SECOND ORDER AND STABILITY ANALYSIS FOR OPTIMAL SPARSE CONTROL OF THE FITZHUGH-NAGUMO EQUATION *

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Abstract. Optimal sparse control problems are considered for the FitzHugh-Nagumo system including the so-called Schlögl model. The non-differentiable objective functional of tracking type includes a quadratic Tikhonov regularization term and the L^1-norm of the control that accounts for the sparsity. Though the objective functional is not differentiable, a theory of second order sufficient optimality conditions is established for Tikhonov regularization parameter $\nu > 0$ and also for the case $\nu = 0$. In this context, also local minima are discussed that are strong in the sense of the calculus of variations. The second order conditions are used as main assumption for proving the stability of locally optimal solutions with respect to $\nu \to 0$ and with respect to perturbations of the desired state functions. The theory is confirmed by numerical examples that are resolved with high precision to confirm that the optimal solution obeys the system of necessary optimality conditions.

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Introduction. This paper contributes to several recent developments in the optimal control of partial differential equations. First, and this is perhaps the main novelty or our work, it improves the theory of second order sufficient optimality conditions for the optimal control of nonlinear evolution equations. We prove results that are not only new for the control of the FitzHugh-Nagumo equations. They are also not yet known for the optimal control theory of standard academic semilinear elliptic or parabolic control problems with smooth objective functionals. We even go beyond this and consider problems with non-smooth functionals in the context of sparse optimal control, where the L^1-norm of the control appears in the objective functional.

Our control system of FitzHugh-Nagumo equations plays an important role in physics, chemistry, and mathematical biology. Here, we continue our research in [8] on the optimal control of wave-type solutions such as travelling waves or spiral waves that appear as typical solutions of this system in unbounded spatial domains. Investigating this class of control problems with its interesting applications, we observed numerical effects that we wanted to confirm by a deeper mathematical analysis. Eventually, we arrived at novel second order conditions that we needed to explain our numerical observations that are examplarily presented at the end of this paper.

Let us detail our main achievements in the theory of second-order sufficient optimality conditions a bit more:

Second order conditions are related to certain critical cones that must be chosen as small as possible. For sparse controls, in Lemma 3.1-3 we introduce the cone of critical directions $C_{\bar{u}_\nu}$ that is smaller than the associated one suggested in [7]. In this

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way, we improve the results of [7] that were developed for a simple class of elliptic equations. Even for this class of simpler problems, our result is new.

Moreover, our paper contains second-order conditions that are sufficient for *strong local minima* in the sense of calculus of variations. So far, almost all contributions to the theory of second-order conditions in the optimal control of PDEs addressed *weak local minima*. To our best knowledge, the first result on strong local minima in PDE control was recently obtained in [1] for the case of semilinear elliptic equations. Although our results are similar to the ones of [1], they are more general, even if they are transferred to the elliptic case discussed there. In particular, we introduce two extended cones of critical directions, $C_{\bar{u}}^\tau$ in (3.14) and $E_{\bar{u}}^\tau$ prior to Theorem 3.13. They can be used to deal with the case of vanishing Tikhonov regularization parameter ν, cf. more detailed remarks in the introduction to Section 3. We do not know papers on the control of PDEs that deal with strong local minima for the case $\nu = 0$.

The case $\nu = 0$ is important for the numerical application of control methods to the FitzHugh-Nagumo equations that we present in the last section. In our preceding paper [8], we observed that the numerical methods worked fairly stable also for very small parameter $\nu > 0$ so that we became interested in the convergence analysis as $\nu \to 0$. To prove the observed stability of optimal controls for $\nu \to 0$, we had to develop our new second order conditions that are applicable to the case $\nu = 0$. This is a situation, where strong local minima are needed. We also discuss the stability of optimal solutions with respect to perturbations of the given desired state functions.

Compared with standard semilinear elliptic or parabolic equations, the analysis of the FitzHugh-Nagumo system is more difficult. Therefore, we have to prove the fairly technical Lemmata 3.8–3.11. Although these results do not directly extend the optimal control theory, their detailed proofs are unavoidable to build our theory and occupy a major part of our paper.

The analysis of optimal control problems for FitzHugh-Nagumo systems was already considered in [3], who investigated associated problems by the Dubovitskij-Milyutin optimality conditions and in [13], who concentrated on the analysis of time-optimal control problems for a linear version of the FitzHugh-Nagumo equations. Later, resolving certain obstacles in the analysis of differentiability, in [8] we proved the first-order optimality conditions for the (nonlinear) FitzHugh-Nagumo equations by showing the differentiability of the control-to-state mapping of any order. Moreover, we reported on a variety of computational results. In this context, we also mention [4], where the analysis and numerical treatment of optimal control problems for traveling wave fronts is discussed for the so-called Schlögl model (also known as Nagumo equation).

The discussion of control and feedback control problems of reaction-diffusion equations has a long tradition in the community of theoretical physics. We mention, for instance, contributions in [15], [17], [20] and the references therein. We also refer to the survey volume by [16]. Our investigations on the control of FitzHugh-Nagumo equations and systems of related type were initiated by these ideas.

1. Control problem. Assumptions and preliminary results. In this paper, we consider optimal control problems for the following two reaction-diffusion

equations:

$$\begin{cases}
\dfrac{\partial y}{\partial t}(x,t) - \Delta y(x,t) + R(y(x,t)) + \alpha\, z(x,t) &= u(x,t) & \text{in } Q \\[2mm]
\partial_n y(x,t) &= 0 & \text{on } \Sigma \\[1mm]
y(x,0) &= y_0(x) & \text{in } \Omega \\[2mm]
\dfrac{\partial z}{\partial t}(x,t) + \beta\, z(x,t) - \gamma\, y(x,t) + \delta &= 0 & \text{in } Q \\[2mm]
z(x,0) &= z_0(x) & \text{in } \Omega.
\end{cases} \qquad (1.1)$$

Here, $\Omega \subset \mathbb{R}^N$, $1 \le N \le 3$, is a bounded Lipschitz domain with boundary Γ; $T > 0$, $Q = \Omega \times (0,T)$, and $\Sigma = \Gamma \times (0,T)$. The parameters α, β, γ, and δ are real numbers with $\beta \ge 0$. Further, R is a cubic polynomial,

$$R(y) = a_R\,(y - y_1)(y - y_2)(y - y_3)$$

with real numbers $a_R > 0$ and $y_1 \le y_2 \le y_3$. Moreover, y_0 and z_0 are given initial states belonging to $L^\infty(\Omega)$. By n and ∂_n, we denote the outward unit normal vector and the associated outward normal derivative on Γ, respectively.

In this system, the partial differential equation for y is said to be the *activator equation*, while the one for z is called the *inhibitor equation*. The function y is the *state* that is to be controlled. In our paper, the inhibitor z has only some auxiliary character with respect to the control. For the choice $\alpha = 0$, both equations decouple and the state function y has to solve the Schlögl equation. We should mention that this equation is also known as Nagumo equation and it is also a particular case of the Allan-Cahn equation. Here, the inhibitor equation is meaningless. For $\alpha = 1$, the FitzHugh-Nagumo system is obtained. These values $\alpha \in \{0,\,1\}$ are the values of our interest, but the analysis for the system (1.1) is the same for any arbitrary real α.

Following the usual notation we set

$$W(0,T) = \left\{ y \in L^2(0,T; H^1(\Omega)) : \frac{\partial y}{\partial t} \in L^2(0,T; H^1(\Omega)^*) \right\}.$$

The following theorem was proved in [8, Theorem 2.1 and Corollary 2.1].

THEOREM 1.1. *Under the previous assumptions, for all $u \in L^p(Q)$ with $p > 5/2$, the equation (1.1) has a unique solution $(y_u, z_u) \in [W(0,T) \cap L^\infty(Q)]^2$. There exists a constant C independent of u such that*

$$\|y_u\|_{L^\infty(Q)} + \|y_u\|_{W(0,T)} + \|z_u\|_{L^\infty(Q)} + \|z_u\|_{W(0,T)}$$
$$\le C(\|u\|_{L^p(Q)} + \|y_0\|_{L^\infty(\Omega)} + \|z_0\|_{L^\infty(\Omega)} + |R(0)|)$$

and we have $(y_u, z_u) \in C(\bar\Omega \times (0,T])^2$. In addition, if y_0 and z_0 are continuous in $\bar\Omega$, then y_u and z_u belong to $C(\bar\Omega \times [0,T])$.

Following again [8], we introduce the mapping $G : L^p(Q) \longrightarrow [W(0,T) \cap L^\infty(Q)]^2$, defined by $G(u) = (y_u, z_u)$. Then, the following differentiability properties were proved in [8, Theorem 2.2].

THEOREM 1.2 (Differentiability of the control-to-state mapping). *The mapping G is of class C^2. The derivative $(\eta_v, \zeta_v) := G'(u)v$ is equal to the pair (η, ζ) solving*

the system

$$\begin{cases} \dfrac{\partial \eta}{\partial t} - \Delta \eta + R'(y_u)\eta + \alpha\,\zeta & = & v & in\ Q \\[2mm] \partial_n \eta & = & 0 & on\ \Sigma \\[2mm] \eta(x,0) & = & 0 & in\ \Omega \\[2mm] \dfrac{\partial \zeta}{\partial t} + \beta\,\zeta - \gamma\,\eta & = & 0 & in\ Q \\[2mm] \zeta(x,0) & = & 0 & in\ \Omega. \end{cases} \qquad (1.2)$$

The second derivative $(\omega_{v_1,v_2}, \chi_{v_1,v_2}) := G''(u)[v_1,v_2]$ *in the directions* $v_1,\ v_2 \in L^p(Q)$ *is given by the pair* (ω,χ) *solving the equation*

$$\begin{cases} \dfrac{\partial \omega}{\partial t} - \Delta \omega + R'(y_u)\omega + \alpha\,\chi & = & -R''(y_u)\,\eta_{v_1}\eta_{v_2} & in\ Q \\[2mm] \partial_n \omega & = & 0 & on\ \Sigma \\[2mm] \omega(x,0) & = & 0 & in\ \Omega \\[2mm] \dfrac{\partial \chi}{\partial t} + \beta\,\chi - \gamma\,\omega & = & 0 & in\ Q \\[2mm] \chi(x,0) & = & 0 & in\ \Omega, \end{cases} \qquad (1.3)$$

where $\eta_{v_i} = G'(u)v_i$, $i = 1,2$.

Remark 1.3. *Though* G *is not a differentiable function from* $L^2(Q)$ *to* $W(0,T)^2$, *it is not difficult to check that* $G'(u)$ *and* $G''(u)$ *can be extended to continuous linear and bilinear mappings from* $L^2(Q)$ *to* $W(0,T)^2$, *for any* $u \in L^p(Q)$ *with* $p > \frac{5}{2}$.

Now, we formulate our control problem

$$(\mathrm{P}_\nu) \begin{cases} \mathrm{Min} & J_\nu(u), \\ u \in \mathcal{U}_{\mathrm{ad}} \end{cases}$$

where $J_\nu(u) = F_\nu(u) + \kappa j(u)$ with

$$F_\nu(u) = \frac{1}{2}\int_Q \{C_Q^Y(x,t)(y_u(x,t) - y_Q(x,t))^2 + C_Q^Z(x,t)(z_u(x,t) - z_Q(x,t))^2\}\,dx\,dt$$

$$+ \frac{1}{2}\int_\Omega \{C_T^Y(x)(y_u(x,T) - y_T(x))^2 + C_T^Z(x)(z_u(x,T) - z_T(x))^2\}\,dx$$

$$+ \frac{\nu}{2}\int_Q u^2(x,t)\,dx\,dt,$$

$$j(u) = \int_Q |u(x,t)|\,dx\,dt.$$

In this setting, we have given constants $\nu \geq 0$, $\kappa > 0$, coefficients $C_Q^Y, C_Q^Z \in L^\infty(Q)$, $C_T^Y, C_T^Z \in L^\infty(\Omega)$, desired state functions $y_Q, z_Q \in L^p(Q)$ with $p > 5/2$, $y_T, z_T \in L^\infty(\Omega)$, and the set of admissible controls

$$\mathcal{U}_{\mathrm{ad}} = \{u \in L^\infty(Q) : a \leq u(x,t) \leq b \text{ for a.a. } (x,t) \in Q\}$$

with $-\infty < a < 0 < b < +\infty$. For the choice $\alpha = 0$ in the state equation (1.1), we fix $C_Q^Z = z_Q = 0$ and $C_T^Z = z_T = 0$. We also assume that the functions C_Q^Y, C_Q^Z, C_T^Y and C_T^Z are nonnegative.

