Final Draft: Boundary feedback stabilization of the Schlögl system (This paper is accepted for publication in Automatica)

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Abstract

The Schlögl system is governed by a nonlinear reaction-diffusion partial differential equation with a cubic nonlinearity that determines three constant equilibrium states. It is a classical example of a chemical reaction system that is bistable. The constant equilibrium that is enclosed by the other two constant equilibrium points is unstable.

In this paper, Robin boundary feedback laws are presented that stabilize the system in a given stationary state or more generally in a given time-dependent desired system orbit. The exponential stability of the closed loop system with respect to the L^2-norm is proved. In particular, it is shown that with the boundary feedback law the unstable constant equilibrium point can be stabilized.

Key words: Lyapunov function, boundary feedback, Robin feedback, parabolic partial differential equation, exponential stability, stabilization of periodic orbits, periodic operation, stabilization of desired orbits, Poincaré-Friedrichs inequality.

1 Introduction

The Schlögl system has been introduced in [19] as a model for chemical reactions for non-equilibrium phase transitions. It describes the concentration of a substance in 1-d. In neurology, the same nonlinear reaction-diffusion system is known under the name Nagumo equation and models an active pulse transmission through an axon ([15], [8]). It is also known as Newell-Whitehead-Segel equation (see [16], [20]). This system is governed by a parabolic partial differential equation with a cubic nonlinearity that determines three constant equilibrium states $u_1 < u_2 < u_3$, where u_2 is unstable. In view of its simplicity, the Schlögl system may serve as a test case for the stabilization of an unstable equilibrium for reaction diffusion equations that generate traveling waves. While this task might appear a little bit academic for the Schögl model, it is of paramount importance for more complicated equations such as the bidomain system in heart medicine, cf. Kunisch and Wagner [12]. Here, the goal of stabilization is to extinguish undesired spiral waves as fast as possible and hereafter to control the system to a desired state. However, there are similarities between these models and it is therefore reasonable to consider the same problem for the Schögl system.

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The control functions can act in the domain (distributed control) or on its boundary. In this paper, the problem of boundary feedback stabilization is studied. Example 1 illustrates that, without the influence of the boundary conditions, the system state approaches exponentially fast a stable equilibrium, even if the initial state is arbitrarily close to the unstable equilibrium.

Also, the more general case of boundary stabilization of time-dependent states of the system is considered in this paper. This includes the stabilization of periodic states that is interesting as a tool to stabilize the periodic operation of reactors, see [21]. This case also includes the stabilization of traveling waves.

In this paper, linear Robin-feedback laws are presented that yield exponential stability with respect to the L^2-norm for desired orbits of the system. The term desired orbit is used to describe a possibly time–dependent solution of the partial differential equation that defines the system. The exponential stabilization is particularly interesting since the boundary feedback allows to stabilize the system in the unstable equilibrium that is enclosed by the other two constant equilibrium points.

To show that the system is exponentially stable, we construct a strict Lyapunov function. The construction of strict Lyapunov functions for semilinear parabolic partial differential equations has also been studied in [14].

In [14], it is assumed that the feedback is space-periodic or the boundary conditions are chosen in such a way that the product of the state and the normal derivative vanishes at the boundary. This assumption implies that the boundary terms that occur after partial integration in the time derivative of the Lyapunov function become nonpositive.

For the state feedback that is presented in this paper, this assumption does not hold. Therefore a different approach is used in the analysis: The Poincaré-Friedrichs inequality is used to show that the Lyapunov function is strict. Note that the Poincaré-Friedrichs inequality is often used to prove the existence or uniqueness of the solution of partial differential equations. However, to our knowledge, up to now it has not been used to show that a Lyapunov function is strict.

The Schlögl system has the interesting property that it allows traveling wave solutions (i.e. uniformly translating solutions moving with a constant velocity) that have the shape of the hyperbolic tangent (see [11]). The traveling wave solutions connect the two stable constant stationary states. The problem to steer associated wave fronts to rest by *distributed* optimal control methods was considered in [5] for the Schlögl model and in [7] for the FitzHugh-Nagumo system, where spiral waves occur. In the present paper, we propose a *boundary* control law that stabilizes the system exponentially fast to a desired orbit.

The boundary control of a linear heat equation via measurement of domain-averaged temperature has been studied in [4], [13]. Results about the control of parabolic partial differential equations with Volterra nonlinearities are given in [25], [26]. In particular, these results are applicable to semilinear parabolic equations. The constructed control laws are expressed by Volterra series. The authors prove the *local* exponential stability.

In [25] [26], a feedback law is proposed to locally stabilize stationary profiles for arbitrarily large reaction coefficients and lengths of the system. Here we show that, under restrictions on the magnitude of the reaction term and the length of the system, a simpler feedback law using boundary values only can globally exponentially stabilize any reference trajectory.

In this paper, a 1-d system of length L is studied. In the reaction-diffusion equation, the diffusion coefficient is normalized to 1. The parameter K determines the size of the reaction term. To show the exponential decay of the solution, it is assumed that $L^2 K$ is sufficiently small. Thus, if the reaction rate K is large, the space interval $[0, L]$ has to be sufficiently short. In this case, Lemma 1 states that the stationary states are uniquely determined by the corresponding boundary value problems. An example illustrates that, if $L^2\, K$ is too large, several stationary states may exist that satisfy the same Robin boundary conditions. Thus, in this situation it is impossible to stabilize the system using these Robin boundary conditions.

This paper has the following structure: In Section 2, the model is defined and a result about the well-posedness is given. Moreover, the stationary states and time-dependent orbits are discussed. In Section 3, the result about two-sided boundary feedback stabilization is presented: if the length L of the reactor is sufficiently small, there is a feedback constant $C > 0$ such that the Robin boundary conditions ensure stability. Numerical experiments illustrate the results. Section 4 contains conclusions.

