A stability theorem for linear-quadratic parabolic control problems

FREDI TRÖLTZSCH Faculty of Mathematics, Technical University of Chemnitz-Zwickau, D-09107 Chemnitz

1 INTRODUCTION

In the past years, Sequential Quadratic Programming (SQP) methods have shown to be very useful for solving optimal control problems. In Tröltzsch (1994 a,b) their convergence was proven for the application to simplified nonlinear parabolic control problems. To show similar results for more general classes of parabolic problems, certain stability results for linear-quadratic approximations of the nonlinear problem are needed. This note presents a stability estimate, which seems to be very useful for the convergence analysis of SQP methods.

To have an application in mind, let us regard the following nonlinear control problem:

(NP) Minimize

$$\begin{aligned} f(\theta, u) &= \int_\Omega \varphi(x, \theta(T,x))dx + \int\int_Q \Psi(t,x,\theta(t,x))dxdt \\ &\quad + \int\int_{S_T} \chi(t,x,\theta(t,x),u(t,x))dSdt \end{aligned} \tag{1.1}$$

subject to the nonlinear initial-boundary value problem

$$\begin{aligned} \theta_t(t,x) &= \Delta\theta(t,x) && \text{on } \Omega \\ \theta(0,x) &= \theta_o(x) && \text{on } \Omega \\ \partial\theta/\partial n &= b(t,x,\theta(t,x),u(t,x)) && \text{on } \Gamma, \end{aligned} \tag{1.2}$$

$t \in [0,T]$, and subject to the constraints on the *control* u,

$$b_1(t,x) \le u(t,x) \le b_2(t,x) \quad \text{on } S_T. \tag{1.3}$$

Here, $\Omega \subset I\!R^n$ is a bounded domain with sufficiently regular boundary S, $S_T = [0,T] \times S$, $Q = [0,T] \times \Omega$, $\theta_o \in C(\bar{\Omega})$, $\varphi = \varphi(x,\theta)$, $\Psi = \Psi(t,x,\theta)$, $\chi = \chi(t,x,\theta,u)$, $b = b(t,x,\theta,u)$ are real-valued functions, twice continuously partially differentiable with respect to θ and u. Moreover, bounded and measurable real-valued functions b_1, b_2 are given on S_T, such that $b_1(t,x) \le b_2(t,x)$ a.e. on S_T.

2 OPTIMALITY CONDITIONS

In the paper, the following notations are used: $\|\cdot\|_{p,M} := \|\cdot\|_{L_p(M)}$, $\|(x,y)\|_{p,M} :=$ $:= \|x\|_{p,M} + \|y\|_{p,M}$, $\|(\theta,u)\|_{V,p} := \|\theta(T)\|_{p,\Omega} + \|\theta\|_{p,Q} + \|\theta\|_{p,S_T} + \|u\|_{p,S_T}$ for $(\theta,u) \in C(\bar{Q}) \times L_\infty(S_T)$. For certain perturbations π we define a norm $\|\pi\|_p$ in section 3. The natural inner product of $L_2(M)$ is written $(\cdot\,;\,\cdot)_M$. Moreover, we introduce the space $W(0,T) = \{\theta \in L_2(0,T;H^1(\Omega)) : \theta_t \in L_2(0,T;H^1(\Omega)^*)\}$ endowed with its natural norm (cf. Lions (1968)). In $W(0,T) \times L_2(S_T)$ we introduce $\|(\theta,u)\|_{W,2}^2 = \|\theta\|_{W(0,T)}^2 + \|u\|_{2,S_T}^2$.

Let a bilinear form a be defined on $H^1(\Omega)^2$ by

$$a(\theta,\eta) = (\nabla\theta\,;\,\nabla\eta)_\Omega.$$

The *Lagrange function* associated to (NP) is

$$\mathcal{L}(\theta,u,y) = f(\theta,u) - \int_0^T [(\theta_t\,;\,y)_\Omega + a(\theta,y)]\,dt + \int_0^T (b(\cdot,\theta,u)\,;\,y)_S\,dt.$$

This function is defined and twice differentiable with respect to (θ,u) in a certain sense. We shall not discuss this and refer to Tröltzsch (1994 a). The linear parts of $\mathcal{L}$ are differentiable as a linear function on $W(0,T)$, while its nonlinear part can be formally differentiated with respect to θ and u in $C(\bar{Q}) \times L_\infty(S_T)$ according to the rules of differentiating Nemytski operators.

