An interior point method for a parabolic optimal control problem with regularized pointwise state constraints

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Abstract A primal-dual interior point method for state-constrained parabolic optimal control problems is considered. By a Lavrentiev type regularization, the state constraints are transformed to mixed control-state constraints which, after a simple transformation, can be handled as control constraints. Existence and convergence of the central path are shown. Moreover, the convergence of a short step interior point algorithm is proven in a function space setting. The theoretical properties of the algorithm are confirmed by numerical examples.

Keywords Parabolic optimal control, pointwise state constraints, Lavrentiev type regularization, interior point method.

AMS-Subjclass 49M15, 49M37

1. Introduction

In this paper, we extend our investigations on interior point methods for elliptic state-constrained optimal control problems in [18] and [13] to the parabolic case.

The main difficulty of the numerical analysis of interior point methods for such problems is the lack of regularity of Lagrange multipliers associated with the state constraints. Therefore, it is helpful to improve the properties of the multipliers by suitable regularization techniques.

For instance, this task can be accomplished by discretization and subsequent application of interior point methods. We mention the work by Bergounioux et al. [1], who carefully compare the performance of primal-dual active set strategies and interior point methods for elliptic problems, Grund and Rösch [5], who solve such problems with maximum norm functional, and Maurer and Mittelmann [16], who handle several state-constrained elliptic control problems by standard interior point codes.

To consider the interior point algorithm in function space, we suggested in [18], [13] a Lavrentiev type regularization. The Lavrentiev regularization of elliptic problems was introduced in [14]. This method ensures regular Lagrange multipliers and preserves, in some sense, the structure of a state-constrained control problem. Moreover, compared with a direct application of interior point methods to state-constrained problems, the regularization improves the performance of the algorithm, [13].

In [26, 27], primal-dual interior point methods are analyzed for ODE problems in an infinite dimensional function space setting, and their computational realization by inexact pathfollowing methods has been suggested. In [18], this method is extended to the optimal control of linear elliptic PDEs with regularized pointwise state constraints, where the analysis is performed in L^∞-spaces. Nonlinear equations are considered in the recent paper [24]. In particular, the convergence of primal-dual interior point methods is shown in L^p-spaces with $p < \infty$ for the control-constrained case.

Today, there exist also several papers on the numerical analysis of interior point methods for parabolic optimal control problems. For instance, trust-region interior point techniques were considered by M. Ulbrich, S. Ulbrich, and Heinkenschloss [25] for the optimal control of semilinear parabolic equations in a function space setting. Affine-scaling interior-point methods are presented for semilinear parabolic boundary control in [23]. Sachs and Leibfritz [10, 9, 8] considered interior point methods in the context of SQP-methods for parabolic optimization problems.

In our paper, we are able to prove the convergence of a conceptual primal interior point method in function space. We confine ourselves to a problem with linear equation and an objective functional with observation at the final time. This seems to be more challenging in the analysis than functionals of tracking type.

The analysis is very similar to the one for the elliptic case that was discussed in [18]. Therefore, we concentrate on those parts of the proofs that need essential modifications for parabolic problems. For parts of the theory that are completely analogous to elliptic problems, we refer to [18].

In the parabolic case, the presence of pointwise state constraints causes stronger restrictions on the dimension of the spatial domain than for elliptic equations. We do not impose control constraints. Therefore, the natural control space is of type L^2. To derive first-order necessary optimality conditions of Karush-Kuhn-Tucker type, the state functions should be continuous. This restricts the theory to distributed problems in one-dimensional domains.

This obstacle is completely overcome by our Lavrentiev regularization, which is crucial for the analysis. After regularization, we obtain Lagrange multipliers for any dimension of the domain. Moreover, we do not need constraint-qualifications. This remarkable advantage of our regularization method is worth mentioning.

The paper is organized as follows: After defining our problem and introducing our main assumptions in Section 2, Section 3 is devoted to known results concerning the parabolic equation. In particular, we regard the properties of the control-to-state mapping.

In Section 4, we introduce the Lavrentiev type regularization. We motivate why the Lagrange multipliers are regular and show that the optimal control of the regularized problem converges towards the optimal control of the original problem. Section 5 is devoted to existence and convergence of the central path defined by the interior point method. In Section 6, we discuss the convergence of a simple interior point algorithm in function space and finally, in Section 7, we confirm our theory by some numerical examples.

2. Problem setting

We consider the optimal control problem

$$\min J(y,u) = \frac{1}{2}\|y(T) - y_d\|_\Omega^2 + \frac{\kappa}{2}\|u\|_Q^2 \tag{1}$$

subject to the parabolic initial boundary value problem

$$\begin{aligned} y_t - \nabla\cdot(A\,\nabla y) + c_0 y &= u \quad \text{in } Q,\\ \partial_n y + \alpha y &= 0 \quad \text{in } \Sigma,\\ y(0) &= 0 \quad \text{in } \Omega, \end{aligned} \tag{2}$$

and to the pointwise state constraints

$$y_a(x,t) \le y(x,t) \le y_b(x,t) \quad \text{for all } (x,t) \in Q. \tag{3}$$

In this setting, $\Omega \subset \mathbb{R}^N$, $N \ge 1$, is a bounded domain with $C^{1,1}$-boundary Γ, and $(0,T)$ is a fixed time interval. We define $Q := \Omega \times (0,T)$ and $\Sigma := \Gamma \times (0,T)$.

$A = (a_{ij}(x))$, $i,j = 1,...,N$, is a symmetric matrix with $a_{ij} \in C^{1,\gamma}(\Omega)$, $\gamma \in (0,1)$. It is assumed to satisfy the following condition of uniform ellipticity: There is an $m > 0$ such that

$$\lambda^\top A(x)\lambda \quad \ge \quad m|\lambda|^2 \quad \text{for all } \lambda \in \mathbb{R}^N \text{ and all } x \in \bar\Omega.$$

Moreover, functions $c_0 \in L^\infty(Q)$, $y_d \in L^\infty(\Omega)$, and y_a, y_b from $C(\bar Q)$ are given that satisfy $y_a(x,t) < y_b(x,t)$ for all $(x,t) \in \bar Q$.

By the continuity of y_a and y_b, there is some $c_Q > 0$, such that it holds

$$y_b(x,t) - y_a(x,t) \ge c_Q \qquad \forall (x,t) \in \bar C. \tag{4}$$

Notations: By $\|\cdot\|_{L^p(M)}$, $M \in \{Q,\, \Sigma,\, \Omega\}$, we denote the standard norm of $L^p(M)$. By $(\cdot,\cdot)_{L^2(M)}$ the inner product of $L^2(M)$ is denoted. In $L^2(Q)$, the norm and the inner product are written without subscript, i.e. $\|\cdot\| := \|\cdot\|_{L^2(Q)}$ and $(\cdot,\cdot) = (\cdot,\cdot)_{L^2(Q)}$ is the associated inner product of $L^2(Q)$. We use $\|B\|_{V\to W}$ for the norm of a linear continuous operator $B : V \to W$. If $V = W = L^2(Q)$ we just write $\|B\|$. Throughout the paper, c is a generic positive constant. To shorten the notation, we write e.g. $B + \frac{\mu}{w-y_a}$ instead of $B + \frac{\mu}{w-y_a}I$, although B is an operator and $\frac{\mu}{w-y_a}$ is a function. By ∂_n we denote the conormal derivative with respect to A, where n is the outward normal direction on Γ.

3. Some facts about the parabolic equation

In this section, we recall some known facts about the parabolic equation defined in (2). For the proof, we refer to [2] and [7], or to the survey in [22].

By $W(0,T)$, we denote the Hilbert space of functions $y \in L^2(0,T;V)$ with time derivative y' in $L^2(0,T;V^*)$, endowed with its standard norm, cf. [11]. For the notion of a weak solution to (2) we refer to [7] or [11].

Theorem 3.1. *The control-to-state mapping $u \mapsto y$ associated with equation (2) is linear and continuous from $L^2(Q)$ to $W(0,T)$.*

With the linearity of the parabolic pde, we can write $y = G_Q u$, where the control-to-state mapping $G_Q :\ L^2(Q) \to W(0,T)$ is continuous in view of Theorem 3.1.

The mapping $u \mapsto y(T)$, considered from $L^2(Q)$ to $L^2(\Omega)$, the "observation" of y at T, is denoted by S. Define $E_T : W(0,T) \to L^2(\Omega)$ by $E_T : y \mapsto y(T)$. Then S is given by $S = E_T G_Q$.

If we consider G_Q with range in $L^2(Q)$, then we denote this operator by G, i.e. $G = EG_Q$, where E is the embedding operator from $W(0,T)$ to $L^2(Q)$.

Corollary 3.2. *The mapping $S : u \mapsto y(T)$ is continuous from $L^2(Q)$ to $L^2(\Omega)$.*

Summarizing up, we have introduced the mappings

$$\begin{aligned} G_Q &: L^2(Q) \to W(0,T), \\ G &: L^2(Q) \to L^2(Q), \\ S &: L^2(Q) \to L^2(\Omega). \end{aligned}$$

Remark 3.3. *Although we have fixed the spaces of L^2-type, where G and S are defined, we shall consider them also in other spaces without changing their notation, as in the next theorem.*

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$-domain and assume $f \in L^r(Q)$ with $r > N/2+1$, $g \in L^s(\Sigma)$ for $s > N+1$ and $y_0 \in C(\bar\Omega)$. Then the weak solution y of*

$$\begin{aligned} y_t - \nabla(A\nabla y) + c_0 y &= f \quad \text{in } Q, \\ \partial_n y + \alpha y &= g \quad \text{on } \Sigma, \\ y(0) &= y_0 \quad \text{in } \Omega \end{aligned}$$

belongs to $C(\bar Q)$ and there is a constant c independent of u, such that

$$\|y\|_{C(\bar Q)} \leq c\left(\|f\|_{L^r(Q)} + \|g\|_{L^s(\Sigma)} + \|y_0\|_{C(\bar\Omega)}\right).$$

Proof. We refer to [2], or [19], cf. also [22], Lemma 7.10. □

For a spatial dimension of $N = 2$, we need $r > 2$ and for $N = 3$ we need $r > 5/2$ to satisfy the assumptions.

Remark 3.5. *We present the theory for homogeneous boundary data and zero initial value. Problems with fixed inhomogeneous data in the parabolic equation,*

$$\begin{aligned} y_t - \nabla\cdot(A\,\nabla y) + c_0 y &= u + f \quad \text{in } Q, \\ \partial_n y + \alpha y &= g \quad \text{in } \Sigma, \\ y(0) &= y_0 \quad \text{in } \Omega, \end{aligned} \tag{5}$$

where $f \in L^r(Q)$, $r > N/2+1$, $g \in L^s(\Sigma)$, $s > N+1$, and $y_0 \in C(\bar\Omega)$ are given, can be easily transformed to a problem of type (1)–(3). One has to separate the fixed part of y associated with (f, g, y_0) and to subtract this part from y_d.

4. Mixed control-state constraints

In this section, we consider the regularized optimal control problem

$$\text{(P)} \qquad \min J(y,u) = \frac{1}{2}\|y(T) - y_d\|_\Omega^2 + \frac{\kappa}{2}\|u\|_Q^2$$

subject to

$$\begin{aligned} y_t - \nabla\cdot(A\,\nabla y) + c_0 &= u \quad \text{in } Q, \\ \partial_n y + \alpha y &= 0 \quad \text{on } \Sigma, \\ y(0) &= 0 \quad \text{in } \Omega, \end{aligned} \tag{6}$$

and to the mixed (ε-regularized) control-state constraints

$$y_a \leq y + \varepsilon u \leq y_b \quad \text{a.e. in } Q. \tag{7}$$

We are able to show that the optimal control u_ε of this problem tends in $L^2(Q)$ to the solution $\bar u$ of the original problem, provided that a Slater type condition is satisfied for the original one. The method of proof is analogous to the one in Hintermüller et al. [6]. We do not prove this result, since we aim at concentrating on the interior point method for problem (P) rather than to discuss the relation to the unregularized problem (1)–(3). Following [14], we transform the mixed control-state constraints into control constraints. By the operator G, introduced in Section 3, we can write

$$y + \varepsilon u = Gu + \varepsilon u = (G + \varepsilon\, I)u.$$

The function $w := y + \varepsilon u$ is considered as a new auxiliary control. Then we have $u = D\,w$, where $D : L^2(Q) \to L^2(Q)$ is defined by

$$D = (G + \varepsilon\, I)^{-1}. \tag{8}$$

D is well defined, as the next result shows:

Lemma 4.1. *For all $\varepsilon \neq 0$, the operator D exists and is continuous in $L^2(Q)$.*

Proof. First we show that the kernel of $G + \varepsilon I$ is trivial. To see this, consider the equation

$$Gu + \varepsilon u = 0.$$

This is equivalent to $u = G(-\varepsilon^{-1}u)$. By the definition of G, u solves the system

$$\begin{aligned} u_t - \Delta u + c_0 u &= -\tfrac{1}{\varepsilon} u && \text{in } Q,\\ \partial_n u + \alpha u &= 0 && \text{on } \Sigma,\\ u(0) &= 0 && \text{in } \Omega. \end{aligned}$$

By taking $(-1/\varepsilon)u$ to the other side of the equation we see that u solves a homogeneous initial-boundary value problem that has only the trivial solution.