Remark 1.4. *We introduced fairly general coefficients in the objective functional F_ν. The reason for that are fairly different applications we have in mind. For instance, in [8] we considered the problem of extinguishing a moving spiral wave. To do so, we observed only the tip of the desired spiral wave that moved from the center to the boundary of the domain Ω. Here, C_Q^Y was the characteristic function of a circle around the tip that moved with the tip through Ω. We also mention that our theory works under the slightly weaker assumption $y_T, z_T \in L^p(\Omega)$, too. However, this would lead to more technicalities, hence we do not use this assumption.*

Thanks to Theorems 1.1 and 1.2, it is easy to prove that for all $\nu \geq 0$ the control problem (P_ν) has at least one solution $\bar{u}_\nu$; see [8, Theorem 3.1].

We finish this section by analyzing the cost functional J_ν that consists of two terms having different smoothness. While the first part is smooth, the second part, $j : L^1(Q) \longrightarrow \mathbb{R}$, is not a differentiable functional, but it is a Lipschitz convex functional. It admits directional derivatives given by the formula

$$j'(u; v) = \int_{Q_+} v(x,t) \, dx \, dt - \int_{Q_-} v(x,t) \, dx \, dt + \int_{Q_0} |v(x,t)| \, dx \, dt \quad \forall u, v \in L^1(Q),$$

(1.4)

where

$$Q_+ = \{(x,t) \in Q : u(x,t) > 0\}, \quad Q_-\{(x,t) \in Q : u(x,t) < 0\},$$

$$Q_0 = \{(x,t) \in Q : u(x,t) = 0\}.$$

Of course, the relations defining these sets are required only almost everywhere. Moreover, we know that a measurable function λ belongs to the subdifferential in the sense of the convex analysis of j at a point u, $\lambda \in \partial j(u)$, if and only if it satisfies for a.a. $(x,t) \in Q$

$$\lambda(x,t) \begin{cases} = +1 & \text{if } u(x,t) > 0 \\ = -1 & \text{if } u(x,t) < 0 \\ \in [-1, +1] & \text{if } u(x,t) = 0. \end{cases}$$

For every $0 < \rho < 1$ we have

$$j(v) - j(u) \geq \frac{j(u + \rho(v - u)) - j(u)}{\rho} \geq j'(u; v - u) = \max_{\lambda \in \partial j(u)} \int_Q \lambda(v - u) \, dx \, dt. \quad (1.5)$$

These inequalities are valid for any convex and locally Lipschitz functional j; see for instance [2].

The first part of J_ν has the following differentiability properties.

THEOREM 1.5. *The function $F_\nu : L^p(Q) \longrightarrow \mathbb{R}$, $p > 5/2$, is of class C^2, and we have the following expressions for the first and second derivatives*

$$F_\nu'(u)v = \int_Q (\varphi_u + \nu u) v \, dxdt \quad \forall u, v \in L^p(Q), \quad (1.6)$$

and for $v_1, v_2 \in L^p(Q)$

$$F_\nu''(u)[v_1, v_2] = \int_Q \left\{ [C_Q^Y - R''(y_u)\varphi_u]\eta_{v_1}\eta_{v_2} + C_Q^Z \zeta_{v_1}\zeta_{v_2} \right\} \, dx \, dt$$

$$+ \int_\Omega \left\{ C_T^Y \eta_{v_1}(T)\eta_{v_2}(T) + C_T^Z \zeta_{v_1}(T)\zeta_{v_2}(T) \right\} \, dx + \nu \int_Q v_1 v_2 \, dx \, dt, \quad (1.7)$$

where φ_u along with ψ_u are the solutions in $W(0,T) \cap L^\infty(Q)$ of the adjoint system

$$\begin{cases} -\dfrac{\partial \varphi_u}{\partial t} - \Delta \varphi_u + R'(y_u)\,\varphi_u + \alpha\,\psi_u & = & C_Q^Y(y_u - y_Q) & in\ Q \\[4pt] \partial_n \varphi_u & = & 0 & on\ \Sigma \\[4pt] \varphi_u(x,T) & = & C_T^Y(y_u(T) - y_T) & in\ \Omega \\[4pt] -\dfrac{\partial \psi_u}{\partial t} + \beta\,\psi_u - \gamma\,\varphi_u & = & C_Q^Z(z_u - z_Q) & in\ Q \\[4pt] \psi_u(x,0) & = & C_T^Z(z_u(T) - z_T) & in\ \Omega. \end{cases} \qquad (1.8)$$

The proof of the existence and uniqueness of the solution $(\varphi_u, \psi_u) \in [W(0,T) \cap L^\infty(Q)]^2$ of the adjoint system and the formula (1.6) can be found in [8, §3.2]. The expression of the second derivative follows from the chain rule, (1.3), and (1.8).

2. First-Order Optimality Conditions. Since the control problem (P_ν) is not convex, we must consider local minima. In this section, the goal is to set up the first-order necessary optimality conditions satisfied by the local minima and to draw some conclusions from the optimality system. We say that $\bar{u}_\nu$ is a local minimum of problem (P_ν) in the $L^p(Q)$ sense, $1 \le p \le +\infty$, if there exists $\varepsilon > 0$ such that

$$J_\nu(\bar{u}_\nu) \le J_\nu(u) \quad \forall u \in \mathcal{U}_{\mathrm{ad}} \cap B_\varepsilon(\bar{u}_\nu),$$

where $B_\varepsilon(\bar{u}_\nu)$ denotes the $L^p(Q)$-ball centered at $\bar{u}_\nu$ with radius ε. Let us mention that the boundedness of $\mathcal{U}_{\mathrm{ad}}$ in $L^\infty(Q)$ implies that $\bar{u}_\nu$ is a local minimum in the $L^2(Q)$ sense if and only if it is a local minimum in the $L^p(Q)$ sense for any $1 \le p < +\infty$. On the other hand, if $\bar{u}_\nu$ is a local minimum in the $L^\infty(Q)$ sense, then it is a local minimum in the $L^p(Q)$ sense for any $1 \le p \le +\infty$. Hereafter, local minima will be always understood as local minima in the $L^2(Q)$ sense.

Remark 2.1. *Minima of this type are, viewed in the sense of calculus of variations, weak local minima. They ensure optimality in a neighborhood of the locally optimal control $\bar{u}_\nu$. Later, we will also investigate conditions for local minima that are* strong *in the sense of calculus of variations. In that case, we have*

$$J_\nu(\bar{u}_\nu) \le J_\nu(u) \quad \forall u \in \mathcal{U}_{\mathrm{ad}} \ \ such\ that\ \ \|y_u - y_{\bar{u}_\nu}\|_{L^\infty(Q)} < \varepsilon.$$

no matter how far u is from $\bar{u}_\nu$.

Now we state the optimality conditions satisfied by a local minimum of (P_ν).

THEOREM 2.2. *Let $\bar{u}_\nu$ be a local minimum of (P_ν). Then, there exist $(\bar{y}_\nu, \bar{z}_\nu)$*

and $(\bar{\varphi}_\nu, \bar{\psi}_\nu)$ in $[W(0,T) \cap L^\infty(Q)]^2$, and $\bar{\lambda}_\nu \in \partial j(\bar{u}_\nu)$ such that

$$\begin{cases} \dfrac{\partial \bar{y}_\nu}{\partial t} - \Delta \bar{y}_\nu + R(\bar{y}_\nu) + \alpha\,\bar{z}_\nu &=& \bar{u}_\nu & in\ Q \\[2mm] \partial_n \bar{y}_\nu &=& 0 & on\ \Sigma \\[1mm] \bar{y}_\nu(x,0) &=& y_0(x) & in\ \Omega \\[3mm] \dfrac{\partial \bar{z}_\nu}{\partial t} + \beta\,\bar{z}_\nu - \gamma\,\bar{y}_\nu + \delta &=& 0 & in\ Q \\[2mm] \bar{z}_\nu(x,0) &=& z_0(x) & in\ \Omega, \end{cases} \tag{2.1}$$

$$\begin{cases} -\dfrac{\partial \bar{\varphi}_\nu}{\partial t} - \Delta \bar{\varphi}_\nu + R'(y_u)\,\bar{\varphi}_\nu + \alpha\,\bar{\psi}_\nu &=& C_Q^Y(\bar{y}_\nu - y_Q) & in\ Q \\[2mm] \partial_n \bar{\varphi}_\nu &=& 0 & on\ \Sigma \\[1mm] \bar{\varphi}_\nu(x,T) &=& C_T^Y(\bar{y}_\nu(T) - y_T) & in\ \Omega \\[3mm] -\dfrac{\partial \bar{\psi}_\nu}{\partial t} + \beta\,\bar{\psi}_\nu - \gamma\,\bar{\varphi}_\nu &=& C_Q^Z(\bar{z}_\nu - z_q) & in\ Q \\[2mm] \bar{\psi}_\nu(x,0) &=& C_T^Z(\bar{z}_\nu(T) - Z_T) & in\ \Omega, \end{cases} \tag{2.2}$$

$$\int_Q (\bar{\varphi}_\nu + \nu\bar{u}_\nu + \kappa\bar{\lambda}_\nu)(u - \bar{u}_\nu)\,dx\,dt \geq 0 \quad \forall u \in \mathcal{U}_{\text{ad}}. \tag{2.3}$$

Proof. Let $\varepsilon > 0$ such that $J_\nu(\bar{u}_\nu) \leq J_\nu(u)$ for every $u \in \mathcal{U}_{\text{ad}} \cap B_\varepsilon(\bar{u}_\nu)$. Then, for any $u \in \mathcal{U}_{\text{ad}}$, $u \neq \bar{u}_\nu$, and for all $0 < \rho < \varepsilon/\|u-\bar{u}_\nu\|_{L^2(Q)}$, we have that $\bar{u}_\nu + \rho(u-\bar{u}_\nu) \in \mathcal{U}_{\text{ad}} \cap B_\varepsilon(\bar{u}_\nu)$. Hence, using the convexity of j, we get

$$0 \leq \frac{J_\nu(\bar{u}_\nu + \rho(u - \bar{u}_\nu)) - J_\nu(\bar{u}_\nu)}{\rho} \leq \frac{F_\nu(\bar{u}_\nu + \rho(u - \bar{u}_\nu)) - F_\nu(\bar{u}_\nu)}{\rho} + \kappa j(u) - \kappa j(\bar{u}_\nu).$$

Now, passing to the limit $\rho \to 0$, we infer

$$F_\nu'(\bar{u}_\nu)(u - \bar{u}_\nu) + \kappa j(u) - \kappa j(\bar{u}_\nu) \geq 0. \tag{2.4}$$

Taking $(\bar{y}_\nu, \bar{z}_\nu)$ and $(\bar{\varphi}_\nu, \bar{\psi}_\nu)$ as the solutions of (2.1) and (2.2), respectively, and using the expression (1.6) of the derivative of F_ν, we deduce from the above inequality that

$$\int_Q (\bar{\varphi}_\nu + \nu\bar{u}_\nu)(u - \bar{u}_\nu)\,dx\,dt + \kappa j(u) - \kappa j(\bar{u}_\nu) \geq 0 \quad \forall u \in \mathcal{U}_{\text{ad}}.$$

Therefore, $\bar{u}_\nu$ solves the convex optimization problem

$$\begin{cases} \text{Min} & \int_Q (\bar{\varphi}_\nu + \nu\bar{u}_\nu)u\,dx\,dt + \kappa j(u) \\ u \in \mathcal{U}_{\text{ad}} \end{cases}$$

that can be considered in $L^1(\Omega)$. Finally, from rules of subdifferential calculus we deduce the existence of $\bar{\lambda}_\nu \in \partial j(\bar{u}_\nu)$ such that (2.3) holds. □

From inequality (2.3), we deduce some interesting relations.

COROLLARY 2.3. *Let $\bar{u}_\nu$, $\bar{\varphi}_\nu$, and $\bar{\lambda}_\nu$ be as in Theorem 2.2, and assume that $\nu > 0$. Then the following relations hold*

$$\begin{cases} \bar{u}_\nu(x,t) = 0 \Leftrightarrow |\bar{\varphi}_\nu(x,t)| \leq \kappa, \\[2mm] \bar{u}_\nu(x,t) = \text{Proj}_{[a,b]}\left(-\dfrac{1}{\nu}[\bar{\varphi}_\nu(x,t) + \kappa\bar{\lambda}_\nu(x,t)]\right), \\[2mm] \bar{\lambda}_\nu(x,t) = \text{Proj}_{[-1,+1]}\left(-\dfrac{1}{\kappa}\bar{\varphi}_\nu(x,t)\right). \end{cases}$$

Moreover, from the last two representation formulas it follows that $\bar{\lambda}_\nu$ is unique and $\bar{u}_\nu, \bar{\lambda}_\nu \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q)$.

