_{L^2} \geq 0 \quad \forall v \in V_{ad},$$

where y^* is the state associated to $v^* \in V_{ad}$ and $p^* = p(v^*)$ is the adjoint state satisfying (3.2) at $(y, v) = (y^*, v^*)$. From this, standard arguments (see, e.g., [23]) yield the following projection formula for a local solution v^*:

$$v^* = P_{[y_a, y_b]}(y^* - \frac{\varepsilon^2}{\alpha} p^*), \tag{3.3}$$

where $P_{[y_a,y_b]}(v)(x) := \min(y_b, \max(y_a, v(x)))$ denotes the projection operator from $L^2(\Omega)$ onto $L^\infty(\Omega, [y_a, y_b])$. Defining the operator $\Psi : L^2(\Omega) \to L^2(\Omega)$ by

$$\Psi(v) := v - P_{[y_a,y_b]}(y(v) - \frac{\varepsilon^2}{\alpha} p(v)). \tag{3.4}$$

then, due to (3.3), the first-order necessary optimality condition for (P^v) can be written as

$$\Psi(v^*) = 0 \quad \text{almost everywhere (a.e.) in } \Omega. \tag{3.5}$$

We next turn to the discrete problem (P_h^v). Similarly to the continuous case, our assumptions guarantee the existence of at least one solution $v_h^* \in V_h$. Analogously, the first-order optimality condition can be written as

$$\Psi_h(v_h^*) = 0 \quad \text{a.e. in } \Omega, \tag{3.6}$$

where $\Psi_h : V_h \to V_h$ denotes the discrete analogue of Ψ. It is defined by

$$\Psi_h(v_h) := v_h - P_{[y_a,y_b]}(\Pi_h y_h(v_h) - \frac{\varepsilon^2}{\alpha}\Pi_h p_h(v_h)).$$

Here, $y_h(v_h) = S_h(v_h)$, and $p_h(v_h) \in Y_h$ denotes the discrete adjoint state satisfying

$$\varepsilon a(p_h, \phi_i) + (\varepsilon d_y(\cdot, y_h(v_h))p_h + p_h, \phi_i)_{L^2} = (y_h(v_h) - y_d + \frac{\alpha}{\varepsilon^2}(y_h(v_h) - v_h), \phi_i)_{L^2}$$

for all $i \in \{1, ..., m_h\}$. This setting is analogous to the one considered in [4].

Since (P^v) is not necessarily convex, the first-order optimality condition (3.4) is not sufficient for local optimality. Under the following second-order *sufficient* optimality condition, the local optimality can be guaranteed.

DEFINITION 3.1 (Second-order sufficient conditions; cf. [9]). *The point $v^* \in L^2(\Omega)$ is said to satisfy the second-order sufficient condition if there exist $\kappa > 0$ and $\tau > 0$ such that*

$$f''(v^*)w^2 \geq \kappa \|w\|_{L^2}^2 \tag{3.7}$$

holds true for all $w \in L^\infty(\Omega)$ satisfying

$$w(x) \begin{cases} \geq 0 & \text{if} \quad v^*(x) = y_a, \\ \leq 0 & \text{if} \quad v^*(x) = y_b, \\ = 0 & \text{if} \quad |p^*(x) + \alpha\varepsilon^{-2}(v^*(x) - y^*(x))| \geq \tau. \end{cases} \tag{3.8}$$

This condition will be essential when proving the mesh-independence principle in section 4. We point out that (3.7) can be expressed in terms of the PDE. It is equivalent to

$$\mathcal{L}''(y^*, v^*, p^*)[y, w]^2 \geq \kappa \|w\|_{L^2}^2$$

for all y solving the linearized PDE

$$Ay + \frac{1}{\varepsilon}y + d_y(\cdot, y^*)y = \frac{1}{\varepsilon}w,$$

where w satisfies the conditions (3.8). In this setting, $\mathcal{L}$ is defined by

$$\mathcal{L}(y, v, p) = \tilde{J}(v, y) - \langle Ay, p \rangle - (d(\cdot, y), p)_{L^2} - \frac{1}{\varepsilon}(y - v, p)_{L^2}.$$

For details we refer to [28, Theorem 4.23]; related techniques for the control of semilinear equations can be found in [8, 10].

3.2. Semismooth Newton algorithm. The nonlinear and nonsmooth operator equation

$$\Psi(v) = 0$$

will now be solved by a semismooth Newton method. Our development is based on the following generalized differentiability notion; compare [11, 17].

DEFINITION 3.2. *Let X, Y be Banach spaces and U be an open domain in X. A function $F : U \to Y$ is said to be semismooth (or Newton differentiable) in U if there exists a (possibly set-valued) mapping $\partial F : U \rightrightarrows \mathscr{L}(X, Y)$ such that*

$$\sup_{V \in \partial F(x+s)} \|F(x+s) - F(x) - Vs\|_Y = o(\|s\|_X) \quad \text{as } \|s\|_X \to 0 \tag{3.9}$$

for all $x \in U$. We call ∂F the Newton differential*, and its elements are referred to as* Newton maps*.*

We point out that in [22] a notion similar to the one in Definition 3.2 is introduced and the name *Newton maps* for elements of the resulting generalized differential is coined.

For an application of this concept, let us consider the operator $H : L^{q_1}(\Omega) \to L^{q_1}(\Omega)$, $H(w) = w - P_{[a,b]}(-C(w))$, where $P_{[a,b]}$ denotes the projection onto $[a, b]$, with $a, b \in \mathbb{R}$, $a < b$, and $C : L^{q_1}(\Omega) \to L^{q_1}(\Omega)$ is a continuously Fréchet differentiable mapping. Moreover, we assume that C is locally Lipschitz continuous from $L^{q_1}(\Omega)$ to $L^{q_2}(\Omega)$ for some $q_2 > q_1 \geq 1$. Then, as demonstrated, e.g., in [17], H is semismooth, i.e., for all $w \in L^{q_1}(\Omega)$ we have

$$\sup_{V \in \partial H(w+s)} \|H(w+s) - H(w) - Vs\|_{L^{q_1}} = o(\|s\|_{L^{q_1}}) \quad \text{as } \|s\|_{L^{q_1}} \to 0.$$

A class of corresponding Newton maps is given by

$$\widehat{\partial H}^{\zeta_1,\zeta_2}(w) = \{\mathrm{id} + D(w)C'(w)\} \subset \partial H(w), \tag{3.10}$$

where $C'(w)$ denotes the Fréchet derivative of C at w, and the operator $D : L^{q_1}(\Omega) \to L^\infty(\Omega)$ satisfies

$$D(w)(x) \begin{cases} = & 0 & \text{if } -C(w)(x) \notin [a, b], \\ \in & [\zeta_1, \zeta_2] & \text{if } -C(w)(x) \in \{a, b\}, \\ = & 1 & \text{if } -C(w)(x) \in (a, b). \end{cases}$$

for some arbitrarily fixed $\zeta_1, \zeta_2 \in \mathbb{R}$ with $\zeta_1 \leq \zeta_2$. In what follows, we frequently use the particular selection $\widehat{\partial H} := \widehat{\partial H}^{0,0}$, which is obtained from (3.10) by setting $\zeta_1 = \zeta_2 = 0$.

A special choice for $H : L^{q_1}(\Omega) \to L^{q_1}(\Omega)$ is given by $w := v$ and

$$H(v) := \Psi(v) = v - P_{[y_a,y_b]}(y(v) - \frac{\varepsilon^2}{\alpha} p(v)) \quad \text{with } q_1 = 2, \tag{3.11}$$

which corresponds to our first order condition (3.4). As we shall see from Lemma 3.1 and 3.2 below, the operator

$$C(v) := \frac{\varepsilon^2}{\alpha} p(v) - y(v) = \frac{\varepsilon^2}{\alpha} p(v) - S(v)$$

satisfies the above requirements such that $\Psi : L^2(\Omega) \to L^2(\Omega)$ turns out to be Newton differentiable in $L^2(\Omega)$. For the state considered as a function of the control variable we have the following result.

LEMMA 3.1. *Let $h > 0$. Then the operators $G : L^2(\Omega) \to H_0^1(\Omega) \cap \mathcal{C}(\bar{\Omega})$ and $G_h : V_h \to Y_h$ are continuously Fréchet differentiable and Lipschitz continuous with Lipschitz constants independent of h. Moreover, for every bounded set $U \subset L^2(\Omega)$, the first Fréchet derivatives $G'(v)$ and $G_h'(v_h)$ are Lipschitz continuous on U and $U \cap V_h$, respectively, with a Lipschitz constant independent of h.*

Proof. In [20, Theorems 7 and 9] the assertion was proved for the semilinear elliptic PDE

$$\tilde{A}y + \tilde{d}(y) = v \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \Gamma,$$

where $\tilde{A}$ is a second order linear elliptic partial differential operator and $\tilde{d}$ is a function with the same properties with respect to the second argument as d from our context. The assertions in [20, Theorems 7 and 9] remain true as long as $\tilde{d} : \Omega \times \mathbb{R} \to \mathbb{R}$ has the same properties as d. Thus, setting $\tilde{A} = \varepsilon A + \mathrm{id}$ and $\tilde{d} = \varepsilon d$, we can apply [20, Theorems 7 and 9] in order to obtain the assertion. ∎

We have an analogous result when considering the adjoint state as a function of the control variable. For its proof, as for the previous lemma, we refer to [20, Theorem 10] with the analogous settings from the proof of Lemma 3.1.