It remains to show that $\varepsilon I + G$ is surjective. Then the Banach theorem on the inverse operator ensures the continuity of $D = (\varepsilon I + G)^{-1}$. Let $w \in L^2(Q)$ be given arbitrarily and consider the equation

$$\varepsilon u + Gu = w.$$

To solve it, we consider the equation

$$\begin{aligned} y_t - \Delta y + c_0 y &= \tfrac{1}{\varepsilon}(w - y) && \text{in } Q,\\ \partial_n y + \alpha y &= 0 && \text{on } \Sigma,\\ y(0) &= 0 && \text{in } \Omega. \end{aligned} \tag{9}$$

Taking $-\frac{1}{\varepsilon}y$ to the other side, we see that this equation has a unique solution $y \in W(0,T)$. Now we define

$$u := \frac{1}{\varepsilon}(w - y). \tag{10}$$

Then $y = Gu$ holds and hence

$$u = \frac{1}{\varepsilon}(w - Gu).$$

Obviously, this u solves the equation $\varepsilon u + Gu = w$ and we have shown the surjectivity. □

4.1. **Regular Lagrange multipliers.** By the technique used in [13] for an elliptic problem, we will motivate the existence of regular multipliers. We do not directly need this result for our convergence analysis. However, it shows how the regularization helps to construct a problem with better properties. In particular, this explains why our numerical method does not have to deal with measures as multipliers. First of all, we transform problem (P) with mixed control-state constraints (7) in a control-constrained problem with new control $w := D^{-1}u$. With S and D, we transform problem (P) to one depending on the control w as

$$\min F(w) = \frac{1}{2}\|SDw - y_d\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|Dw\|^2 \tag{11}$$

subject to

$$y_a \le w \le y_b \quad \text{a.e. in } Q. \tag{12}$$

This transformation of our control problem (P) will be used for the analysis of the interior point algorithm, while all computations are performed with the original form of (P).

The functional F is continuously Fréchet-differentiable on $L^2(Q)$. Its Fréchet derivative is represented by

$$F'(w)v = ((SD)^*(SDw - y_d), v) + \kappa\,(D^*Dw, v)\,.$$

We can identify it with the function

$$g := (SD)^*(SDw - y_d) + \kappa D^*Dw \in L^2(Q),$$

the Riesz representation of the derivative. Using the same arguments as in [13, 14], we define Lagrange multipliers η_a and $\eta_b \in L^2(Q)$ by

$$\begin{aligned} \eta_a(x,t) &= g(x,t)_+, \\ \eta_b(x,t) &= g(x,t)_-, \end{aligned}$$

so that $g = g_+ - g_- = \eta_a - \eta_b$.

Remark 4.2. *In all what follows, a bar as in $\bar{u}$, $\bar{y}$, or $\bar{w}$ etc. indicates optimality.*

The optimal solution $\bar{w}$ fulfills, together with η_a and η_b, the following necessary and (by convexity) sufficient optimality conditions:

$$S^*(SD\bar{w} - y_d) + \kappa D\bar{w} + (D^*)^{-1}(\eta_b - \eta_a) \;=\; 0, \tag{13}$$

together with the complementary conditions

$$\begin{array}{rclrcll} (\eta_a, \bar{w} - y_a) &=& 0\,, & (\eta_b, y_b - \bar{w}) &=& 0 & \\ \eta_a(x,t) &\geq& 0\,, & \eta_b(x,t) &\geq& 0 & \text{a.e. in } Q \\ \bar{w}(x,t) - y_a(x,t) &\geq& 0\,, & y_b(x,t) - \bar{w}(x,t) &\geq& 0 & \text{a.e. in } Q. \end{array} \tag{14}$$

Following the same steps as in [15], η_a, η_b are verified to be the Lagrange multipliers associated with the mixed constraints (7).

4.2. **Transformation in terms of PDEs.** By $D^{-1} = \varepsilon I + G$ we can write (13) in the form

$$S^*(SD\bar{w} - y_d) + \kappa D\bar{w} + \varepsilon(\eta_b - \eta_a) + G^*(\eta_b - \eta_a) \;=\; 0. \tag{15}$$

Re-substituting $D\bar{w} = \bar{u}$, and defining an adjoint state p by

$$p = G^*(\eta_b - \eta_a) + S^*(S\bar{u} - y_d), \tag{16}$$

we obtain the optimality conditions

$$\bar{y} \;=\; G\bar{u}, \tag{17}$$

$$p + \kappa\bar{u} \;=\; \varepsilon(\eta_a - \eta_b), \tag{18}$$

together with the complementarity conditions (14), where we resubstitute $\bar{w} := \varepsilon\bar{u} + \bar{y}$.

The adjoint state p defined by (16) is the unique solution of the following adjoint equation:

$$\begin{array}{rcll} -p_t - \nabla\cdot(A\,\nabla p) + c_0 p &=& \eta_b - \eta_a & \text{in } Q, \\ \partial_n p + \alpha p &=& 0 & \text{on } \Sigma, \\ p(T) &=& \bar{y}(T) - y_d & \text{in } \Omega. \end{array} \tag{19}$$

The adjoint equation has a unique solution $p \in W(0,T)$. It holds

$$\|p\|_{W(0,T)} \le c_w \left(\|\eta_b - \eta_a\|_{L^2(Q)} + \|\bar{y}(T) - y_d\|_{L^2(\Omega)}\right)$$

with some c_w not depending on the given data. This follows from Theorem 3.1 after the transformation of time $\tau := T - t$.

Remark 4.3. *The case $\varepsilon = 0$ is formally covered by the optimality system* (14)–(18)*, too. Here, possibly, η_a, η_b belong to $\mathcal{M}(\bar{Q})$, the space of regular Borel measures defined at $\bar{Q}$. Then equation* (19) *is a parabolic PDE with measures on the righthand-side, which may even appear in the boundary and terminal conditions, we refer to Casas* [2]*. In this case, our theory does not work, since the operator D is unbounded and not defined on the whole space $L^2(Q)$.*

In summary, we have derived the following theorem:

Theorem 4.4. *For all $\varepsilon \neq 0$, problem (P) has a unique optimal control $\bar{u}_\varepsilon$ with associated state $\bar{y}_\varepsilon$. There exist non-negative Lagrange multipliers $\eta_a \in L^2(Q)$ and $\eta_b \in L^2(Q)$ and an associated adjoint state $p \in W(0,T)$, such that the optimality system* (14)–(18) *is satisfied.*

The existence of the optimal control follows in particular from the fact that the equation $\varepsilon u + Gu = y_a$ is solvable for all nonzero ε. Therefore, the admissible set is never empty. Due to the convexity of the objective functional F, the necessary optimality conditions are also sufficient for optimality.

5. Interior-point method in function space

By the interior point method, the constrained problem (11)–(12) is transformed into a formally unconstrained problem by adding a logarithmic barrier term to the objective functional F. In this section, we show that the transformed problems are solvable and that the associated *central path* exists.

In terms of PDE, the problem (P) is converted to the following one:

$$\min J_\mu(y,u) := \frac{1}{2}\|y(T) - y_d\|_\Omega^2 + \frac{\kappa}{2}\|u\|_Q^2 - \mu \iint\limits_Q \ln\left(y + \varepsilon u - y_a\right) + \ln\left(y_b - \varepsilon u - y\right) dxdt$$

subject to the equation (2).

Let us first state the associated necessary optimality conditions. In a standard way, e.g. by the formal Lagrange-technique explained in [22], we obtain the adjoint equation

$$\text{(20)} \qquad \begin{aligned} -p_t - \nabla\cdot(A\nabla p) + c_0 p &= -\tfrac{\mu}{y+\varepsilon u - y_a} + \tfrac{\mu}{y_b - \varepsilon u - y} && \text{in } Q,\\ \partial_n p + \alpha p &= 0 && \text{on } \Sigma,\\ p(T) &= y(T) - y_d && \text{in } \Omega, \end{aligned}$$

and the gradient equation

$$\text{(21)} \qquad p + \kappa u - \frac{\varepsilon\mu}{y + \varepsilon u - y_a} + \frac{\varepsilon\mu}{y_b - \varepsilon u - y} = 0 \quad \text{a.e in } Q.$$

The solution (u_μ, y_μ, p_μ), if it exists, is expected to converge to the solution of Problem (P) as $\mu \downarrow 0$. We prove the existence of this and the optimality conditions by considering the penalized version of Problem (11)–(12), i.e.,

(P$_\mu$)

$$\min F_\mu(w) := \frac{1}{2}\|SDw - y_d\|_\Omega^2 + \frac{\kappa}{2}\|Dw\|_Q^2 - \mu \iint\limits_Q \ln(w - y_a) + \ln(y_b - w)\,dx\,dt,$$

where $\mu > 0$ is the path parameter that will tend to zero. This is a formally unconstrained problem, but the logarithmic barrier term can only be finite for $w \in L^2(Q)$ with $y_a < w < y_b$ a.e. in Q. Therefore, the admissible set of (P$_\mu$) is open in some sense. Notice that $F_\mu(w)$ is a convex functional.

To prove the existence of a solution of problem (P$_\mu$), we apply a method that has been introduced in [18]. It considers the minimization of F_μ in a closed subset and, at the same time, finally permits to show that the solution w_μ has some positive distance to the bounds: We have $y_a + \tau \le w_\mu \le y_b - \tau$ for some sufficiently small $\tau > 0$ that depends on μ.

5.1. **Existence.** For fixed $\tau > 0$, we consider the auxiliary problem

(Aux)
$$\min_{y_a+\tau\le w\le y_b-\tau} F_\mu(w).$$

The admissible set of this problem is closed, and the functional F_μ is bounded.

We define the following admissible sets:

$$W := \{w \in L^2(Q)\,|\, y_a \le w \le y_b \ \text{ a.e. in } Q\}, \tag{22}$$
$$W_\tau := \{w \in L^2(Q)\,|\, y_a + \tau \le w \le y_b - \tau \text{ a.e. in } Q\}. \tag{23}$$

Theorem 5.1. *For every $0 < \tau < c_Q/3$, c_Q defined in (4) and for all $\mu > 0$, problem (Aux) has a unique solution $w_{\tau,\mu}$. There is a bound c not depending on τ and μ such that it holds $\|w_{\tau,\mu}\|_{L^\infty(Q)} \le c$.*

Proof. It is clear that W_τ is non-empty, convex, closed and bounded. F_μ is strictly convex and continuous on W_τ, and hence weakly lower semicontinuous. Therefore, standard arguments show the existence of a unique solution of (Aux). The uniform boundedness of the solution is an obvious consequence of the boundedness of $W_\tau \subset W$ in $L^\infty(Q)$. □

In the case of one-sided constraints $y + \varepsilon u \le y_b$ or $y_a \le y + \varepsilon u$, Theorem 5.1 cannot be shown in this way, since the associated set W_τ is not bounded. Here, the following Lemma applies that can be proven completely analogous to Lemma 3.2 in [18].

Lemma 5.2. *For all $\mu \ge 0$, it holds that $F_\mu(w) \to \infty$ if $\|w\| \to \infty$ and $w \ge y_a$ or $w \le y_b$, respectively.*

The function F_μ is directionally differentiable at $w_{\tau,\mu}$ in all directions $w - w_{\tau,\mu}$ with $w \in W_\tau$. The optimality of $w_{\tau,\mu}$ gives

$$F'_\mu(w_{\tau,\mu})(w - w_{\tau,\mu}) \quad \ge \quad 0 \quad \forall\, w \in W_\tau,$$

where F'_μ denotes the directional derivative of F_μ. According to the definition of F_μ, we obtain the variational inequality

$$(g_{\tau,\mu}, w - w_{\tau,\mu})_Q \quad \ge \quad 0 \quad \forall\, w \in W_\tau, \tag{24}$$

where the function $g_{\tau,\mu} \in L^2(Q)$ is defined by

$$g_{\tau,\mu} := (SD)^*(SDw_{\tau,\mu} - y_d) + \kappa D^* D w_{\tau,\mu} - \frac{\mu}{w_{\tau,\mu} - y_a} + \frac{\mu}{y_b - w_{\tau,\mu}}. \tag{25}$$

Next, we define two auxiliary functions, namely

$$p_{\tau,\mu} := (SD)^*(SDw_{\tau,\mu} - y_d) \quad \text{and} \quad q_{\tau,\mu} := \kappa D^* D w_{\tau,\mu}.$$

We show that they are bounded in $L^\infty(Q)$, uniformly with respect to τ and μ. To this aim, we need the following result.

Lemma 5.3. *The operators D and D^* are continuous in $L^\infty(Q)$.*

Proof. To find $u = Dw$, we have to solve the equation $\varepsilon u + Gu = w$. In view of (9) and (10), this is equivalent to the following two steps: We solve first (9) to find y. Next, we obtain u by formula (10). Thanks to Theorem 3.4, the mapping $w \mapsto y$ is linear and continuous in $L^\infty(Q)$. Therefore, the same holds true for the mapping $w \mapsto u = \varepsilon^{-1}(w - y(w))$. This shows the continuity of D.

The proof for D^* is analogous, since G^* is related to an adjoint parabolic equation that has the same properties as equation (9). □

The following Lemma asserts the L^∞-boundedness of p and q.

Lemma 5.4. *There is a positive constant $c_{p,q}$ such that*

$$\|p_{\tau,\mu}\|_{L^\infty(Q)} + \|q_{\tau,\mu}\|_{L^\infty(Q)} \le c_{p,q}$$

holds true for all $0 < \tau < c_Q/3$ and all $\mu > 0$.