Let $\bar{v} = (\bar{\theta},\bar{u})$ denote a fixed locally optimal reference pair for (NP). The standard first order necessary optimality conditions are

$$\mathcal{L}_\theta(\bar{\theta},\bar{u},\bar{y}) = 0 \qquad (2.1)$$

$$\mathcal{L}_u(\bar{\theta},\bar{u},\bar{y})(u-\bar{u}) \ge 0 \qquad \forall u \in U^{ad}, \qquad (2.2)$$

where $U^{ad} \subset L_\infty(S_T)$ is defined according to (1.3). We shall denote derivatives of φ, Ψ, etc. at the pair $\bar{v}$ by a bar and by according subscripts, for instance $\bar{\varphi}_{\theta\theta}(x) = \frac{\partial^2}{\partial\theta^2}\varphi(x,\bar{\theta}(T,x))$ etc. With this notation, formula (2.1), the *adjoint equation* for $y = \bar{y}$, reads

$$\begin{aligned} -y_t &= \Delta y + \bar{\Psi}_\theta \\ y(T) &= \bar{\varphi}_\theta \\ \partial y/\partial n &= \bar{b}_\theta\, y + \bar{\Psi}_\theta. \end{aligned} \qquad (2.3)$$

The derivative $\mathcal{L}_u$ can be identified with a function of $L_\infty(S_T)$, which is denoted by $\mathcal{L}_u(t,x)$. We have

$$\mathcal{L}_u(t,x) = \bar{\chi}_u(t,x) + \bar{b}_u(t,x)\,\bar{y}(t,x).$$

Thus the first order condition (2.2) implies $\bar{u}(t,x) = b_1(t,x)$, if $\mathcal{L}_u(t,x) > 0$, $\bar{u}(t,x) = b_2(t,x)$, if $\mathcal{L}_u(t,x) < 0$. Here, we mention also the form of $f'(\bar{v})v$, namely

$$f'(\bar{v})v = (\bar{\phi}_\theta\,;\,\theta(T))_\Omega + (\bar{\psi}_\theta\,;\,\theta)_Q + (\bar{\chi}_\theta\,;\,\theta)_{S_T} + (\bar{\chi}_u\,;\,u)_{S_T} \qquad (2.4)$$

for $v = (\theta, u) \in C(\bar{Q}) \times L_\infty(S_T)$. Moreover, we need the second order derivative $\mathcal{L}_{vv}(\bar{v}, \bar{y})[v_1, v_2] =: \bar{\mathcal{L}}_{vv}[v_1, v_2]$,

$$\bar{\mathcal{L}}_{vv}[v_1, v_2] := f_{vv}(\bar{v})[v_1, v_2] + (\bar{y}\,;\,\bar{b}_{vv}[v_1, v_2])_{S_T}, \tag{2.5}$$

where f_{vv} is defined through the expression

$$\begin{aligned} f_{vv}(\bar{v})[v_1, v_2] &= \int_\Omega \bar{\varphi}_{\theta\theta}(x)\theta_1(T,x)\theta_2(T,x)dx + \int\int_Q (\bar{\psi}_{\theta\theta}\theta_1\theta_2)(t,x)dxdt \\ &\quad + \int\int_{S_T} [\bar{\chi}_{\theta\theta}\theta_1\theta_2 + \bar{\chi}_{\theta u}(\theta_1 u_2 + \theta_2 u_1) + \bar{\chi}_{uu} u_1 u_2](t,x)dSdt \\ \bar{b}_{vv}[v_1, v_2] &= \bar{b}_{\theta\theta}\theta_1\theta_2 + \bar{b}_{\theta u}(\theta_1 u_2 + \theta_2 u_1) + \bar{b}_{uu} u_1 u_2. \end{aligned}$$

Let us take $\sigma > 0$ arbitrarily small but fixed and define $I_\sigma = \{(t,x) \in S_T : |\mathcal{L}_u(t,x)| > 0\}$. We assume the *sufficient second order condition* (SSC): There is a $\delta > 0$ such that

$$\mathcal{L}_{vv}(\bar{\theta}, \bar{u}, \bar{y})[v, v] \geq \delta \|u\|^2_{2,S_T} \tag{2.6}$$

for all $v = (\theta, u)$ with $u(t,x) = 0$ on I_σ and

$$\begin{aligned} \theta_t &= \Delta\theta \\ \theta(0) &= \theta_o \\ \partial\theta/\partial n &= \bar{b}_\theta \theta + \bar{b}_u u. \end{aligned} \tag{2.7}$$

The linear mapping $u \mapsto \theta$ is continuous from $L_2(S_T)$ to $W(0,T)$, hence (2.6) and (2.7) imply

$$\mathcal{L}_{vv}(\bar{\theta}, \bar{u}, \bar{y}) \geq \tilde{\delta}(\|\theta\|^2_{W(0,T)} + \|u\|^2_{2,S_T}) = \tilde{\delta}\|(\theta, u)\|^2_{W,2} \tag{2.8}$$

with some $\tilde{\delta} > 0$.

From now on we regard the following perturbed *linear-quadratic approximation* of (NP), which is related to the investigation of sequential quadratic programming methods: Let $\pi = (d, e) = (d_\Omega, d_Q, d_S, d_u, e) \in \Pi := C(\bar{\Omega}) \times L_\infty(Q) \times L_\infty(S_T)^3$ be a given perturbation.