LEMMA 3.2. *Let U be a bounded set in $L^2(\Omega)$ and $h > 0$. Then the adjoint states considered as function of the control v and its discrete counterpart v_h, i.e., $p : L^2(\Omega) \to H_0^1(\Omega)$ and $p_h : V_h \to Y_h$, are Lipschitz continuous and bounded on U and $U \cap V_h$, respectively, with a Lipschitz constant and a bound independent of h. Moreover, the first Fréchet derivatives $p'(v)$ and $p_h'(v)$ are Lipschitz continuous on U and $U \cap V_h$, respectively, with a Lipschitz constant independent of h.*

As noted above, we have that Ψ is semismooth, and a particular selection of Newton maps is given by

$$\widehat{\partial \Psi}^{\zeta_1,\zeta_2}(v) = \{\mathrm{id} + z(v)(-S'(v) + \frac{\varepsilon^2}{\alpha} p'(v))\}, \tag{3.12}$$

with $z : L^2(\Omega) \to L^\infty(\Omega)$ satisfying

$$z(v)(x) \begin{cases} = & 0 & \text{if } (S(v) - \frac{\varepsilon^2}{\alpha} p(v))(x) \notin [y_a, y_b], \\ \in & [\zeta_1, \zeta_2] & \text{if } (S(v) - \frac{\varepsilon^2}{\alpha} p(v))(x) \in \{y_a, y_b\}, \\ = & 1 & \text{if } (S(v) - \frac{\varepsilon^2}{\alpha} p(v))(x) \in [y_a, y_b]. \end{cases}$$

Analogously, we obtain that Ψ_h is semismooth, and a selection of Newton maps is given by

$$\widehat{\partial \Psi}_h^{\zeta_1,\zeta_2}(v_h) = \{\mathrm{id}_h + z_h(v_h)(-\Pi_h S_h'(v_h) + \frac{\varepsilon^2}{\alpha} \Pi_h p_h'(v_h)\}, \tag{3.13}$$

with $z_h : V_h \to V_h$ satisfying

$$z_h(v_h)(x) \begin{cases} = & 0 & \text{if } (S_h(v_h) - \frac{\varepsilon^2}{\alpha} p_h(v_h))(x) \notin [y_a, y_b], \\ \in & [\zeta_1, \zeta_2] & \text{if } (S_h(v_h) - \frac{\varepsilon^2}{\alpha} p_h(v_h))(x) \in \{y_a, y_b\}, \\ = & 1 & \text{if } (S_h(v_h) - \frac{\varepsilon^2}{\alpha} p_h(v_h))(x) \in [y_a, y_b]. \end{cases}$$

Now we have all the ingredients at hand for defining the semismooth Newton methods for solving

$$\Psi(v) = 0 \quad \text{and} \quad \Psi_h(v_h) = 0,$$

respectively.

ALGORITHM 3.1 (Semismooth Newton method; continuous version).

(i) Choose $v^0 \in L^2(\Omega)$ *and set* $k = 0$.

(ii) Compute $V^k \in \widehat{\partial\Psi}(v^k)$.

(iii) Find $v^{k+1} \in L^2(\Omega)$ *such that* $V^k(v^{k+1} - v^k) = -\Psi(v^k)$,

(iv) If $\Psi(v^{k+1}) = 0$ *then stop, else set* $k := k + 1$ *and return to (ii).*

We point out that in step (ii) we could have chosen $\partial\Psi(v^k)$ instead of $\widehat{\partial\Psi}(v^k)$. Then the following theory would still hold true. Our choice in step (ii), however, is motivated by the fact that in this case the semismooth Newton algorithm for solving $\Psi(v) = 0$ becomes a primal-dual active-set strategy, i.e., the special Newton map for the projection operator implies an active and inactive set prediction. In this context we call $\{x \in \Omega : v^*(x) \in (y_a, y_b)\}$ the inactive set at a local solution v^* of (P^v). Its complement in Ω is called the active set at v^*. The primal-dual active-set method has several numerical advantages including the fact that in every Newton step a linear system has to be solved only on the currently inactive set. To explain this in more detail, note that the generalized linearization of (3.11) can be decomposed according to the following partition of Ω into the active and inactive set estimates $\mathcal{A}^k = \mathcal{A}_a^k \cup \mathcal{A}_b^k$ and $\mathcal{I}^k$:

$$\begin{aligned}
\mathcal{A}_a^k &:= \{x \in \Omega : -C(v^k)(x) < y_a\}, \\
\mathcal{A}_b^k &:= \{x \in \Omega : -C(v^k)(x) > y_b\}, \\
\mathcal{I}^k &:= \Omega \setminus \mathcal{A}^k.
\end{aligned}$$

Then, due to the structure of the mapping Ψ and the definition of $\widehat{\partial\Psi}(v^k)$, we infer

$$v^{k+1} = y_a \text{ a.e. on } \mathcal{A}_a^k, \quad v^{k+1} = y_b \text{ a.e. on } \mathcal{A}_b^k$$

and further

$$v^{k+1} + C'(v^k)(v^{k+1} - v^k) = -C(v^k) \text{ on } \mathcal{I}^k.$$

Hence, on the active set estimate $\mathcal{A}^k$ the new control v^{k+1} is obtained by setting the variable to either the upper or lower bound. On the current approximation of the inactive set a linear equation has to be solved. For more details on the active-set technique, we refer the reader to [17].

Analogously, we solve the discrete problem by the following discrete semismooth Newton method.

ALGORITHM 3.2 (Semismooth Newton method; discrete version).

(i) Choose $v_h^0 \in V_h$ *and set* $k = 0$.

(ii) Compute $V_h^k \in \widehat{\partial\Psi}_h(v_h^k)$.

(iii) Find $v_h^{k+1} \in V_h$ *such that* $V_h^k(v_h^{k+1} - v_h^k) = -\Psi_h(v_h^k)$,

(iv) If $\Psi_h(v_h^{k+1}) = 0$ *then stop, else set* $k := k + 1$ *and return to (ii).*

Of course, the stopping conditions $\Psi(v^{k+1}) = 0$ and $\Psi_h(v_h^{k+1}) = 0$ of Algorithm 3.1 and 3.2, respectively, are of theoretical importance only. In our numerical practice we stop Algorithm 3.2 as soon as $\|\Psi_h(v_h^{k+1})\|_{L^2}$ drops below a user-specified stopping tolerance.

Concerning the local convergence of Algorithm 3.1 and Algorithm 3.2, respectively, we remark that a straightforward application of the convergence result [17, Theorem 1.1] yields the locally superlinear convergence provided the Newton maps are invertible with uniformly bounded inverses in a neighborhood of local solutions to $\Psi(v)=0$ and $\Psi_h(v_h)=0$, respectively.

We end this section by providing a sufficient condition which guarantees uniformly bounded inverses of the Newton maps in $\widehat{\partial\Psi}(v)$ for all v in a sufficiently small neighborhood of a local solution v^* of $\Psi(v)=0$. In fact, suppose that (P^v) satisfies the following second order sufficient condition.

Definition 3.3. *For a local solution v^* of $\Psi(v)=0$, $v\in L^2(\Omega)$, there exists a constant $\hat\kappa>0$ such that*

$$f''(v^*)w^2 \geq \hat\kappa\|w\|_{L^2}^2 \quad \text{for all } w\in L^2(\Omega). \tag{3.14}$$

This condition is stronger than the second order sufficient condition of Definition 3.1. We use this stronger version to have its stability with respect to L^2-perturbations of v^*; see (3.18) below. Note that due to (3.1) we have

$$f''(v^*)w^2 = (w,[p'(v^*)-\frac{\alpha}{\varepsilon^2}S'(v^*)+\frac{\alpha}{\varepsilon^2}\,\mathrm{id}]w)_{L^2}. \tag{3.15}$$

Assuming that (3.14) holds true, then after multiplying (3.15) by $\alpha^{-1}\varepsilon^2$ we obtain

$$(w,[\frac{\varepsilon^2}{\alpha}p'(v^*)-S'(v^*)+\mathrm{id}]w)_{L^2} \geq \frac{\varepsilon^2\hat\kappa}{\alpha}\|w\|_{L^2}^2 \quad \forall w\in L^2(\Omega). \tag{3.16}$$

Hence, the left hand side in (3.16) induces a positive definite bilinear form on $L^2(\Omega)\times L^2(\Omega)$. From [28, Lemma 4.24] we infer

$$|(f''(v^*)-f''(v))w^2| \leq c\|v-v^*\|_{L^2}\|w\|_{L^2}^2 \quad \forall w\in L^2(\Omega) \tag{3.17}$$

for all $v\in L^2(\Omega)$ sufficiently close to $v^*\in L^2(\Omega)$. Here, c denotes a positive constant. We point out that general problems require a L^2-L^∞-norm gap, i.e., $v^*,v\in L^\infty(\Omega)$ and, hence, $\|v-v^*\|_{L^2}$ replaced by $\|v-v^*\|_{L^\infty}$ in (3.17). This is due to the fact that we cannot expect twice differentiability of f in L^2 in these cases; see [21] or [28, Lemma 4.24]. In our context, however, it is well known that the solution operators of the state as well as the adjoint equation are continuous from $L^2(\Omega)$ to $\mathcal{C}(\bar\Omega)$ for $\Omega\subset\mathbb{R}^n$ with $n<4$. As a result, here we can work with (3.17). In conclusion we have that there exist a radius $r>0$ and a constant $\kappa_r>0$ such that for all $v\in\{v\in L^2(\Omega):\|v-v^*\|_{L^2}\leq r\}$ the following condition holds true:

$$(w,[\frac{\varepsilon^2}{\alpha}p'(v)-S'(v)+\mathrm{id}]w)_{L^2} \geq \kappa_r\|w\|_{L^2}^2 \quad \forall w\in L^2(\Omega). \tag{3.18}$$

Therefore, from the structure of the Newton maps of $\widehat\Psi(v)$ (cf. (3.12) with $\zeta_1=\zeta_2=0$) and the Lax-Milgram lemma we infer that in a neighborhood of a local solution v^* satisfying the second order conditions in Definition 3.3 step (iii) of Algorithm 3.1 is well-defined, i.e., the Newton maps are uniformly invertible in a neighborhood of v^*.