Proof. We have

$$\|p_{\tau,\mu}\|_{L^\infty(Q)} + \|q_{\tau,\mu}\|_{L^\infty(Q)} \le \|(SD)^*(SDw_{\tau,\mu} - y_d)\|_{L^\infty(Q)} + \kappa \|D^* D w_{\tau,\mu}\|_{L^\infty(Q)}.$$

In view of Theorem 3.4 and Lemma 5.3, all operators appearing in this formula are continuous in L^∞-spaces on associated domains. Moreover, we have assumed that $y_d \in L^\infty(\Omega)$. Therefore, the result of the Lemma is an immediate conclusion of Theorem 5.1. □

The main result of this section, the existence of the central path, can be shown completely analogous to the elliptic case discussed in [18]. Nevertheless, we briefly sketch the proof for convenience of the reader. To this aim, we define the sets

$$\begin{aligned} M_+(\tau,\mu) &:= \{(x,t) \in Q \,|\, g_{\tau,\mu}(x,t) > 0\}, \\ M_-(\tau,\mu) &:= \{(x,t) \in Q \,|\, g_{\tau,\mu}(x,t) < 0\}, \\ M_0(\tau,\mu) &:= \{(x,t) \in Q \,|\, g_{\tau,\mu}(x,t) = 0\}. \end{aligned}$$

Lemma 5.5. *For all $\mu > 0$, there are positive numbers $\tau_+(\mu)$ and $\tau_-(\mu)$ such that, for all $0 < \tau < \tau(\mu)$, the sets $M_+(\tau)$ and $M_-(\tau)$ have measure zero.*

Proof. A standard evaluation of (24) yields for almost all $(x,t) \in Q$ that

$$w_{\tau,\mu}(x,t) = \begin{cases} y_a(x,t) + \tau, & (x,t) \in M_+(\tau,\mu) \\ y_b(x,t) - \tau, & (x,t) \in M_-(\tau,\mu). \end{cases}$$

Almost everywhere on $M_+(\tau,\mu)$, Lemma 5.4 implies

$$\begin{aligned} 0 \;<\; g_{\tau,\mu}(x,t) &= p_{\tau,\mu}(x,t) + q_{\tau,\mu}(x,t) - \frac{\mu}{\tau} + \frac{\mu}{y_b(x,t) - y_a(x,t) - \tau} \\ &\le c_{p,q} - \frac{\mu}{\tau} + 2\frac{\mu}{c_Q}. \end{aligned} \tag{26}$$

For $\tau \downarrow 0$, the right hand side tends to $-\infty$, a contradiction for all sufficiently small $\tau > 0$, say $\tau < \tau_+(\mu)$. Consequently, $M_+(\tau,\mu)$ is of measure zero for these τ. Analogously, $M_-(\tau,\mu)$ can be handled. Define now $\tau(\mu) := \min\{\tau_+(\mu), \tau_-(\mu)\}$, we have found the bound $\tau(\mu)$, for that the Lemma holds. □

Now we can formulate the main result of this section.

Theorem 5.6. *For all $\mu > 0$ and all $0 < \tau < \tau(\mu)$, the solution $w_{\tau,\mu}$ of* (Aux) *is the unique solution w_μ of problem* (P_μ).

Proof. Since $Q = M_0(\tau) \cup M_+(\tau) \cup M\_(\tau)$ and the set $M_+(\tau) \cup M\_(\tau)$ has measure zero for $0 < \tau < \min\{\tau_+(\mu), \tau_-(\mu)\}$, we have

$$g_{\tau,\mu}(x,t) \quad = \quad 0 \qquad \text{a.e. in } Q.$$

Therefore, it holds that

$$F_\mu^{'}(w_{\tau,\mu})h \quad = \quad \iint_Q g_{\tau,\mu}(x,t)\,h(x,t)\,dxdt = 0 \quad \forall h \in L^2(Q),$$

and hence $w_{\tau,\mu}$ satisfies the necessary optimality condition for problem (P_μ). By convexity, the necessary conditions are sufficient for optimality. Strong convexity yields uniqueness (notice that $\kappa > 0$). In view of this, $w_{\tau,\mu}$ is the unique solution w_μ of (P_μ). □

Corollary 5.7. *Let be $\mu_0 > 0$ an initial value. Then, for all $0 < \mu < \mu_0$, the solution w_μ of (P_μ) satisfies*

$$y_a(x,t) + \tau(\mu) \le w_\mu(x,t) \le y_b(x,t) - \tau(\mu), \qquad \text{a.e. in } Q, \tag{27}$$

where $\tau(\mu) = c_\tau \mu > 0$ and c_τ is given by

$$c_\tau = \frac{1}{c_{p,q} + 2\frac{\mu_0}{c_Q}}.$$

Proof. By Theorem 5.6, we have $w_{\tau,\mu} = w_\mu$ for all $0 < \tau < \tau(\mu)$. In view of the definition of (Aux), $w_{\tau,\mu}$ satisfies

$$y_a(x,t) + \tau(\mu) \le w_\mu(x,t) \le y_b(x,t) - \tau(\mu)$$

for all $\tau < \tau(\mu)$, i.e., inequality (27) is satisfied. Let us quantify $\tau(\mu)$. From (26), we get $\frac{\mu}{\tau} \le c_{p,q} + 2\frac{\mu}{c_Q}$, hence

$$\tau \ge \frac{\mu}{c_{p,q} + 2\frac{\mu}{c_Q}} \ge \frac{\mu}{c_{p,q} + 2\frac{\mu_0}{c_Q}} = c_\tau \mu, \tag{28}$$

where $c_\tau = \frac{1}{c_{p,q}+2\frac{\mu_0}{c_Q}}$, what gives us $\tau \ge \tau(\mu) = c_\tau \mu$. □

Remark 5.8. *Collorary 5.7 shows that minimizing J_μ generates solutions in the interior of the feasible set, so that the name "interior point method" is justified.*

After having solved the problem of existence, let us verify and re-formulate the optimality conditions (20)–(21). We denote by u_μ the optimal control with state y_μ given by $\varepsilon u_\mu + y_\mu = w_\mu$. The associated adjoint state is p_μ. Define $\eta_{a,\mu}$ and $\eta_{b,\mu}$ by

$$\eta_{a,\mu} = \frac{\mu}{y_\mu + \varepsilon u_\mu - y_a}, \qquad \eta_{b,\mu} = \frac{\mu}{y_b - \varepsilon u_\mu - y_\mu}. \tag{29}$$

Multiplying (25) by $(D^*)^{-1} = (\varepsilon I + G^*)$, we obtain in view of $Dw_\mu = u_\mu$ and $Su_\mu = y_\mu$ that

$$S^*(y_\mu(T) - y_d) + \kappa u_\mu + \varepsilon(\eta_{b,\mu} - \eta_{a,\mu}) + G^*(\eta_{b,\mu} - \eta_{a,\mu}) = 0. \tag{30}$$

This is the counterpart to (15). We set

$$p_\mu := S^*(y_\mu(T) - y_d) + G^*(\eta_{b\mu} - \eta_{a,\mu}). \tag{31}$$

Then, analogous to (19), p_μ, y_μ, u_μ solve the adjoint equation (20). Moreover, (30) becomes

$$p_\mu + \kappa u_\mu + \varepsilon(\eta_{b,\mu} - \eta_{a,\mu}) = 0.$$

This is equivalent to (21). Summarizing up, we get the optimality system for y_μ, u_μ, and p_μ,

$$\begin{array}{rcll} (y_\mu)_t - \nabla(A\,\nabla y_\mu) + c_0 y_\mu &=& u_\mu & \text{in } Q,\\ \partial_n y_\mu + \alpha y_\mu &=& 0 & \text{on } \Sigma,\\ y_\mu(0) &=& 0 & \text{in } \Omega, \end{array} \tag{32}$$

$$\begin{array}{rcll} -(p_\mu)_t - \nabla(A\,\nabla p_\mu) + c_0 p_\mu &=& \eta_{b,\mu} - \eta_{a,\mu} & \text{in } Q,\\ \partial_n p_\mu + \alpha p_\mu &=& 0 & \text{on } \Sigma,\\ p_\mu(T) &=& y_\mu - y_d & \text{in } \Omega, \end{array} \tag{33}$$

$$p_\mu + \kappa u_\mu + \varepsilon(\eta_{b,\mu} - \eta_{a,\mu}) \;=\; 0 \text{ a.e. in } Q, \tag{34}$$

$$\begin{array}{rcll} \eta_{a,\mu} \ge 0,\ y_\mu + \varepsilon u_\mu - y_a \ge 0,\ \eta_{a,\mu}(y_\mu + \varepsilon u_\mu - y_a) &=& \mu & \text{a.e. in } Q,\\ \eta_{b,\mu} \ge 0,\ y_b - \varepsilon u_\mu - y_\mu \ge 0,\ \eta_{b,\mu}(y_b - \varepsilon u_\mu - y_\mu) &=& \mu & \text{a.e. in } Q. \end{array} \tag{35}$$

Notice that (29) can be rewritten as $\mu = \eta_{a,\mu}(y_\mu+\varepsilon u_\mu - y_a)$, $\mu = \eta_{b,\mu}(y_b - y_\mu - \varepsilon u_\mu)$.

5.2. **Convergence.** In Section 5.1, we established the existence of the central path $\mu \mapsto w_\mu$ for all fixed $\mu > 0$. Now we proceed with proving the continuity of the mapping $\mu \mapsto w_\mu$ and the convergence towards a solution $\bar{w}$ of (11)–(12).

The unique minimizer w_μ of (P$_\mu$) is the solution of $F_\mu^{'}(w) = 0$, hence

$$\begin{aligned} H(w_\mu;\mu) \;&:=\; (SD)^*(SDw_\mu - y_d) + \kappa D^* D w_\mu - \frac{\mu}{w_\mu - y_a} + \frac{\mu}{y_b - w_\mu}\\ &=\; (D^*S^*SD + \kappa D^*D)w_\mu - D^*S^*y_d - \frac{\mu}{w_\mu - y_a} + \frac{\mu}{y_b - w_\mu} = 0. \end{aligned}$$

By Corollary 5.7, we know $w_\mu - y_a \ge \tau(\mu)$ and $y_b - w_\mu \ge \tau(\mu)$ for all sufficiently small $\mu > 0$. H is Fréchet-differentiable in all directions $w \in L^\infty(Q)$ for all $\mu > 0$. Let $\partial_\mu H$ denote the derivative of H with respect to μ and let $\partial_w H$ be the derivative of H with respect to w. The derivative $\partial_w H$ is

$$\partial_w H(w;\mu) \;=\; D^*S^*SD + \kappa D^*D + \frac{\mu}{(w - y_a)^2} + \frac{\mu}{(y_b - w)^2}. \tag{36}$$

It satisfies the estimate

$$(v, \partial_w H(w;\mu)v) = (SDv, SDv)_\Omega + \kappa\,(Dv, Dv) + \mu\left(\frac{v}{w - y_a}, \frac{v}{w - y_a}\right) + \mu\left(\frac{v}{y_b - w}, \frac{v}{y_b - w}\right) \ge \kappa\frac{1}{\|D^{-1}\|^2}\|v$$

By Lemma 5.3, $\partial_w H$ is continuous in $L^\infty(Q)$ for all $w \in L^\infty(Q)$ with $y_a \le w \le y_b$ a.e. in Q. We show the boundedness of the inverse $(\partial_w H)^{-1}$ in $L^\infty(Q)$.

Theorem 5.9. *For all $\mu > 0$, the mapping $\partial_w H(w;\mu) : L^\infty(Q) \to L^\infty(Q)$ is a bijection. Its inverse is uniformly bounded for all $\mu > 0$; i.e. there exists a constant $c_{inv} > 0$ such that*

$$\|\partial_w H(w;\mu)^{-1}\|_{L^\infty(Q)\to L^\infty(Q)} \le c_{inv} \text{ for all } \mu > 0 \text{ and for all } w \in W.$$

The proof is the same as the one for Lemma 4.1 in [18], cf. the argumentation there.

Theorem 5.10. *For $\mu \downarrow 0$, w_μ converges towards the solution $\bar{w}$ of (11)–(12). There is a constant $c_{path} > 0$, such that*

$$\|w_\mu - \bar{w}\|_{L^\infty(Q)} \quad \le \quad c_{path}\sqrt{\mu}$$

holds for all sufficiently small μ.

The proof is analogous to the one of Theorem 4.3 in [18] for the elliptic case with a unilateral constraint. However it is more technical in view of our bilateral constraints. The main idea of the proof is to estimate $\|w'\|_{L^\infty}$. In Lemma 6.5 we will derive a similar result for a scaled norm that shows the technique of this proof.

6. An interior point algorithm

A conceptual interior point algorithm in function space is given by the following steps.

Algorithm 1.
Choose $0 < \sigma < 1$, $0 < eps$, and an initial function $w^0 \in L^\infty$ such that $y_a + \tau \le w^0 \le y_b - \tau$ holds for some $\tau > 0$. Take $\mu_0 > 0$, and set $k = 0$.

while $\mu_k > eps$ do {

$\mu_{k+1} = \sigma\mu_k$,
$d^{k+1} = -\partial H_w(w^k;\mu_{k+1})^{-1} H(w^k;\mu_{k+1})$
$w^{k+1} = w^k + d^{k+1}$
$k = k+1$

}

The code-sequence in the while-loop performs one classical Newton step for the equation $H(w^{k+1};\mu_{k+1}) = 0$ with fixed μ_{k+1}.

In the numerical analysis, we have to impose certain restrictions on σ to guarantee convergence.

In the following, we denote the solutions of (P_μ) associated with the parameter μ_k by subscripts, i.e. w_{μ_k} is a point on the central path and solves $H(w_{\mu_k};\mu_k) = 0$. On the other hand, let w^k , $k = 1, 2, ...$ denote the iterates of Algorithm 1 associated with the parameter μ_k. Figure 1 illustrates the situation.

Under our assumptions, the Newton method provides, for fixed μ_k, a unique solution w_{μ_k}. It converges quadratically to w_{μ_k}, if the starting point (in the Figure, we choose w^{k-1}) is sufficiently close to w_{μ_k}.

To prove the convergence of our method in function space, we show that

$$\|w^k - w_{\mu_k}\| \le c\sqrt{\mu_k} \text{ and } \|w^k - \bar{w}\| \quad \le \quad c\sigma^k$$

holds for some constant $c > 0$. Clearly, it holds that

$$w^{k+1} - w^k = d^{k+1}.$$

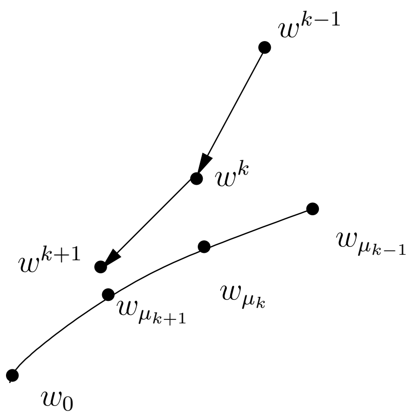


FIGURE 1. Some iterates of Algorithm 1 and the associated points on the central path.