2 The model

2.1 Definition of the model

In this section, the Schlögl model is defined.

Let real numbers $u_1 \le u_2 \le u_3$ be given. Define the polynomial

$$\varphi(u) = (u - u_1)(u - u_2)(u - u_3). \tag{1}$$

Due to its definition, φ has the property

$$m_\varphi = \inf_{u \in (-\infty, \infty)} \varphi'(u) > -\infty, \tag{2}$$

that is the derivative of φ is bounded below. The infimum $m_\varphi < 0$ is attained at the point $(u_1 + u_2 + u_3)/3$.

The system that is considered in this paper is governed by the semilinear parabolic partial differential equation

$$u_t = u_{xx} - K\varphi(u) \tag{3}$$

with a constant $K > 0$ complemented by appropriate initial and boundary conditions. In the reaction diffusion equation (3), the diffusion coefficient is equal to 1 and the constant K determines the size of the reaction term. If K equals zero, the reaction term vanishes and the partial differential equation (3) models a pure diffusion process.

Let the length $L > 0$ be given. Let $u^{stat} \in H^2(0, L)$ denote a stationary solution of (3), that is $u = u^{stat}$ solves the equation

$$u_{xx}(x) = K\varphi(u(x)),\ x \in [0, L]. \tag{4}$$

To define a feedback law, introduce a real constant $C > 0$. For the stabilization of (3), for $(t, x) \in [0, \infty) \times [0, L]$, consider the Robin boundary conditions

$$u_x(t, 0) = C(u(t, 0) - u^{stat}(0)) + u_x^{stat}(0), \tag{5}$$
$$u_x(t, L) = -C(u(t, L) - u^{stat}(L)) + u_x^{stat}(L). \tag{6}$$

Notice that the boundary values $u_x^{stat}(0)$ and $u_x^{stat}(L)$ are well defined since $u_x^{stat} \in H^1(0, L)$.

With the feedback laws (5), (6), if

$$L^2\, K < \frac{1}{2\,|m_\varphi|},$$

the Lyapunov function presented in Theorem 3 decays exponentially.

2.2 Existence and uniqueness of the solutions

In [5], the well-posedness of the system governed by (3) is studied for homogeneous Neumann boundary conditions. It is shown that for initial data in $L^\infty(0,L)$, the system has a unique weak solution that is continuous for $t>0$. If the initial state is continuous, the solution of the system is continuous for all times. The same result extends to the Robin boundary conditions (5),(6). In the associated theorem below, the standard Sobolev space

$$W(0,T) = L^2(0,T,H^1(0,L)) \cap H^1(0,T;H^1(0,L)')$$

is used. Moreover, the notation

$$Q_T = (0,T)\times(0,L)$$

is used.

Theorem 1 *Suppose that it holds $K \ge 0$ and $u_1 < u_2 < u_3$. Then, for all $f \in L^2(Q_T)$, $u_0 \in L^\infty(0,L)$, $g_i \in L^p(0,T)$, $i=1,2$, $p>2$, the parabolic initial-boundary value problem*

$$\begin{aligned} u_t(t,x) - K\,u_{xx}(t,x) + \varphi(u(t,x)) &= f(t,x) \quad \text{in } Q_T\\ u_x(t,0) - Cu(t,0) &= g_1(t) \quad \text{in } (0,T)\\ u_x(t,L) + Cu(t,L) &= g_2(t) \quad \text{in } (0,T)\\ u(0,x) &= u_0(x) \quad \text{in } (0,L) \end{aligned} \tag{7}$$

has a unique solution u in

$$L^\infty(Q_T)\cap W(0,T)\cap C((0,T]\times[0,L]).$$

If $u_0 \in C[0,1]$, then u is also continuous on $[0,T]\times[0,L]$.

Proof. The derivative φ' is bounded from below by $m_\varphi < 0$. Define $\mu = |m_\varphi|$ and introduce a new unknown function v by

$$u(t,x) = e^{\mu t} v(t,x).$$

Inserting this expression in (7), an easy calculation yields

$$e^{\mu t} v_t - K e^{\mu t} v_{xx} + \varphi(e^{\mu t} v) + e^{\mu t}\mu v = f$$

and finally the new initial-boundary value problem

$$\begin{aligned} v_t(t,x) - K v_{xx}(t,x) + e^{-\mu t}\varphi(e^{\mu t} v) + \mu v &= e^{-\mu t} f(t,x) \\ &\quad \text{in } Q_T \end{aligned} \tag{8}$$

$$\begin{aligned} v_x(t,0) - Cv(t,0) &= e^{-\mu t} g_1(t) \quad \text{in } (0,T)\\ v_x(t,L) + Cv(t,L) &= e^{-\mu t} g_2(t) \quad \text{in } (0,T)\\ v(0,x) &= u_0(x) \quad \text{in } (0,L). \end{aligned}$$

For each fixed t, the function $v \mapsto e^{-\mu t}\varphi(e^{\mu t} v) + \mu v$ is monotone non-decreasing and differentiable. Moreover, it is continuous w.r. to t for all fixed v. Therefore, the monotonicity and Carathèodory conditions are satisfied that are needed for existence and uniqueness of a solution v. Since $\Omega = (0,L)$ is one-dimensional, the equation (8) admits for all given $f \in L^2(Q_T)$, $g_i \in L^p(0,T)$ with $p>2$, and $u_0 \in L^\infty(0,L)$ a unique solution $v \in L^\infty(Q_T)\cap W(0,T)\cap C((0,T]\times[0,L])$. If $u_0 \in C[0,L]$, then there even holds $v \in C([0,T]\times[0,L])$. For this result on existence, uniqueness and regularity, we refer to Casas [6], Raymond and Zidani [18] or to the exposition in [22], Theorem 5.5. Associated with v, we also obtain in turn a unique solution u with the same regularity as v. □

This result will not be needed here in its full generality. In (3), (5), (6), it holds $f=0$ while the g_i are constant. In this case, $u(\cdot,t) \in H^2(0,L)$ holds for all $t>0$. Again, we slightly extend a result of [5].