Finally, the properties of our discretization (see also (A1)–(A3) in the next section) imply that for sufficiently small mesh sizes h step (iii) of the discrete method, Algorithm 3.2, is also well-defined in a neighborhood of a local solution v_h of the discrete problem $\Psi_h(v_h)=0$ satisfying $\|v_h-v^*\|_{L^2}\to 0$ as $h\to 0$. This, again, yields uniformly bounded (also with respect to h) inverses of the discrete Newton maps in a sufficiently small neighborhood of v_h.

4. Mesh-independence principle. In [20] the mesh-independence of semismooth Newton methods for solving control constrained optimal control problems of semilinear elliptic PDEs was proven. Due to the transformation of the mixed control-state constrained problem (P) into a purely control constrained problem (P^v), for proving the mesh-independence of Algorithm 3.2 we can exploit the results of [20]. However, let us point out that our transformed problem is more complicated since the nonlinear solution operator occurs in both terms of the objective function.

For the subsequent discussion let v^* be a local solution of (P^v) that satisfies the second order sufficient optimality condition given in Definition 3.1. Then, following [4], we can find a sequence $(v_h)_{h>0}$ of locally optimal controls for (P^v_h) converging to the local solution v^* as h tends to zero. In what follows, we consider only this sequence $(v_h)_{h>0}$. Furthermore, we impose the following assumption:

$$\text{(SC)} \qquad \begin{aligned} &\operatorname{meas}\{x \in \Omega \mid y_a - y^*(x) + \tfrac{\varepsilon^2}{\alpha} p^*(x) = 0\} = 0, \\ &\operatorname{meas}\{x \in \Omega \mid y^*(x) - \tfrac{\varepsilon^2}{\alpha} p^*(x) - y_b = 0\} = 0. \end{aligned}$$

Observe that (SC) requires v^* to satisfy a strict complementarity condition, i.e., on the set where v^* hits one of the bounds y_a or y_b, the quantity $p^* + \alpha\,\varepsilon^{-2}(v^* - y^*)$, which is related to the Lagrange multiplier for the pointwise control constraint in (P^v), can only vanish on a set of measure zero. In fact, for the first set in (SC) note that $v^*(x) = y_a$ implies

$$p^*(x) + \frac{\alpha}{\varepsilon^2}(y_a - y^*(x)) = 0 \quad \Leftrightarrow \quad y_a - y^*(x) + \frac{\varepsilon^2}{\alpha} p^*(x) = 0,$$

which corresponds to the definition of the first set in (SC). A similar reasoning yields the second set.

The mesh-independence result in [20] relies on the following assumptions: For some $p > 2$ there holds

- (A_1) $\lim_{h\to 0} \|C_h(v_h) - C(v^*)\|_{L^p} = 0$;
- (A_2) the discretization family is *locally Lipschitz uniform*, i.e., there exist positive real numbers h_0, δ_0 and L_C such that

  $$\|C(v^1) - C(v^2)\|_{L^p} \le L_C \|v^1 - v^2\|_{L^2} \ \forall v^1, v^2 \in L^2(\Omega) \text{ with } \|v^1 - v^2\|_{L^2} \le \delta_0$$

  and

  $$\|C_h(v_h^1) - C_h(v_h^2)\|_{L^p} \le L_C \|v_h^1 - v_h^2\|_{L^2} \ \forall v_h^1, v_h^2 \in V_h \text{ with } \|v_h^1 - v_h^2\|_{L^2} \le \delta_0,$$

  for all $h \le h_0$;
- (A_3) the discretization family has the *uniform linear approximation property*, i.e. C and C_h, $h \le h_0$, are Fréchet differentiable in a neighborhood of v^* and v_h, respectively, and there exists a function $\rho : [0, \delta_0) \to [0, \infty)$ such that

  $$\begin{aligned} &\lim_{t\to 0} \frac{\rho(t)}{t} = 0, \\ &\|C(v) - C(v^*) - C'(v)(v - v^*)\|_{L^2} \le \rho(\|v - v^*\|_{L^2}) \\ &\qquad\qquad \forall v \in L^2(\Omega) \text{ with } \|v - v^*\|_{L^2} \le \rho_0, \\ &\|C_h(v_h) - C_h(v_h^*) - C_h'(v_h)(v_h - v_h^*)\|_{L^2} \le \rho(\|v_h - v_h^*\|_{L^2}) \\ &\qquad\qquad \forall v_h \in L^2(\Omega) \text{ with } \|v_h - v_h^*\|_{L^2} \le \rho_0, h \le h_0. \end{aligned}$$

Under these assumptions, in [20] the following mesh-independence result for the Algorithms 3.1 and 3.2 was shown.

THEOREM 4.1. *Let the assumptions (SC) and* $(A1)$–$(A3)$ *be satisfied. Further suppose that there exist* $\delta > 0$, $\kappa_V > 0$ *and* $h_1 \leq h_0$ *such that*

$$\sup\{\|V_h^{-1}\|_{L^2\to L^2} | V_h \in \widehat{\partial\Psi}(v_h + s_h) \text{ with } \|s_h\|_{L^2} \leq \delta\} \leq \kappa_V \quad \forall h \leq h_1. \tag{4.1}$$

Then, for any given $\theta \in (0,1)$, *there exist real numbers* $\bar{\delta} > 0$ *and* $\bar{h} > 0$ *such that the iterates of Algorithm 3.1 and Algorithm 3.2 satisfy*

$$\|v^{k+1} - v^*\|_{L^2} \leq \theta \|v^k - v^*\|_{L^2}, \tag{4.2}$$
$$\|v_h^{k+1} - v_h\|_{L^2} \leq \theta \|v_h^k - v_h\|_{L^2} \quad \text{for all } 0 < h \leq \bar{h} \tag{4.3}$$

provided that $\max\{\|v^0 - v^*\|_{L^2}, \|v_h^0 - v_h\|_{L^2}\} \leq \bar{\delta}$.

Our goal is now to verify the assumptions $(A1)$–$(A3)$ in the present context. Then, we can immediately apply Theorem 4.1 and, provided the solution to the continuous problem (P^v) satisfies (SC) (recall that (4.1) was discussed at the end of the previous section), we can conclude that Algorithm 3.2 satisfies the above mesh-independence principle. For this purpose we make use of the following result which was established in [4, Theorems 4.2 and 5.1]. Here and below, c typically denotes a generic positive constant which can take different values on different occasions.

THEOREM 4.2. *For sufficiently small* $h > 0$ *the following estimates holds true:*

$$\|y(v^*) - y_h(v_h)\|_{H^1} + \|p(v^*) - p_h(v_h)\|_{H^1} \leq c(h + \|v^* - v_h\|_{L^2}), \tag{4.4}$$
$$\|v^* - v_h\|_{L^2} \leq ch. \tag{4.5}$$

With these tools at hand, we can now verify assumptions $(A1)$–$(A3)$.