In contrast to the Algorithm 5.1 in [18], the Newton corrector is assumed to be exact for simplicity, i.e. we assume to compute d^{k+1} exactly. Certainly, this is not realistic for a practical implementation. However, we do not aim at estimating here all errors that occur in a real computation.

6.1. **Scaled norms.** Local norms are a valuable tool in the analysis of interior point methods. Here we will use the affine scaled norm

$$\|w\|_\mu \quad := \quad \|\phi(\mu)w\|_{L^\infty(Q)}\,,$$

where $\phi(\mu)$ is defined by

$$\phi(\mu) \quad = \quad \sqrt{\frac{\kappa}{\varepsilon^2} + \frac{\mu}{(w_\mu - y_a)^2} + \frac{\mu}{(y_b - w_\mu)^2}}. \tag{37}$$

For the theory of affine scaled norms, we refer to [3]. First, we provide some results on this scaled norm.

Lemma 6.1. *For all $w \in L^\infty(Q)$, the norm $\|\cdot\|_\mu$ satisfies the estimate*

$$\|w\|_{L^\infty(Q)} \le \frac{\varepsilon}{\sqrt{\kappa}}\|w\|_\mu.$$

Proof. It holds by the definition of $\|\cdot\|_\mu$ and $\phi(\mu)$ that

$$\|w\|_\mu = \|\phi(\mu)w\|_{L^\infty(Q)} \ge \|\sqrt{\frac{\kappa}{\varepsilon^2}}w\|_{L^\infty(Q)} = \sqrt{\frac{\kappa}{\varepsilon^2}}\|w\|_{L^\infty(Q)}.$$

□

Lemma 6.2. *For every $w \in L^\infty(Q)$, the norm $\|\cdot\|_\mu$ fulfills*

$$c_1\sqrt{\mu}\|w\|_{L^\infty(Q)} \le \|w\|_\mu \le \frac{c_2}{\sqrt{\mu}}\|w\|_{L^\infty(Q)}.$$

Proof. It holds that

$$\begin{aligned}\|w\|_\mu &= \|\phi(\mu)w\|_{L^\infty(Q)} = \left\|\sqrt{\frac{\kappa}{\varepsilon^2}+\frac{\mu}{(w_\mu-y_a)^2}+\frac{\mu}{(y_b-w_\mu)^2}}w\right\|_{L^\infty(Q)}\\ &\le \sqrt{\frac{\kappa}{\varepsilon^2}+\frac{\mu}{\tau(\mu)^2}+\frac{\mu}{\tau(\mu)^2}}\|w\|_{L^\infty(Q)} \le \sqrt{\frac{\kappa}{\varepsilon^2}+\frac{2\mu}{(c_\tau\mu)^2}}\|w\|_{L^\infty(Q)}\\ &\le \sqrt{\frac{\kappa}{\varepsilon^2}\mu_0+\frac{2}{c_\tau^2}}\frac{1}{\sqrt{\mu}}\|w\|_{L^\infty(Q)} = \frac{c_2}{\sqrt{\mu}}\|w\|_{L^\infty(Q)},\end{aligned}$$

where we have used Corollary 5.7. On the other hand, we have

$$\begin{aligned}\|w\|_\mu &= \|\sqrt{\frac{\kappa}{\varepsilon^2}+\frac{\mu}{(w_\mu-y_a)^2}+\frac{\mu}{(y_b-w_\mu)^2}}w\|_{L^\infty(Q)}\\ &\ge \|\sqrt{\frac{\mu}{(w_\mu-y_a)^2}+\frac{\mu}{(y_b-w_\mu)^2}}w\|_{L^\infty(Q)}.\end{aligned}$$

Now, $y_a, y_b \in L^\infty(Q)$ yields

$$\max\{\|w-y_a\|_{L^\infty(Q)}, \|y_b-w\|_{L^\infty(Q)}\} < \max_{x,t\in Q}\{y_b(x,t)-y_a(x,t)\} =: c_{max},$$

hence

$$\|w\|_\mu \ge \|\sqrt{\frac{\mu}{(w_\mu-y_a)^2}+\frac{\mu}{(y_b-w_\mu)^2}}w\|_{L^\infty(Q)} > \frac{\sqrt{2\mu}}{c_{max}}\|w\|_{L^\infty(Q)},$$

so that the constant $c_1 := \frac{\sqrt{2}}{c_{max}}$ satisfies the statement of the lemma. □

Lemma 6.3. *Let $0 < \theta < c_1 c_\tau$ be given, where c_τ is defined in Corollary 5.7 and c_1 is given by Lemma 6.2. Then it holds for all $w \in B_\mu(w_\mu; \theta\sqrt{\mu}) := \{w \in L^\infty(Q) : \|w-w_\mu\|_\mu \le \theta\sqrt{\mu}\}$*

$$\|\partial_w H(w;\mu)^{-1}\eta\|_\mu \quad\le\quad c_\phi\|\phi(\mu)^{-1}\eta\|_{L^\infty(Q)} \quad \forall\,\eta \in L^\infty(Q),$$

where c_ϕ is defined by $c_\phi = 1 + \frac{\varepsilon^2}{\kappa}|Q|\|K\|_{L^2(Q)\to L^\infty(Q)} < \infty$.

Proof. Let us first discuss the form of the operator D. We have by its definition $D = (G+\varepsilon I)^{-1}$. Take $w = Dz$, then $z = D^{-1}w = (G+\varepsilon I)w = Gw + \varepsilon w$. On the other hand, $\varepsilon w = z - Gw = z - GDz = (I-GD)z$. Altogether, we have $D = \frac{1}{\varepsilon}(I-GD)$.

From that, we get the representation

$$D^*D = \frac{1}{\varepsilon^2}(I-GD)^*(1-GD) = \frac{1}{\varepsilon^2}(I-(GD)^*-GD+(D^*G^*GD)).$$

We define $K = D^*S^*SD + \frac{\kappa}{\varepsilon}(G^*D^*DG - (G^*D^* + GD))$. This operator assembles the constant parts of $\partial_w H$. The derivative of $\partial_w H(w;\mu) = D^*S^*SD + \kappa D^*D + \frac{\mu}{(w-y_a)^2} + \frac{\mu}{(y_b-w)^2}$ reads now

$$\begin{aligned}\partial_w H(w;\mu) &= D^*S^*SD + \frac{\kappa}{\varepsilon^2}\left(I-(GD)^*-GD+(D^*G^*GD)\right) + \frac{\mu}{(w-y_a)^2} + \frac{\mu}{(y_b-w)^2}\\ &= D^*S^*SD + \frac{\kappa}{\varepsilon^2}\left((GD)^*-GD+(D^*G^*GD)\right) + \frac{\kappa}{\varepsilon^2} + \frac{\mu}{(w-y_a)^2} + \frac{\mu}{(y_b-w)^2}\\ &= K + \frac{\kappa}{\varepsilon^2} + \frac{\mu}{(w-y_a)^2} + \frac{\mu}{(y_b-w)^2} \;=\; K + \phi^2(\mu).\end{aligned}$$

Here, we have used the feasibility of w given by $w \in B_\mu(w_\mu; \theta\sqrt{\mu})$ and Corollary 5.7. Scaling the inverse of $\partial_w H$ by $\phi(\mu)$, the identity

$$\begin{aligned}
\phi(\mu)\partial_w H(w;\mu)^{-1}\phi(\mu) &= \phi(\mu)\left(K + \phi^2(\mu)\right)^{-1}\phi(\mu) \\
&= \left(\phi(\mu)^{-1}\right)^{-1}\left((K + \phi^2(\mu))\right)^{-1}\left(\phi(\mu)^{-1}\right)^{-1} \\
&= \left(\phi(\mu)^{-1}\left(K + \phi^2(\mu)\right)\phi(\mu)^{-1}\right)^{-1} \\
&= \left(\phi(\mu)^{-1}K\phi(\mu)^{-1} + I\right)^{-1}
\end{aligned}$$

holds. From that we get the L^2-estimate

$$\begin{aligned}
\|\phi(\mu)\partial_w H(w;\mu)^{-1}\eta\| &= \|\phi(\mu)\partial_w H(w;\mu)^{-1}\phi(\mu)\phi(\mu)^{-1}\eta\| \\
&= \|(\phi(\mu)^{-1}K\phi(\mu)^{-1} + I)^{-1}\phi(\mu)^{-1}\eta\| \\
&\leq \|\phi^{-1}(\mu)\eta\|, \qquad (38)
\end{aligned}$$

where we used that K is a positive definite operator and $\|\phi^{-1}(\mu)K\phi^{-1}(\mu) + I\|_{L^2(Q)\to L^2(Q)} \geq 1$.

Setting $\omega := \partial_w H(w;\mu)^{-1}\eta$, we have $\eta = (K + \phi^2(\mu))\omega$, hence $\phi^2(\mu)\omega = \eta - K\omega$ and

$$\begin{aligned}
\|\phi(\mu)\partial_w H(w;\mu)^{-1}\eta\|_{L^\infty(Q)} &= \|\phi(\mu)\omega\|_{L^\infty(Q)} = \|\phi(\mu)\phi(\mu)^{-2}(\eta - K\omega)\|_{L^\infty(Q)} \\
&= \|\phi(\mu)^{-1}(\eta - K\omega)\|_{L^\infty(Q)} \\
&\leq \|\phi(\mu)^{-1}\eta\|_{L^\infty(Q)} + \|\phi(\mu)^{-1}K\omega\|_{L^\infty(Q)} \\
&\leq \|\phi(\mu)^{-1}\eta\|_{L^\infty(Q)} + \|\phi(\mu)^{-1}\|_{L^\infty(Q)}\|K\omega\|_{L^\infty(Q)} \\
&\leq \|\phi(\mu)^{-1}\eta\|_{L^\infty(Q)} + \frac{\varepsilon}{\sqrt{\kappa}}\|K\|_{L^2(Q)\to L^\infty(Q)}\|\omega\| \\
&= \|\phi(\mu)^{-1}\eta\|_{L^\infty(Q)} + \frac{\varepsilon}{\sqrt{\kappa}}\|K\|_{L^2(Q)\to L^\infty(Q)}\|\phi(\mu)^{-1}\phi(\mu)\partial_w H(w;\mu)^{-1}\eta\| \\
&\leq \|\phi(\mu)^{-1}\eta\|_{L^\infty(Q)} + \frac{\varepsilon}{\sqrt{\kappa}}\|K\|_{L^2(Q)\to L^\infty(Q)}\|\phi(\mu)^{-1}\|\|\phi(\mu)\partial_w H(w;\mu)^{-1}\eta\| \\
&\leq \|\phi(\mu)^{-1}\eta\|_{L^\infty(Q)} + \frac{\varepsilon}{\sqrt{\kappa}}\|K\|_{L^2(Q)\to L^\infty(Q)}\|\phi(\mu)^{-1}\|\|\phi(\mu)^{-1}\eta\| \\
&\leq \|\phi(\mu)^{-1}\eta\|_{L^\infty(Q)} + \frac{\varepsilon}{\sqrt{\kappa}}\|K\|_{L^2(Q)\to L^\infty(Q)}\frac{\varepsilon}{\sqrt{\kappa}}|Q|\|\phi(\mu)^{-1}\eta\|_{L^\infty(Q)}, \qquad (39)
\end{aligned}$$

where we inserted the L^2-estimate (38) and used $\|w\|_{L^2(Q)} \leq \sqrt{|Q|}\|w\|_{L^\infty}$. With help of (39), we can finally estimate

$$\begin{aligned}
\|\partial_w H(w;\mu)^{-1}\eta\|_\mu &= \|\phi(\mu)\partial_w H(w;\mu)^{-1}\eta\|_{L^\infty(Q)} \\
&\leq \left(1 + \frac{\varepsilon^2}{\kappa}|Q|\|K\|_{L^2(Q)\to L^\infty(Q)}\right)\|\phi(\mu)^{-1}\eta\|_{L^\infty(Q)},
\end{aligned}$$

which gives us the constant c_ϕ to obtain the desired result. □

Lemma 6.4. *For all $0 < \theta < 1$ and all $w \in B_\mu(w_\mu; \theta\sqrt{\mu})$, the following estimates hold true:*

$$w - y_a \geq (1-\theta)(w_\mu - y_a) \text{ a.e. in } Q$$

and

$$y_b - w \geq (1-\theta)(y_b - w_\mu) \text{ a.e. in } Q.$$

Proof. By its definition, the diagonal preconditioner $\phi(\mu)$ satisfies a.e. in Q

$$\sqrt{\frac{\mu}{(w_\mu - y_a)^2}} \leq \phi(\mu)$$

and

$$\sqrt{\frac{\mu}{(y_b - w_\mu)^2}} \leq \phi(\mu)$$

for all $y_a < w_\mu < y_b$. For all $w \in B_\mu(w_\mu; \theta\sqrt{\mu})$ we obtain

$$\begin{aligned} \left\| \frac{w - w_\mu}{w_\mu - y_a} \right\|_{L^\infty(Q)} &= \left\| \frac{w - w_\mu}{\sqrt{\mu}} \sqrt{\frac{\mu}{(w_\mu - y_a)^2}} \right\|_{L^\infty(Q)} \\ &\leq \frac{1}{\sqrt{\mu}} \|\phi(\mu)(w - w_\mu)\|_{L^\infty(Q)} \\ &= \frac{1}{\sqrt{\mu}} \|w - w_\mu\|_\mu < \theta < 1, \end{aligned}$$

since $w \in B_\mu(w_\mu, \theta\sqrt{\mu})$. From $w_\mu - y_a > 0$ a.e. in Q we therefore get

$$\pm(w - w_\mu) \leq \theta(w_\mu - y_a) \text{ a.e. in } Q,$$

hence, multiplying the minus-version by (-1) and adding on both sides $(w_\mu - y_a)$,

$$w - y_a \geq (1 - \theta)(w_\mu - y_a).$$

By the same argumentation we obtain the estimate $y_b - w \geq (1 - \theta)(y_b - w_\mu)$. □

Lemma 6.5. *Let* $w_\mu^{'} = \frac{\partial w_\mu}{\partial \mu} = -\partial_w H(w; \mu)^{-1} \partial_\mu H(w; \mu)$ *be the derivative of* w_μ *with respect to* μ*. Then the* μ*-norm of* $w_\mu^{'}$ *is bounded by a constant depending on* $\sqrt{\mu}$*, i.e.*

$$\|w_\mu^{'}\|_\mu \leq \frac{\sqrt{2} c_\phi}{\sqrt{\mu}}$$

for all $0 < \mu \leq \mu_0$.