(LQ_π) Minimize

$$F(v, d) = f'(\bar{v})(v - \bar{v}) + \frac{1}{2}\bar{\mathcal{L}}_{vv}[v - \bar{v}, v - \bar{v}] - d(v - \bar{v}) \tag{2.9}$$

subject to $v = (\theta, u)$ and

$$\begin{aligned} \theta_t &= \Delta\theta \\ \theta(0) &= \theta_o \\ \partial\theta/\partial n &= b(\bar{\theta}, \bar{u}) + \bar{b}_\theta\,(\theta - \bar{\theta}) + \bar{b}_u\,(u - \bar{u}) + e, \end{aligned} \tag{2.10}$$

$u \in U^{ad}$, and $u(t,x) = \bar{u}(t,x)$ for all $(t,x) \in I_\sigma$ (d, e are given fixed).

The linear functional d in (2.9) is defined by

$$d(v) = (d_\Omega\,;\,\theta(T))_\Omega + (d_Q\,;\,\theta)_Q + (d_S\,;\,\theta)_{S_T} + (d_u\,;\,u)_{S_T}.$$

The derivatives f' and $\mathcal{L}_{vv}$ are introduced formally through the expressions (2.4), (2.5). We do not have to discuss their sense. In view of our assumptions, the derivatives $\bar{b}_\theta$, $\bar{b}_{\theta\theta}$ etc. are bounded and measurable functions. The following result is a standard conclusion of strong convexity due to (SSC):

THEOREM 1 *For all $\pi \in \Pi$ the problem (LQ_π) admits a unique minimizer v_π.*

3 STABILITY THEOREM

First, we show a stability result with respect to the L_2-norm.

THEOREM 2 *Let perturbations $\pi_1,\ \pi_2\ \in \Pi$ be given, v_1, v_2 be the associated unique solutions of (LQ_π). There is a positive constant L, which does not depend on $\pi_1,\ \pi_2$ such that*

$$\|v_1 - v_2\|_{W,2} \leq L\,\|\pi_1 - \pi_2\|_2. \tag{3.1}$$

Before proving this theorem, we introduce the Lagrange function associated to (LQ_π) by

$$\begin{aligned}\mathcal{L}(\theta,u,y,\pi) &= F(\theta,u,d) - \int_0^T [(\theta_t\,;\,y)_\Omega + a(\theta,y)]\,dt \\ &\quad +(\bar{b} + \bar{b}_\theta\,(\theta-\bar{\theta}) + \bar{b}_u\,(u-\bar{u}) + e\,;\,y)_{S_T}.\end{aligned} \tag{3.2}$$

From $\mathcal{L}_\theta = 0$ the associated adjoint states $y_1\,, y_2$ are obtained,

$$\begin{aligned} -y_{i,t} &= \Delta y_i + \bar{\Psi}_{\theta\theta}\cdot(\theta_i - \bar{\theta}) \\ y_i(T) &= \bar{\phi}_\theta - d_{i,\Omega} + \bar{\phi}_{\theta\theta}\cdot(\theta_i(T) - \bar{\theta}(T)) \\ \partial y_i/\partial n &= \bar{b}_\theta\, y_i + \bar{\chi}_\theta - d_{i,S} + (\bar{\chi}_{\theta\theta} + \bar{y}\bar{b}_{\theta\theta})\cdot(\theta_i - \bar{\theta}) \\ &\quad +(\bar{\chi}_{\theta u} + \bar{y}\bar{b}_{\theta u})\cdot(u_i - \bar{u}),\end{aligned} \tag{3.3}$$

$i = 1, 2$.

Proof of Theorem 2: From the first order necessary condition,

$$\begin{aligned} 0 &\leq \mathcal{L}_v(\theta_i, u_i, y_i, \pi_i)(v - v_i) \\ &= (f'(\bar{v}) - d_i)(v - v_i) + \bar{\mathcal{L}}_{vv}[v_i - \bar{v}, v - v_i] \\ &\quad -\int_0^T \{(\theta - \theta_{i,t}\,;\,y_i)_\Omega + a(\theta - \theta_i, y_i) - (\bar{b}_\theta(\theta - \theta_i) + \bar{b}_u(u - u_i)\,;\,y_i)_S\}dt,\end{aligned} \tag{3.4}$$

$i = 1, 2$. We insert $v = v_2$ in the first and $v = v_1$ in the second variational inequality. After adding them, we arrive at

$$\begin{aligned} 0 &\leq (d_2 - d_1)(v_2 - v_1) + \bar{\mathcal{L}}_{vv}[v_1 - v_2, v_2 - v_1] \\ &\quad -\int_0^T \{(\theta_{2,t} - \theta_{1,t}\,;\,y_1 - y_2)_\Omega + a(y_1 - y_2, \theta_2 - \theta_1) \\ &\quad -(y_1 - y_2\,;\,\bar{b}_\theta(\theta_2 - \theta_1) + \bar{b}_u(u_2 - u_1))_S\}dt \\ &= (d_2 - d_1)(v_2 - v_1) - \bar{\mathcal{L}}_{vv}[v_1 - v_2, v_1 - v_2] - (e_2 - e_1\,;\,y_1 - y_2)_{S_T}.\end{aligned} \tag{3.5}$$