PROPOSITION 4.1. *Under the assumptions of section 2.1,* $(A1)$ *is satisfied.*

Proof. The assertion can easily be shown by utilizing the inequalities (2.2) and

(2.3) as well as Theorem 4.2. In fact, for sufficiently small $h > 0$ we have

$$
\begin{aligned}
\|C_h(v_h) - C(v^*)\|_{L^p} &= \|\Pi_h \frac{\varepsilon^2}{\alpha} p_h(v_h) - \Pi_h y_h(v_h) - \frac{\varepsilon^2}{\alpha} p(v^*) + y(v^*)\|_{L^p} \\
&\leq \frac{\varepsilon^2}{\alpha} \|\Pi_h(p_h(v_h) - p(v^*))\|_{L^p} + \|\Pi_h(y(v^*) - y_h(v_h))\|_{L^p} \\
&\quad + \|y(v^*) - \frac{\varepsilon^2}{\alpha} p(v^*) - \Pi_h(y(v^*) - \frac{\varepsilon^2}{\alpha} p(v^*))\|_{L^p} \\
&\leq \max\{1, \frac{\varepsilon^2}{\alpha}\}(\|p_h(v_h) - p(v^*)\|_{L^p} + \|y(v^*) - y_h(v_h)\|_{L^p}) \\
&\quad + \|y(v^*) - \frac{\varepsilon^2}{\alpha} p(v^*) - \Pi_h(y(v^*) - \frac{\varepsilon^2}{\alpha} p(v^*))\|_{L^2}^{\frac{2}{p}} \cdot \\
&\qquad \cdot \|y(v^*) - \frac{\varepsilon^2}{\alpha} p(v^*) - \Pi_h(y(v^*) - \frac{\varepsilon^2}{\alpha} p(v^*))\|_{L^\infty}^{\frac{p-2}{p}} \\
&\leq \max\{1, \frac{\varepsilon^2}{\alpha}\} s_p c(h + \|v^* - v_h\|_{L^2}) \\
&\quad + c h^{\frac{2}{p}} \|y(v^*) - \frac{\varepsilon^2}{\alpha} p(v^*)\|_{H^1}^{\frac{2}{p}} \|y(v^*) - \frac{\varepsilon^2}{\alpha} p(v^*) - \Pi_h(y(v^*) - \frac{\varepsilon^2}{\alpha} p(v^*))\|_{L^\infty}^{\frac{p-2}{p}} \\
&\leq c\, s_p \max\{1, \frac{\varepsilon^2}{\alpha}\}(h + ch) + c h^{\frac{2}{p}} \|y(v^*) - \frac{\varepsilon^2}{\alpha} p(v^*)\|_{H^1}^{\frac{2}{p}} \cdot \\
&\quad \cdot \|y(v^*) - \frac{\varepsilon^2}{\alpha} p(v^*) - \Pi_h(y(v^*) - \frac{\varepsilon^2}{\alpha} p(v^*))\|_{L^\infty}^{\frac{p-2}{p}},
\end{aligned}
$$

which converges to zero as $h \to 0$. Note that here we also use (2.2). $\square$

We continue with $(A2)$.

Proposition 4.2. *Under the assumptions of section 2.1,* $(A2)$ *is satisfied.*

Proof. We start by choosing a mesh-size $h_0 > 0$, a radius $\delta_0 > 0$ and a bounded set $U \subset L^2(\Omega)$ such that

$$
\begin{aligned}
&\|v - v^*\|_{L^2} \leq \delta_0 \Rightarrow v \in U, \text{ and} \\
&\|v - v_h\|_{V_h} \leq \delta_0 \Rightarrow v \in U \cap V_h \quad \forall\, 0 < h \leq h_0.
\end{aligned}
$$

Next, by Lemma 3.1 and Lemma 3.2, we can find a constant $c > 0$ independent of the choice of h such that for $j \in \{1, 2\}$

$$
\begin{aligned}
\|y(v^1) - y(v^2)\|_{H^1} &\leq c\|v^1 - v^2\|_{L^2} \quad \text{for all } v^j \in U, \\
\|y_h(v_h^1) - y_h(v_h^2)\|_{Y_h} &\leq c\|v_h^1 - v_h^2\|_{V_h} \quad \text{for all } v_h^j \in U \cap V_h, \\
\|p(v^1) - p(v^2)\|_{H^1} &\leq c\|v^1 - v^2\|_{L^2} \quad \text{for all } v^j \in U, \\
\|p_h(v_h^1) - p_h(v_h^2)\|_{Y_h} &\leq c\|v_h^1 - v_h^2\|_{V_h} \quad \text{for all } v_h^j \in U \cap V_h.
\end{aligned}
$$

Hence, for all $v^j \in U$, $v_h^j \in U \cap V_h$, $j \in \{1, 2\}$, and $0 < h \leq h_0$, we get

$$
\begin{aligned}
\|C(v^1) - C(v^2)\|_{L^p} &= \|\frac{\varepsilon^2}{\alpha} p(v^1) - y(v^1) - \frac{\varepsilon^2}{\alpha} p(v^2) + y(v^2)\|_{L^p} \\
&\leq s_p \frac{\varepsilon^2}{\alpha}(\|p(v^1) - p(v^2)\|_{H^1} + \|y(v^2) - y(v^1)\|_{H^1}) \\
&\leq s_p\, c(1 + \frac{\varepsilon^2}{\alpha})\|v^1 - v^2\|_{L^2}.
\end{aligned}
$$

Similarly, for the discrete setting we obtain

$$\begin{aligned}\|C_h(v_h^1) - C_h(v_h^2)\|_{L^p} &\le s_p\, \|\Pi_h(\frac{\varepsilon^2}{\alpha} p_h(v_h^1) - y_h(v_h^1) - \frac{\varepsilon^2}{\alpha} p_h(v_h^2) + y_h(v_h^2))\|_{Y_h} \\ &\le s_p\, c(1+\frac{\varepsilon^2}{\alpha})\|v_h^1 - v_h^2\|_{L^2},\end{aligned}$$

where we used (2.2) to obtain the last inequality above. □

Finally, we verify assumption $(A3)$.

PROPOSITION 4.3. *Under the assumptions of section 2.1,* $(A3)$ *is satisfied.*

Proof. For the verification of $(A3)$ we utilize the subset $U \subset L^2(\Omega)$ defined in the proof of the preceding proposition. Again, by Lemma 3.1 and Lemma 3.2, there exists a constant $c > 0$ independent of the choice of h such that for $j \in \{1,2\}$

$$\begin{aligned}\|S'(v^1) - S'(v^2)\|_{H^1} &\le c\|v^1 - v^2\|_{L^2} \quad \text{for all } v^j \in U,\\ \|S_h'(v_h^1) - S_h'(v_h^2)\|_{Y_h} &\le c\|v_h^1 - v_h^2\|_{V_h} \quad \text{for all } v_h^j \in U \cap V_h,\\ \|p'(v^1) - p'(v^2)\|_{H^1} &\le c\|v^1 - v^2\|_{L^2} \quad \text{for all } v^j \in U,\\ \|p_h'(v_h^1) - p_h'(v_h^2)\|_{Y_h} &\le c\|v_h^1 - v_h^2\|_{V_h} \quad \text{for all } v_h^j \in U \cap V_h.\end{aligned}$$

As a consequence, for arbitrary $v \in U$ and $v_h \in U \cap V_h$ we obtain

$$\begin{aligned}&\|C(v) - C(v^*) - C'(v)(v - v^*)\|_{L^2}\\ &= \|\int_0^1 (C'(v^* + t(v - v^*)) - C'(v))(v - v^*)\ dt\|_{L^2}\\ &\le \int_0^1 \|\frac{\varepsilon^2}{\alpha} p'(v^* + t(v-v^*)) - S'(v^* + t(v - v^*)) - \frac{\varepsilon^2}{\alpha} p'(v) + S'(v)\|_{L^2} \|v - v^*\|_{L^2} dt\\ &\le \frac{\varepsilon^2}{\alpha}\int_0^1 \|p'(v^* + t(v-v^*)) - p'(v)\|_{H^1}\|v - v^*\|_{L^2}\ dt\\ &\qquad + \int_0^1 \|S'(v) - S'(v^* + t(v - v^*))\|_{H^1}\|v - v^*\|_{L^2}\ dt\\ &\le (1+\frac{\varepsilon^2}{\alpha})c\int_0^1 (1-t)\|v - v^*\|_{L^2}^2\ dt\\ &= \frac{c}{2}(1+\frac{\varepsilon^2}{\alpha})\|v - v^*\|_{L^2}^2\\ &=: \rho(\|v - v^*\|_{L^2}).\end{aligned}$$

This yields $\rho(t) := \frac{c}{2}(1 + \alpha^{-1}\varepsilon^2)t^2$. By using (2.2), for the approximation in the discrete case a similar calculation can be carried out yielding the same choice for ρ with a possibly different constant c. Now, taking the larger of the two constants completes the proof. □

In conclusion, we have shown that $(A1)$–$(A3)$ hold true for (P^v) respectively (P_h^v). Hence, if v^* satisfies (SC), then, from Theorem 4.1, we conclude that the convergence behavior of Algorithm 3.2 is mesh-independent.

5. Passage to the limit $\varepsilon \to 0$. In the previous sections we focused on the investigation of (P) and its reformulation (P^v) for a fixed Lavrentiev-parameter $\varepsilon > 0$. Now we study the convergence behavior of the sequence of optimal controls $\{u_\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \downarrow 0$. For this purpose we find it convenient to refer to (P) as (P_ε) in order to make the dependence of the problem on ε transparent. The convergence for $\varepsilon \downarrow 0$ has not been studied for a semilinear state equation in the literature so far, and it is complicated by the involved nonlinearity and the mixing of control and state variables within the explicit inequality constraints. Our main result states that, as $\varepsilon \downarrow 0$, $\{u_\varepsilon\}$ converges strongly in $L^2(\Omega)$ to an optimal solution of the original unregularized state constrained problem

$$(P_0) \qquad \begin{cases} \text{minimize } J(u,y) := \frac{1}{2}\|y-y_d\|_{L^2}^2 + \frac{\alpha}{2}\|u\|_{L^2}^2 \\ \text{over } (u,y) \in L^2(\Omega) \times H_0^1(\Omega) \cap \mathcal{C}(\bar{\Omega}) \\ \text{subject to } Ay + d(\cdot, y) = u \quad \text{in } \Omega, \\ \qquad\qquad y_a \le y(x) \le y_b \quad \text{for almost all (f.a.a.) } x \in \Omega, \end{cases}$$

where we assume that $y_b \ge 0 \ge y_a$ and $y_b > y_a$. Moreover we suppose that $d(\cdot, 0) = 0$. Alternatively, the latter assumption can be replaced by the requirement that there exists a feasible point $(\hat{u}, \hat{y}) \in L^2(\Omega) \times H_0^1(\Omega) \cap \mathcal{C}(\bar{\Omega})$ for (P_ε) for all sufficiently small $\varepsilon \ge 0$.