Proof. We follow the proof in [17], Theorem 5.9. First, we have $\|w_\mu^{'}\|_{L^\infty(Q)} \leq \frac{c}{\sqrt{\mu}}$. We use Lemma 6.3 and estimate

$$\begin{aligned} \|w_\mu^{'}\|_\mu &= \|\partial_w H^{-1}(w_\mu; \mu) \partial_\mu H(w_\mu; \mu)\|_\mu \\ &\leq c_\phi \|\phi^{-1}(\mu) \partial_\mu H(w_\mu; \mu)\|_{L^\infty(Q)} \\ &= c_\phi \left\| \phi^{-1}(\mu) \left(\frac{1}{y_b - w_\mu} - \frac{1}{w_\mu - y_a} \right) \right\|_{L^\infty(Q)}. \end{aligned}$$

From the definition of $\phi(\mu)$ we see immediately

$$\phi(\mu) = \sqrt{\frac{\kappa}{\varepsilon^2} + \frac{\mu}{(w_\mu - y_a)^2} + \frac{\mu}{(y_b - w_\mu)^2}} > \sqrt{\frac{\mu}{(w_\mu - y_a)^2} + \frac{\mu}{(y_b - w_\mu)^2}} \quad .$$

This gives us

$$
\begin{aligned}
\|w_\mu^{'}\|_\mu &\leq c_\phi \left\| \left(\sqrt{\frac{\mu}{(w_\mu - y_a)^2} + \frac{\mu}{(y_b - w_\mu)^2}} \right)^{-1} \left(\frac{1}{y_b - w_\mu} - \frac{1}{w_\mu - y_a} \right) \right\|_{L^\infty(Q)} \\
&= c_\phi \left\| \frac{1}{\sqrt{\mu}} \sqrt{\frac{(w_\mu - y_a)^2 (y_b - w_\mu)^2}{(y_b - w_\mu)^2 + (w_\mu - y_a)^2}} \frac{(w_\mu - y_a) - (y_b - w_\mu)}{(w_\mu - y_a)(y_b - w_\mu)} \right\|_{L^\infty(Q)} \\
&\leq c_\phi \left\| \frac{1}{\sqrt{\mu}} \frac{(w_\mu - y_a)(y_b - w_\mu)}{\sqrt{(y_b - w_\mu)^2 + (w_\mu - y_a)^2}} \frac{(w_\mu - y_a) + (y_b - w_\mu)}{(w_\mu - y_a)(y_b - w_\mu)} \right\|_{L^\infty(Q)} \\
&= \frac{c_\phi}{\sqrt{\mu}} \left\| \frac{(w_\mu - y_a) + (y_b - w_\mu)}{\sqrt{(y_b - w_\mu)^2 + (w_\mu - y_a)^2}} \right\|_{L^\infty(Q)} \leq \frac{c_\phi \sqrt{2}}{\sqrt{\mu}}
\end{aligned}
$$

where we used, that $(x+y)/\sqrt{(x^2+y^2)} < \sqrt{2}$ for all $x, y > 0$. □

Moreover, we need the following relation between the μ_k- and the μ_{k+1}-norms.

Lemma 6.6. *For all $w \in L^\infty(Q)$, and all $0 < \sigma < 1$, $0 < \mu < \mu_0$, it holds that*

$$\|w\|_{\sigma\mu} \leq c_{\sigma\mu} \|w\|_\mu.$$

where $c_\sigma = 3\sigma^{1/2 - \sqrt{2} c_\phi}$.

Proof. First, we observe

$$\|w\|_{\sigma\mu} = \|\phi(\sigma\mu) w\|_{L^\infty(Q)} = \left\| \frac{\phi(\sigma\mu)}{\phi(\mu)} \phi(\mu) w \right\|_{L^\infty(Q)} \leq \left\| \frac{\phi(\sigma\mu)}{\phi(\mu)} \right\|_{L^\infty(Q)} \|w\|_\mu.$$

Now, the main work of the proof is to estimate $\left\| \frac{\phi(\sigma\mu)}{\phi(\mu)} \right\|_{L^\infty(Q)} = \left\| \frac{\frac{\kappa}{\varepsilon^2} + \frac{\sigma\mu}{(w_{\sigma\mu} - y_a)^2} + \frac{\sigma\mu}{(y_b - w_{\sigma\mu})^2}}{\frac{\kappa}{\varepsilon^2} + \frac{\mu}{(w_\mu - y_a)^2} + \frac{\mu}{(y_b - w_\mu)^2}} \right\|_{L^\infty(Q)}^{1/2}$.

We define two functions $\upsilon_a(\sigma)$, $\upsilon_b(\sigma) \in L^\infty(Q)$

$$(\upsilon_a(\sigma))(x,t) = \left(\frac{w_\mu - y_a}{w_{\sigma\mu} - y_a} \right)(x,t) \quad \text{and} \quad (\upsilon_b(\sigma))(x,t) = \left(\frac{y_b - w_\mu}{y_b - w_{\sigma\mu}} \right)(x,t).$$

Because $\mu \mapsto w_\mu$ is a differentiable mapping from $\mathbb{R}_+$ to $L^\infty(Q)$, the functions υ_a, υ_b map $(0,1]$ differentiably into $L^\infty(Q)$. We derive a bound for the function υ_a, the proof of boundedness for υ_b is completely analogous.

First, we estimate the derivative of υ_a with respect to σ:

$$
\begin{aligned}
\|\upsilon_a^{'}(\sigma)\|_{L^\infty(Q)} &= \left\| \frac{w_\mu - y_a}{(w_{\sigma\mu} - y_a)^2} w_{\sigma\mu}^{'} \mu \right\|_{L^\infty(Q)} \leq \left\| \frac{w_\mu - y_a}{w_{\sigma\mu} - y_a} \right\|_{L^\infty(Q)} \left\| \frac{1}{w_{\sigma\mu} - y_a} w_{\sigma\mu}^{'} \right\|_{L^\infty(Q)} \mu \\
&= \left\| \frac{w_\mu - y_a}{w_{\sigma\mu} - y_a} \right\|_{L^\infty(Q)} \left\| \frac{\sqrt{\sigma\mu}}{w_{\sigma\mu} - y_a} w_{\sigma\mu}^{'} \right\|_{L^\infty(Q)} \frac{\mu}{\sqrt{\sigma\mu}} \\
&\leq \left\| \frac{w_\mu - y_a}{w_{\sigma\mu} - y_a} \right\|_{L^\infty(Q)} \left\| \sqrt{\frac{\kappa}{\varepsilon^2} + \frac{\sigma\mu}{(w_{\sigma\mu} - y_a)^2} + \frac{\sigma\mu}{(y_b - w_{\sigma\mu})^2}} w_{\sigma\mu}^{'} \right\|_{L^\infty(Q)} \frac{\mu}{\sqrt{\sigma\mu}} \\
&= \left\| \frac{w_\mu - y_a}{w_{\sigma\mu} - y_a} \right\|_{L^\infty(Q)} \left\| w_{\sigma\mu}^{'} \right\|_{\sigma\mu} \frac{\mu}{\sqrt{\sigma\mu}} \leq \|\upsilon_a(\sigma)\|_{L^\infty(Q)} \frac{\sqrt{2} c_\phi}{\sigma}. \qquad (40)
\end{aligned}
$$

by Lemma 6.5. With the identity $\upsilon_a(\sigma) = \upsilon(1) - \int_\sigma^1 \upsilon'(\gamma)\, d\gamma$ we find by (40)

$$\|\upsilon_a(\sigma)\|_{L^\infty(Q)} \leq \|\upsilon_a(1)\|_{L^\infty(Q)} + \left\| \int_\sigma^1 \upsilon_a'(\gamma)\, d\gamma \right\|_{L^\infty(Q)} \leq 1 + \int_\sigma^1 \frac{\sqrt{2}c_\phi}{\gamma} \|\upsilon_a(\gamma)\|_{L^\infty(Q)} d\gamma.$$

Now we use the Gronwall Lemma to conclude

$$\|\upsilon_a(\sigma)\|_{L^\infty(Q)} = \left\| \frac{w_\mu - y_a}{w_{\sigma\mu} - y_a} \right\|_{L^\infty(Q)} \leq \sigma^{-\sqrt{2}c_\phi}. \tag{41}$$

In an analogous manner, we get the estimate

$$\|\upsilon_b(\sigma)\|_{L^\infty(Q)} = \left\| \frac{y_b - w_\mu}{y_b - w_{\sigma\mu}} \right\|_{L^\infty(Q)} \leq \sigma^{-\sqrt{2}c_\phi}. \tag{42}$$

Next, we notice the following auxiliary result: For all positive numbers c, a_1, a_2, b_1, b_2. it holds

$$\frac{c + a_1 + a_2}{c + b_1 + b_2} \leq 1 + \frac{a_1}{b_1} + \frac{a_2}{b_2}. \tag{43}$$

To verify this, we discuss three cases. If $a_1 + a_2 \leq a_2 + b_2$, then the left-hand side is less or equal than one so that the inequality is true.

If $a_1 > b_1$ and $a_2 > b_2$, then we have

$$\frac{c + a_1 + a_2}{c + b_1 + b_2} = 1 + \frac{a_1 - b_1}{c + b_1 + b_2} + \frac{a_2 - b_2}{c + b_1 + b_2} \leq 1 + \frac{a_1}{b_1} + \frac{a_2}{b_2}.$$

If $a_1 + a_2 > b_1 + b_2$ and $a_1 > b_1$ but $a_2 < b_2$, then $\frac{a_2 - b_2}{c + b_1 + b_2}$ is negative, hence

$$\frac{c + a_1 + a_2}{c + b_1 + b_2} \leq 1 + \frac{a_1 - b_1}{c + b_1 + b_2} \leq 1 + \frac{a_1}{b_1} \leq 1 + \frac{a_1}{b_1} + \frac{a_2}{b_2}.$$

The case $a_1 < b_1$ but $a_2 > b_2$ is analogous. Now we can proceed to estimate $\left\| \frac{\phi(\sigma\mu)}{\phi(\mu)} \right\|_{L^\infty(Q)}$. For fixed $(x, t) \in Q$, we distinguish between two cases:

(i) $(\phi(\sigma\mu))(x, t) \leq (\phi(\mu))(x, t)$. Then $\left(\frac{\phi(\sigma\mu)}{\phi(\mu)} \right)(x, t) \leq 1$ is satisfied.

(ii) If $(\phi(\sigma\mu))(x, t) > (\phi(\mu))(x, t)$, then by combining (43), (42), and (41) we can estimate

$$\begin{aligned}
1 < \left(\frac{\phi(\sigma\mu)}{\phi(\mu)} \right)(x, t) &= \left(\frac{\sqrt{\frac{\kappa}{\varepsilon^2} + \frac{\sigma\mu}{(w_{\sigma\mu} - y_a)^2} + \frac{\sigma\mu}{(y_b - w_{\sigma\mu})^2}}}{\sqrt{\frac{\kappa}{\varepsilon^2} + \frac{\mu}{(w_{\sigma\mu} - y_a)^2} + \frac{\mu}{(y_b - w_{\sigma\mu})^2}}} \right)(x, t) \\
&\leq \left(\sqrt{1 + \sigma \frac{(w_\mu - y_a)^2}{(w_{\sigma\mu} - y_a)^2} + \frac{(y_b - w_\mu)^2}{(y_b - w_{\sigma\mu})^2}} \right)(x, t) \\
&\leq \left(1 + \sqrt{\sigma \frac{(w_\mu - y_a)^2}{(w_{\sigma\mu} - y_a)^2}} + \sqrt{\sigma \frac{(y_b - w_\mu)^2}{(y_b - w_{\sigma\mu})^2}} \right)(x, t) \\
&= \left(1 + \sqrt{\sigma} \frac{w_\mu - y_a}{w_{\sigma\mu} - y_a} + \sqrt{\sigma} \frac{y_b - w_\mu}{y_b - w_{\sigma\mu}} \right)(x, t) \\
&\leq 1 + 2\sigma^{1/2 - \sqrt{2}c_\phi} \leq 3\sigma^{1/2 - \sqrt{2}c_\phi},
\end{aligned}$$

where we used $c_\phi \geq 1$, so that $1/2-\sqrt{2}c_\phi < 0$, and hence it holds $\sigma^{1/2-\sqrt{2}c_\phi} > 1$ for all c_ϕ. Clearly (i) and (ii) imply $\|\frac{\phi(\sigma\mu)}{\phi(\mu)}\|_{L^\infty(Q)} \leq \max\{1, 3\sigma^{1/2-\sqrt{2}c_\phi}\}$. By $\sigma < 1$, $c_\phi \geq 1$, we see that the maximum is $3\sigma^{1/2-\sqrt{2}c_\phi}$. □

Lemma 6.7. *(Lipschitz-Condition) For all* $0 < \theta < 1$ *and all* w, $\hat{w} \in B_\mu(w_\mu, \theta\sqrt{\mu})$, *the following Lipschitz condition holds:*

$$\|\partial_w H(w;\mu)^{-1}(\partial_w H(w;\mu) - \partial_w H(\hat{w};\mu))(w-\hat{w})\|_\mu \leq \frac{2\sqrt{2}c_\phi}{(1-\theta)^3\sqrt{\mu}}\|w-\hat{w}\|_\mu^2. \tag{44}$$

Proof. The main idea of the proof is analogous to the proof of Lemma 5.5 in [18] for unilateral constraints. The difficulty here is the more complicated structure of $\phi(\mu)$, what results in a more technical proof. For convenience of the reader, we perform it here in detail.