Theorem 2 *Let $u_0 \in L^\infty(0,L)$ be a given initial function and assume that $f=0$ and that $g_i(t) = c_i$ for all $t \in [0,T]$. Then the solution u of (7) exhibits the regularity $u \in C((0,T],H^2(0,L))$. Therefore, for all $t \in (0,T]$, the function $u(t,\cdot)$ belongs to $H^2(0,L)$.*

Proof By Theorem 1, u is bounded, i.e. $u \in L^\infty(Q_T)$. Now Theorem 4 by Di Benedetto [3] can be applied that ensures Hölder continuity of bounded solutions to parabolic equations. By this theorem, we obtain $u \in C^{0,\alpha}([\varepsilon,T],C^{0,\alpha}[0,L])$ for arbitrarily small $\varepsilon>0$ with some Hölder constant $\alpha \in (0,1)$ that may depend on ε.

Next, the initial boundary value problem with non-homogeneous but constant boundary values

$$\begin{aligned} u_x(t,0) - Cu(t,0) &= c_1\\ &= -Cu^{stat}(0) + u_x^{stat}(0) \quad \text{in } (0,T)\\ u_x(t,L) + Cu(t,L) &= c_2\\ &= Cu^{stat}(L) + u_x^{stat}(L) \quad \text{in } (0,T) \end{aligned}$$

is transformed to one with homogeneous boundary data. To this aim, write u as $u(t,x) = v(t,x) + w(x)$, where $w(x) = \alpha_1 x + 0.5\,\alpha_2 x^2$ is constructed such that the conditions

$$\begin{aligned} w_x(0) - Cw(0) &= c_1\\ w_x(L) + Cw(L) &= c_2 \end{aligned}$$

are satisfied. Then the function v satisfies the system

$$\begin{aligned} v_t(t,x) - K\,v_{xx}(t,x) - \alpha_2 + \varphi(v(t,x) + w(x)) &= 0\\ &\quad \text{in } Q_T \end{aligned} \tag{9}$$

$$v_x(t,0) - Cv(t,0) = 0 \quad \text{in } (0,T)$$
$$v_x(t,L) + Cv(t,L) = 0 \quad \text{in } (0,T)$$
$$v(0,x) + w(x) = u_0(x) \text{ in } (0,L).$$

The Hölder continuity of u extends also to v, hence the function $F : (t,x) \mapsto -\alpha_2 + \varphi(v(t,x) + w(x))$ is also Hölder continuous on $[\varepsilon, T] \times [0,L]$ with some constant $\tilde{\alpha} \in (0,1)$. Considering v on the interval $[t_0, T]$ with $t_0 := \varepsilon$ and starting with $v_\varepsilon := v(t_0, \times) \in C[0,L] \subset L^2(0,L)$ yields

$$v_t(t,x) - Kv_{xx}(t,x) = -F(t,x), \quad t \geq t_0,$$

where the right-hand side F belongs to

$$C^{0,\tilde{\alpha}}([t_0,T], C^{0,\tilde{\alpha}}[0,L]),$$

hence also to $C^{0,\tilde{\alpha}}([t_0,T], L^2(0,L))$. Now Theorem 1.2.1 in Amann [2] can be applied, where the differential operator $A = -\partial_{xx}$ with domain $D(A) = \{v \in H^2(0,L) : v_x(0) - Cv(0) = v_x(L) + Cv(L) = 0\}$ and spaces $E_0 = L^2(0,L)$, $E_1 = H^2(0,L)$ is used.

This theorem ensures that the solution v belongs to $C((t_0,T], H^2(0,L))$. Therefore, $v(t)$ is in $H^2(0,L)$ for all $t \geq t_0 + \varepsilon = 2\varepsilon$. Since ε can be taken arbitrarily small, the claim follows immediately for v and hence also for u. □

Example 1 *Here, a solution of (3) is constructed that is independent of x. Assume that $u_1 = -1$, $u_2 = 0$ and $u_3 = 1$. Let $C_0 < 1$ be given. Define the function*

$$u(t,x) = \frac{1}{\sqrt{1 - C_0 \exp(-2Kt)}}.$$

Then, for all $t > 0$, u satisfies $u_t = -Ku(u^2 - 1)$, hence u solves (3). Note that $-u$ also solves (3).

For $\varepsilon > 0$ and $C_0 = 1 - \frac{1}{\varepsilon^2}$ we have $u(0,x) = \varepsilon$. The number $\varepsilon > 0$ can be chosen arbitrarily small, so the initial state can be arbitrarily close to the stationary state zero, but still $\lim_{t\to\infty} u(t,x) = 1$. Moreover, the state converges to 1 exponentially fast and the convergence rate is determined by the reaction rate K. This illustrates that the stationary state u_2 is unstable.

2.3 Stationary States

One of the important targets of control is to reach the stationary state $u^{stat} = u_2$ that is unstable, if $u_1 < u_2 < u_3$. For instance, in the case $u_2 = 0$, the extinction of wave type solutions to the Schögl system is of interest.