In what follows, let $G_0 : L^2(\Omega) \to H_0^1(\Omega) \cap \mathcal{C}(\bar{\Omega})$ denote the solution operator of the state equation in (P_0) and $S_0 = \imath_0 G_0$. It can be shown that G_0 and $G_{0,h}$ satisfy the analogue of Lemma 3.1. Further, let $u^* \in L^2(\Omega)$ be an optimal solution of the unregularized problem (P_0).

Definition 5.1 (Linearized Slater condition). *We say that $u^* \in L^2(\Omega)$ satisfies the linearized Slater condition, if there exists a function $u_0 \in L^\infty(\Omega)$ such that*

$$y_a + \delta \le G_0(u^*) + G_0'(u^*)u_0 \le y_b - \delta \quad \forall x \in \Omega, \tag{5.1}$$

for some fixed $\delta > 0$.
Notice the L^2-L^∞-norm gap involved in the definition of the linearized Slater condition. It is necessitated by the pointwise constraint and will be of use in the proof of Theorem 5.1 below. Based on this definition, for the remainder of this section we invoke the following assumption.

Assumption 5.1. *There exists an optimal solution $u^* \in L^2(\Omega)$ of (P_0), which satisfies the linearized Slater condition. Let $u_0 \in L^\infty(\Omega)$ denote the corresponding interior (Slater) point.*

Now, let $\{\varepsilon_n\}$ be a sequence of positive numbers converging to zero. By $\{u_n\}$ we denote a sequence of global solutions associated to (P_{ε_n}). This sequence is bounded in $L^2(\Omega)$. In fact, due to our assumptions, $(\hat{y}, \hat{u}) = (0,0) \in H_0^1(\Omega) \times L^2(\Omega)$ is feasible for (P_{ε_n}) for all $n \in \mathbb{N}$ (and for (P_0) as well). Hence, we have

$$\frac{\alpha}{2}\|u_n\|_{L^2}^2 \le J(u_n, y_n) \le \frac{1}{2}\|y_d\|_{L^2}^2 \quad \forall n \in \mathbb{N},$$

where y_n denotes the state pertinent to u_n. Thus, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging weakly in $L^2(\Omega)$ to $\bar{u} \in L^2(\Omega)$. By a standard argument, we assume without loss of generality that $u_n \rightharpoonup \bar{u}$.

Lemma 5.1. *The weak limit $\bar{u}$ of $\{u_n\}$ is feasible for (P_0), i.e., it satisfies $y_a \le S_0(\bar{u}) \le y_b$ a.e. in Ω.*

Proof. First, standard arguments yield

$$S_0(u_n) \to S_0(\bar{u}) \quad \text{strongly in } L^2(\Omega) \text{ as } n \to \infty.$$

Since $\|u_n\|_{L^2}$ is uniformly bounded, we infer $\varepsilon_n\|u_n\|_{L^2} \to 0$ for $n \to \infty$. Hence, there exists a subsequence, which we denote–without loss of generality–again by $\varepsilon_n u_n$, such that $\varepsilon_n u_n \to 0$ a.e. in Ω. Since u_n solves (P_{ε_n}), we have

$$y_a \le \varepsilon_n u_n + S_0(u_n) \le y_b \quad \text{a.e. in } \Omega \quad \forall n \in \mathbb{N}.$$

From this we obtain

$$y_a \le S_0(\bar{u}) \le y_b \quad \text{a.e. in } \Omega.$$

This implies the feasibility of $(\bar{u}, \bar{y})$, with $\bar{y} = G_0(\bar{u}) \in H_0^1(\Omega) \cap \mathcal{C}(\bar{\Omega})$, for (P_0). ◻

Next we show that u_n converges strongly to $\bar{u}$, which turns out to be a global solution of the optimal control problem with pure state constraints.

THEOREM 5.1. *The sequence $\{u_n\}$ converges strongly in $L^2(\Omega)$ to $\bar{u}$, and $\bar{u}$ is a global solution of the unregularized state constrained problem (P_0).*

Proof. Since $\mathcal{C}_0(\bar{\Omega})$ is dense in $L^2(\Omega)$, there exists a sequence $\{u_t\}_{0<t<1}$ in $\mathcal{C}_0(\bar{\Omega})$ such that

$$\|u_t - u^*\|_{L^2} \le t \quad \forall\, 0 < t < 1. \tag{5.2}$$

Hence $\{u_t\}$ converges strongly in $L^2(\Omega)$ to u^* as $t \to 0$. Further, by the analogue of Lemma 3.1 there exists a real number $c_1 > 0$ such that

$$\|G_0'(u^*)(u_t - u^*)\|_{\mathcal{C}(\bar{\Omega})} \le c_1 \|u_t - u^*\|_{L^2} \le c_1 t \quad \forall\, 0 < t < 1. \tag{5.3}$$

Let us now define an auxiliary sequence $\{u_t^0\}_{0<t<1}$ in $L^\infty(\Omega)$ by

$$u_t^0 := u_t + \frac{3c_1}{\delta} t u_0, \quad 0 < t < 1, \tag{5.4}$$

where $u_0 \in L^\infty(\Omega)$ is the interior Slater point according to Definition 5.1. Due to (5.2), (5.4) implies

$$\|u_t^0 - u^*\|_{L^2} \le \|u_t - u^*\|_{L^2} + \frac{3c_1}{\delta} t \|u_0\|_{L^\infty} \le t \left(1 + \frac{3c_1}{\delta} \|u_0\|_{L^\infty}\right) \tag{5.5}$$

and, hence,

$$\lim_{t \to 0} \|u_t^0 - u^*\|_{L^2} = 0.$$

In the following we use $c_2 := 3c_1\delta^{-1}$. Our next goal is to show that for every $t \in (0, \bar{t}]$, with $\bar{t} = \min\{1, c_2^{-1}\}$, there exists $n_t \in \mathbb{N}$ such that u_t^0 is feasible for (P_{ε_n}) for all $n \ge n_t$. For this purpose consider for $0 < t \le \bar{t}$

$$\begin{aligned}
\varepsilon_n u_t^0 + G_0(u_t^0) &= \varepsilon_n u_t^0 + G_0(u^*) + G_0'(u^*)(u_t + c_2\, t\, u_0 - u^*) + o_{\mathcal{C}(\bar{\Omega})}(\|u_t^0 - u^*\|_{L^2}) \\
&= \varepsilon_n u_t^0 + c_2\, t(G_0(u^*) + G_0'(u^*)u_0) + (1 - c_2\, t)G_0(u^*) + G_0'(u^*)(u_t - u^*) \\
&\qquad + o_{\mathcal{C}(\bar{\Omega})}(\|u_t^0 - u^*\|_{L^2}) \\
&\le c_2\, t(y_b - \delta) + (1 - c_2\, t)y_b + \varepsilon_n \|u_t^0\|_{L^\infty} + c_1 t + o_{\mathcal{C}(\bar{\Omega})}(t) \\
&= y_b - 2c_1 t + \varepsilon_n \|u_t^0\|_{L^\infty} + o_{\mathcal{C}(\bar{\Omega})}(t) =: \kappa(t; \varepsilon_n)
\end{aligned}$$

as $t \to 0$, where we used $c_2\, \delta = 3c_1$, the feasibility of u^* and the linearized Slater condition. If necessary, we reduce $\bar{t} > 0$ such that $o_{\mathcal{C}(\bar{\Omega})}(t) \le c_1 t$ for all $0 < t \le \bar{t}$. Then we obtain

$$\kappa(t; \varepsilon_n) \le y_b - c_1 t + \varepsilon_n \|u_t^0\|_{L^\infty} \quad \forall\, 0 < t \le \bar{t}.$$

Since $\varepsilon_n \to 0$ for $n \to \infty$, there exists $n_t \in \mathbb{N}$ such that

$$\kappa(t; \varepsilon_n) \le y_b \quad \forall\, n \ge n_t.$$

This implies

$$\varepsilon_n u_t^0 + G_0(u_t^0) \le y_b \quad \forall\, n \ge n_t.$$

From an analogous argument we infer $\varepsilon_n u_t^0 + G_0(u_t^0) \ge y_a$ for all sufficiently large n. In conclusion we have shown that there exists $\bar{t} > 0$ such that for all $0 < t \le \bar{t}$ there exists $n_t \in \mathbb{N}$ such that

$$y_a \le \varepsilon_n u_t^0 + G_0(u_t^0) \le y_b \quad \forall\, n \ge n_t.$$

Since u_n is optimal for (P_{ε_n}), we have

$$J(u_n, y_n) \le J(u_t^0, y_t^0) \quad \forall\, n \ge n_t,$$

where $y_n = G_0(u_n)$ and $y_t^0 = G_0(u_t^0)$. Then, due to the weak lower semicontinuity of J, we find

$$J(\bar{u}, \bar{y}) \le \liminf_{n\to\infty} J(u_n, y_n) \le \limsup_{n\to\infty} J(u_n, y_n) \le J(u_t^0, y_t^0). \tag{5.6}$$