Now $\bar{\mathcal{L}}_{vv}$ is added on both sides of (3.5), and (SSC) applies to obtain

$$\begin{aligned} \delta\|v_1 - v_2\|^2_{W,2} &\leq c\|\pi_1 - \pi_2\|_2\,\|v_2 - v_1\|_{V,2} + \|e_2 - e_1\|_{2,S_T}\,\|y_1 - y_2\|_{2,S_T} \\ &\leq c\|\pi_1 - \pi_2\|_2\,\|v_1 - v_2\|_{W,2} + \|\pi_1 - \pi_2\|_{2,S_T}\,\|y_1 - y_2\|_{2,S_T},\end{aligned} \tag{3.6}$$

as $|d(v)| \leq c\|d\|_2\,\|v\|_{V,2}$ and $\|v\|_{V,2} \leq c\,\|v\|_{W,2}$. From (3.3), parabolic L_2-regularity, and the uniform boundedness of $\bar{\phi}_\theta, \bar{\Psi}_{\theta\theta}$ etc. we derive

$$\begin{aligned} \|y_1 - y_2\|_{2,S_T} &\leq c_1\,\|v_1 - v_2\|_{V,2} + c_2\,\|\pi_1 - \pi_2\|_2 \\ &\leq c_1\,\|v_1 - v_2\|_{W,2} + c_2\,\|\pi_1 - \pi_2\|_2,\end{aligned} \tag{3.7}$$

thus (3.6) implies

$$\delta \, \|v_1 - v_2\|^2_{W,2} \le \tilde{c}_1 \, \|\pi_1 - \pi_2\|_2 \, \|v_1 - v_2\|_{W,2} + \tilde{c}_2 \|\pi_1 - \pi_2\|^2_2. \tag{3.8}$$

If $\|\pi_1 - \pi_2\|_2 \le \|v_1 - v_2\|_{W,2}$, then (3.1) follows from (3.8). In the opposite case we have $\|v_1 - v_2\|_{W,2} \le 1 \cdot \|\pi_1 - \pi_2\|_2$. Therefore, (3.1) is an easy conclusion. □

The main aim of this paper is to derive (3.1) in the L_∞-norm. This can be done combining the full regularity of parabolic operators with a detailed discussion of necessary optimality conditions. In the proof of the next theorem, we shall employ regularity properties in domains with sufficiently smooth boundary S. They are based on the well known *variation of constants formula*

$$\theta(t) = S(t)\theta_o + \int_0^t AS(t-s)N(\bar{b}_\theta \, \theta(s) + \bar{b}_u \, u(s) + b(\bar{\theta}, \bar{u}))ds \tag{3.9}$$

and the estimates

$$\|AS(t)N\|_{L_p(S) \to W_p^\sigma(\Omega)} \le c \, t^{-(1-(\sigma'-\sigma)/2)} \tag{3.10}$$

$0 < \sigma < \sigma' < 1 + 1/p$,

$$\|S(t)\|_{L_p(\Omega) \to W_p^\sigma(\Omega)} \le c \, t^{-\sigma/2}, \tag{3.11}$$

where $A = -\Delta + I$, defined on $\{\theta \in W_p^2(\Omega) : \partial\theta/\partial n = 0 \text{ on } S\}$, N is the so-called Neumann operator and $S(t) := e^t \, exp(-At)$ in $L_p(\Omega)$. We refer for the details to Amann (1986). (the equation (2.7) can be transformed by $\theta(t) = e^t w(t)$ to $w_t = \Delta w - w$). Further, we make use of properties of the weakly singular integral operator K,

$$(Kz)(t) = \int_0^t k(t,s) z(s) \, ds, \tag{3.12}$$

where k is a continuous real function on $0 \le s < t \le T$, $|k(t,s)| \le c \, (t-s)^{-\alpha}$, $\alpha \in (0,1)$: K transforms continuously $L_p(0,T)$ into $L_{p'}(0,T)$, if

$$\frac{1}{p'} > \alpha - 1 + \frac{1}{p} \tag{3.13}$$

and $L_p(0,T)$ into $C[0,T]$, if

$$p > \frac{1}{1-\alpha}. \tag{3.14}$$

The following *generalized Legendre-Clebsch condition* is needed to derive the L_∞-estimate: There is a constant $\lambda > 0$ such that

$$\bar{\chi}_{uu}(t,x) + \bar{y}(t,x)\bar{b}_{uu}(t,x) \ge \lambda \quad \forall (t,x) \in S_T. \tag{3.15}$$

THEOREM 3 *Suppose that $\bar{v} = (\bar{\theta}, \bar{u})$ satisfies the first order conditions (2.1), (2.2) together with (SSC) and the generalized Legendre-Clebsch condition (3.15). Suppose further that the boundary S of Ω is sufficiently smooth and $\theta_o \in C(\bar{\Omega})$. Then there are positive constants ε_L, c_L such that*

$$\|v_1 - v_2\|_{C(\bar{Q}) \times L_\infty(S_T)} \le c_L \, \|\pi_1 - \pi_2\|_\infty \tag{3.16}$$

holds for arbitrary perturbations $\pi_i \in \Pi$ with $\|\pi_i\|_\infty < \varepsilon_L$, $i = 1,2$.