There are several types of stationary solutions. The system has three constant solutions, namely $u^{stat} = u_1$, $u^{stat} = u_2$ and $u^{stat} = u_3$.

If $u^{stat} > u_3$, $u^{stat}_{xx} = K\varphi(u^{stat}) > 0$ hence u^{stat} is a strictly convex function. On the other hand, if $u^{stat} < u_1$, $u^{stat}_{xx} = K\varphi(u^{stat}) < 0$ hence u^{stat} is a strictly concave function.

In general stationary continuous solutions of (3) that do not attain the value u_1, u_2 or u_3 are either convex or concave.

Example 2 *Assume that $\sqrt{K}\, L < \pi$. An example for a convex stationary state with $u_1 = -1$, $u_2 = 0$ and $u_3 = 1$ is*

$$u_c^{stat}(x) = \frac{\sqrt{2}}{\cos\left(\sqrt{K}(x - \frac{L}{2})\right)} \tag{10}$$

with $u^{stat}(x) \geq u^{stat}(L/2) = \sqrt{2} > 1$.

An example for a concave stationary state with $u_1 = -1$, $u_2 = 0$ and $u_3 = 1$ is $-u_c^{stat}$ with u_c^{stat} from (10).

Observe that stationary solutions that attain the value u_1, u_2 or u_3 and are not constant are neither convex nor concave.

Example 3 *For $u_1 = -1$, $u_2 = 0$ and $u_3 = 1$, solutions can be found that are neither convex nor concave and have the representation*

$$u_1^{stat}(x) = \tanh\left(C_1 - \sqrt{\frac{K}{2}}x\right). \tag{11}$$

Here, $u_1 < u_1^{stat}(x) < u_3$ and if the constant C_1 is chosen appropriately, $u_1^{stat}(x)$ attains the value $u_2 = 0$.

Now consider the constant stationary state

$$u^{stat}(x) = u_2.$$

This state satisfies the Robin boundary conditions

$$u_x(t,0) = C(u(t,0) - u_2), \tag{12}$$
$$u_x(t,L) = -C(u(t,L) - u_2). \tag{13}$$

In Lemma 1 it is stated that, if L^2K is sufficiently small and the feedback parameter C is sufficiently large, the boundary value problem (4) with the Robin boundary conditions (12) and (13) only has the constant solution u_2.

Example 4 *Consider the Jacobi elliptic function $sn(x, m)$ with $m = (0.5)^2$ (see [1]). The function $u_j^{stat}(x) = sn(x, 0.25)$ solves the differential equation*

$$(u_j^{stat})_{xx} = \frac{1}{2}\, u_j^{stat}\left((u_j^{stat})^2 - \frac{5}{2}\right).$$

Thus u_j^{stat} is a nonconstant stationary state for $K = 1/2$, $u_1 = -\sqrt{5}/\sqrt{2}$, $u_2 = 0$ and $u_3 = \sqrt{5}/\sqrt{2}$. Fig. 1 shows u_j^{stat}. Note that u_j^{stat} is periodic with respect to x (see Example 8).

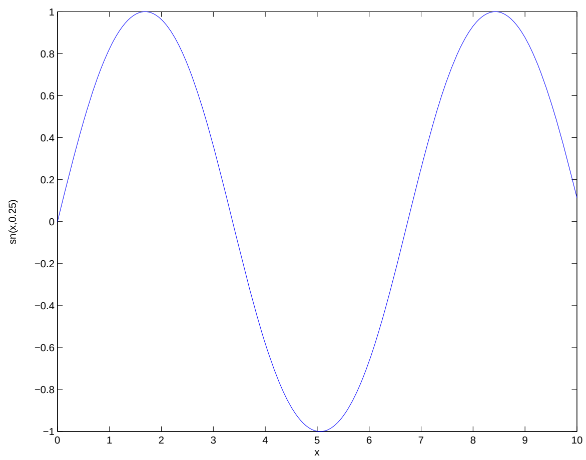


Fig. 1. The stationary state $u_j^{stat}(x)$ from Example 4

Example 5 *Now the case $u_1 = u_2 = u_3 = 0$ is considered. Choose a constant $\alpha > 0$ or $\alpha < -L$. Then*

$$u_+^{stat}(x) = \frac{\sqrt{2}}{\sqrt{K}\,(x+\alpha)},\; u_-^{stat}(x) = -\frac{\sqrt{2}}{\sqrt{K}\,(x+\alpha)}$$

are stationary states because $(u_\pm^{stat})_{xx} = K(u_\pm^{stat})^3$.

2.4 Instationary Orbits

Often in the applications, time-dependent orbits are desired. For example, time-periodic states in the periodic operation of chemical reactor systems are of interest. Here, periodic control can generate higher conversion than the steady state mode of operation. Therefore, in this section we consider time-dependent orbits, for example periodic states with period $T > 0$, that is solutions $u = u^{peri}$ of (3) with $u(t+T,x) = u(t,x)$ for all $t > 0$ and all $x \in [0,L]$. Notice that in general these states will have non homogeneous boundary data.

Periodic states can be constructed by exact controllability results for the semilinear heat equation. If the system is exactly controllable, any given stationary state can be steered to any other stationary state belonging to the same connected component of the set of stationary states. Thus, to obtain a periodic state, we can move the system periodically between the two stationary states.

The global steady-state controllability of one-dimensional semilinear heat equations has been studied in [10]. In [10], a function $y \in C^2[0,L]$ is called a steady state of (3) if $y(0) = 0$ and $u = y$ satisfies (4). Note that for our system, if $u_3 = -u_1$ and $u_2 = 0$, there holds $\varphi(u) = u(u^2 - u_1^2)$, thus $\varphi(-u) = -\varphi(u)$, i.e. φ is odd. It is stated in Proposition 3.1 in [10] that in this case the set of steady states is connected.