COROLLARY 2.4. *Let $\bar{u}_\nu$, $\bar{\varphi}_\nu$ and $\bar{\lambda}_\nu$ be as in Theorem 2.2 and assume that $\nu = 0$. Then the following relations are satisfied*

$$\begin{cases} |\bar{\varphi}_\nu(x,t)| < \kappa \Rightarrow \bar{u}_\nu(x,t) = 0, \\ \bar{\varphi}_\nu(x,t) > +\kappa \Rightarrow \bar{u}_\nu(x,t) = a, \\ \bar{\varphi}_\nu(x,t) < -\kappa \Rightarrow \bar{u}_\nu(x,t) = b, \\ \bar{\varphi}_\nu(x,t) = +\kappa \Rightarrow \bar{u}_\nu(x,t) \leq 0, \\ \bar{\varphi}_\nu(x,t) = -\kappa \Rightarrow \bar{u}_\nu(x,t) \geq 0, \\ \bar{\lambda}_\nu(x,t) = \text{Proj}_{[-1,+1]} \left(-\frac{1}{\kappa} \bar{\varphi}_\nu(x,t) \right). \end{cases}$$

The last representation formula yields that $\bar{\lambda}_\nu \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q)$ and $\bar{\lambda}_\nu$ is unique for a fixed local minimum $\bar{u}_\nu$.

For the proofs of these relations, the reader is referred to [7, Corollary 3.2] and [6, Theorem 3.1] that deal with the cases $\nu > 0$ and $\nu = 0$, respectively, in a similar situation. The relations in the corollary show that local minima $\bar{u}_\nu$ can be zero in large regions, the support of $\bar{u}_\nu$ being monitored by κ.

In particular, a value $\kappa_0 < \infty$ exists such that for $\kappa > \kappa_0$ we have only one local minimum, namely $\bar{u}_\nu = 0$. Indeed, since $\mathcal{U}_{\text{ad}}$ is bounded in $L^\infty(Q)$, the set $\{(y_u, z_u)\}_{u \in \mathcal{U}_{\text{ad}}}$ is bounded in $L^\infty(Q)$, and hence from (1.8) we deduce that the set $\{(\varphi_u, \psi_u)\}_{u \in \mathcal{U}_{\text{ad}}}$ is also bounded in $L^\infty(Q)$. Then, our statement holds obviously for

$$\kappa_0 = \sup\{\|\varphi_u\|_{L^\infty(Q)} : u \in \mathcal{U}_{\text{ad}}\}.$$

In the case $\nu = 0$, the control $\bar{u}_\nu$ can admit values outside of $\{a, 0, b\}$ on a set of positive measure only if the set $\{(x,t) \in Q : |\bar{\varphi}_\nu(x,t)| = \kappa\}$ has a positive Lebesgue measure. If the measure of this set is zero then $\bar{u}_\nu(x,t)$ must belong to $\{a, 0, b\}$ for almost every $(x,t) \in Q$. Such optimal controls $\bar{u}_\nu(x,t)$ are of "bang-bang-bang" type.

3. Second Order Optimality Conditions. In this section, we carry out the second order analysis for (P_ν). First, we establish second order necessary conditions. Distinguishing between the cases $\nu > 0$ and $\nu = 0$, we consider sufficient conditions. The reader might be surprised how different both cases can be.

Let us give at this point a short orientation on the second order conditions. To deal with the different situations depending on ν, we will introduce three different cones of critical directions, namely the cone $C_{\bar{u}}$ and the cones $C_{\bar{u}}^\tau$ and $E_{\bar{u}}^\tau$. The cone $C_{\bar{u}}$ is quite standard and well known. It will be used in the case $\nu > 0$. In Thm. 3.4 we prove that sufficient second order conditions based upon $C_{\bar{u}}$ ensure (weak) local optimality. However, we show just as a corollary that then $\bar{u}_\nu$ yields even a strong local minimum.

The two other cones are needed for the case $\nu = 0$. The second order conditions based on $E_{\bar{u}}^\tau$ imply that $\bar{u}$ is a strong local minimum in the sense of calculus of variations, cf. Theorem 3.13 and relation (3.38). This property of strong local optimality cannot be deduced, if we replace $E_{\bar{u}}^\tau$ by $C_{\bar{u}}^\tau$.

Section 4 is devoted to the stability of locally optimal solutions with respect to perburbations. For $\nu > 0$ the cone $C_{\bar{u}}^\tau$ can be used to show Lipschitz stability of optimal controls cf. Theorem 4.2. Here, weak local minima are useful.

In §4.1.2, we discuss the stability of locally optimal solutions for $\nu = 0$. Here, we invoke the second-order conditions based on $E_{\bar{u}}^\tau$ and prove the stability of the optimal

states with respect to perturbations of the desired state. Now, stability can be proved only if $\bar{u}$ is a strong minimum. For weak minima we cannot prove this property.

We should mention that this strong minimum property would also be needed if we want to prove error estimates for the optimal states when the control problem is discretized. Indeed, we only have weak convergence of the optimal discrete controls to the continuous ones, but the corresponding states have strong convergence properties, which allows the use of the sufficient second order condition.

3.1. The case $\nu > 0$. First, we introduce a cone of critical directions associated with a control $\bar{u}_\nu \in \mathcal{U}_{\mathrm{ad}}$ satisfying the optimality system (2.1)-(2.3) along with the state $(\bar{y}_\nu, \bar{z}_\nu)$ and the adjoint state $(\bar{\varphi}_\nu, \bar{\psi}_\nu)$. This cone is an extension of the analogous one for finite dimensional optimization problems to the infinite-dimensional case,

$$C_{\bar{u}_\nu} = \{v \in L^2(Q) : v \text{ satisfies the sign conditions (3.1) and } F'_\nu(\bar{u}_\nu)v + \kappa j'(\bar{u}_\nu; v) = 0\},$$

$$v(x,t) \begin{cases} \geq 0 & \text{if } \bar{u}_\nu(x,t) = a, \\ \leq 0 & \text{if } \bar{u}_\nu(x,t) = b. \end{cases} \tag{3.1}$$

The next proposition establishes some properties of $C_{\bar{u}_\nu}$.

LEMMA 3.1.

1 – If $v \in L^2(Q)$ satisfies (3.1), then $(\bar{\varphi}_\nu + \nu\bar{u}_\nu + \kappa\bar{\lambda}_\nu)v \geq 0$ a.e. in Q.
2 – If $v \in C_{\bar{u}_\nu}$, then

$$(\bar{\varphi}_\nu + \nu\bar{u}_\nu + \kappa\bar{\lambda}_\nu)v = 0, \tag{3.2}$$

$$\int_{Q_0} |v|\, dx\, dt = \int_{Q_0} \bar{\lambda}_\nu v\, dx\, dt \Rightarrow v(x,t) \begin{cases} = 0 & \text{if } |\bar{\lambda}_\nu(x,t)| < 1, \\ \geq 0 & \text{if } \bar{\lambda}_\nu(x,t) = +1, \\ \leq 0 & \text{if } \bar{\lambda}_\nu(x,t) = -1, \end{cases} \tag{3.3}$$

where $Q_0 = \{(x,t) \in Q : \bar{u}_\nu(x,t) = 0\}$.
3 – An element $v \in L^2(Q)$ belongs to $C_{\bar{u}_\nu}$ if and only if the following properties hold

$$v(x,t) \begin{cases} = 0 & \text{if } |\bar{\varphi}_\nu(x,t) + \nu\bar{u}_\nu(x,t)| \neq \kappa, \\ \geq 0 & \text{if } \bar{u}_\nu(x,t) = a \text{ or } (\bar{\varphi}_\nu(x,t) = -\kappa \text{ and } \bar{u}_\nu(x,t) = 0), \\ \leq 0 & \text{if } \bar{u}_\nu(x,t) = b \text{ or } (\bar{\varphi}_\nu(x,t) = +\kappa \text{ and } \bar{u}_\nu(x,t) = 0). \end{cases} \tag{3.4}$$

is fulfilled.

Proof. (1) The first statement is an immediate consequence of Corollary 2.3. Indeed, if $(\bar{\varphi}_\nu + \nu\bar{u}_\nu + \kappa\bar{\lambda}_\nu)(x,t) > 0$, then $\bar{u}_\nu(x,t) > -\frac{1}{\nu}(\bar{\varphi}_\nu + \kappa\bar{\lambda}_\nu)(x,t)$ holds. According to the representation formula for $\bar{u}_\nu$, this inequality is possible only if $\bar{u}_\nu(x,t) = a$. Then (3.1) implies that $v(x,t) \geq 0$. Analogously, we proceed when $(\bar{\varphi}_\nu + \nu\bar{u}_\nu + \kappa\bar{\lambda}_\nu)(x,t) < 0$.

(2) If $v \in C_{\bar{u}_\nu}$, then the identity $F'(\bar{u}_\nu)v + \kappa j'(\bar{u}_\nu; v) = 0$ holds. Hence, we have with (1)

$$0 \leq \int_Q (\bar{\varphi}_\nu + \nu\bar{u}_\nu + \kappa\bar{\lambda}_\nu)v\, dx\, dt \leq \int_Q (\bar{\varphi}_\nu + \nu\bar{u}_\nu)v\, dx\, dt + \kappa j'(\bar{u}_\nu; v) = 0.$$

Using again (1), we deduce (3.2) and

$$\int_Q \bar{\lambda}_\nu v\, dx\, dt = j'(\bar{u}_\nu; v).$$

Then the formula (1.4) for $j'(\bar{u}_\nu; v)$ and this identity imply (3.3).

(3) First, we assume that $v \in C_{\bar{u}_\nu}$. Then v satisfies (3.1) and we have to discuss 3 cases in (3.4).

– If $\bar{\varphi}_\nu(x,t) = +\kappa$ and $\bar{u}_\nu(x,t) = 0$, then Corollary 2.3 implies that $\bar{\lambda}_\nu(x,t) = -1$. This identity along with $\bar{u}_\nu(x,t) = 0$ and (3.3) yields $v(x,t) \leq 0$.

– Analogously, we prove that $v(x,t) \geq 0$ if $\bar{\varphi}_\nu(x,t) = -\kappa$ and $\bar{u}_\nu(x,t) = 0$.

– Let us now analyze the case $|\bar{\varphi}_\nu(x,t) + \nu\bar{u}_\nu(x,t)| \neq \kappa$. We distinguish between two situations. In the first, we suppose that $|\bar{\varphi}_\nu(x,t) + \nu\bar{u}_\nu(x,t)| > \kappa$. Since $|\bar{\lambda}_\nu| \leq 1$, we infer that $\bar{\varphi}_\nu + \nu\bar{u}_\nu + \kappa\bar{\lambda}_\nu \neq 0$; hence (3.2) implies that $v(x,t) = 0$.

In the other case, we assume that $|\bar{\varphi}_\nu(x,t) + \nu\bar{u}_\nu(x,t)| < \kappa$. If $\bar{u}_\nu(x,t) \neq 0$, then $|\bar{\lambda}_\nu(x,t)| = 1$ and consequently $\bar{\varphi}_\nu + \nu\bar{u}_\nu + \kappa\bar{\lambda}_\nu \neq 0$. Once again, (3.2) implies that $v(x,t) = 0$. Finally, we assume in this second case that $\bar{u}_\nu(x,t) = 0$ and $|\bar{\varphi}_\nu(x,t)| < \kappa$. Then $\bar{\lambda}_\nu(x,t)| < 1$ and (3.3) leads to $v(x,t) = 0$.