Proof: a) Preparations

We first mention the variational inequality $\mathcal{L}_u(\theta_i, u_i, y_i, \pi_i)(u - u_i) \geq 0$ for all $u \in U^{ad}$:

$$(\bar{\chi}_u - d_u + y_i \bar{b}_u + (\bar{\chi}_{\theta u} + \bar{y}\bar{b}_{\theta u})(\theta_i - \bar{\theta}) + (\bar{\chi}_{uu} + \bar{y}\bar{b}_{uu})(u_i - \bar{u})\,;\, u - u_i)_{S_T} \geq 0 \qquad (3.17)$$

for all $u \in U^{ad}$. A standard pointwise discussion yields

$$u_i(t,x) = P(t,x)(\bar{u} - \frac{\bar{\chi}_u - d_u + (\bar{\chi}_{\theta u} + \bar{y}\bar{b}_{\theta u})(\theta_i - \bar{\theta}) + \bar{b}_u y_i}{\bar{\chi}_{uu} + \bar{y}\bar{b}_{uu}})(t,x) \qquad (3.18)$$

with the projection operator $P(t,x) : I\!R \to [b_1(t,x), b_2(t,x)]$. Note that P is Lipschitz with constant 1. Moreover, $P(t,x)u(t,x)$ is measurable, if u is measurable.

b) $L_2(S_T)$-estimate $\to$ $L_p(S_T)$-estimate with some $p > 2$:

From (3.1),

$$\|u_1 - u_2\|_{2,S_T} \leq l\,\|\pi_1 - \pi_2\|_2\,. \qquad (3.19)$$

The difference of states $\theta_1 - \theta_2$ satisfies

$$\begin{aligned} (\theta_1 - \theta_2)_t &= \Delta(\theta_1 - \theta_2) \\ (\theta_1 - \theta_2)(0) &= 0 \\ \frac{\partial(\theta_1 - \theta_2)}{\partial n} - \bar{b}_\theta(\theta_1 - \theta_2) &= \bar{b}_u(u_1 - u_2) + (e_1 - e_2). \end{aligned} \qquad (3.20)$$

By parabolic regularity, right hand sides in the boundary condition contained in $L_2(S_T)$ are transformed continuously into $L_p(0,T; W_2^\sigma(\Omega))$, provided that

$$\frac{1}{p} > \frac{\sigma}{2} - \frac{1}{4} \qquad (3.21)$$

(cf. section 4). To achieve boundary data of $\theta_1 - \theta_2$ in $L_p(S_T)$ we need $W_2^\sigma(\Omega) \subset W_p^{\frac{1}{p}+\varepsilon}(\Omega)$ ($\varepsilon > 0$ arbitrarily small), hence we require according to the Sobolev embedding theorem

$$\sigma - \frac{n}{2} \geq \frac{1}{p} + \varepsilon - \frac{n}{p}\,. \qquad (3.22)$$

From (3.21), (3.22) the chain $2/p + 1/2 > \sigma > 1/p - n/p + n/2$ follows. It is satisfied, if

$$p < \frac{n+1}{n-1}\,. \qquad (3.23)$$

Summarizing up the first step,

$$\|\theta_1 - \theta_2\|_{p,S_T} \leq c\,(\|u_1 - u_2\|_{2,S_T} + \|e_1 - e_2\|_{2,S_T}) \qquad (3.24)$$

is obtained (in what follows, c denotes a generic constant). In the adjoint equation, the term $(\theta_1 - \theta_2)(T)$ must be defined in $L_p(\Omega)$, hence we need $\theta_1 - \theta_2 \in C([0,T], W_2^\sigma(\Omega))$. This holds true for $\sigma < 1/2$ (cf. section 4) and $W_2^\sigma(\Omega) \subset L_p(\Omega)$. The latter takes place, if $\sigma = 1/2 - \varepsilon \geq n/2 - n/p$, i.e.