From this we obtain for $t \to 0$

$$J(\bar{u}, \bar{y}) \le \lim_{t\to 0} J(u_t^0, y_t^0) = J(u^*, y^*) \le J(\bar{u}, \bar{y}), \tag{5.7}$$

where $y^* = G_0(u^*)$. Hence, (5.6) yields

$$J(\bar{u}, \bar{y}) \le \liminf_{n\to\infty} J(u_n, y_n) \le \limsup_{n\to\infty} J(u_n, y_n) = J(\bar{u}, \bar{y}) = J(u^*, y^*) \tag{5.8}$$

and, thus, the optimality of $(\bar{u}, \bar{y})$ for (P_0). The strong convergence (in L^2) of u_n to $\bar{u}$ now follows from

$$\lim_{n\to\infty} \|S_0(u_n) - y_d\|_{L^2}^2 = \|S_0(\bar{u}) - y_d\|_{L^2}^2 \quad \text{and} \quad \lim_{n\to\infty} \|u_n\|_{L^2}^2 = \|\bar{u}\|_{L^2}^2,$$

which is due to (5.8), and the fact that weak convergence and convergence in norm imply the strong convergence of $\{u_n\}$ to $\bar{u}$ in $L^2(\Omega)$. ◻

6. Numerical experiments. Our goal in this section is threefold: (i) We verify our theoretical fast convergence and mesh-independence results numerically. (ii) Based on the nested iteration concept well known from multigrid methods, we investigate an acceleration technique for our algorithm. (iii) In the case where one is interested in solving the state constrained problem

$$(P_0) \qquad \begin{cases} \text{minimize } J(u,y) := \frac{1}{2}\|y - y_d\|_{L^2}^2 + \frac{\alpha}{2}\|u\|_{L^2}^2 \\ \text{over } (u,y) \in L^2(\Omega) \times H_0^1(\Omega) \cap \mathcal{C}(\bar{\Omega}) \\ \text{subject to } Ay + d(\cdot, y) = u \quad \text{in } \Omega, \\ \qquad\qquad y_a \le y(x) \le y_b \quad \text{f.a.a. } x \in \Omega, \end{cases}$$

based on the convergence result of Theorem 5.1 an efficient continuation procedure with respect to the Lavrentiev-parameter ε will be introduced.

For all computations reported on below we used $\Omega = (0,1)^2$ which was triangulated with a regular mesh of mesh-size h (Friedrichs-Keller triangulation). Further, in all test cases we chose $A = -\Delta$.

6.1. Mesh-independence. For the numerical validation of Theorem 4.1 we initialize the algorithm with $v_h^0 = 0, y_h^0 \equiv y_b$. The iterations are stopped as soon as $\|v_h^{k+1} - v_h^k\|_{L^2} \leq 10^{-8}$ holds true. Moreover, the reference optimal solution on a grid of mesh-size h is approximated by v_h, the solution obtained by our algorithm with a stopping tolerance of 10^{-14}. In the subsequent tables, for every k the quantity $\mathrm{res}_h^k = \|v_h^k - v_h\|_{L^2}$ represents the distance of the current iterate to the reference optimal solution. Further, for monitoring locally superlinear convergence we use

$$\theta_h^k = \frac{\|v_h^k - v_h\|_{L^2}}{\|v_h^{k-1} - v_h\|_{L^2}}.$$

We consider the following test problem.

Example 1. *The nonlinear function d is defined as*

$$d(x, y(x)) = y(x)^3 + \exp(10y(x)) + y(x),$$

and the desired state is given by

$$y_d(x) = \frac{\cos(\pi x_1)\cos(\pi x_2)\exp(x_1)}{2}.$$

The cost parameter is $\alpha = 10^{-4}$, and the bounds are $y_a = -10^{-2}$, $y_b = 0$. The Lavrentiev-parameter is $\varepsilon = 10^{-3}$.

Table 6.1 displays the required number of iterations for various mesh-sizes h. We clearly observe the mesh-independent behavior of our Algorithm 3.2. The fact that the algorithm always needs 6 iterations, regardless of the mesh-size of discretization, is known as strong mesh independence; see [2] for Newton's method in the case of smooth operator equations.

Table 6.1
Example 1. Number of iterations for various mesh-sizes h.

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
#It.	6	6	6	6	6	6

Table 6.2
Example 1. Stabilizing effect of $\mathrm{res}_h^k = \|v_h^k - v_h\|_{L^2}$ for decreasing mesh-sizes h (rows).

k	$\mathrm{res}_{1/8}^k$	$\mathrm{res}_{1/16}^k$	$\mathrm{res}_{1/32}^k$	$\mathrm{res}_{1/64}^k$	$\mathrm{res}_{1/128}^k$	$\mathrm{res}_{1/256}^k$
1	2.8413e-02	8.7660e-02	1.5073e-01	1.8255e-01	1.9271e-01	1.9543e-01
2	1.4956e-03	1.5224e-03	1.3222e-03	1.3423e-03	1.4368e-03	1.4891e-03
3	2.4336e-04	5.1859e-04	3.8196e-04	2.1770e-04	1.9034e-04	1.9798e-04
4	2.9861e-05	2.1424e-04	1.1693e-04	2.7537e-05	2.2058e-05	2.5520e-05
5	2.5470e-06	6.4071e-05	5.0923e-06	1.6677e-06	8.9492e-07	1.2724e-06
6	4.6975e-13	1.7286e-09	8.5543e-12	7.1109e-13	8.5501e-10	1.3787e-08

In the Tables 6.2 and 6.3 we provide a detailed insight into the convergence behavior of Algorithm 3.2. Each row in Table 6.2 presents the changes of the residuals res_h^k along the iteration sequence with respect to decreasing h. Following the rows, one detects a stabilizing effect as the mesh is refined. This fact reflects the asymptotic mesh-independence of the algorithm.

Finally, the mesh-independent locally superlinear convergence is shown in Table 6.3. For fixed h, it is known from finite dimensional approaches [14] that the method converges locally at a superlinear rate. This can be seen by following the columns in Table 6.3. The rows, however, indicate the stable behavior with respect to decreasing h. This effect is not contained in our theoretical result Theorem 4.1. Here, it numerically augments our theoretical findings.

Table 6.3
Example 1. Stable fast local convergence of Algorithm 3.2.

k	$\theta^k_{1/8}$	$\theta^k_{1/16}$	$\theta^k_{1/32}$	$\theta^k_{1/64}$	$\theta^k_{1/128}$	$\theta^k_{1/256}$
2	5.2638e-02	1.7368e-02	8.7717e-03	7.3530e-03	7.4557e-03	7.6194e-03
3	1.6271e-01	3.4063e-01	2.8889e-01	1.6218e-01	1.3248e-01	1.3295e-01
4	1.2270e-01	4.1312e-01	3.0613e-01	1.2649e-01	1.1588e-01	1.2890e-01
5	8.5296e-02	2.9907e-01	4.3551e-02	6.0561e-02	4.0572e-02	4.9859e-02
6	1.8443e-07	2.6979e-05	1.6798e-06	4.2639e-07	9.5541e-04	1.0835e-02

In Figure 6.1 we display the discrete optimal state y_h and the corresponding transformed control v_h, respectively, in the upper row. The lower row contains the adjoint state p_h and the optimal control u_h. The mesh-size is $h = 1/256$.

Next we study the behavior of Algorithm 3.2 for $\varepsilon = 10^{-3.5}$ (instead of $\varepsilon = 10^{-3}$ in the previous case). The number of iterations until successful termination is given in Table 6.4. As before, one can clearly observe the mesh-independent convergence of

Table 6.4
Example 1 with $\varepsilon = 10^{-3.5}$. Number of iterations for various mesh-sizes h.

h	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
#It.	9	12	11	11	10

our algorithm. The behavior of the residuals res_h^k for various mesh-sizes h is shown in the left plot of Figure 6.2 (semi-logarithmic scale on vertical axis). The results indicate a stabilizing effect as h is refined which reflects the mesh-independence of our method.

Table 6.5 contains the results for $\varepsilon = 10^{-4}$ analogous to the ones in Table 6.4. Again, we can observe the mesh-independence of the method. However, in contrast to the previous test runs the number of iterations is subject to some fluctuations. This effect can be explained by the increasing ill-conditioning for decreasing ε and the occurrence of the factor $\alpha/2\varepsilon^2$ in the objective function of the transformed problem $(\tilde{P^v})$ which also adversely affects the conditioning of the problem for small ε. The corresponding residuals res_h^k are depicted in the right plot of Figure 6.2.