Second, we consider an element $v \in L^2(Q)$ satisfying (3.4). Then v obeys (3.1) and it remains to prove that $F'(\bar{u}_\nu)v + \kappa j'(\bar{u}_\nu; v) = 0$. The equality

$$\int_{Q_0} |v|\, dx\, dt = \int_{Q_0} \bar{\lambda}_\nu v\, dx\, dt$$

is a consequence of (3.4) and the projection formula for $\bar{\lambda}_\nu$ given in Corollary 2.3. Therefore, the identity $(\bar{\lambda}_\nu, v) = j'(\bar{u}_\nu; v)$ holds and consequently we have

$$\int_Q (\bar{\varphi}_\nu + \nu\bar{u}_\nu + \kappa\bar{\lambda}_\nu)v\, dx\, dt = F'_\nu(\bar{u}_\nu)v + \kappa j'(\bar{u}_\nu; v).$$

If $(\bar{\varphi}_\nu + \nu\bar{u}_\nu + \kappa\bar{\lambda}_\nu)(x,t) > 0$, then $\bar{u}_\nu(x,t) > -\frac{1}{\nu}(\bar{\varphi}_\nu + \kappa\bar{\lambda}_\nu)(x,t)$. From this inequality and the projection formula for $\bar{u}_\nu$ established in Corollary 2.3, we infer that $\bar{u}_\nu(x,t) = a$, hence $\bar{\lambda}_\nu(x,t) = -1$ and $(\bar{\varphi}_\nu + \nu\bar{u}_\nu)(x,t) > \kappa$. Then, (3.4) implies that $v(x,t) = 0$. Analogously, we deduce that $v(x,t) = 0$ if $(\bar{\varphi}_\nu + \nu\bar{u}_\nu + \kappa\bar{\lambda}_\nu)(x,t) < 0$. From these properties and the above identity we deduce that $F'(\bar{u}_\nu)v + \kappa j'(\bar{u}_\nu; v) = 0$. $\blacksquare$

The proof of the following theorem follows the lines of the proofs of Proposition 3.4 and Theorem 3.7 of [7] with obvious modifications.

THEOREM 3.2. *The set $C_{\bar{u}_\nu}$ is a convex and closed cone in $L^2(Q)$. Furthermore, if $\bar{u}_\nu$ is a local minimum for* (P_ν), *then $F''_\nu(\bar{u}_\nu)v^2 \geq 0$ $\forall v \in C_{\bar{u}_\nu}$.*

The reader should observe that the contribution of the term $j(u)$ in the second order optimality conditions is through the critical cone $C_{\bar{u}_\nu}$, but only $F''_\nu(\bar{u}_\nu)v^2$ is involved in the second order approximation. In some sense, j is piecewise linear and there is no second order contribution from it. Next we consider the sufficient second order conditions.

Let us now start the second order analysis. The presence of the so-called Tikhonov term $\frac{\nu}{2}\|u\|^2_{L^2(Q)}$ in the cost functional is extremely important. Due to this term, the second order sufficient conditions for the local optimality of $\bar{u}_\nu$ are a straightforward extension of the corresponding conditions for a finite dimensional optimization problem. Actually, it is well known that the equivalence stated in the following theorem is not in general true for infinite-dimensional optimization problems. However, it works perfectly in this case because of the presence of the Tikhonov term.

THEOREM 3.3. *The following statements are equivalent*

1. $F''_\nu(\bar{u}_\nu)v^2 > 0$ $\forall v \in C_{\bar{u}_\nu} \setminus \{0\}$.
2. *There exists $\sigma > 0$ such that*

$$F''_\nu(\bar{u}_\nu)v^2 \geq \sigma\|v\|^2_{L^2(Q)} \quad \forall v \in C_{\bar{u}_\nu}. \tag{3.5}$$

We omit the proof of this result, because it was proved for a certain class of parabolic partial differential equations in [9, Th. 4.11]. For the FitzHugh-Nagumo system, the situation is slightly more involved, but the extension of the proof in [9] to this system is more or less straightforward.

Finally, we are able to prove the following result.

THEOREM 3.4. *Let* $\bar{u}_\nu \in \mathcal{U}_{\mathrm{ad}}$ *satisfy the optimality system* (2.1)-(2.3) *along with the state* $(\bar{y}_\nu, \bar{z}_\nu)$ *and the adjoint state* $(\bar{\varphi}_\nu, \bar{\psi}_\nu)$. *We also assume that* $F_\nu''(\bar{u}_\nu)v^2 > 0$ $\forall v \in C_{\bar{u}_\nu} \setminus \{0\}$. *Then, there exist* $\delta > 0$ *and* $\varepsilon > 0$ *such that*

$$J_\nu(\bar{u}_\nu) + \frac{\delta}{2}\|u - \bar{u}_\nu\|_{L^2(Q)}^2 \leq J_\nu(u) \quad \forall u \in \mathcal{U}_{\mathrm{ad}} \cap B_\varepsilon(\bar{u}_\nu), \tag{3.6}$$

where $B_\varepsilon(\bar{u}_\nu)$ *is the* $L^2(Q)$ *ball centered at* $\bar{u}_\nu$ *with radius* ε.

Proof. This proof follows the classical one that is performed by contradiction. The only difference to take into account is the non-differentiable term j involved in the cost functional and in the definition of the cone of critical directions. Let us sketch some details of the proof. Arguing by contradiction, we assume that there exists a sequence $\{u_k\}_{k=1}^\infty \subset \mathcal{U}_{\mathrm{ad}}$ such that

$$\|u_k - \bar{u}_\nu\|_{L^2(Q)} < \frac{1}{k} \quad \text{and} \quad J_\nu(u_k) < J_\nu(\bar{u}_\nu) + \frac{1}{2k}\|u_k - \bar{u}_\nu\|_{L^2(Q)}^2 \quad \forall k \geq 1. \tag{3.7}$$

Now, we take

$$\rho_k = \|u_k - \bar{u}_\nu\|_{L^2(Q)}, \ v_k = \frac{1}{\rho_k}(u_k - \bar{u}_\nu), \quad \text{hence} \ \|v_k\|_{L^2(Q)} = 1 \text{ and } v_k \rightharpoonup v \text{ in } L^2(Q), \tag{3.8}$$

the last convergence after possibly selecting a subsequence of $\{v_k\}_{k=1}^\infty$. The proof is split into three parts.

(i) $v \in C_{\bar{u}_\nu}$. Since every v_k satisfies the sign conditions of the definition of $C_{\bar{u}_\nu}$, we deduce that v also satisfies them. Let us prove that $F_\nu'(\bar{u}_\nu)v + \kappa j'(\bar{u}_\nu; v) = 0$. From (1.5) and (2.3) we get that

$$F_\nu'(\bar{u}_\nu)v + \kappa j'(\bar{u}_\nu; v) \geq F_\nu'(\bar{u}_\nu)v + \kappa \int_Q \bar{\lambda}_\nu v \, dx \, dt$$

$$= \lim_{k \to \infty} \frac{1}{\rho_k} \int_Q (\bar{\varphi}_\nu + \nu \bar{u}_\nu + \kappa \lambda_\nu)(u_k - \bar{u}) \, dx \, dt \geq 0.$$

To prove the converse inequality, we apply the mean value theorem and use (3.7),

$$F_\nu(\bar{u}_\nu) + \rho_k F_\nu'(\bar{u}_\nu + \theta_k \rho_k v_k)v_k + \kappa j(\bar{u}_\nu + \rho_k v_k) < F_\nu(\bar{u}_\nu) + \kappa j(\bar{u}_\nu) + \frac{\rho_k^2}{2k}$$

with some $0 \leq \theta_k \leq 1$. Therefore, with (1.5) we obtain from above

$$\rho_k\{F_\nu'(\bar{u}_\nu + \theta_k(u_k - \bar{u}_\nu))v_k + \kappa j'(\bar{u}_\nu; v_k)\} < \frac{\rho_k^2}{2k}.$$

Now by convexity and continuity of the mapping $L^1(Q) \ni v \mapsto j(\bar{u}_\nu; v)$, we get

$$F_\nu'(\bar{u}_\nu)v + \kappa j'(\bar{u}_\nu; v) = \liminf_{k \to \infty}\{F_\nu'(\bar{u}_\nu + \theta_k(u_k - \bar{u}_\nu))v_k + \kappa j'(\bar{u}_\nu; v_k)\} \leq \liminf_{k \to \infty} \frac{\rho_k}{2k} = 0.$$

Here, we have used that $F_\nu : L^p(Q) \longrightarrow \mathbb{R}$ is of class C^2 for $p > 5/2$ and $u_k \to \bar{u}_\nu$ strongly in $L^q(Q)$ for all $q < \infty$.

(ii) $v = 0$. Performing a Taylor expansion and using (3.7) again, we find

$$F_\nu(\bar{u}_\nu) + \rho_k F_\nu'(\bar{u}_\nu)v_k + \frac{\rho_k^2}{2}F_\nu''(\bar{u}_\nu + \theta_k\rho_k v_k)v_k^2 + \kappa j(\bar{u}_\nu + \rho_k v_k) < F_\nu(\bar{u}_\nu) + \kappa j(\bar{u}_\nu) + \frac{\rho_k^2}{2k}.$$

From (1.5), we obtain

$$\rho_k\{F_\nu'(\bar{u}_\nu)v_k + \kappa j'(\bar{u}_\nu; v_k)\} + \frac{\rho_k^2}{2}F_\nu''(\bar{u}_\nu + \theta_k(u_k - \bar{u}_\nu))v_k^2 < \frac{\rho_k^2}{2k}.$$

Invoking (1.5) and (2.3) we deduce that $F_\nu'(\bar{u}_\nu)v_k + \kappa j'(\bar{u}_\nu; v_k) \geq 0$, then $F_\nu''(\bar{u}_\nu + \theta_k(u_k - \bar{u}_\nu))v_k^2 < \frac{1}{2k}$. Thanks to the expression of F_ν'' given in (1.7), it is easy to pass to the limit with the aid of the compactness of the linear operator $L^2(Q) \ni v \mapsto (\eta_v, \zeta_v) \in L^2(Q)^2$ and $L^2(Q) \ni v \mapsto (\eta_v(T), \zeta_v(T)) \in L^2(\Omega)^2$ to deduce that $F_\nu''(\bar{u}_\nu)v^2 \leq 0$. In view of the assumptions of the theorem, since $v \in C_{\bar{u}_\nu}$, this is possible only if $v = 0$.

(iii) Contradiction. Since $v_k \rightharpoonup 0$ in $L^2(Q)$, it holds $(\eta_{v_k}, \zeta_{v_k}) \rightharpoonup (0,0)$ in $W(0,T)$. Hence all the terms in $F''(\bar{u}_\nu)v_k^2$ tend to zero except the Tikhonov term $\|v_k\|^2_{L^2(Q)} = 1$. Thus, we have $\nu \leq \liminf_{k\to\infty} F_\nu''(\bar{u}_\nu)v_k^2 = 0$. □

Now we prove the surprising fact that the local optimality ensured by Theorem 3.4 is even strong.