$$p < 2\frac{n}{n-1}\,. \qquad (3.25)$$

We take $p = p_1 := 2 + 2/(n-1)$. Then (3.23), (3.25) are jointly satisfied, $p_1 > 2$, and

$$\|\theta_1 - \theta_2\|_{C([0,T],L_{p_1}(\Omega))} \le c\,(\|u_1 - u_2\|_{2,S_T} + \|e_1 - e_2\|_{2,S_T}) \tag{3.26}$$

holds. Note that our gain of smoothness is $\gamma = p_1 - 2 = 2/(n-1)$. Next we discuss $y_1 - y_2$. From (3.3),

$$\begin{aligned} -(y_1 - y_2)_t &= \Delta(y_1 - y_2) - (d_{1,Q} - d_{2,Q}) + \bar\psi_{\theta\theta} \cdot (\theta_1 - \theta_2) \\ (y_1 - y_2)(T) &= -(d_{1,\Omega} - d_{2,\Omega}) + \bar\phi_{\theta\theta} \cdot (\theta_1(T) - \theta_2(T)) \\ \tfrac{\partial(y_1 - y_2)}{\partial n} - \bar b_\theta(y_1 - y_2) &= -(d_{1,S} - d_{2,S}) + (\bar\chi_{\theta\theta} + \bar y \bar b_{\theta\theta})(\theta_1 - \theta_2) \\ &\quad + (\bar\chi_{\theta u} + \bar y \bar b_{\theta u})(u_1 - u_2). \end{aligned} \tag{3.27}$$

Let us write $y_1 - y_2 = y_Q + y_\Omega + y_S$, where y_Q belongs to the right hand side of the heat equation (3.27) with $y_Q(T) = 0, \partial y_Q/\partial n - \bar b_\theta y_Q = 0$, y_Ω solves the homogeneous heat equation in (3.27) with homogeneous boundary data, and y_S is the remaining part (belonging to the inhomogeneous boundary data). Parabolic regularity applies to show

$$\|y_\Omega\|_{L_{p_1}(0,T;W_{p_1}^{\frac{1}{p_1}+\varepsilon}(\Omega))} \le c\,(\|\bar\phi_{\theta\theta}\|_{\infty,\Omega}\|(\theta_1 - \theta_2)(T)\|_{p_1,\Omega} + \|d_{2,\Omega} - d_{2,\Omega}\|_{p_1,\Omega}). \tag{3.28}$$

In the same way,

$$\|y_Q\|_{L_{p_1}(0,T;W_{p_1}^{\frac{1}{p_1}+\varepsilon}(\Omega))} \le c\,(\|\theta_1 - \theta_2\|_{p_1,Q} + \|d_{1,Q} - d_{2,Q}\|_{p_1,Q}) \tag{3.29}$$

$$\begin{aligned} \|y_S\|_{L_{p_1}(0,T;W_{p_1}^{\frac{1}{p_1}+\varepsilon}(\Omega))} &\le c\,(\|\theta_1 - \theta_2\|_{p_1,S_T} + \|u_1 - u_2\|_{2,S_T} \\ &\quad + \|d_{1,S} - d_{2,S}\|_{p_1,S_T}) \end{aligned} \tag{3.30}$$

holds. (3.29) can be derived from the usual variation of constants formula for parabolic equations with inhomogeneous right hand side. Thus

$$\begin{aligned} \|y_1 - y_2\|_{p_1,S_T} \le\ & c\,(\|(\theta_1 - \theta_2)(T)\|_{p_1,\Omega} + \|\theta_1 - \theta_2\|_{p_1,Q} \\ & + \|\theta_1 - \theta_2\|_{p_1,S_T} + \|u_1 - u_2\|_{2,S_T} \\ & + \|\pi_1 - \pi_2\|_{p_1}). \end{aligned} \tag{3.31}$$

In view of (3.18), (3.15),

$$|(u_1 - u_2)(t,x)| \le \frac{1}{\lambda}(|d_{1,u} - d_{2,u}| + c|\theta_1 - \theta_2| + c|y_1 - y_2|)(t,x) \tag{3.32}$$

holds on S_T, hence

$$\|u_1 - u_2\|_{p_1,S_T} \le c\,(\|\pi_1 - \pi_2\|_{p_1} + \|\theta_1 - \theta_2\|_{p_1,S_T} + \|y_1 - y_2\|_{p_1,S_T}). \tag{3.33}$$

Now

$$\|u_1 - u_2\|_{p_1,S_T} \le c_1\,\|\pi_1 - \pi_2\|_{p_1} \tag{3.34}$$

follows from (3.24), (3.31), (3.26), (3.19), and $\|\cdot\|_{2;M} \le c\,\|\cdot\|_{p_1,M}$ for any set M.

c) $L_{p_k}(S_T)$-estimate $\to L_{p_{k+1}}(S_T)$-estimate for $p_{k+1} > p_k$

We have shown

$$\|u_1 - u_2\|_{p_k,S_T} \le c_k\,\|\pi_1 - \pi_2\|_{p_k} \tag{3.35}$$

for some $p_k \geq 2+\gamma$. This estimate yields together with the system (3.20)

$$\|\theta_1-\theta_2\|_{C([0,T],W^{\sigma}_{p_k}(\Omega))} \leq c\,\|\pi_1-\pi_2\|_{p_k}, \tag{3.36}$$

if

$$\sigma < 1-\frac{1}{p_k} \tag{3.37}$$

(cf. section 4). To get traces in $L_{p_{k+1}}(S_T)$, where $p_{k+1} > p_k$, we need $W^{\sigma}_{p_k}(\Omega) \subset W^{\frac{1}{p_{k+1}}+\varepsilon}_{p_{k+1}}(\Omega)$, hence $\sigma - n/p_k \geq 1/p_{k+1}+\varepsilon - n/p_{k+1}$, i.e.