By Lemma 6.3 we obtain

$$\begin{aligned}
&\|\partial_w H(w;\mu)^{-1}(\partial_w H(w;\mu) - \partial_w H(\hat{w};\mu))(w-\hat{w})\|_\mu \\
&\qquad \leq c_\phi \|\phi(\mu)^{-1}(\partial_w H(w;\mu) - \partial_w H(\hat{w};\mu))(w-\hat{w})\|_{L^\infty(Q)} \\
&\qquad = c_\phi \left\|\phi(\mu)^{-1}\left(\frac{\mu}{(w-y_a)^2} + \frac{\mu}{(y_b-w)^2} - \frac{\mu}{(\hat{w}-y_a)^2} - \frac{\mu}{(y_b-\hat{w})^2}\right)\cdot(w-\hat{w})\right\|_{L^\infty(Q)}
\end{aligned}$$

since the constant parts of $\partial_w H(w;\mu)$ are compensated by the constant parts of $\partial_w H(\hat{w};\mu)$, cf. the definition of $\partial_w H(w;\mu)$ in (36). By Lemma 6.4, we get $w - y_a \geq (1-\theta)(w_\mu - y_a)$ and $y_b - w \geq (1-\theta)(y_b - w_\mu)$. The same holds for $\hat{w}$. Because the Lipschitz constant of x^{-2} for $x \geq a > 0$ is $2a^{-3}$, we can estimate

$$\begin{aligned}
&\|\partial_w H(w;\mu)^{-1}(\partial_w H(w;\mu) - \partial_w H(\hat{w};\mu))(w-\hat{w})\|_\mu \\
&\qquad \leq c_\phi \|\phi(\mu)^{-1}\left(\frac{2\mu}{(1-\theta)^3(w_\mu-y_a)^3} + \frac{2\mu}{(1-\theta)^3(y_b-w_\mu)^3}\right)(w-\hat{w})^2\|_{L^\infty(Q)} \\
&\qquad = \frac{\sqrt{2}c_\phi}{(1-\theta)^3}\left\|\left(\frac{\mu}{\phi(\mu)^3(w_\mu-y_a)^3} + \frac{\mu}{\phi(\mu)^3(y_b-w_\mu)^3}\right)\phi(\mu)^2(w-\hat{w})^2\right\|_{L^\infty(Q)} \\
&\qquad \leq \frac{\sqrt{2}c_\phi}{(1-\theta)^3}\left\|\frac{\mu}{\phi(\mu)^3(w_\mu-y_a)^3} + \frac{\mu}{\phi(\mu)^3(y_b-w_\mu)^3}\right\|_{L^\infty(Q)}\|w-\hat{w}\|_\mu^2 \\
&\qquad \leq \frac{\sqrt{2}c_\phi}{(1-\theta)^3}\left(\left\|\frac{\mu}{\phi(\mu)^3(w_\mu-y_a)^3}\right\|_{L^\infty(Q)} + \left\|\frac{\mu}{\phi(\mu)^3(y_b-w_\mu)^3}\right\|_{L^\infty(Q)}\right)\|w-\hat{w}\|_\mu^2.
\end{aligned}$$

We show that $\frac{\mu}{\phi(\mu)^3(w_\mu-y_a)^3}$ and $\frac{\mu}{(\phi(\mu)^3(y_b-w_\mu)^3}$ are essentially bounded by $1/\sqrt{\mu}$. First, we find

$$\begin{aligned}
\phi(\mu)^3(w_\mu-y_a)^3 &= \left(\sqrt{\frac{\kappa}{\varepsilon^2}(w_\mu-y_a)^2 + \frac{\mu(w_\mu-y_a)^2}{(w_\mu-y_a)^2} + \frac{\mu(w_\mu-y_a)^2}{(y_b-w_\mu)^2}}\right)^3 \\
&\geq (\sqrt{\mu})^3.
\end{aligned}$$

From that we get $\left\|\frac{\mu}{\phi(\mu)^3(w_\mu-y_a)^3}\right\|_{L^\infty(Q)} \le \mu/\mu^{3/2} = 1/\sqrt{\mu}$. The same holds for the term containing y_b. Altogether, this yields the Lipschitz condition

(45)
$$\|\partial_w H(w;\mu)^{-1}(\partial_w H(w;\mu)-\partial_w H(\hat{w};\mu))(w-\hat{w})\|_\mu \le \frac{2\sqrt{2}c_\phi}{(1-\theta)^3\sqrt{\mu}}\|w-\hat{w}\|_\mu^2, \quad k=1,2,...$$

□

Lemma 6.8. *Let* $\|w^0 - w_{\mu_0}\|_{\mu_0} \le \theta\sqrt{\mu_0}$ *with* $\mu_0 > 0$, $0 < \theta < 1/32c_\phi$ *and let be* σ *given by* $\left(\frac{\theta+1}{\frac{4}{3}\theta+1}\right)^{1/2c_\phi} < \sigma < 1$. *Then the iterates of Algorithm 1 obey*

$$\|w^k - w_{\mu_{k+1}}\|_{\mu_{k+1}} \le \theta\sqrt{\mu_{k+1}} = \theta\sqrt{\mu_0}\sigma^{k/2}.$$

Proof. We proceed by induction. Assume that $\|w^k - w_{\mu_k}\| \le \theta\sqrt{\mu_k}$ holds for some $k \in \mathbb{N}\cup\{0\}$. It holds

$$\begin{aligned}\|w^k - w_{\mu_{k+1}}\|_{\mu_{k+1}} &\le \|w^k - w_{\mu_k}\|_{\mu_{k+1}} + \|w_{\mu_k} - w_{\mu_{k+1}}\|_{\mu_{k+1}} \\ &\le \|w^k - w_{\mu_k}\|_{\mu_{k+1}} + \left\|\int_{\mu_{k+1}}^{\mu_k} w_\tau' \, d\tau\right\|_{\mu_{k+1}}.\end{aligned}$$

By Lemma 6.3, the first item can be estimated as

$$\|w^k - w_{\mu_k}\|_{\mu_{k+1}} \le 3\sigma^{1/2-\sqrt{2}c_\phi}\|w^k - w_{\mu_k}\|_{\mu_k} \le 3\sigma^{1/2-\sqrt{2}c_\phi}\theta\sqrt{\mu_k}.$$

The term containing the integral can be estimated in the following way: Setting $\mu_{k+1} = \bar{\sigma}\tau$, $\bar{\sigma} = \frac{\mu_{k+1}}{\tau}$, we obtain

$$\begin{aligned}\left\|\int_{\mu_{k+1}}^{\mu_k} w_\tau' \, d\tau\right\|_{\mu_{k+1}} &\le \int_{\mu_{k+1}}^{\mu_k} \left(3\left(\mu_{k+1}/\tau\right)^{1/2-\sqrt{2}c_\phi}\right)\|w_\tau'\|_\tau \, d\tau \le 3\mu_{k+1}^{1/2-\sqrt{2}c_\phi}\int_{\mu_{k+1}}^{\mu_k} \tau^{-(1/2-\sqrt{2}c_\phi)}\frac{\sqrt{2}c_\phi}{\sqrt{\tau}}d\tau \\ &= 3\sqrt{2}c_\phi\mu_{k+1}^{1/2-\sqrt{2}c_\phi}\int_{\sigma\mu_k}^{\mu_k} \tau^{-1+\sqrt{2}c_\phi}\,d\tau = 3\sqrt{2}c_\phi\mu_{k+1}^{1/2-\sqrt{2}c_\phi}\left(\left.\frac{\tau^{\sqrt{2}c_\phi}}{\sqrt{2}c_\phi}\right|_{\sigma\mu_k}^{\mu_k}\right) \\ &= 3\mu_{k+1}^{1/2-\sqrt{2}c_\phi}\left(\mu_k^{\sqrt{2}c_\phi} - \sigma^{\sqrt{2}c_\phi}\mu_k^{\sqrt{2}c_\phi}\right) = 3\sigma^{1/2-\sqrt{2}c_\phi}\mu_k^{1/2-\sqrt{2}c_\phi}\left(1-\sigma^{\sqrt{2}c_\phi}\right)\mu_k^{\sqrt{2}c_\phi} \\ &= 3\sigma^{1/2-\sqrt{2}c_\phi}\left(1-\sigma^{\sqrt{2}c_\phi}\right)\mu_k^{1/2},\end{aligned}$$

where we used Lemma 6.6 and Lemma 6.5.

Summarizing up, we obtain

$$\begin{aligned}\|w^k - w_{\mu_{k+1}}\|_{\mu_{k+1}} &\le 3\sigma^{1/2-\sqrt{2}c_\phi}\theta\mu_k^{1/2} + 3\sigma^{1/2-\sqrt{2}c_\phi}\left(1-\sigma^{\sqrt{2}c_\phi}\right)\mu_k^{1/2} \\ &= 3\sigma^{1/2-\sqrt{2}c_\phi}\left(\theta+1-\sigma^{\sqrt{2}c_\phi}\right)\mu_k^{1/2} \\ &= 3\sigma^{-\sqrt{2}c_\phi}\left(\theta+1-\sigma^{\sqrt{2}c_\phi}\right)\sqrt{\mu_{k+1}}, \qquad (46)\end{aligned}$$

what gives us the constant $c(\theta,\sigma,c_\phi) := 3\sigma^{-\sqrt{2}c_\phi}\left(\theta+1-\sigma^{\sqrt{2}c_\phi}\right)$.

Now we choose σ such that $c(\theta,\sigma,c_\phi) \le 4\theta$ holds. Later, we need this result to perform one Newton step in direction w^{k+1}, where we use, that the initial value

w^k for that step is in a 4θ-ball around the associated w_{μ_k} on the central path. The desired inequality is equivalent with

$$\theta + 1 - \sigma^{\sqrt{2}c_\phi} \leq \frac{4}{3}\theta\sigma^{\sqrt{2}c_\phi}.$$

Resolving for $\sigma^{\sqrt{2}c_\phi}$, we obtain

$$\frac{\theta+1}{\frac{4}{3}\theta+1} \leq \sigma^{\sqrt{2}c_\phi}$$

and hence

$$\sigma \quad \geq \quad \left(\frac{\theta+1}{\frac{4}{3}\theta+1}\right)^{1/\sqrt{2}c_\phi}.$$

Now we have found a suitable $0 < \sigma < 1$, such that $3\sigma^{-\sqrt{2}c_\phi}\left(\theta + 1 - \sigma^{\sqrt{2}c_\phi}\right) \leq 4\theta$. Since the right-hand side is less then one, hence we can find $\sigma < 1$ satisfying this inequality. For our choice of σ, the result (46) reads

$$\|w^k - w_{\mu_{k+1}}\|_{\mu_{k+1}} \leq 4\theta\sqrt{\mu_{k+1}}. \tag{47}$$

Next, we perform one Newton-step in the direction $w_{\mu_{k+1}}$. With $c_\phi \geq 1$ in mind, we can choose e.g. $\theta = 1/32c_\phi < 1/32$, where we have $4\theta < 1/8$. In the ball $B_{\mu_{k+1}}(w_{\mu_{k+1}}, 4\theta\sqrt{\mu_{k+1}})$ we obtain by Lemma 6.7 the Lipschitz-constant

$$\omega = \frac{2\sqrt{2}c_\phi}{(1-4\theta)^3\sqrt{\mu_{k+1}}}.$$

By our assumption on w^0 and by our choice of σ and θ, we have $w^k \in B_{\mu_{k+1}}(w_{\mu_{k+1}}, 4\theta\sqrt{\mu_{k+1}}) \subset B_{\mu_{k+1}}(w_{\mu_{k+1}}, 2/\omega)$, cf.(47). Now the assumptions of the Newton-Mysovskii-Theorem are fulfilled, cf. Theorem 1.2 in [4]. (We have an "affine invariant" Lipschitz constant ω and w^k is close to $w_{\mu_{k+1}}$, i.e. $w^k \in B_{\mu_{k+1}}(w_{\mu_{k+1}}, 2/\omega)$.) The theorem provides now

$$\|w^{k+1} - w_{\mu_{k+1}}\|_{\mu_{k+1}} \leq \frac{\omega}{2}\|w^k - w_{\mu_{k+1}}\|^2_{\mu_{k+1}} < \frac{\sqrt{2}c_\phi}{(1-4\theta)^3\sqrt{\mu_{k+1}}}\|w^k - w_{\mu_{k+1}}\|^2_{\mu_{k+1}}.$$

By our choice of σ, we have by (47)

$$\|w^k - w_{\mu_{k+1}}\|^2_{\mu_{k+1}} \leq 16\theta^2\mu_{k+1},$$

and finally, we get the estimate

$$\|w^{k+1} - w_{\mu_{k+1}}\|_{\mu_{k+1}} \leq \frac{16\sqrt{2}c_\phi\theta^2}{(1-4\theta)^3\sqrt{\mu_{k+1}}}\mu_{k+1} = \frac{16\sqrt{2}c_\phi\theta^2}{(1-4\theta)^3}\sqrt{\mu_{k+1}}.$$