Thus, in this case one can move back and forth between any two steady states: one starts with a steady state u_0 at the time $t = 0$ and controls the system to a steady state u_1 at the time T_1. Then the system is controlled back to u_0 in the time T_2. Repeating the process yields a periodic state with period $T = T_1 + T_2$.

We also consider the case of general time-dependent desired orbits u^{desi} that satisfy (3).

Example 6 *Traveling waves solutions of (3) with speed v satisfy $u_t = -v u_x$. Hence (3) implies that they satisfy the ordinary differential equation*

$$u_{xx} + v\,u_x - K\,\varphi(u) = 0.$$

For real numbers $D > L$ and $\alpha > 0$,

$$y(z) = \frac{1}{1 - \exp(\alpha(z - D))}$$

solves

$$y'' + 3\alpha\,y' - 2\alpha^2\,y\,(y^2 - 1) = 0.$$

So, for $u_1 = -1$, $u_2 = 0$, $u_3 = 1$ and

$$K = 2\alpha^2,$$

the traveling waves solution $u(t,x) = y(x - 3\,\alpha\,t)$ is obtained with speed $v = 3\alpha$. In Section 3.2, the traveling waves solution

$$u^{desi}(t,x) = \frac{1}{1 - \exp(\alpha(x - 3\alpha t - D))}$$

will be considered as an example for a desired orbit.

3 Two-sided boundary feedback stabilization

3.1 Exponential Stability

In this section, we present our main result about the exponential stability in the L^2–sense of our system. A boundary feedback law is constructed that stabilizes the system around a given desired orbit u^{desi}, for example a stationary state u^{stat}, a periodic state u^{peri} or a non-periodic desired instationary orbit. To construct a stabilizing Robin-feedback, only the boundary values $u^{desi}(t,0)$, $u^{desi}(t,L)$, $u_x^{desi}(t,0)$, and $u_x^{desi}(t,L)$ of the given desired state u^{desi} are used.

An essential tool in the analysis is the 1-d POINCARÉ-FRIEDRICHS inequality (see also [22] for the general case) in the following form: let $L > 0$ be given. For all $u \in H^1(0,L)$, the following inequality holds:

$$\int_0^L u^2(x)\,dx \tag{14}$$

$$\leq L\,\left[u(0)^2 + u(L)^2\right] + 2L^2 \int_0^L (\partial_x u(x))^2\,dx.$$

Proof: Consider the inequality

$$u^2(x) = \frac{1}{2}\left(u(0) + \int_0^x \partial_x u(s)\,ds\right)^2 + \frac{1}{2}\left(u(L) - \int_x^L \partial_x u(s)\,ds\right)^2 \le \left(u(0)^2 + \left(\int_0^L |\partial_x u(s)|\,ds\right)^2\right) + \left(u(L)^2 + \left(\int_0^L |\partial_x u(s)|\,ds\right)^2\right) \le u(0)^2 + u(L)^2 + 2L\int_0^L (\partial_x u(s))^2\,ds.$$

Integrating on the interval $[0, L]$ yields (14). □

Now the stabilization result for desired orbits is given.

Theorem 3 (Exponential Stability) *Assume $L > 0$ is sufficiently small in the sense that*

$$L^2\,K < \frac{1}{2\,|m_\varphi|}. \tag{15}$$

Let a desired state $u^{desi} \in H^2(Q_T)$ be given that satisfies (3). Let a feedback parameter $C \ge \frac{1}{2L}$ and an initial state $u_0 \in L^\infty(0, L)$ be given.

Then the solution u of (3) subject to

$$u_x(t,0) = C(u(t,0) - u^{desi}(t,0)) + u_x^{desi}(t,0) \tag{16}$$

$$u_x(t,L) = -C(u(t,L) - u^{desi}(t,L)) + u_x^{desi}(t,L) \tag{17}$$

converges exponentially fast in the L^2-sense to u^{desi} in the sense that for

$$\mu = \frac{1}{L^2} - 2\,K\,|m_\varphi|$$

there holds the inequality

$$\int_0^L \left(u(t,x) - u^{desi}(t,x)\right)^2\,dx \le \int_0^L \left(u_0(x) - u^{desi}(0,x)\right)^2\,dx\ \exp\left(-\mu t\right). \tag{18}$$

The function $V(t) = \frac{1}{2}\int_0^L \left(u(t,x) - u^{desi}(t,x)\right)^2\,dx$ is a strict Lyapunov function for the system (3), (16), (17) in the sense that it satisfies the inequality

$$V'(t) \le -\mu V(t).$$

Proof. Define the function

$$V(t) = \frac{1}{2}\int_0^L \left(u(t,x) - u^{desi}(t,x)\right)^2\,dx. \tag{19}$$

For initial data u_0 in $H^2(0, L)$, the time-derivative of V obeys

$$\begin{aligned} &V'(t) \\ &= \textstyle\int_0^L \left(u(t,x) - u^{desi}(t,x)\right)\left(u_t(t,x) - u_t^{desi}(t,x)\right)\,dx \\ &= \textstyle\int_0^L \left(u(t,x) - u^{desi}(t,x)\right)\left(u_{xx}(t,x) - u_{xx}^{desi}(t,x)\right)\,dx \\ &-K\textstyle\int_0^L \left(u(t,x) - u^{desi}(t,x)\right) \\ &\qquad\left(\varphi(u(t,x)) - \varphi(u^{desi}(t,x))\right)\,dx \\ &= -\textstyle\int_0^L \left(u_x(t,x) - u_x^{desi}(t,x)\right)^2\,dx \\ &+ \left[\left(u(t,x) - u^{desi}(t,x)\right)\left(u_x(t,x) - u_x^{desi}(x)\right)\right]|_{x=0}^L \\ &-K\textstyle\int_0^L \left(u(t,x) - u^{desi}(t,x)\right) \\ &\qquad\left(\varphi(u(t,x)) - \varphi(u^{desi}(t,x))\right)dx. \end{aligned}$$