COROLLARY 3.5. *Let $\bar{u}_\nu$ satisfy the assumptions of Theorem 3.4. Then, there exist $\delta' > 0$ and $\varepsilon' > 0$ such that*

$$J_\nu(\bar{u}_\nu) + \frac{\delta'}{2}\|u - \bar{u}_\nu\|^2_{L^2(Q)} \leq J_\nu(u) \ \forall u \in \mathcal{U}_{\mathrm{ad}} : \|y_u - \bar{y}_\nu\|_{L^\infty(Q)} < \varepsilon'. \qquad (3.9)$$

Proof. Let us assume that (3.9) does not hold for any δ' and ε'. Then, for any integer $k \geq 1$, we can find a control $u_k \in \mathcal{U}_{\mathrm{ad}}$ with $\|y_{u_k} - \bar{y}_\nu\|_{L^\infty(Q)} < 1/k$ such that

$$J_\nu(u_k) < J_\nu(\bar{u}_\nu) + \frac{1}{2k}\|u_k - \bar{u}_\nu\|^2_{L^2(Q)}. \qquad (3.10)$$

We can take a subsequence, denoted in the same way, such that $\{u_k\}_{k\geq 1}$ is weakly convergent in $L^2(Q)$. Since $y_{u_k} \to \bar{y}_\nu$ in $L^\infty(Q)$, we deduce that $u_k \rightharpoonup \bar{u}_\nu$ in $L^2(Q)$. Indeed, from the boundedness of $\{u_k\}_{k\geq 1}$ in $L^\infty(Q)$ and Theorem 1.1 we infer the boundedness of $\{(y_k, z_k)\}_{k\geq 1}$ in $W(0,T)$. Therefore, taking again a subsequence if necessary, we can assume the weak convergence of $\{(y_k, z_k)\}_{k\geq 1}$ in $W(0,T)$ and of $\{u_k\}_{k\geq 1}$ in $L^p(Q)$ for every $p < +\infty$. The weak limit of $\{y_k\}_{k\geq 1}$ is obviously $\bar{y}_\nu$. From the state equation satisfied by the functions z_k we deduce that $z_k \rightharpoonup \bar{z}_\nu$ in $W(0,T)$. Finally, passing to the limit in the first equation of the system (1.1) we conclude that $u_k \rightharpoonup \bar{u}_\nu$ in $L^2(Q)$. From $y_{u_k} \to \bar{y}_\nu$ in $L^\infty(Q)$ and the last two equations of (1.1) we also obtain that $z_{u_k} \to \bar{z}_\nu$ in $L^\infty(Q)$. Now, we deduce from (3.10)

$$\frac{\nu}{2}\|u_k\|^2_{L^2(Q)} + \kappa j(u_k) \leq \frac{\nu}{2}\|\bar{u}_\nu\|^2_{L^2(Q)} + \kappa j(\bar{u}_\nu) + F_0(\bar{u}_\nu) - F_0(u_k) + \frac{1}{2k}\|u_k - \bar{u}_\nu\|^2_{L^2(Q)}.$$

Passing to the limit, we get

$$\frac{\nu}{2}\|\bar{u}_\nu\|^2_{L^2(Q)} + \kappa j(\bar{u}_\nu) \leq \liminf_{k\to\infty}\{\frac{\nu}{2}\|u_k\|^2_{L^2(Q)} + \kappa j(u_k)\}$$

$$\leq \limsup_{k\to\infty}\{\frac{\nu}{2}\|u_k\|^2_{L^2(Q)} + \kappa j(u_k)\} \leq \frac{\nu}{2}\|\bar{u}_\nu\|^2_{L^2(Q)} + \kappa j(\bar{u}_\nu).$$

Notice that $F_0(u_k) \to F_0(\bar{u}_\nu)$ follows from the strong convergence of y_{u_k} and the circumstance that F_0 does not explicitly depend on u.

This implies that $\frac{\nu}{2}\|u_k\|^2_{L^2(Q)} + \kappa j(u_k) \to \frac{\nu}{2}\|\bar{u}_\nu\|^2_{L^2(Q)} + \kappa j(\bar{u}_\nu)$. From here, we obtain the convergence $\|u_k\|_{L^2(Q)} \to \|\bar{u}_\nu\|_{L^2(Q)}$. Due to the presence of the term $\kappa j(u_k)$, this is not completely standard and we proceed as follows:

$$
\begin{aligned}
\frac{\nu}{2}\|\bar{u}_\nu\|^2 &\leq \liminf_{k\to 0} \frac{\nu}{2}\|u_k\|^2 \leq \limsup_{k\to 0}\frac{\nu}{2}\|u_k\|^2 = \limsup_{k\to 0}(\frac{\nu}{2}\|u_k\|^2 + \kappa j(u_k) - \kappa j(u_k)) \\
&\leq \limsup_{k\to 0}(\frac{\nu}{2}\|u_k\|^2 + \kappa j(u_k)) + \kappa \limsup_{k\to 0}(-j(u_k)) \\
&= \frac{\nu}{2}\|\bar{u}_\nu\|^2 + \kappa j(\bar{u}_\nu) - \kappa \liminf_{k\to 0} j(u_k) \leq \frac{\nu}{2}\|\bar{u}_\nu\|^2.
\end{aligned}
\tag{3.11}
$$

Hence, we infer that $u_k \to \bar{u}_\nu$ strongly in $L^2(Q)$. Therefore, given $\epsilon > 0$ such that (3.6) holds, we have that $\|u_k - \bar{u}_\nu\|_{L^2(Q)} < \varepsilon$ for all k sufficiently large. Then (3.10) contradicts (3.6). ∎

3.2. The case $\nu = 0$. In this case, we write (P) instead of (P$_0$), J and F instead of J_0 and F_0, and $\bar{u}$, $\bar{y}$, $\bar{\varphi}$, etc., instead of $\bar{u}_0$, $\bar{y}_0$, $\bar{\varphi}_0$, etc. In general, for an infinite-dimensional optimization problem, the strict positivity of the second derivative of the functional on the critical cone is not enough for local optimality. The reader can find an example for this fact in [12]. Therefore, we have to consider an extended cone. Given $0 < \tau < \kappa$, we define the cone $C^\tau_{\bar{u}}$ as the set of elements $v \in L^2(Q)$ satisfying

$$
v(x,t) \begin{cases} = 0 & \text{if } |\,|\bar{\varphi}(x,t)| - \kappa| \geq \tau, \\ \geq 0 & \text{if } \bar{u}(x,t) = a \text{ or } (\bar{\varphi}(x,t) = -\kappa \text{ and } \bar{u}(x,t) = 0), \\ \leq 0 & \text{if } \bar{u}(x,t) = b \text{ or } (\bar{\varphi}(x,t) = +\kappa \text{ and } \bar{u}(x,t) = 0). \end{cases}
\tag{3.12}
$$

PROPOSITION 3.6. *The extended cone $C^\tau_{\bar{u}}$ has the following properties:*
1 – $C_{\bar{u}} \subset C^\tau_{\bar{u}}$ for all $0 < \tau < \kappa$.
2 – For every $v \in L^2(Q)$ satisfying the sign conditions (3.1), the inequality

$$
F'(\bar{u})v + \kappa j'(\bar{u};v) \geq \tau\|v\|_{L^1(Q_v)}
\tag{3.13}
$$

is fulfilled, where Q_v denotes the set of points $(x,t) \in Q$ such that (3.12) is not satisfied by $v(x,t)$.

Proof. The first statement is an immediate consequence of Lemma 3.1-3. Let us prove the second statement. If $|\,|\bar{\varphi}(x,t)| - \kappa| \geq \tau$ and $\bar{u}(x,t) \neq 0$, then $|\bar{\lambda}(x,t)| = 1$ and the projection formula implies hence $|\bar{\varphi}(x,t)| \geq \kappa$. Therefore, $|\bar{\varphi}(x,t)| \geq \kappa + \tau$ holds. If $\bar{\varphi}(x,t) \geq \kappa + \tau$, then Corollary 2.4 implies that $\bar{u}(x,t) = a$ and therefore $v(x,t) \geq 0$. Here, we have $\bar{\varphi}(x,t) - \kappa \geq \tau$ and hence $(\bar{\varphi}(x,t) - \kappa)v(x,t) \geq \tau|v(x,t)|$. Analogously, we prove that $(\bar{\varphi}(x,t) + \kappa)v(x,t) \geq \tau|v(x,t)|$, if $-\bar{\varphi}(x,t) \leq \kappa + \tau$.

Now, we assume that $|\,|\bar{\varphi}(x,t)| - \kappa| \geq \tau$ and $\bar{u}(x,t) = 0$. Hence, from Corollary 2.4 we deduce that $|\bar{\varphi}(x,t)| < \kappa$ (here, $|\bar{\varphi}(x,t)| = \kappa$ is not possible) and continue by

$$
\bar{\varphi}(x,t)v(x,t) + \kappa|v(x,t)| \geq (\kappa - |\bar{\varphi}(x,t)|)|v(x,t)| \geq \tau|v(x,t)|.
$$

Finally, we assume that $|\bar{\varphi}(x,t)| = \kappa$ and $\bar{u}(x,t) = 0$. From Corollary 2.4 we get that $|\bar{\lambda}(x,t)| = 1$. If $(x,t) \notin Q_v$, then $v(x,t)$ satisfies the sign conditions established in (3.12) and hence $\bar{\varphi}(x,t)v(x,t) + \kappa|v(x,t)| = 0$. Conversely, if $(x,t) \in Q_v$, then we have that $\bar{\varphi}(x,t)v(x,t) + \kappa|v(x,t)| = 2\kappa|v(x,t| > \tau|v(x,t)|$.

The established inequalities prove (3.13). ∎

From (3.4) and (3.12) we infer that $C_{\bar{u}}^{\tau}$ is a small relaxation of $C_{\bar{u}}$ when $\nu = 0$. In the finite dimensional case, both cones are the same for all sufficiently small τ. Now, one might be tempted to formulate the second order sufficient conditions as

$$\exists \tau > 0 \;\; \text{and} \;\; \exists \sigma > 0 \;\; \text{such that} \;\; F''(\bar{u})v^2 \geq \sigma \|v\|_{L^2(Q)}^2 \;\; \forall v \in C_{\bar{u}}^{\tau}. \qquad (3.14)$$

However, for $\nu = 0$, this condition can only hold in exceptional cases as we will prove at the end of this section. In fact, it is enough to take a look at the expression (1.7) of $F''(\bar{u})v^2$ to understand why the condition (3.14) cannot be expected for $\nu = 0$. The next theorem provides the correct sufficient condition.

THEOREM 3.7. *Let $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$ satisfy the optimality system (2.1)-(2.3) along with the state $(\bar{y}, \bar{z})$ and the adjoint state $(\bar{\varphi}, \bar{\psi})$. We also assume that*

$$\exists \tau > 0 \; and \; \exists \sigma > 0 : F''(\bar{u})v^2 \geq \sigma \left(\|\eta_v\|_{L^2(Q)}^2 + \|\eta_v(T)\|_{L^2(\Omega)}^2 \right) \; \forall v \in C_{\bar{u}}^{\tau}. \qquad (3.15)$$

Then, there exists $\varepsilon > 0$ such that

$$J(\bar{u}) + \frac{\sigma}{16} \left(\|\eta_{u-\bar{u}}\|_{L^2(Q)}^2 + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}^2 \right) \leq J(u) \;\; \forall u \in \mathcal{U}_{\mathrm{ad}} \cap B_{\varepsilon}(\bar{u}), \qquad (3.16)$$

where $B_{\varepsilon}(\bar{u})$ is the ball of $L^2(Q)$ centered at $\bar{u}$ with radius ε.

Before proving this theorem, we derive some auxiliary results.

LEMMA 3.8. *There exist constants $C_{a,b}$, C_1, C_2, C_3, and C_{∞} such that for all $u \in \mathcal{U}_{\mathrm{ad}}$ we have the estimates*

$$\|y_u\|_{W(0,T)} + \|y_u\|_{L^{\infty}(Q)} + \|\varphi_u\|_{W(0,T)} + \|\varphi_u\|_{L^{\infty}(Q)} \leq C_{a,b} \qquad (3.17)$$

$$\|y_u - \bar{y}\|_{W(0,T)} + \|\varphi_u - \bar{\varphi}\|_{W(0,T)} \leq C_2 \|u - \bar{u}\|_{L^2(Q)} \qquad (3.18)$$

$$\|\varphi_u - \bar{\varphi}\|_{L^{\infty}(Q)} \leq C_{\infty} \|y_u - \bar{y}\|_{L^{\infty}(Q)} \leq C_3 \|u - \bar{u}\|_{L^3(Q)}, \qquad (3.19)$$

$$\|\eta_v\|_{L^2(Q)}^2 + \|\eta_v(T)\|_{L^2(\Omega)}^2 \leq C_1 \|v\|_{L^3(Q)} \|v\|_{L^1(Q)} \;\; \forall v \in L^3(Q). \qquad (3.20)$$

Proof. The first estimate follows from Theorem 2.1 and Corollary 2.1 of [8]. To prove the second and third estimates, we introduce the family of operators

$$K_{\mu}(w)(x,t) = \int_0^t \gamma e^{-(\beta+\mu)(t-s)} w(x,s) ds$$

for $\mu \geq 0$. Now, we take $w(x,t) = e^{-\mu t}(y_u - \bar{y})(x,t)$. Subtracting the equations satisfied by y_u and $\bar{y}$ and observing that $(z_u - \bar{z})(x,t) = K_0(y_u - \bar{y})(x,t)$, we obtain the following equation for w

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w + R'(\hat{y}_u)w + \mu w + \alpha K_{\mu}(w) & = & e^{-\mu t}(u - \bar{u}) & \text{in } Q \\ \partial_n w & = & 0 & \text{on } \Sigma \\ w(x,0) & = & 0 & \text{in } \Omega, \end{cases} \qquad (3.21)$$

where $\hat{y}_u(x,t) = \bar{y}(x,t) + \theta(x,t)(y_u(x,t) - \bar{y}(x,t))$ and $0 \leq \theta(x,t) \leq 1$. If μ is sufficiently large, [8, Lemma 2.3] directly leads to the estimate for $y_u - \bar{y}$ in (3.18).