Table 6.5
Example 1 with $\varepsilon = 10^{-4}$. Number of iterations for various mesh-sizes h.

h	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
#It.	9	16	23	21	19

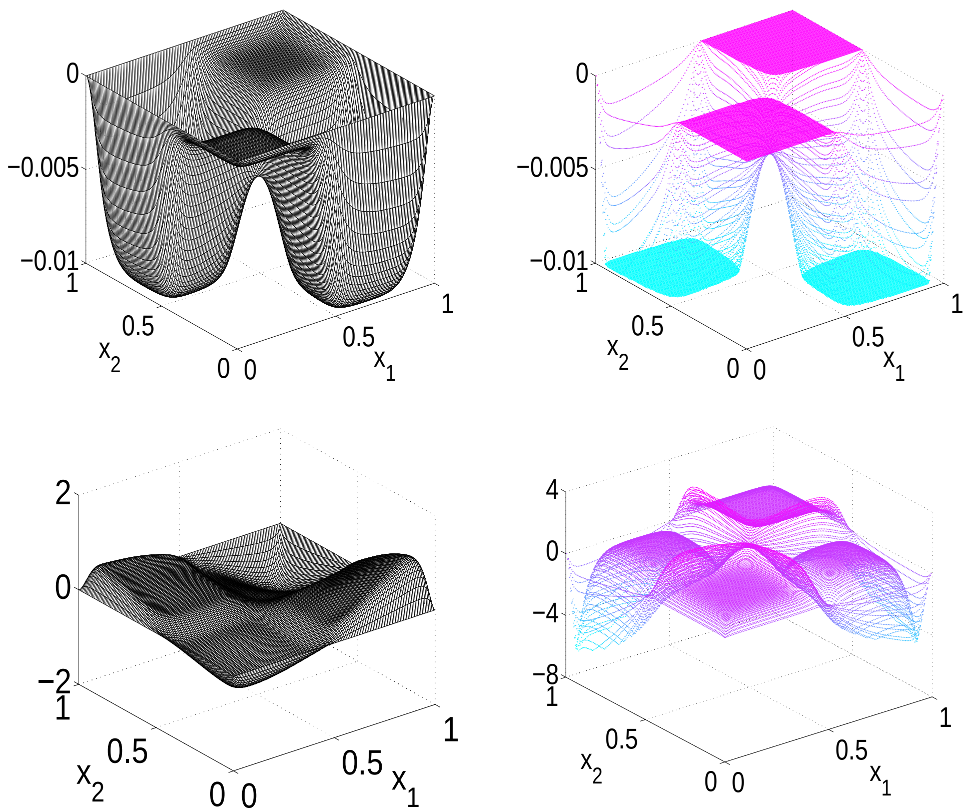

FIG. 6.1. *Example 1. Discrete solution for $h = 1/256$: Optimal state (upper left), optimal transformed control v_h (upper right), adjoint state (lower left) and optimal control u_h (lower right).*

6.2. Nested iteration. Next we briefly report on the speed-up of Algorithm 3.2 obtained by a coarse-to-fine grid sweep. By solving the problem on a coarse grid and using the prolongated solution as the initial guess on the fine grid, one expects to stay within or at least very close to the basin of fast local convergence of our semismooth Newton method. On the coarsest grid we initialize by setting $v_h^0 = 0$ and $y_h^0 \equiv y_b$. For the prolongation of the coarse grid solution y_H to the refined mesh with mesh-size $h < H$ we utilize the nine-point-prolongation scheme; see [15] for details. This yields the initial state y_h^0 on the refined mesh. From y_h^0 we then find v_h^0 by inserting y_h^0 in the state equation. In Table 6.6 we report on the result for the following problem.

Example 2. *We choose $d(y) = y^5$, $\alpha = 10^{-4.5}$, $\varepsilon = 10^{-4.5}$, $y_a = -10^{-2}$ and $y_b = 0$.*

The row entitled *Prolongation* in Table 6.6 shows the results when using Algorithm 3.2 within the nested iteration environment as described above. Below this row we display the CPU-time required on the respective grid. Under *Fixed mesh* we report on the number of iterations needed when the initial point on every grid is chosen as $v_h^0 = 0$ and $y_h^0 \equiv y_b$ (the qualitative behavior remains true for other initial choices). The next row, again, shows the CPU-time consumed on the respective grid. From the iteration count as well as from the CPU-time comparison, compared to the conventional run on a fixed mesh one finds a significant speed-up when combining

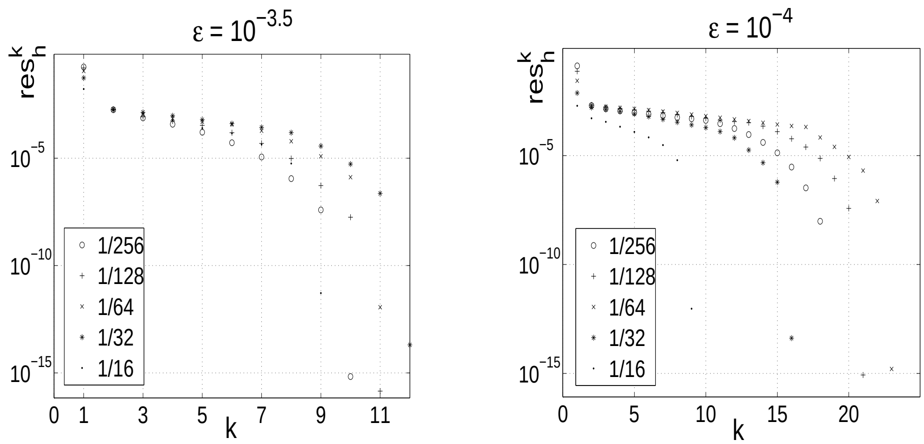

FIG. 6.2. *Example 1. Stabilizing behavior of* res_h^k *for* $\varepsilon = 10^{-3.5}$ *(left plot) and* $\varepsilon = 10^{-4}$ *(right plot) for various mesh-sizes h.*

Algorithm 3.2 with a coarse-to-fine mesh sweep.

TABLE 6.6
Example 2. Speed-up under a coarse-to-fine mesh sweep.

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
Prolongation	7	2	3	14	16	12
CPU-time	6.00e-02	9.00e-02	5.90e-01	1.13e+01	1.20e+02	1.48e+03
Fixed mesh	7	9	17	27	39	34
CPU-time	6.00e-02	2.70e-01	2.31e00	2.13e+01	2.60e+02	3.25e+03

6.3. Extrapolation. We are now concerned with the continuation as the Lavrentiev-parameter ε tends to zero. Hence, our goal is to solve the original state constrained optimal control problem (P_0). Since the regularized problems (P_ε) become increasingly more complicated as $\varepsilon \to 0$, we are interested in a stable and efficient continuation with respect to ε. For this purpose we propose here an extrapolation technique which, based on a solution of (P_ε) for $\varepsilon = \varepsilon_n$, aims at providing a good initial guess for the subsequent solution of the regularized problem $(P_{\varepsilon_{n+1}})$ with $\varepsilon_{n+1} < \varepsilon_n$. We point out that we will only examine linear extrapolation. Higher order techniques are conceivable.

In the following, assume that $y_n = y(\varepsilon_n)$ is the optimal state corresponding to (P_{ε_n}), and let $\varepsilon_{n+1} < \varepsilon_n$ denote the next Lavrentiev-parameter. Without loss of generality, we assume that ε_{n+1} is given by

$$\varepsilon_{n+1} = (1 - \kappa_n)\varepsilon_n \quad \text{with } 0 < \kappa_n < 1.$$

Assuming that $y(\varepsilon)$ varies smoothly with ε, the solution $y_{n+1} = y(\varepsilon_{n+1})$ is now approximated by

$$y_{n+1} \approx y_n + \frac{\kappa_n}{\kappa_{n-1}}(1 - \kappa_{n-1})(y_n - y_{n-1}). \tag{6.1}$$

Notice that (6.1) is obtained from the assumption that $y(\cdot)$ is smooth enough, the remainder term in a Taylor expansion of y about ε_n is small, and $\dot{y}(\varepsilon_n)$, the (directional) derivative of y with respect to ε evaluated at ε_n, is approximated by the difference quotient

$$\dot{y}(\varepsilon_n) \approx \frac{y(\varepsilon_{n+1}) - y(\varepsilon_n)}{\varepsilon_{n+1} - \varepsilon_n}.$$

We point out that the rigorous investigation of the regularity of $y(\varepsilon)$ as a function of ε goes beyond the scope of the present work. It is the subject of ongoing research.

The findings reported on below are based on the following test problem.

Example 3. *We choose the nonlinearity* $d(y) = y^3$, *the bounds* $y_a = -2.2$, $y_b = 4.5$, *the cost of the control* $\alpha = 10^{-2}$, *and the desired state* $y_d = \sin(2\pi x_1 x_2)\exp(7x_1)$.

The test runs are initialized with the feasible pair $(u_h^0, y_h^0) = (0, 0)$. In Table 6.7 we provide the results for various Lavrentiev-parameter choices and for various mesh-sizes h. As expected, the smaller ε becomes, the more challenging the solution of the respective problem becomes. This effect is more pronounced on fine meshes.

TABLE 6.7
Example 3. Number of iterations for several Lavrentiev-parameter choices and mesh-sizes h.