For $\theta = \frac{1}{32c_\phi} \leq \frac{1}{32}$ we have

$$\begin{aligned}\frac{16\sqrt{2}c_\phi\theta^2}{(1-4\theta)^3} &= \frac{\frac{16\sqrt{2}c_\phi}{32^2c_\phi^2}}{\left(1-\frac{1}{8c_\phi}\right)^3} \\ &= \frac{\sqrt{2}}{64c_\phi}\left(\frac{8c_\phi - 1}{8c_\phi}\right)^3 \leq \frac{1}{32c_\phi} = \theta.\end{aligned}$$

Altogether we have

$$\|w^{k+1} - w_{\mu_{k+1}}\|_{\mu_{k+1}} \leq \theta\sqrt{\mu_{k+1}}$$

for $\theta = 1/32c_\phi$ and for all $\sigma > \left(\frac{\theta+1}{\frac{4}{3}\theta+1}\right)^{1/2c_\phi}$. $\square$

Theorem 6.9. *Assume that* $\|w^0 - w_{\mu_0}\|_{\mu_0} \le \theta\sqrt{\mu_0}$, *where* $0 < \sigma < 1$ *and* $0 < \theta < 1$ *are given by Lemma 6.8. Then the iterates* w^k *of Algorithm 1 converge linearly towards the solution* $\bar{w}$ *of problem (P): There is some* $c > 0$, *such that*

$$\|w^k - \bar{w}\|_{L^\infty(Q)} \le c\sigma^{k/2}, \quad k = 0, 1, 2, ...$$

Proof. We have

$$\begin{aligned} \|w^k - \bar{w}\|_{L^\infty(Q)} &\le \|w^k - w_{\mu_k}\|_{L^\infty(Q)} + \|w_{\mu_k} - \bar{w}\|_{L^\infty(Q)} \\ &\le \frac{\varepsilon}{\sqrt{\kappa}}\|w^k - w_{\mu_k}\|_{\mu_k} + \|w_{\mu_k} - \bar{w}\|_{L^\infty(Q)}, \end{aligned} \tag{48}$$

where the constant $\frac{\varepsilon}{\sqrt{\kappa}}$ results from the transition from the L^∞-norm to the μ-norm, cf. Lemma 6.1. The first item can be estimated by Lemma 6.8 as

$$\|w^k - w_{\mu_k}\|_{\mu_k} \le \theta\sqrt{\mu_k} \; = \; \theta\sqrt{\mu_0}\sigma^{k/2}.$$

The second item of (48) can be estimated by the length of a segment of the central path: Theorem 5.10 yields $\|w_{\mu_k} - \bar{w}\|_{L^\infty(Q)} \le c_{path}\sqrt{\mu_k} = c_{path}\sqrt{\mu_0}\sigma^{k/2}$. Together with (48), we arrive at

$$\|w^k - \bar{w}\|_{L^\infty(Q)} \le \left(\frac{\varepsilon}{\sqrt{\kappa}}\theta + c_{path}\right)\sqrt{\mu_0}\sigma^{k/2} =: c\sigma^{k/2}.$$

$\square$

7. Numerical examples

7.1. **Discretization of the optimality system.** In Section 5.1, we have introduced the optimality system (32)–(35) for our problem with state equation (2). In view of our test examples, we will use now the extended form (5) of the state equation, for which the theory works as well, cf. Remark 3.5. In (35) we write $\eta_{a,\mu} = \frac{\mu}{\varepsilon u + y - y_a}$ and $\eta_{b,\mu} = \frac{\mu}{y_b - \varepsilon u - y}$ and we have to solve the optimality system

$$\begin{aligned} y_t - \nabla\cdot(A\nabla y) + c_0 y &= u + f && \text{in } Q, \\ \partial_n y + \alpha y &= g && \text{on } \Sigma, \\ y(0) &= y_0 && \text{in } \Omega, \end{aligned} \tag{49}$$

$$\begin{aligned} -p_t - \nabla\cdot(A\nabla p) + c_0 p &= -\tfrac{\mu}{\varepsilon u + y - y_a} + \tfrac{\mu}{y_b - \varepsilon u - y} && \text{in } Q, \\ \partial_n p + \alpha p &= 0 && \text{on } \Sigma, \\ p(T) &= y(T) - y_d && \text{in } \Omega, \end{aligned} \tag{50}$$

$$\kappa u + p - \frac{\varepsilon\mu}{\varepsilon u + y - y_a} + \frac{\varepsilon\mu}{y_b - \varepsilon u - y} \;=\; 0 \qquad \text{a.e. in } Q. \tag{51}$$

Our test examples are defined in one-dimensional domains $\Omega = (a, b)$. Let $0 = t_0 < t_1 < ... < t_n = T$ be a partion of $[0, T]$, and denote by $\delta_k = t_k - t_{k-1}$ the time steps. Define $y_k = y(\cdot, t_k)$, $u_k = u(\cdot, t_k)$, $p_k = p(\cdot, t_k)$, $g_k = g(\cdot, t_k)$,$(y_a)_k = y_a(\cdot, t_k)$, $(y_b)_k = y_b(\cdot, t_k)$, $(y_d)_k = y_d(\cdot, t_k)$, $k = 0, 1, ..., n$. Using an implicit Euler scheme for discretizing (49) and (50) in time, we have to solve a sequence of elliptic problems

$$\begin{aligned} -\nabla\cdot(A\nabla y_{k+1}) + \frac{1 + \delta_{k+1}c_0}{\delta_{k+1}}y_{k+1} &= \frac{1}{\delta_{k+1}}y_k + u_{k+1} + f_{k+1}, \\ \partial_n y_{k+1} + \alpha y_{k+1} &= g_{k+1} \end{aligned} \tag{52}$$

for $k = 0, ..., n-1$, starting at

$$y(\cdot, 0) = y_0.$$

To get a fully discrete system, we use linear finite elements to discretize the elliptic subproblems. Let $a = x_0 < x_1 < ... < x_n = b$ be a partition of $(a, b) = \Omega \subset \mathbb{R}$ with $h_i = x_{i+1} - x_i$, $i = 0, ..., n-1$. By using standard hat functions with $\varphi_i(x_j) = \delta_{ij}$, $i, j \in I$, where $I \subset \mathbb{N}$ is the set of indices of the nodes x_i, we can identify the coefficients of the FEM approximation of a function by the values of the function f in the nodes, $f(x) \approx \sum_{i \in I} f(x_i)\varphi_i(x)$. In all what follows, we identify the functions f, y, u, etc. by their coeffecent vectors $(f(x_i))$, $(y(x_i))$, $(u(x_i))$ and denote them by the same symbols, i.e., we will write f instead $(f(x_i))$ etc.

By the stiffness matrix

$$\mathbf{K} = (K_{ij}), \quad K_{ij} = \int_\Omega (a_{ij} \nabla \varphi_j) \cdot (\nabla \varphi_i)\, dx,$$

the mass matrices

$$\mathbf{M}_{k+1} = (M_{ij})_{k+1}, \quad M_{ij,k+1} = \int_\Omega \frac{1 + \delta_{k+1} c_0}{\delta_{k+1}} \varphi_j \varphi_i \, dx,$$

$$\bar{\mathbf{M}} = (\bar{M}_{ij}), \quad \bar{M}_{ij} = \int_\Omega \varphi_j \varphi_i \, dx,$$

and the matrices associated with the boundary Γ,

$$\mathbf{Q} = (Q_{ij}), \quad Q_{ij} = \int_\Gamma \alpha \varphi_j \varphi_i \, ds,$$

$$\mathbf{G} = (G_i), \quad G_i = \int_\Gamma g \varphi_i \, ds,$$

the FEM representation of the elliptic subproblems is given by

$$(\mathbf{K} + \mathbf{M}_{k+1} + \mathbf{Q}) y_{k+1} = \frac{1}{\delta_{k+1}} \bar{\mathbf{M}} y_k + \bar{\mathbf{M}}(u_{k+1} + f_{k+1}) + \mathbf{G}_{k+1}, \tag{53}$$

$k = 0, 1, ..., n-1$. Analogously, the adjoint equation is discretized by

$$\begin{aligned} (\mathbf{K} + \mathbf{M}_k + \mathbf{Q}) p_k \;\; &= \;\; \bar{\mathbf{M}} \left(\frac{\mu}{y_k + \varepsilon u_k - (y_a)_k} \right) \\ &\quad - \bar{\mathbf{M}} \left(\frac{\mu}{(y_b)_k - y_k - \varepsilon u_k} \right) + \frac{1}{\delta_k} \bar{\mathbf{M}} p_{k+1} \end{aligned} \tag{54}$$

for $k = n-1, ..., 0$ with terminal condition

$$p_n \;\; = \;\; y_n - y_d.$$

The vectors $\frac{\mu}{y_k + \varepsilon u_k - (y_a)_k}$ and $\frac{\mu}{(y_b)_k - y_k - \varepsilon u_k}$ are defined by

$$\left(\frac{\mu}{y_k + \varepsilon u_k - (y_a)_k} \right)_i = \frac{\mu}{(y_k)_i + \varepsilon (u_k)_i - ((y_a)_k)_i}$$

and

$$\left(\frac{\mu}{(y_b)_k - y_k - \varepsilon u_k} \right)_i = \frac{\mu}{((y_b)_k)_i - (y_k)_i - \varepsilon (u_k)_i},$$

for $i = 0, ..., n$, respectively. These equations are coupled through the discrete version of the gradient equation

$$\kappa u_k + p_k + \frac{\varepsilon \mu}{(y_b)_k - y_k - \varepsilon u_k} - \frac{\varepsilon \mu}{y_k + \varepsilon u_k - (y_a)_k} = 0, \tag{55}$$

for $k = 0, ..., n$.

We arrange the coefficient vectors as follows:

$$z = [y_0^T, y_1^T, \dots, y_n^T, u_0^T, u_1^T, \dots, u_n^T, p_0^T, p_1^T, \dots, p_n^T]^T.$$

The identities $y_0^T = y(0)$ and $p_n^T = y_n^T - y_d^T$ are implemented by identity matrices in the discrete optimality system. We write now the optimality conditions as a nonlinear system

$$F(z; \mu) := \Xi z + \Psi(z) + \Phi = 0,$$

where Ξ is a large, sparse matrix, essentially built of blocks $\mathbf{K} + \mathbf{M}_k + \mathbf{Q}$ on the diagonal and $\bar{\mathbf{M}}$ on the subdiagonal. Ψ is a function that covers the nonlinearity and Φ is a vector that contains the constant parts of the equations (53)–(55).

One difficulty in the Algorithm 1 is to find a suitable initial function z^0. The following steps provide a feasible initial function that can be expected sufficiently close to z_{μ_0}. Moreover, the time and space discretizations can be adapted during the computations.

Algorithm 2. *(Computation of z^0 on an adapted grid)*

(i) *Define equidistant initial partitions $\mathbf{T}_0 = \{t_0, t_0 + \delta_t, ..., T\}$ of $[0, T]$ and $\mathbf{\Omega}_0 = \{a = x_0, x_0 + h, ..., x_n = b\}$ of $\Omega = (a, b)$, where δ_t and h are the fixed initial stepsizes in time and space, respectively.*

(ii) *Choose $z_0 = (y_0^T, u_0^T, p_0^T)^T$ feasible, i.e. $y_a \le y_0 + \varepsilon u_0 \le y_b$, while p_0 can be taken arbitrarily.*

(iii) *Assemble the matrices $\boldsymbol{K}$, $\boldsymbol{M}_k$, $\bar{\boldsymbol{M}}$, $\boldsymbol{Q}$, and the vector $\boldsymbol{G}$.*

(iv) *Choose $\mu_0 > 0$. Compute a solution of*

$$F(z; \mu_0) = 0$$

by the Newton Method.

(v) *Refine the space and time grids by suitable methods.*

(vi) *Reassemble all matrices and compose the associated system matrix Ξ. Interpolate z onto the new grids.*

Remark 7.1. *After step (iv) of Algorithm 2, we have determined a solution of a discrete Newton system of PDEs. In principle, this solution might be taken as the starting value for Algorithm 1. However, our numerical experience showed that the discretization error may dominate the entire error, so that Algorithm 1 fails. Therefore, an adaptive refinement of the grid turned out to be necessary. This step is the main aim of Algorithm 2.*

The spatial grids may change between the different time steps. After Algorithm 2 is finished, the joint refinement of all spatial grids is taken as the fixed spatial grid for Algorithm 1. The discretized version of Algorithm 1 is started with z^0. For all computations, we used Matlab 7.1.0 R14 on a Pentium IV machine with 1GB memory. The linear subproblems are solved by direct methods. For refinig the meshes in Algorithm 2, we used for the time refinement ode15s with the setting RelTol = 1e-3, MaxOrder = 1, and BDF=on. For the grid refinement in space, we applied an error indicator function similar to the one described in [12]. The spatial grid is fixed in all time steps.

7.2. **Examples.**

Example 1. *We tested our method by the problem*

$$\min J(y,u) := \frac{1}{2}\|y(T) - y_d\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|u\|^2_{L^2(Q)}$$

subject to

$$\begin{aligned} y_t - \Delta y &= u && \text{in } Q, \\ \partial_n y + 10y &= 0 && \text{on } \Sigma, \\ y(0) &= y_0 && \text{in } \Omega, \end{aligned}$$

and to the mixed control-state constraints

$$y + \varepsilon u \geq y_a := \max\{-100(t(t-1) + x(x-1)) - 49.0, 0.5\} \qquad \text{a.e. in } Q.$$

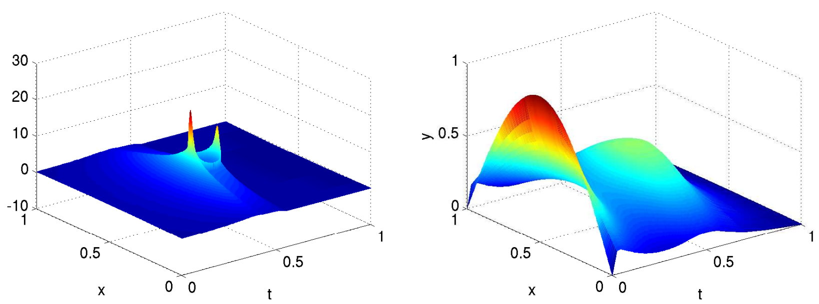
We take $\Omega = (0,1) \subset \mathbb{R}$, $T = 1$. *Further, let there be given* $y_d \equiv 0$ *and* $y_0 = \sin(\pi x)$. *Obviously, this problem fits in our general setting with* $\alpha = 10$.