$$\sigma > \frac{1-n}{p_{k+1}}+\frac{n}{p_k}. \tag{3.38}$$

Put $p_{k+1} = l\,p_k$. (3.37), (3.38) have a non-void intersection, if

$$p_k > n+1-\frac{n-1}{l}. \tag{3.39}$$

If $p_k \geq n+1$, then this holds for all $l>0$. In the case $p_k > n+1$ we are able to finish our bootstrapping procedure (cf. d)). Assume $p_k < n+1$. We know $p_k > 2$, hence (3.39) holds true for $l=1$, and we may satisfy it for some $l>1$. From (3.39),

$$l < \frac{n-1}{n+1-p_k}, \tag{3.40}$$

hence

$$\begin{aligned} p_{k+1}-p_k &= l\,p_k - p_k < \frac{(n-1)p_k}{n+1-p_k} - p_k \\ &= p_k\frac{p_k-2}{n+1-p_k}. \end{aligned} \tag{3.41}$$

On the other hand, the right end of this chain is greater than $2\gamma/(n+1-2)$, thus the optimal gain of smoothness would be $p_{k+1}-p_k = 2\gamma/(n-1)$. We may ensure at least the gain

$$p_{k+1}-p_k \geq \frac{\gamma}{n-1}. \tag{3.42}$$

In the case $p_k = n+1$ we may take l and thus p_{k+1} arbitrarily large. Continuing our bootstrapping process, from (3.36), (3.38),

$$\|\theta_1-\theta_2\|_{C([0,T],W^{\frac{1}{p_{k+1}}+\varepsilon}_{p_{k+1}}(\Omega))} \leq c\,\|\pi_1-\pi_2\|_{p_k}, \tag{3.43}$$

is found. Next we may improve (3.31) to

$$\begin{aligned} \|y_1-y_2\|_{p_{k+1},S_T} \leq\ & c\,(\|(\theta_1-\theta_2)(T)\|_{p_{k+1},\Omega} + \|\theta_1-\theta_2\|_{p_{k+1},Q} \\ &+\|\theta_1-\theta_2\|_{p_{k+1},S_T} + \|u_1-u_2\|_{p_k,S_T} \\ &+\|\pi_1-\pi_2\|_{p_{k+1},S_T}). \end{aligned} \tag{3.44}$$

Continuing as before, in view of (3.15), (3.18), (3.43), and (3.44)

$$\|u_1-u_2\|_{p_{k+1},S_T} \leq c_{k+1}\,\|\pi_1-\pi_2\|_{p_{k+1}}. \tag{3.45}$$

is derived.

d) $p_k > n+1$

This case is obtained after finitely many steps. In (3.37), all $\sigma < 1 - \frac{1}{n+1+\varepsilon}$ can be taken. The inequality $\sigma > n/p_k$ can be fulfilled in this case, ensuring $W^{\sigma}_{p_k}(\Omega)) \subset C(\bar{\Omega})$. In view of this, controls from $L_{p_k}(S_T)$ are transformed into the state-space $C([0,T], C(\bar{\Omega}))$. Now (3.36) admits the form

$$\|\theta_1 - \theta_2\|_{\infty,\bar{\Omega}} \leq c\,\|\pi_1 - \pi_2\|_{p_k} \leq c\,\|\pi_1 - \pi_2\|_{\infty}. \tag{3.46}$$

The estimation of $y_1 - y_2$ is along the lines of the preceding steps. Owing to the maximum principle (proved by means of the integral equation method),

$$\|y_\Omega\|_{C(\bar{Q})} \leq c\,(\|\bar{\phi}_{\theta\theta}\|_{\infty,\bar{\Omega}}\,\|(\theta_1 - \theta_2)(T)\|_{\infty,\bar{\Omega}} + \|d_{2,\Omega} - d_{2,\Omega}\|_{\infty,\bar{\Omega}}) \tag{3.47}$$

follows from (3.27), cf. (3.28). This leads to

$$\begin{aligned} \|y_1 - y_2\|_{\infty,S_T} \;\leq\;& c\,(\|(\theta_1 - \theta_2)(T)\|_{\infty,\bar{\Omega}} + \|\theta_1 - \theta_2\|_{\infty,Q} + \|\theta_1 - \theta_2\|_{\infty,S_T} \\ & + \|u_1 - u_2\|_{p_k,S_T} + \|\pi_1 - \pi_2\|_{\infty}). \end{aligned} \tag{3.48}$$