	# It. for various h			
ε	1/16	1/32	1/64	1/128
10^{-1}	8	8	8	8
10^{-2}	8	8	8	8
10^{-3}	12	15	15	15
10^{-4}	16	26	35	45
10^{-5}	17	31	58	89
10^{-6}	17	32	66	129
10^{-8}	17	30	68	130

In Figure 6.3 we depict the discrete optimal state y_h and the corresponding optimal control u_h for $h = 1/128$. We recall that as long as $\varepsilon > 0$ the results in [29] guar-

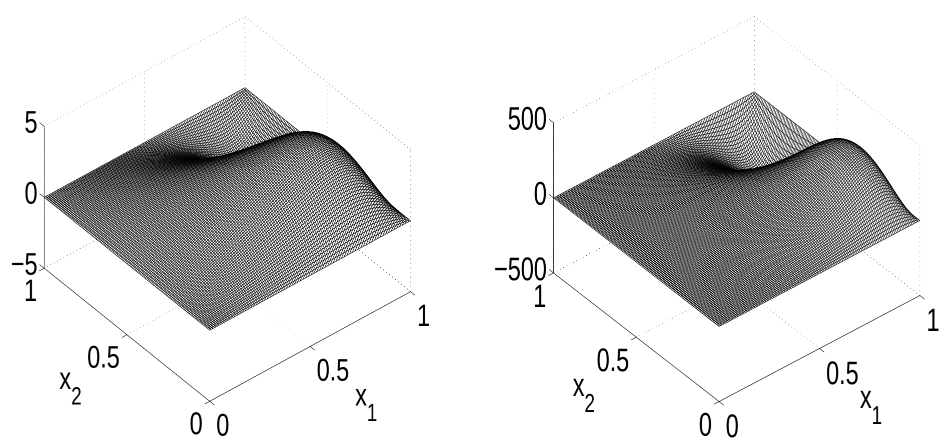

FIG. 6.3. *Example 3. Optimal state (left plot) and corresponding control u_h (right plot) for $h = 1/128$.*

antee that the Lagrange multiplier associate with the pointwise inequality constraints in (P_ε) is a function in $L^2(\Omega)$. For our test problem and due to our construction, as $\varepsilon \to 0$ the Lagrange multipliers for the upper and lower bound, respectively, approach a Dirac measure concentrated in a single point. This fact is shown in Figures 6.4–6.9 for the Lagrange multiplier associated with the lower bound.

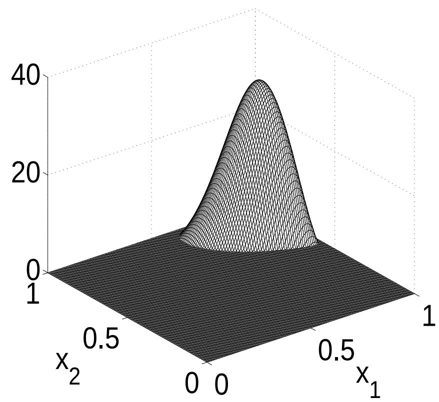

FIG. 6.4. $\varepsilon = 10^{-1}$.

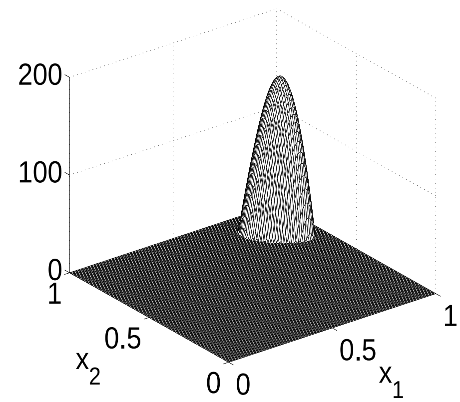

FIG. 6.5. $\varepsilon = 10^{-2}$.

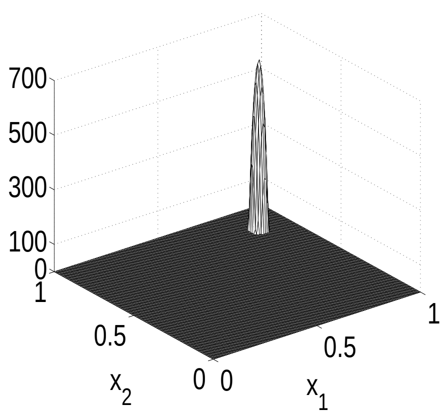

FIG. 6.6. $\varepsilon = 10^{-3}$.

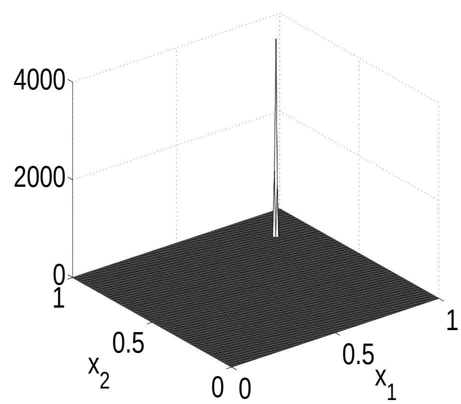

FIG. 6.7. $\varepsilon = 10^{-4}$.

From the result in Table 6.7 we find that for the initial choice (y_h^0, u_h^0) the performance of the Algorithm deteriorates for $\varepsilon \leq 10^{-3}$. Therefore, we employ our extrapolation technique in this case in order to benefit from improved initial guesses. We set $\kappa_n = 1 - 10^{-1.5}$ and $\varepsilon_0 = 10^{-2}$. The solution of (P_{ε_1}) is computed with initial guess $(y_{\varepsilon_0}, u_{\varepsilon_0})$, the solution of (P_{ε_0}). Table 6.8 displays the number of iterations required for successful termination of Algorithm 3.2. The results show that the extrapolation technique yields a significant acceleration of the solution process for (P_ε) with small Lavrentiev-parameter.

Finally, we are interested in combining the nested iteration concept with the extrapolation technique. For reasons of comparison, in Table 6.9 we provide the iteration numbers when solving Example 3 by solely using the nested iteration technique. Upon

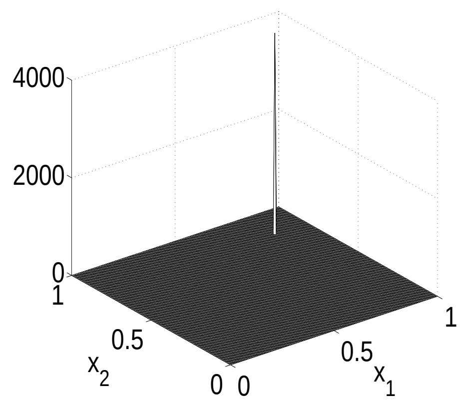

FIG. 6.8. $\varepsilon = 10^{-6}$.

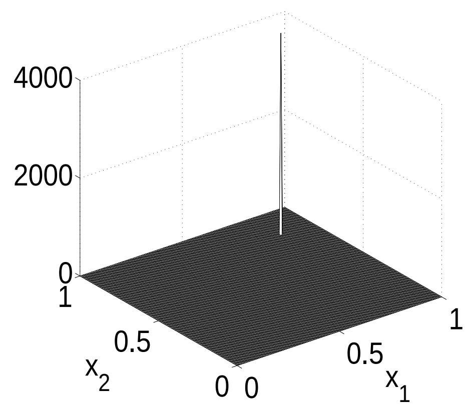

FIG. 6.9. $\varepsilon = 10^{-9}$.

TABLE 6.8
Example 3. Number of iterations for the extrapolation technique.

ε	# It. for various h			
	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
10^{-2}	8	8	8	8
$10^{-3.5}$	5	6	8	9
10^{-5}	2	3	5	8
$10^{-6.5}$	2	3	6	9
10^{-8}	1	3	6	10

comparing the results in Table 6.7 with the ones in Table 6.9, as for the test cases in section 6.2 we find that the nested iteration approach yields a significant speed-up of Algorithm 3.2. As noticed earlier, the regularized problems are the harder to solve the smaller ε becomes. Further, the effect is more pronounced on fine grids. Hence, it is of interest to combine the nested iteration with our extrapolation concept in order to benefit from both acceleration techniques. For this purpose, we invoke the following procedure: We start with $\varepsilon_0 = 10^{-2}$ on our coarsest grid ($h = 1/8$) and solve the regularized problem. Then we prolongate this solution to the next finer mesh ($h = 1/16$), reduce ε to $\varepsilon_1 = 10^{-3.5}$ and solve (P_{ε_1}). We extrapolate the solution for ε_1 and prolongate the result to the next finer mesh level ($h = 1/32$) where it serves as the initial guess for our algorithm when solving (P_{ε_2}) with $\varepsilon_2 = 10^{-5}$. The extrapolation and prolongation for obtaining initial guesses is repeated until the desired mesh size and Lavrentiev-parameter ε are reached. In Table 6.10 we provide the corresponding results. The design of the table emphasizes the diagonalization process by simultaneously tuning h and ε. Again, compared to the results in Table 6.7 we clearly detect a strong acceleration of Algorithm 3.2 by employing our combined nested iteration and extrapolation technique. Further, we can see that the pure nested iteration and the combined strategy require about the same number of iterations. With respect to the numerical stability of the computations, in our tests we find that the combined concept is advantageous over the pure nested iteration. This is due to the ill-conditioning of the problems because of small regularization parameters for the pure nested iteration concept.

TABLE 6.9
Example 3. Nested iteration technique for various Lavrentiev-parameters ε.

ε	h					
	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
10^{-2}	7	4	4	3	3	2
$10^{-3.5}$	7	5	4	4	5	4
10^{-5}	8	4	4	3	5	5
$10^{-6.5}$	8	4	4	3	6	5
10^{-8}	8	4	4	3	6	5

TABLE 6.10
Example 3. Combined extrapolation and nested iteration approach.

ε	h				
	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
10^{-2}	7	-	-	-	-
$10^{-3.5}$	-	5	-	-	-
10^{-5}	-	-	3	-	-
$10^{-6.5}$	-	-	-	6	-
10^{-8}	-	-	-	-	5

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