Thanks to the boundary feedback conditions (16), (17), this yields

$$\begin{aligned} &V'(t) = -\textstyle\int_0^L \left(u_x(t,x) - u_x^{desi}(t,x)\right)^2\,dx \\ &-C\left(u(t,0) - u^{desi}(t,0)\right)^2 \\ &-C\left(u(t,L) - u^{desi}(t,L)\right)^2 \\ &-K\textstyle\int_0^L \left(u(t,x) - u^{desi}(t,x)\right) \\ &\qquad\left(\varphi(u(t,x)) - \varphi(u^{desi}(t,x))\right)dx. \end{aligned}$$

Due to (2), for all $v_1,\, v_2 \in (-\infty, \infty)$ it holds

$$(v_2 - v_1)(\varphi(v_2) - \varphi(v_1)) \ge (v_2 - v_1)^2 m_\varphi.$$

This yields the inequality

$$\begin{aligned} &V'(t) \le -\textstyle\int_0^L \left(u_x(t,x) - u_x^{desi}(t,x)\right)^2\,dx \\ &-C\left(u(t,0) - u^{desi}(t,0)\right)^2 \\ &-C\left(u(t,L) - u^{desi}(t,L)\right)^2 \\ &-K\,m_\varphi\textstyle\int_0^L \left(u(t,x) - u^{desi}(t,x)\right)^2\,dx. \end{aligned}$$

Since $C \ge \frac{1}{2L}$, due to the Poincaré-Friedrichs inequality (14) this implies

$$\begin{aligned} &V'(t) \le -\tfrac{1}{2L^2}\textstyle\int_0^L \left(u(t,x) - u^{desi}(t,x)\right)^2\,dx \\ &+K\,|m_\varphi|\textstyle\int_0^L \left(u(t,x) - u^{desi}(t,x)\right)^2\,dx \\ &= -\left(\tfrac{1}{2L^2} - K\,|m_\varphi|\right)\textstyle\int_0^L \left(u(t,x) - u^{desi}(t,x)\right)^2\,dx \\ &= -\left(\tfrac{1}{L^2} - 2K\,|m_\varphi|\right)V(t) \\ &= -\mu\,V(t). \end{aligned}$$

Since $H^2(0,L)$ is dense in $L^\infty(0,L)$, the same estimate remains true (by continuous extension) for any initial state u_0 in $L^\infty(0,L)$. Thus V is a strict Lyapunov function and the assertion (18) follows (see for example [9]).

□

Example 7 *Let $u_1 = -1$, $u_2 = 0$ and $u_3 = 1$. Then $m_\varphi = -1$ and Theorem 3 states that, if*

$$L^2\, K < \frac{1}{2},$$

the Lyapunov function decays exponentially, i.e. the state converges exponentially fast in the L^2-sense to a desired orbit u^{desi} with the rate $\mu = \frac{1}{L^2} - 2K$.

As proved below, an interesting consequence of Theorem 3 is the following uniqueness result: the stationary state u^{stat} that solves the equation (4) subject to the Robin boundary conditions (5) and (6) with given values $u^{stat}(0)$, $u^{stat}_x(0)$, $u^{stat}(L)$, $u^{stat}_x(L)$ as parameters in the boundary conditions, is uniquely determined provided that C is sufficiently large and $L^2\, K > 0$ is sufficiently small.

Lemma 1 (Uniqueness) *Let real numbers s_{00}, s_{01}, s_{L0}, s_{L1} be given.*

If $L^2\, K < \frac{1}{2\,|m_\varphi|}$ and $C \geq \frac{1}{2L}$, then the boundary value problem $u_{xx} = K\varphi(u)$, with the Robin boundary conditions

$$u_x(0) = C(u(0) - s_{00}) + s_{01} \tag{20}$$
$$u_x(L) = -C(u(L) - s_{L0}) + s_{L1} \tag{21}$$

has at most one solution. In particular, this implies that for $j \in \{1,2,3\}$ the boundary value problem with the Robin boundary conditions

$$u_x(0) = C(u(0) - u_j) \tag{22}$$
$$u_x(L) = -C(u(L) - u_j) \tag{23}$$

only has the constant solution u_j.

Proof. Suppose that the stationary states u^{stat}_1 and u^{stat}_2 solve the boundary value problem. In particular, both satisfy the boundary conditions (20), (21). Choose the initial state $u_0 = 0$ and $u^{desi} = u^{stat}_1$. Then the state u that solves the initial boundary value problem (3), (16), (17) considered in Theorem 3 is well-defined. Then u satisfies the boundary conditions

$$u_x(t,0) = C(u(t,0) - u^{stat}_1(0)) + (u^{stat}_1)_x(0) \tag{24}$$
$$u_x(t,L) = -C(u(t,L) - u^{stat}_1(L)) + (u^{stat}_1)_x(L). \tag{25}$$