In view of (3.15) (3.32), (3.46), (3.48), (3.35)

$$\begin{aligned} \|u_1 - u_2\|_{\infty,S_T} &\leq c\,(\|\pi_1 - \pi_2\|_{\infty} + \|u_1 - u_2\|_{p_k,S_T}) \\ &\leq c\,(\|\pi_1 - \pi_2\|_{\infty} + \|\pi_1 - \pi_2\|_{p_k}) \\ &\leq c\,\|\pi_1 - \pi_2\|_{\infty} \end{aligned} \tag{3.49}$$

holds. (3.49), (3.46) yield the desired result. □

4 REMARKS

It can be shown that the second order condition (SSC) considering the active set of $\bar{u}$ is sufficient for local optimality of $\bar{u}$ for (NP) with respect to the topology of L_∞. This information was not needed to prove Theorem 2. The theorem expresses more or less the following simple fact: (SSC) implies local optimality of $\bar{u}$ for the problem (NP) with constraint $b_1 \leq u \leq b_2$, where b_1 and b_2 are redefined such that $b_1 = b_2 = \bar{u}$ on I_σ.

However, we are able to show more: The (global) solution v_π of (LQ_π) remains *locally* optimal for this problem (local in L_∞-sense), if the restriction $u(t,x) = \bar{u}(t,x)$ is deleted in (LQ_π), provided that $\|\pi\|_\infty$ is sufficiently small. Moreover, it is the only local minimizer in a sufficiently small L_∞-neighborhood of v_π. The radius of this neighborhood is uniform with respect to all sufficiently small perturbations π. Therefore, the main stability theorem 3 is a result concerning a set of local minimizers of (LQ_π) without the restricton $u(t,x) = \bar{u}(t,x)$ on I_σ. The discussion of these facts would go beyond the scope of this paper.

Let us comment some of the estimates used in the proof of Theorem 3. First, we derive (3.21): Apply (3.10) for $p = 2$ and $\sigma = 3/2 - 2\varepsilon$, where ε is arbitrarily small. Then

$$\begin{aligned} \|AS(t)N\|_{L_2(S)\to W_2^\sigma(\Omega)} &\leq c\,t^{-(1-(3/2-2\varepsilon-\sigma)/2)} \\ &= t^{-\alpha}, \end{aligned} \tag{4.1}$$

where $\alpha = 1 + \sigma/2 - 3/4 + \varepsilon$. According to the transformation property (3.13) of K, the (Bochner) integral operator

$$u(t) \mapsto \int_0^t AS(t-s)Nu(s)\,ds \tag{4.2}$$

transforms continuously $L_2(0,T;L_2(S))$ into $L_p(0,T;W_2^\sigma(\Omega))$, if

$$\frac{1}{p} > \alpha - 1 + \frac{1}{2} = \frac{\sigma}{2} - \frac{1}{4} + \varepsilon.$$

This inequality is satisfied for sufficiently small ε, if $1/p > \sigma/2 - 1/4$, i.e. (3.21).

Moreover, we comment (3.37). Regard the operator (4.2), defined on $L_{p_k}(0,T;L_{p_k}(S))$. Then (3.10) applies,

$$\|AS(t)N\|_{L_{p_k}(S)\to W^\sigma_{p_k}(\Omega)} \le c\,t^{-\alpha},$$

where $\alpha = 1 - (1 + 1/p_k - 2\varepsilon - \sigma)/2 = 1/2 - 1/(2p_k) + \sigma/2 + \varepsilon$. The operator (4.2) is continuous from $L_{p_k}(0,T;L_{p_k}(S))$ to $C([0,T],W^\sigma_{p_k}(\Omega))$, if

$$p_k > \frac{1}{1-\alpha}$$

(cf. (3.14)). This amounts to $1 - 1/p_k > \sigma$.

Finally, it should be mentioned that the theory holds true for any uniformly elliptic differential operator A instead of $-\Delta$, which allows to derive the estimates (3.10), (3.11).

REFERENCES

[1] H. Amann (1986). "Parabolic evolution equations with nonlinear boundary conditions", In *Proc. Sympos. Pure Math. 45*, Part I, Nonlinear Functional Analysis, pp. 17–27, (F.E. Browder, ed.).

[2] H. Goldberg and F. Tröltzsch (1993). Second order sufficient optimality conditions for a class of nonlinear parabolic boundary control problems, *SIAM J. Control and Optimization 31*: 1007–1025.

[3] J.L. Lions (1968). Contrôle optimal de systèmes gouvernès par des équations aux dérivées partielles, Dunod, Paris.

[4] F. Tröltzsch (1994). An SQP method for the optimal control of a nonlinear heat equation, *Control and Cybernetics 23*: 267–288.

[5] F. Tröltzsch. "Convergence of an SQP-method for a class of nonlinear parabolic boundary control problems", In *Control and Estimation of Distributed Parameter Systems: Nonlinear Phenomena*, ISNM Vol. 118, pp. 343–358 (W. Desch, F. Kappel, K. Kunisch, eds.), Birkhäuser , Basel 1994.