The estimate of $\varphi_u - \bar{\varphi}$ in (3.19) is obtained by considering the difference of the adjoint equations for φ_u and $\bar{\varphi}$. Notice that the right-hand side of the associated adjoint PDE of this difference is just $C_Q^Y(y_u - \bar{y})$, while in the terminal condition

the term $C_T^Y(y_u(T) - \bar{y}(T))$ appears. Here, we use that $\|y_u(T) - \bar{y}(T)\|_{L^\infty(\Omega)} \leq \|y_u - \bar{y}\|_{L^\infty(Q)}$. The estimation of $\varphi_u - \bar{\varphi}$ is performed for the version of the adjoint equation where ψ_u and $\bar{\psi}$ are eliminated on using the adjoint integral operator K_μ^*. We omit the related computations, because they are straightforward.

Finally, we prove the estimate (3.20). From [8, (10)], we know the estimate $\|K_\mu\|_{\mathcal{L}(L^2(Q))} \leq |\gamma|\sqrt{\frac{T}{2(\beta+\mu)}}$. Now, we set $w(x,t) = e^{-\mu t}\eta_v(x,t)$; then, w satisfies (3.21) with $e^{-\mu t}v$ in the right hand side. Next, we fix μ large enough such that $R'(\bar{y}) + \mu + \alpha|\gamma|\sqrt{\frac{T}{2(\beta+\mu)}} \geq 1$ in Q. Multiplying the equation by w and integrating in Q we infer

$$\frac{1}{2}\|w(T)\|_{L^2(\Omega)}^2 + \|\nabla w\|_{L^2(Q)}^2 + \|w\|_{L^2(Q)}^2 \leq \int_Q e^{-\mu t}vw\,dx\,dt \leq \|v\|_{L^1(Q)}\|w\|_{L^\infty(Q)}.$$

From here we get

$$\|w(T)\|_{L^2(\Omega)}^2 + \|w\|_{L^2(Q)}^2 \leq C\|v\|_{L^1(Q)}\|v\|_{L^3(Q)}.$$

Writing $\eta_v(x,t) = e^{\mu t}w(x,t)$ we deduce (3.20). $\square$

LEMMA 3.9. *There exist a constant C_η and $\varepsilon > 0$ such that*

$$\|\eta_{u,v} - \eta_v\|_{W(0,T)} \leq C_\eta\|y_u - \bar{y}\|_{L^\infty(Q)}\|\eta_v\|_{L^2(Q)} \quad \forall v \in L^2(Q), \ \forall u \in \mathcal{U}_{\mathrm{ad}}, \qquad (3.22)$$

where from now on, differing from the notation in Theorem 1.2, $\eta_{u,v}$ and η_v are $\eta_{u,v}$ and η_v are the first components of $G'(u)v$ and $G'(\bar{u})v$, respectively. Moreover, there is a constant C_0 such that

$$\|y_u - \bar{y} - \eta_{u-\bar{u}}\|_{W(0,T)} \leq C_0\|y_u - \bar{y}\|_{L^\infty(Q)}\|y_u - \bar{y}\|_{L^2(Q)} \quad \forall u \in \mathcal{U}_{\mathrm{ad}}. \qquad (3.23)$$

Proof. We introduce the operator K_μ as in the proof of Lemma 3.8 and define $w(x,t) = e^{-\mu t}(\eta_{u,v} - \eta_v)$. Now, we subtract the equation satisfied by η_v from that for $\eta_{u,v}$ and apply the mean value theorem to get

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w + R'(y_u)w + \mu w + \alpha K_\mu(w) &=& -e^{-\mu t}R''(\tilde{y}_u)(y_u - \bar{y})\eta_v & \text{in } Q \\ \partial_n w &=& 0 & \text{on } \Sigma \\ w(x,0) &=& 0 & \text{in } \Omega, \end{cases} \qquad (3.24)$$

where $\tilde{y}_u(x,t) = \bar{y}(x,t) + \vartheta(x,t)(y_u(x,t) - \bar{y}(x,t))$, $0 \leq \vartheta(x,t) \leq 1$, and μ is sufficiently large. Invoking again [8, Lemma 2.3], we obtain (3.22).

To show (3.23), we proceed similarly. We set $\tilde{w} = e^{-\mu t}(y_u - \bar{y} - \eta_{u-\bar{u}})$ and perform the Taylor expansion

$$R(y_u) = R(\bar{y}) + R'(\bar{y})(y_u - \bar{y}) + \frac{1}{2}R''(\bar{y} + \theta(y_u - \bar{y}))(y_u - \bar{y})^2 \quad \text{with} \ 0 \leq \theta(x,t) \leq 1.$$

From the PDEs for y_u, $\bar{y}$, and $\eta_{u-\bar{u}}$, we deduce analogously to (3.24)

$$\frac{\partial\tilde{w}}{\partial t} - \Delta\tilde{w} + R'(\bar{y})\tilde{w} + \mu\tilde{w} + \alpha K_\mu(\tilde{w}) = -e^{-\mu t}R''(\bar{y} + \theta(y_u - \bar{y}))(y_u - \bar{y})^2 \quad \text{in } Q$$

subject to the same homogeneous boundary and initial conditions as for w above. Then the estimate (3.23) for $\tilde{w} = y_u - \bar{u} - \eta_{u-\bar{u}}$ follows as the one for w. □

LEMMA 3.10. *There exists a constant $M_{a,b}$ such that $\forall u \in \mathcal{U}_{\mathrm{ad}}$ and $\forall v_1, v_2 \in L^2(Q)$ the following estimate holds*

$$|F''(u)(v_1, v_2)| \le M_{a,b} \left(\|\eta_{v_1}\|_{L^2(Q)} \|\eta_{v_2}\|_{L^2(Q)} + \|\eta_{v_1}(T)\|_{L^2(\Omega)} \|\eta_{v_2}(T)\|_{L^2(\Omega)} \right).$$
(3.25)

Proof. This estimate follows easily from the expression (1.7) of F'', (3.17), and the fact that the functions C_Q^Y, C_T^Y, C_Q^Z and C_T^Z are bounded. In addition, we have to recall that $\zeta_v(x,t) = K_0(\eta_v)(x,t)$, where K_0 is the operator introduced in the proof of Lemma 3.8. From this identity we get

$$\|\zeta_v\|_{L^2(Q)} \le \gamma T \|\eta_v\|_{L^2(Q)} \quad \text{and} \quad \|\zeta_v(T)\|_{L^2(\Omega)} \le \gamma\sqrt{T} \|\eta_v\|_{L^2(Q)}.$$
(3.26)

□

Next, we prove our last lemma.

LEMMA 3.11. *For every $\rho > 0$ there exists $\varepsilon > 0$ such that*

$$|[F''(u) - F''(\bar{u})]v^2| \le \rho \left(\|\eta_v\|_{L^2(Q)}^2 + \|\eta_v(T)\|_{L^2(\Omega)}^2 \right)$$
(3.27)

holds for all $v \in L^2(Q)$ and for all $u \in \mathcal{U}_{\mathrm{ad}}$ such that $\|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon$.

Proof. From (1.7), it follows

$$
\begin{aligned}
[F''(u) - F''(\bar{u})]v^2 &= \int_Q C_Q^Y (\eta_{u,v}^2 - \eta_v^2)\, dx\, dt + \int_Q \{R'(\bar{y})\bar{\varphi}\eta_v^2 - R'(y_u)\varphi_u \eta_{u,v}^2\}\, dx\, dt \\
&\quad + \int_\Omega C_T^Y (\eta_{u,v}^2(T) - \eta_v^2(T))\, dx + \int_\Omega C_T^Z (\zeta_{u,v}^2(T) - \zeta_v^2(T))\, dx \\
&\quad + \int_Q C_Q^Z (\zeta_{u,v}^2 - \zeta_v^2)\, dx\, dt = I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

Let us indicate how all these terms can be estimated. First, from (3.22) we deduce that

$$\|\eta_{u,v} - \eta_v\|_{L^2(Q)} \le \|\eta_{u,v} - \eta_v\|_{W(0,T)} \le C_\eta \|y_u - \bar{y}\|_{L^\infty(Q)} \|\eta_v\|_{L^2(Q)} \tag{3.28}$$
$$\|\eta_{u,v}(T) - \eta_v(T)\|_{L^2(\Omega)} \le C\|\eta_{u,v} - \eta_v\|_{W(0,T)} \le CC_\eta \|y_u - \bar{y}\|_{L^\infty(Q)} \|\eta_v\|_{L^2(Q)}. \tag{3.29}$$

From here, we also get

$$\|\eta_{u,v}\|_{L^2(Q)} \le \left(1 + C_\eta \|y_u - \bar{y}\|_{L^\infty(Q)}\right) \|\eta_v\|_{L^2(Q)} \tag{3.30}$$
$$\|\eta_{u,v}(T)\|_{L^2(\Omega)} \le \|\eta_{u,v}(T) - \eta_v(T)\|_{L^2(\Omega)} + \|\eta_v(T)\|_{L^2(\Omega)}$$
$$\le CC_\eta \|y_u - \bar{y}\|_{L^\infty(Q)} \|\eta_v\|_{L^2(Q)} + \|\eta_v(T)\|_{L^2(\Omega)}. \tag{3.31}$$

We use (3.28) and (3.30) to estimate I_1

$$|I_1| \le \|C_Q^Y\|_{L^\infty(Q)} \|\eta_{u,v} + \eta_v\|_{L^2(Q)} \|\eta_{u,v} - \eta_v\|_{L^2(Q)}$$

$$\le \|C_Q^Y\|_{L^\infty(Q)} \left(2 + C_\eta \|y_u - \bar{y}\|_{L^\infty(Q)}\right) C_\eta \|y_u - \bar{y}\|_{L^\infty(Q)} \|\eta_v\|_{L^2(Q)}^2.$$

I_3 is estimated from (3.29) and (3.31) as follows

$$|I_3| \le \|C_T^Y\|_{L^\infty(\Omega)} \|\eta_{u,v}(T) + \eta_v(T)\|_{L^2(\Omega)} \|\eta_{u,v}(T) - \eta_v(T)\|_{L^2(\Omega)}$$

$$\leq \|C_T^Y\|_{L^\infty(\Omega)} C^2 C_\eta^2 \|y_u - \bar{y}\|_{L^\infty(Q)}^2 \|\eta_v\|_{L^2(Q)}^2$$

$$+\|C_T^Y\|_{L^\infty(\Omega)} C C_\eta \|y_u - \bar{y}\|_{L^\infty(Q)} 2\|\eta_v\|_{L^2(Q)} \|\eta_v(T)\|_{L^2(\Omega)}$$

$$\leq \|C_T^Y\|_{L^\infty(\Omega)} \|y_u - \bar{y}\|_{L^\infty(Q)} \left\{ C^2 C_\eta^2 (1 + \|y_u - \bar{y}\|_{L^\infty(Q)}) \|\eta_v\|_{L^2(Q)}^2 + \|\eta_v(T)\|_{L^2(\Omega)}^2 \right\}.$$

To deduce the estimates for I_4 and I_5 we use again the operator K_0. Then, from (3.26) we obtain

$$\|\zeta_{u,v}\|_{L^2(Q)} \leq \gamma T \|\eta_{u,v}\|_{L^2(Q)},$$
$$\|\zeta_{u,v}(T)\|_{L^2(\Omega)} \leq \gamma \sqrt{T} \|\eta_{u,v}(T)\|_{L^2(\Omega)},$$
$$\|\zeta_{u,v} - \zeta_v\|_{L^2(Q)} \leq \gamma T \|\eta_{u,v} - \eta_v\|_{L^2(Q)},$$
$$\|\zeta_{u,v}(T) - \zeta_v(T)\|_{L^2(\Omega)} \leq \gamma \sqrt{T} \|\eta_{u,v}(T) - \eta_v(T)\|_{L^2(\Omega)}.$$

Now, we proceed in the same way as for the estimates I_1 and I_3. Finally, we estimate I_2 as follows

$$\begin{aligned}|I_2| \leq &\|R'(\bar{y}) - R'(y_u)\|_{L^\infty(Q)} \|\bar{\varphi}\|_{L^\infty(Q)} \|\eta_v\|_{L^2(Q)}^2 \\ &+\|R'(y_u)\|_{L^\infty(Q)} \|\bar{\varphi} - \varphi_u\|_{L^\infty(Q)} \|\eta_v\|_{L^2(Q)}^2 \\ &+\|R'(y_u)\|_{L^\infty(Q)} \|\varphi_u\|_{L^\infty(Q)} \|\eta_v - \eta_{u,v}\|_{L^2(Q)}^2.\end{aligned}$$