Thus Theorem 3 implies

$$\int_0^L \left(u(t,x) - u^{stat}_1(x)\right)^2\, dx \tag{26}$$

$$\leq \int_0^L \left(u^{stat}_1(x)\right)^2\, dx\; \exp\left(-\mu t\right).$$

Moreover, u also satisfies the boundary conditions

$$\begin{aligned} u_x(t,0) - Cu(t,0) &= (u^{stat}_1)_x(0) - Cu^{stat}_1(0) \\ &= (u^{stat}_2)_x(0) - Cu^{stat}_2(0), \\ u_x(t,L) + Cu(t,L) &= (u^{stat}_1)_x(L) + Cu^{stat}_1(L) \\ &= (u^{stat}_2)_x(L) + Cu^{stat}_2(L), \end{aligned}$$

hence there holds

$$u_x(t,0) = C(u(t,0) - u^{stat}_2(0)) + (u^{stat}_2)_x(0), \tag{27}$$
$$u_x(t,L) = -C(u(t,L) - u^{stat}_2(L)) + (u^{stat}_2)_x(L). \tag{28}$$

Thus Theorem 3 implies

$$\int_0^L \left(u(t,x) - u^{stat}_2(x)\right)^2\, dx \tag{29}$$

$$\leq \int_0^L \left(u^{stat}_2(x)\right)^2\, dx\; \exp\left(-\mu t\right).$$

In particular, this implies that for all $t > 0$ we have

$$\begin{aligned} &\left(\int_0^L \left(u^{stat}_1(x) - u^{stat}_2(x)\right)^2\, dx\right)^{1/2} \\ \leq &\left(\int_0^L \left(u(t,x) - u^{stat}_1(x)\right)^2\, dx\right)^{1/2} \\ + &\left(\int_0^L \left(u(t,x) - u^{stat}_2(x)\right)^2\, dx\right)^{1/2} \\ \leq &\left(\int_0^L \left(u^{stat}_1(x)\right)^2\, dx\right)^{1/2} \exp\left(-\frac{\mu}{2}t\right) \\ + &\left(\int_0^L \left(u^{stat}_2(x)\right)^2\, dx\right)^{1/2} \exp\left(-\frac{\mu}{2}t\right). \end{aligned}$$

Since t can be chosen arbitrarily large, this yields

$$\left(\int_0^L \left(u^{stat}_1(x) - u^{stat}_2(x)\right)^2\, dx\right) = 0$$

and thus $u^{stat}_1 = u^{stat}_2$ holds almost everywhere, which yields the uniqueness. □

The following examples illustrate that, if L^2K is too large, i.e. if the assumptions of Lemma 1 do not hold, the boundary value problem $u_{xx} = K\varphi(u)$, (20), (21) in general does not have a unique solution. In this case, Robin boundary feedback stabilization is not possible, but for local stabilization the approach from [25], [26] can be used.

Example 8 *Consider Example 4. In this case, we have* $K = 1/2$, $u_1 = -\sqrt{5/2}$, $u_2 = 0$, $u_3 = \sqrt{5/2}$ *and* $m_\varphi = -5/2$, *thus (15) and Lemma 1 require* $L \leq \sqrt{\frac{2}{5}}$.

For the nonconstant stationary state $u_j^{stat} = sn(\cdot, m)$, *where* $sn(\cdot, m)$ *denotes the Jacobi elliptic function with* $m = (0.5)^2$ *(see Fig. 1), we have* $u_j^{stat}(0) = 0$ *and* $(u_j^{stat})_x(0) > 0$.

Since $u_j^{stat}(x)$ *is oscillating, one can find a number* L *such that* $u_j^{stat}(L) = 0$ *and* $(u_j^{stat})_x(L) = -(u_j^{stat})_x(0)$. *Then, for*

$$C := -\frac{(u_j^{stat})_x(0)}{u_1} = \frac{(u_j^{stat})_x(L)}{u_1} > 0,$$

the function u_j^{stat} *satisfies (22) and (23) for the constant stationary state* u_1. *In particular, in this case, the corresponding boundary value problem does* not *have a unique solution. The smallest possible choice of* L, *where this construction works, is the smallest strictly positive root of* $u_j^{stat}(x)$ *which is strictly greater than* $\sqrt{\frac{2}{5}}$.

3.2 Numerical Experiments

In our numerical experiments, we select $u_1 = -1$, $u_2 = 0$, and $u_3 = 1$. Then we have

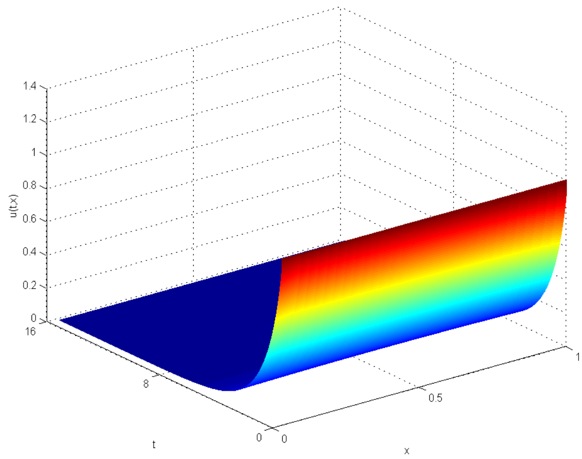
$$\varphi(u) = u(u^2 - 1)$$

and it holds $m_\varphi = -1$. Let $L = 1$ and $C = \frac{1}{2}$; the constant K will be specified below.