Uncontrolled solutions of the heat equation are known to decay exponentially in time. The constraints are chosen to form an obstacle for this decay such that a control action is needed to fulfil them. In this way, a reasonable active set is expected. Although we do not know the exact solution of this problem, the computations confirmed this behaviour.

In our examples, there is no upper bound y_b*, but it is clear that our method covers the one-sided case as well, cf. our comments before Lemma 5.2. In contrast to the next example, here the exact optimal control* $\bar{u}$ *and the associated functions* $\bar{y}$, p *and* η_a *are unknown.*

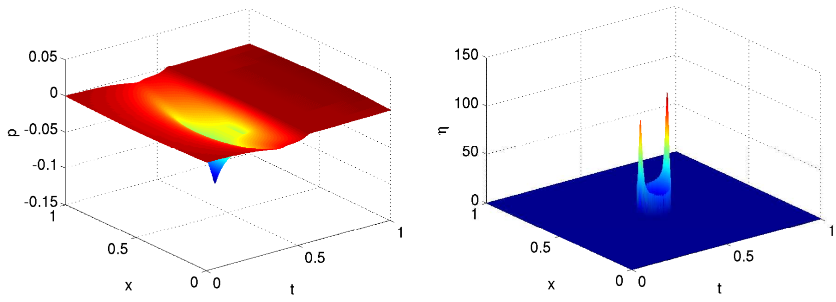
The initial vector for Algorithm 2 was z^0 *with all entries equal to zero and the initial stepsizes were* $h = 0.01$ *and* $\delta_t = 0.005$. *In Algorithm 1, we choose* $\sigma = 0.8$, $\mu_0 = 10^{-3}$, *and* $eps = 10^{-5}$. *Figures 2 and 3 show the computed optimal solutions* $\bar{y}$, p , $\bar{u}$ *and* $\eta_a = \frac{\mu}{\varepsilon u + y - y_a}$ *for the regularized problem with* $\varepsilon = 10^{-3}$ *and* $\kappa = 10^{-3}$.

In contrast to the next example, we only provide the figures of the final result, since the distance to the optimal solution cannot be estimated. In this example, we stopped Algorithm 2 after two outer iteration to refine the time and space grids. The interior-point algorithm needed up to 40 inner iterations for decreasing μ.



(a) optimal control (b) optimal state

FIGURE 2. Solutions to Example 1, control and state.



(a) adjoint state (b) Lagrange multiplier

FIGURE 3. Solutions to Example 1, adjoint and approximation on the multiplier.

Example 2. *Here, we consider the slightly modified problem*

$$\min J(y,u) := \frac{1}{2}\|y(T) - y_\Omega\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|u\|^2_{L^2(Q)} + \iint\limits_Q y_Q y \, dxdt,$$

subject to

$$(56) \qquad \begin{aligned} y_t - \Delta y &= u + f && \text{in } Q, \\ y &= 0 && \text{on } \Sigma, \\ y(0) &= y_0 && \text{in } \Omega, \end{aligned}$$

and to the mixed control-state constraints

$$y_a \le y + \varepsilon(u+f) \qquad \text{a.e. in } Q.$$

The last term in the objective function was added to construct an example with explicitly known optimal solution. This term does not change our theory. We simply have to add its derivative y_Q to the right hand side of the adjoint equation.

We construct an optimal solution which fulfills the optimality conditions (49)–(51) for the unregularized problem, i.e., for $\varepsilon = 0$.

Remark 7.2. *In Section 5, it was shown that the Lavrentiev-regularization is essential for our theory. We consider here an unregularized problem, because our aim is to construct an example with a regular Borel measure as Lagrange multiplier. In Section 4 we have shown that for $\varepsilon > 0$ the Lagrange multipliers are function from L^2, only for $\varepsilon = 0$ we will get measures. On the other hand, in some recent papers, e.g. in* [21], [14], [18], *Section 7, and* [20], *the convergence of the optimal control $\bar{u}_\varepsilon$ of the regularized problem to the optimal control $\bar{u}$ of the unregularized one is shown with order $\sqrt{\mu}$. For sufficiently small ε, e.g. $\varepsilon < 10^{-6}$, we can expect that the regularization error can be neglected in comparison with the error $\|u_\mu - \bar{u}\|$ measured in the L^2-norm. Indeed, this is our numerical observation.*

The integral $\iint\limits_Q y_Q y\, dxdt$ *in the objective function leads to the adjoint equation*

$$\begin{aligned} -p_t + \Delta p &= y_Q - \tfrac{\mu}{y - y_a} && \text{in } Q, \\ p &= 0 && \text{on } \Sigma, \\ p(T) &= y(T) - y_d && \text{in } \Omega \end{aligned}$$

instead of (50).

Construction of the optimal solution. *We choose* $\Omega = (0, \pi)$, $T = 1$, *and define the optimal state by* $\bar{y}(x,t) := e^{-t}\sin(x)$. *Together with* $\bar{y}(x,0) = \sin(x)$ *and* $\bar{y}(x,T) = e^{-1}\sin(x)$ *we obtain from (56) and* $y_t - \Delta y = 0$ *the condition* $\bar{u} + f = 0$.

From the gradient equation (51) and $\varepsilon = 0$ *we therefore get* $f = \frac{1}{\kappa}p$. *Next, we construct the state constraint such that* $\bar{y}$ *touches the bound* y_a *only on a set* $(t_1, t_2) \times \{\frac{\pi}{2}\}$. *This set has measure zero, so that we construct a Lagrange multiplier as a regular Borel measure. We take* $t_1 = 0.3$ *and* $t_2 = 0.6$. *The bound* y_a *is fixed by* $y_a(x,t) = \eta(t)\theta(x)$ *with*

$$\eta(t) = \begin{cases} \frac{1}{2}\frac{t-t_1}{t_0-t_1} + e^{-t_1}\frac{t}{t_1}, & t \in (0, t_1), \\ e^{-t}, & t \in (t_1, t_2), \\ e^{-t_2}\frac{t-1}{t_2-1} + \frac{1}{8}\frac{t-t_2}{1-t_2}, & t \in (t_2, 1), \end{cases}$$

and

$$\theta(x) = \begin{cases} \frac{3}{\pi} - 0.5, & x \in (0, \pi/2), \\ 2.5 - \frac{3}{\pi}, & x \in (\pi/2, \pi). \end{cases}$$

The adjoint state is constructed by the ansatz $p = \phi(t)v(x)$. *To this aim, let*

$$\phi(t) = \begin{cases} -\sin^2\left(\frac{\pi}{t_2-t_1}(t - t_1)\right), & t \in (t_1, t_2), \\ 0 & \textit{else}. \end{cases}$$

The derivative of ϕ is given by the continuous function

$$\phi'(t) = \begin{cases} -\frac{2\pi}{t_2-t_1}\cos\left((\frac{\pi}{t_2-t_1})(t-t_1)\right)\sin\left(\frac{\pi}{t_2-t_1}(t-t_1)\right), & t \in (t_1, t_2) \\ 0 & \text{else}. \end{cases}$$

Moreover, we introduce the continuous piecewise linear function

$$v(x) = \begin{cases} \frac{2}{\pi}x & x \in [0, \frac{\pi}{2}] \\ 2 - \frac{2}{\pi}x & x \in [\frac{\pi}{2}, \pi]. \end{cases}$$

The second derivative of $v(x)$ with respect to x is a multiple of the Dirac measure concentrated at $\pi/2$:

$$v_{xx} = -\frac{4}{\pi}\delta_{\frac{\pi}{2}}.$$

The adjoint equation gives

$$-p_t - p_{xx} = -\mu + y_Q,$$

so we can set

$$\mu = \phi(t)v_{xx} = -\phi(t)\frac{4}{\pi}\delta_{\frac{\pi}{2}} \geq 0$$

and

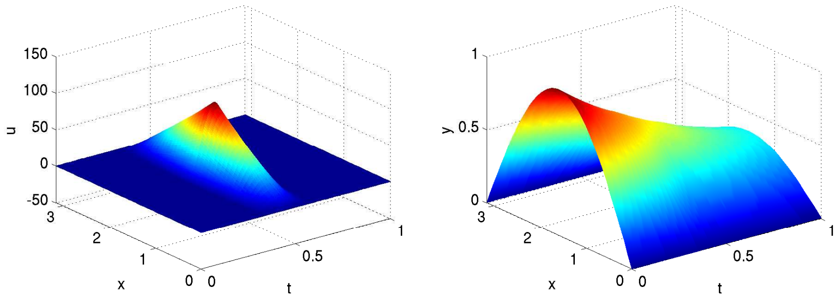
$$y_Q = -\phi'(t)v(x).$$

Obviously, μ and $y - y_a$ fulfill the complementary slackness conditions

$$\iint\limits_Q (y - y_a)\, d\mu(x,t) = 0,$$

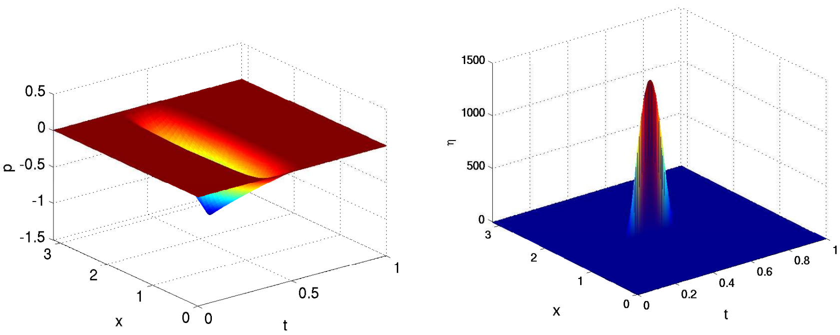
$$y - y_a \geq 0 \text{ a.e. in } Q, \qquad \mu \geq 0.$$

Having the exact optimal solutions, we are able to confirm the convergence rates for u_μ as $\mu \to 0$. We fix $\kappa = 10^{-2}$, $\varepsilon = 10^{-6}$, $\sigma = 0.8$, $\mu_0 = 10^{-3}$, and $eps = 10^{-5}$. Figures 4 and 5 show the numerical solutions.



(a) optimal control (b) optimal state

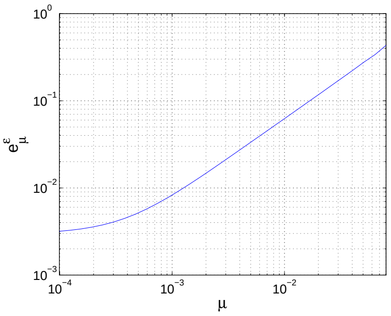
FIGURE 4. Solutions to Example 2, control and state.



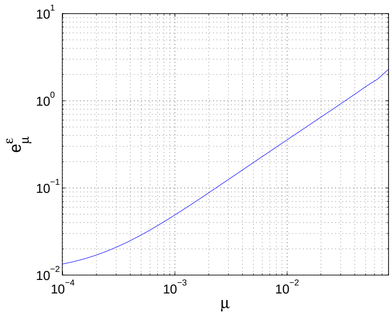
(a) adjoint state (b) Lagrange multiplier

FIGURE 5. Solutions to Example 2, adjoint and approximation on the multiplier.

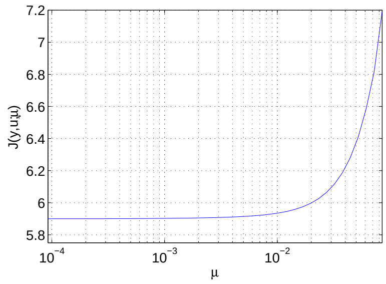
With the given exact solutions of the unregularized problem and our choice of ε, we observe linear convergence in u and y. Notice that ε is very small compared with the expected discretization error and also compared with μ. Therefore, it is reasonable to consider the distance to the exact solution at $\varepsilon = 0$ rather than to the one corresponding to $\varepsilon = 10^{-6}$. Figure 6(c) shows the value of the objective function J_μ.



(a) relative error $\|u_\mu - \bar{u}\|/\|\bar{u}\|$



(b) relative error $\|y_\mu - \bar{y}\|/\|\bar{y}\|$



(c) value of the objective function J_μ

FIGURE 6. Convergence for $\mu \to 10^{-4}$. μ-axis scaled logarithmically.

In Table 1, we present the errors of the solutions and the value of the objective function for Example 2 for selected values of μ.

μ	$\|y_\mu - \bar{y}\|/\|\bar{y}\|$	$\|u_\mu - \bar{u}\|/\|\bar{u}\|$	$\|p_\mu - \bar{u}\|/\|\bar{p}\|$	$J(y,u;\mu)$
8.0^{-2}	2.2954	4.3332^{-1}	4.3332^{-1}	7.3130
4.3980^{-3}	1.7467^{-2}	2.9738^{-2}	2.9738^{-2}	6.1299
7.3787^{-4}	3.8415^{-2}	6.6231^{-3}	6.6234^{-3}	6.1211
3.0223^{-4}	2.0936^{-2}	4.0684^{-3}	4.0685^{-3}	6.1204
9.9035^{-5}	1.3354^{-2}	3.1801^{-3}	3.1799^{-3}	6.1202

TABLE 1. Relative errors in y, u, and p , and values of $J(y,u)$ depending on μ.

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