By (3.19), we can estimate the first two terms by $C'\|y_u - \bar{y}\|_{L^\infty(Q)} \|\eta_v\|_{L^2(Q)}^2$ for some constant C'. The third term is estimated as I_1. The statement of the lemma is a straightforward consequence of the obtained estimates. $\square$

Proof of Theorem 3.7. By Lemma 3.11, we have the existence of $\varepsilon_0 > 0$ such that, for all $u \in \mathcal{U}_{ad}$ with $\|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon_0$,

$$|[F''(u) - F''(\bar{u})]v^2| \leq \frac{\sigma}{16} \left\{ \|\eta_v\|_{L^2(Q)}^2 + \|\eta_v(T)\|_{L^2(\Omega)}^2 \right\} \quad \forall v \in L^2(Q). \qquad (3.32)$$

Using Lemma 3.8, we deduce that

$$\|y_u - \bar{y}\|_{L^\infty(Q)} \leq \frac{C_3}{C_\infty} \|u - \bar{u}\|_{L^3(Q)} \leq \frac{C_3}{C_\infty} (b-a)^{1/3} \|u - \bar{u}\|_{L^2(Q)}^{2/3}.$$

Let us take

$$0 < \varepsilon_1 \leq \left(\frac{C_\infty \varepsilon_0}{C_3 (b-a)^{1/3}} \right)^{3/2}.$$

Then $\|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon_0$ if $\|u - \bar{u}\|_{L^2(Q)} < \varepsilon_1$. Hence, (3.32) is true in $\mathcal{U}_{ad} \cap B_{\varepsilon_1}(\bar{u})$,

$$|[F''(u) - F''(\bar{u})]v^2| \leq \frac{\sigma}{16} \left\{ \|\eta_v\|_{L^2(Q)}^2 + \|\eta_v(T)\|_{L^2(\Omega)}^2 \right\} \quad \forall v \in L^2(Q), u \in \mathcal{U}_{ad} \cap B_{\varepsilon_1}(\bar{u}).$$
$$(3.33)$$

Next, we prove that (3.16) holds with $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, where ε_2 will be defined below. Let us take $u \in \mathcal{U}_{ad} \cap B_\varepsilon(\bar{u})$. We denote by Q_v the set of points $(x,t) \in Q$ such that (3.12) is not satisfied by $(u - \bar{u})(x,t)$. Now we set

$$v = (u - \bar{u})\chi_{Q_v},$$

where χ_{Q_v} is the characteristic function of Q_v, and $w = (u - \bar{u}) - v$. By definition $w(x,t)$ satisfies (3.12) for every $(x,t) \in Q$, and hence $w \in C_{\bar{u}}^\tau$.

From (1.5), (3.13), and (3.33) we infer

$$
\begin{aligned}
J(u) - J(\bar{u}) &\geq F'(\bar{u})(u - \bar{u}) + \frac{1}{2}F''(\bar{u} + \theta(u - \bar{u}))(u - \bar{u})^2 + \kappa j'(\bar{u}; u - \bar{u}) \\
&\geq \tau\|v\|_{L^1(Q_v)} + \frac{1}{2}F''(\bar{u})(u - \bar{u})^2 + \frac{1}{2}[F''(\bar{u} + \theta(u - \bar{u})) - F''(\bar{u})](u - \bar{u})^2 \\
&\geq \tau\|v\|_{L^1(Q)} + \frac{1}{2}F''(\bar{u})w^2 + \frac{1}{2}F''(\bar{u})v^2 + F''(\bar{u})(v, w) \\
&\quad - \frac{\sigma}{16}\left\{\|\eta_{u-\bar{u}}\|_{L^2(Q)}^2 + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}^2\right\}.
\end{aligned}
\tag{3.34}
$$

From (3.20) we get

$$
\begin{aligned}
\|\eta_v\|_{L^2(Q)}^2 + \|\eta_v(T)\|_{L^2(\Omega)}^2 &\leq C_1(b-a)^{1/3}\|u - \bar{u}\|_{L^2(Q)}^{2/3}\|v\|_{L^1(Q)} \\
&\leq C_1(b-a)^{1/3}\varepsilon_2^{2/3}\|v\|_{L^1(Q)}.
\end{aligned}
$$

From here it follows

$$
\|v\|_{L^1(Q)} \geq \frac{1}{C_1(b-a)^{1/3}\varepsilon_2^{2/3}}\left\{\|\eta_v\|_{L^2(Q)}^2 + \|\eta_v(T)\|_{L^2(\Omega)}^2\right\}.
$$

Inserting this inequality in (3.34) and using (3.15) and (3.25) it follows

$$
\begin{aligned}
J(u) - J(\bar{u}) &\geq \frac{\tau}{C_1(b-a)^{1/3}\varepsilon_2^{2/3}}\left\{\|\eta_v\|_{L^2(Q)}^2 + \|\eta_v(T)\|_{L^2(\Omega)}^2\right\} \\
&\quad + \frac{\sigma}{2}\left\{\|\eta_w\|_{L^2(Q)}^2 + \|\eta_w(T)\|_{L^2(\Omega)}^2\right\} - \frac{M_{a,b}}{2}\left\{\|\eta_v\|_{L^2(Q)}^2 + \|\eta_v(T)\|_{L^2(\Omega)}^2\right\} \\
&\quad - M_{a,b}\left\{\|\eta_v\|_{L^2(Q)}\|\eta_w\|_{L^2(Q)} + \|\eta_v(T)\|_{L^2(\Omega)}\|\eta_w(T)\|_{L^2(\Omega)}\right\} \\
&\quad - \frac{\sigma}{16}\left\{\|\eta_{u-\bar{u}}\|_{L^2(Q)}^2 + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}^2\right\}
\end{aligned}
$$

and, with Young's inequality,

$$
\begin{aligned}
&\geq \left(\frac{\tau}{C_1(b-a)^{1/3}\varepsilon_2^{2/3}} - \frac{M_{a,b}}{2} - \frac{M_{a,b}^2}{\sigma}\right)\left\{\|\eta_v\|_{L^2(Q)}^2 + \|\eta_v(T)\|_{L^2(\Omega)}^2\right\} \\
&\quad + \frac{\sigma}{4}\left\{\|\eta_w\|_{L^2(Q)}^2 + \|\eta_w(T)\|_{L^2(\Omega)}^2\right\} - \frac{\sigma}{16}\left\{\|\eta_{u-\bar{u}}\|_{L^2(Q)}^2 + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}^2\right\}.
\end{aligned}
\tag{3.35}
$$

We take $\varepsilon_2 > 0$ satisfying

$$
\frac{\tau}{C_1(b-a)^{1/3}\varepsilon_2^{2/3}} - \frac{M_{a,b}}{2} - \frac{M_{a,b}^2}{\sigma} \geq \frac{\sigma}{4}.
$$

Inserting this inequality in (3.35) we obtain

$$J(u) - J(\bar{u}) \geq \frac{\sigma}{4} \left\{ \|\eta_v\|_{L^2(Q)}^2 + \|\eta_v(T)\|_{L^2(\Omega)}^2 + \|\eta_w\|_{L^2(Q)}^2 + \|\eta_w(T)\|_{L^2(\Omega)}^2 \right\}$$

$$- \frac{\sigma}{16} \left\{ \|\eta_{u-\bar{u}}\|_{L^2(Q)}^2 + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}^2 \right\}$$

$$\geq \frac{\sigma}{8} \left\{ \|\eta_{v+w}\|_{L^2(Q)}^2 + \|\eta_{v+w}(T)\|_{L^2(\Omega)}^2 \right\} - \frac{\sigma}{16} \left\{ \|\eta_{u-\bar{u}}\|_{L^2(Q)}^2 + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}^2 \right\}$$

$$= \frac{\sigma}{16} \left\{ \|\eta_{u-\bar{u}}\|_{L^2(Q)}^2 + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}^2 \right\}$$

(notice that $v + w = u - \bar{u}$). Hence, (3.16) holds with $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. □

COROLLARY 3.12. *Under the assumptions of Theorem 3.7, there exist $\varepsilon > 0$ and $\delta > 0$ such that*

$$J(\bar{u}) + \frac{\delta}{2} \left\{ \|y_u - \bar{y}\|_{L^2(Q)}^2 + \|y_u(T) - \bar{y}(T)\|_{L^2(\Omega)}^2 \right\} \leq J(u) \quad \forall u \in \mathcal{U}_{ad} \cap B_\varepsilon(\bar{u}). \quad (3.36)$$

Proof. We define $w = e^{-\mu t}(y_u - \bar{y} - \eta_{u-\bar{u}})$; then we know the estimate (3.23),

$$\|w\|_{W(0,T)} \leq C_1 \|y_u - \bar{y}\|_{L^\infty(Q)} \|y_u - \bar{y}\|_{L^2(Q)} \quad \forall u \in \mathcal{U}_{ad}.$$

Let now ε be as in Theorem 3.7 and take $\varepsilon_1 \leq \varepsilon$ such that $e^{\mu T} C_1 \varepsilon_0 < \frac{1}{2}$. Then, for $\|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon_0$ we have

$$\|y_u - \bar{y}\|_{L^2(Q)} \leq e^{\mu T} \|w\|_{L^2(Q)} + \|\eta_{u-\bar{u}}\|_{L^2(Q)} \leq \frac{1}{2} \|y_u - \bar{y}\|_{L^2(Q)} + \|\eta_{u-\bar{u}}\|_{L^2(Q)},$$

hence, moving the term $\frac{1}{2}\|y_u - \bar{y}\|_{L^2(Q)}$ to the other side,

$$\|y_u - \bar{y}\|_{L^2(Q)} \leq 2\|\eta_{u-\bar{u}}\|_{L^2(Q)} \quad \forall u \in \mathcal{U}_{ad} : \|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon_0.$$

Moreover, we have

$$\|y_u(T) - \bar{y}(T)\|_{L^2(\Omega)} \leq e^{\mu T} \|w(T)\|_{L^2(\Omega)} + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}$$

$$\leq C_2 e^{\mu T} \|w\|_{W(0,T)} + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}$$

$$\leq \frac{C_2}{2} \|y_u - \bar{y}\|_{L^2(Q)} + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)} \leq C_2 \|\eta_{u-\bar{u}}\|_{L^2(Q)} + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}.$$

Therefore, we get from the last two estimates

$$\|y_u - \bar{y}\|_{L^2(Q)}^2 + \|y_u(T) - \bar{y}(T)\|_{L^2(\Omega)}^2 \leq C_3 \left\{ \|\eta_{u-\bar{u}}\|_{L^2(Q)}^2 + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}^2 \right\}$$

$\forall u \in \mathcal{U}_{ad}$ such that $\|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon$. Finally, we take $0 < \delta \leq \frac{\sigma}{8C_3}$ and $0 < \varepsilon \leq \varepsilon_0$ so that $\|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon_0$ for every $u \in \mathcal{U}_{ad} \cap B_\varepsilon(\bar{u})$. Then, (3.36) follows from (3.16). □

The growth condition (3.36) is valid in a ball around $\bar{u}$. Therefore, we analyze if a result similar to Corollary 3.5 can be proved for the case $\nu = 0$. This would be applicable in a ball around $\bar{y}$, hence in a possibly larger neighborhood of $\bar{u}$.