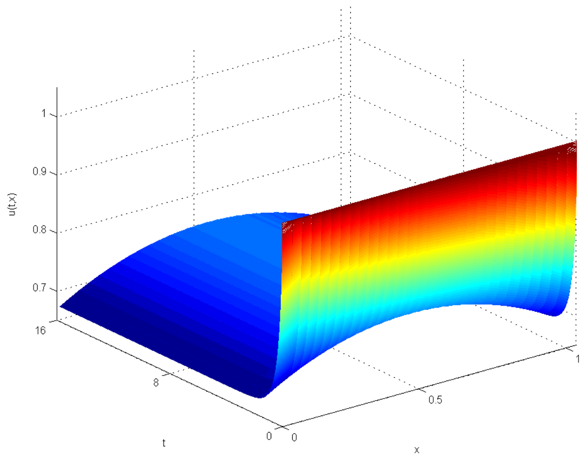
In the Examples 9 to 11, the stationary state $u^{stat} = 0$ is considered. The corresponding Robin feedback is given by

$$u_x(t, 0) = Cu(t, 0), \quad (30)$$
$$u_x(t, L) = -Cu(t, L). \quad (31)$$

Example 9 *Select* $K = \frac{1}{4}$. *Then (15) holds; fix* $u_0 = 1$. *We have discretized the system by the method of lines on an equidistant grid of* 31 *nodes. In Fig. 2a, the resulting approximation of the state* u *is displayed on the time interval* $[0, 16]$. *As theoretically predicted, the state converges exponentially fast to zero with respect to the time.*



(a) The generated state for Example 9



(b) The generated state for Example 10

Fig. 2. Pictures of the states for Examples 9 and 10

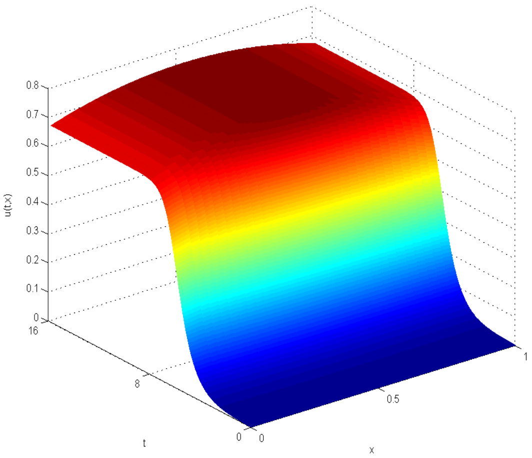
Example 10 *Consider now* $K = 2$. *Then the inequality (15) is not satisfied. Let again* $u_0 = 1$. *Fig. 2b displays the resulting approximation of the state* $u(t, x)$ *on the time interval* $[0, 16]$.

Note that the state converges with time to a nonconstant concave stationary state with values between zero and one, and not to the zero state. This nonconstant stationary state is compatible with the homogeneous Robin boundary conditions (30), (31). The assumptions of the uniqueness Lemma 1 are not fulfilled here. So this example illustrates that if K *is too large, in general the boundary value problem considered in Lemma 1 does not have a unique solution.*

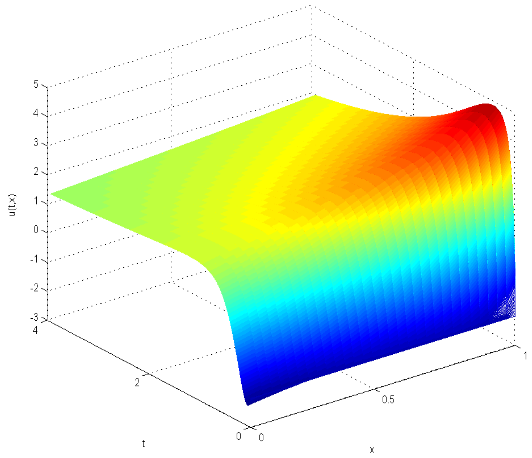
Example 11 *As in Example 10 fix* $K = 2$; *then (15) does not hold. Here we start with the inital state* $u_0 = 0.001$ *that is quite close to zero. Fig. 3a shows the resulting approximation of the state* $u(t, x)$ *on the time interval* $[0, 16]$.

Note that the state converges with time to the nonconstant concave stationary state that is also the limit in Example 10, although this time the initial state is very close to the zero state that is also stationary.

Example 12 *Now the desired orbit* u^{desi} *defined in Example 6 is considered. Select* $\alpha = \frac{1}{4}$ *and* $K = \frac{1}{8}$. *Then (15) holds. We choose* $D = 2$ *and start with* $u_0 = -2$.



(a) The generated state for Example 11



(b) The generated state for Example 12

Fig. 3. Pictures of the states for Examples 11 and 12

The boundary feedback is defined in (16), (17). Fig. 3b illustrates that the system state approaches the desired traveling wave very fast, as we could expect from Theorem 3.

4 Conclusion

In this paper, boundary feedback laws have been provided that stabilize the Schlögl system globally to a given stationary state. It was shown that, with a similar feedback of Robin type, the system can also be stabilized to a given desired instationary orbit.

The case of feedback on both ends of the interval was considered. In the feedback laws, the values and the derivatives of the stationary states at the boundary points are used. A strict Lyapunov function was constructed to show the exponential stability of the resulting closed-loop system in the L^2-sense. In fact, it can be shown that also the velocity decays exponentially. In the proof of the exponential stability, the 1-d Poincaré-Friedrichs inequality was used.

For these results, it was assumed that L^2K is sufficiently small and C is sufficiently large. This assumption also ensures that the stationary states of our system are uniquely determined by the corresponding Robin boundary conditions. It was illustrated by an example that, if the assumptions are violated, the boundary conditions do not in general uniquely determine a stationary state.

For one-sided feedback with a homogeneous Neumann boundary condition at the free end, a similar analysis is possible for the stabilization to stationary states: if L^2K is sufficiently small (in the sense of (15)) and if $C \geq 1/(2L)$, then V decays exponentially.

The extension of the results to the case of chemical reactors that can be modeled by coupled systems of reaction diffusion equations is an open problem. This would allow the stabilization of the periodic operation of such reactors, which is discussed for example in [21].

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