We have not been able to establish such a result based on the cone $C_{\bar{u}}^\tau$. To deal with this problem we introduce a different extended cone defined by

$$E_{\bar{u}}^\tau = \big\{ v \in L^2(Q) \text{ satisfying (3.1) and}$$

$$F'(\bar{u})v + \kappa j'(\bar{u}; v) \leq \tau\{\|\eta_v\|_{L^2(Q)} + \|\eta_v(T)\|_{L^2(\Omega)}\} \big\}.$$

From Lemma 3.1-1 and (1.5) we infer that $C_{\bar{u}} \subset E_{\bar{u}}^\tau$ for every $\tau > 0$. Thus the cone $E_{\bar{u}}^\tau$ can be considered as an small extension of $C_{\bar{u}}$. We are able to prove the following result on second order sufficiency that is based on $E_{\bar{u}}^\tau$:

THEOREM 3.13. *Let $\bar{u} \in \mathcal{U}_{ad}$ satisfy the optimality system (2.1)-(2.3) along with the state $(\bar{y}, \bar{z})$ and the adjoint state $(\bar{\varphi}, \bar{\psi})$. Assume also that*

$$\exists \tau > 0 \ \text{ and } \ \exists \sigma > 0 \ \text{ such that } \ F''(\bar{u})v^2 \geq \sigma \left(\|\eta_v\|_{L^2(Q)}^2 + \|\eta_v(T)\|_{L^2(\Omega)}^2 \right) \ \forall v \in E_{\bar{u}}^\tau. \tag{3.37}$$

Then, there exists $\varepsilon > 0$ such that

$$J(\bar{u}_\nu) + \frac{\delta}{2} \left\{ \|\eta_{u-\bar{u}}\|_{L^2(Q)}^2 + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}^2 \right\} \leq J(u) \ \forall u \in \mathcal{U}_{ad} : \|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon. \tag{3.38}$$

Proof. First, we prove that there is a constant $M > 0$ such that

$$\|\eta_{u-\bar{u}}\|_{L^2(Q)} + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)} \leq M \|y_u - \bar{y}\|_{L^\infty(Q)} \ \forall u \in \mathcal{U}_{ad}. \tag{3.39}$$

With Lemma 3.9, (3.23) and (3.17) we obtain

$$\begin{aligned}
\|\eta_{u-\bar{u}}\|_{L^2(Q)} + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)} &\leq \|\eta_{u-\bar{u}} - (y_u - \bar{y})\|_{L^2(Q)} \\
&+ \|\eta_{u-\bar{u}}(T) - (y_u - \bar{y})(T)\|_{L^2(\Omega)} + \|y_u - \bar{y}\|_{L^2(Q)} + \|(y_u - \bar{y})(T)\|_{L^2(\Omega)} \\
&\leq C\|\eta_{u-\bar{u}} - (y_u - \bar{y})\|_{W(0,T)} + (|Q|^{1/2} + |\Omega|^{1/2})\|y_u - \bar{y}\|_{L^\infty(Q)} \\
&\leq [CC_0 2C_{a,b} + |Q|^{1/2} + |\Omega|^{1/2}]\|y_u - \bar{y}\|_{L^\infty(Q)}.
\end{aligned}$$

Hence, (3.39) holds with $M = CC_0 2C_{a,b} + |Q|^{1/2} + |\Omega|^{1/2}$. Let us set

$$\varepsilon_1 = \frac{4\tau}{M(\sigma + 2M_{a,b})},$$

where $M_{a,b}$ was introduced in Lemma 3.10. From Lemma 3.11, we deduce the existence of $\varepsilon_2 > 0$ such that $\forall u \in \mathcal{U}_{ad}$ with $\|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon_2$

$$|[F''(u) - F''(\bar{u})]v^2| \leq \frac{\sigma}{2} \left\{ \|\eta_v\|_{L^2(Q)}^2 + \|\eta_v(T)\|_{L^2(\Omega)}^2 \right\} \ \forall v \in L^2(Q). \tag{3.40}$$

Next, we prove that (3.38) holds with $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. We take $u \in \mathcal{U}_{ad}$ such that $\|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon$ and distinguish two cases.

CASE I: $u - \bar{u} \notin E_{\bar{u}}^\tau$. Here we have

$$\begin{aligned}
J(u) - J(\bar{u}) &\geq F'(\bar{u})(u - \bar{u}) + \frac{1}{2}F''(\bar{u} + \theta(u - \bar{u}))(u - \bar{u})^2 + \kappa j'(\bar{u}; u - \bar{u}) \\
&> \tau \left\{ \|\eta_{u-\bar{u}}\|_{L^2(Q)} + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)} \right\} - \frac{M_{a,b}}{2} \left\{ \|\eta_{u-\bar{u}}\|_{L^2(Q)}^2 + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}^2 \right\} \\
&\geq \frac{\tau}{M\varepsilon} \left\{ \|\eta_{u-\bar{u}}\|_{L^2(Q)} + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)} \right\}^2 - \frac{M_{a,b}}{2} \left\{ \|\eta_{u-\bar{u}}\|_{L^2(Q)}^2 + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}^2 \right\} \\
&\geq \left(\frac{\tau}{M\varepsilon} - \frac{M_{a,b}}{2} \right) \left\{ \|\eta_{u-\bar{u}}\|_{L^2(Q)}^2 + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}^2 \right\} \\
&\geq \frac{\sigma}{4} \left\{ \|\eta_{u-\bar{u}}\|_{L^2(Q)}^2 + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}^2 \right\}.
\end{aligned}$$

Notice that (3.39) yields $(\|\eta_{u-\bar{u}}\|_{L^2(Q)} + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)})^{-1} \geq (M\varepsilon)^{-1}$.

CASE II: $u - \bar{u} \in E_{\bar{u}}^\tau$. This time, we use (1.5), (2.3), (3.37) and (3.40)

$$J(u) - J(\bar{u}) \geq F'(\bar{u})(u - \bar{u}) + \frac{1}{2}F''(\bar{u} + \theta(u - \bar{u}))(u - \bar{u})^2 + \kappa j'(\bar{u}; u - \bar{u})$$

$$\geq \int_Q (\bar{\varphi} + \nu\bar{u} + \kappa\bar{\lambda})(u - \bar{u}) \, dx \, dt + \frac{1}{2}F''(\bar{u} + \theta(u - \bar{u}))(u - \bar{u})^2$$

$$\geq \frac{1}{2}F''(\bar{u})(u - \bar{u})^2 + \frac{1}{2}[F''(\bar{u} + \theta(u - \bar{u})) - F''(\bar{u})](u - \bar{u})^2$$

$$\geq \frac{\sigma}{2}\left\{\|\eta_{u-\bar{u}}\|_{L^2(Q)}^2 + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}^2\right\} - \frac{\sigma}{4}\left\{\|\eta_{u-\bar{u}}\|_{L^2(Q)}^2 + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}^2\right\}$$

$$= \frac{\sigma}{4}\left\{\|\eta_{u-\bar{u}}\|_{L^2(Q)}^2 + \|\eta_{u-\bar{u}}(T)\|_{L^2(\Omega)}^2\right\}.$$

☐

COROLLARY 3.14. *Let $\bar{u}$ satisfy the assumptions of Theorem 3.13. Then, there exist $\delta > 0$ and $\varepsilon > 0$ such that, for all $u \in \mathcal{U}_{\mathrm{ad}}$ with $\|y_u - \bar{y}\|_{L^\infty(Q)} < \varepsilon$,*

$$J(\bar{u}) + \frac{\delta}{2}\left\{\|y_u - \bar{y}\|_{L^2(Q)}^2 + \|y_u(T) - \bar{y}(T)\|_{L^2(\Omega)}^2\right\} \leq J(u). \tag{3.41}$$

The proof of this corollary is almost the same as the one of Corollary 3.12. Corollary 3.12 also holds under the second order condition (3.37). This is a consequence of (3.41) and (3.19). By Corollary 3.14 the local optimality of $\bar{u}$ is strong in the sense of calculus of variations.

We finish this section by proving that the second order condition (3.14) can only hold in some exceptional cases.

THEOREM 3.15. *Let $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$ satisfy the optimality system (2.1)-(2.3) along with the state $(\bar{y}, \bar{z})$ and the adjoint state $(\bar{\varphi}, \bar{\psi})$. If $\nu = 0$ and $\bar{u} \not\equiv 0$, $\bar{u} \not\equiv a$, $\bar{u} \not\equiv b$, $\bar{\varphi} \not\equiv \kappa$ and $\bar{\varphi} \not\equiv -\kappa$, then (3.14) cannot hold.*

Proof. We argue again by contradiction: Assume that (3.14) holds. Then, arguing as in the proof of Theorem 3.7, we deduce the existence of $\delta > 0$ and $\varepsilon > 0$ such that

$$J(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}_\nu\|_{L^2(Q)}^2 \leq J(u) \quad \forall u \in \mathcal{U}_{\mathrm{ad}} \cap B_\varepsilon(\bar{u}).$$

Then, $\bar{u}$ is a solution of the problem

$$\text{(Q)} \quad \begin{cases} \underset{u \in \mathcal{U}_{\mathrm{ad}} \cap B_\varepsilon(\bar{u})}{\text{Min}} \quad I(u) := J(u) - \frac{\delta}{2}\|u - \bar{u}\|_{L^2(Q)}^2. \end{cases}$$

The Hamiltonian of this control problem is given by

$$\mathcal{H}(x, t, y, z, \varphi, u) = L(x, t, y, z) + \varphi u + \kappa|u| - \frac{\delta}{2}(u - \bar{u}(x, t))^2 = L(x, t, y, z) + H(x, t, \varphi, u),$$

where

$$L(x, t, y, z) = C_Q^Y(x, t)(y - y_Q(x, t))^2 + C_Q^Z(x, t)(z - z_Q(x, t))^2$$

and H denotes the part of the Hamiltonian involving u. To shorten the notation we set $\bar{H}(x, t, u) = H(x, t, \bar{\varphi}(x, t), u)$. According to Pontryagin's principle

$$\bar{H}(x, t, \bar{u}(x, t)) = \min_{u \in [a,b]} \bar{H}(x, t, u) \quad \text{for a.a. } (x, t) \in Q; \tag{3.42}$$

see, for instance, [5] or [14, Chapter 4] for the proof of Pontryagin's principle for control problems associated with evolution partial differential equations. However, (3.42) contradicts the following inequalities

$$
\begin{aligned}
&\text{if } \kappa < \bar{\varphi}(x,t) < \kappa + \frac{\delta}{2}|a| && \Rightarrow \bar{H}(x,t,0) < \bar{H}(x,t,a) \\
&\text{if } \kappa < -\bar{\varphi}(x,t) < \kappa + \frac{\delta}{2}b && \Rightarrow \bar{H}(x,t,0) < \bar{H}(x,t,b) \\
&\text{if } 0 < \bar{\varphi}(x,t) < \kappa \text{ and } \kappa - \bar{\varphi}(x,t) < \frac{\delta}{2}|a| && \Rightarrow \bar{H}(x,t,a) < \bar{H}(x,t,0) \\
&\text{if } -\kappa < \bar{\varphi}(x,t) < 0 \text{ and } \kappa + \bar{\varphi}(x,t) < \frac{\delta}{2}b && \Rightarrow \bar{H}(x,t,b) < \bar{H}(x,t,0).
\end{aligned}
\tag{3.43}
$$

Indeed, let us prove that the first and the third statements contradict (3.42). The other two statements are analyzed similarly. If $\kappa < \bar{\varphi}(x,t)$, then Corollary 2.4 implies that $\bar{u}(x,t) = a$. Now, using that $\bar{\varphi}(x,t) < \kappa + \frac{\delta}{2}|a|$, we get

$$
\bar{H}(x,t,\bar{u}(x,t)) = \bar{\varphi}(x,t)\bar{u}(x,t) + \kappa|\bar{u}(x,t)| = \bar{\varphi}(x,t)a + \kappa|a|
$$
$$
= (\kappa - \bar{\varphi}(x,t))|a| > -\frac{\delta}{2}a^2 = \bar{H}(x,t,0).
$$

Let us consider the case $0 < \bar{\varphi}(x,t) < \kappa$. Then Corollary 2.4 implies that $\bar{u}(x,t) = 0$. On the other hand, using the inequality $\kappa - \bar{\varphi}(x,t) < \frac{\delta}{2}|a|$, we obtain

$$
\bar{H}(x,t,\bar{u}(x,t)) = \bar{\varphi}(x,t)\bar{u}(x,t) + \kappa|\bar{u}(x,t)| = 0
$$
$$
> (\kappa - \bar{\varphi}(x,t))|a| - \frac{\delta}{2}a^2 = \bar{H}(x,t,a).
$$

Finally, since $\bar{\varphi} \in C(Q)$, and $\bar{u} \not\equiv 0$, $\bar{u} \not\equiv a$, $\bar{u} \not\equiv b$, $\bar{\varphi} \not\equiv \kappa$ and $\bar{\varphi} \not\equiv -\kappa$, we deduce from Corollary 2.4 that the sets

$$
Q_\kappa^+ = \{(x,t) \in Q : \bar{\varphi}(x,t) > \kappa\}, \quad Q_\kappa^- = \{(x,t) \in Q : \bar{\varphi}(x,t) < -\kappa\},
$$
$$
\text{and } Q_\kappa = \{(x,t) \in Q : |\bar{\varphi}(x,t)| < \kappa\}
$$

are strict open subsets of Q, and at least one of them is not empty. Hence, the set of points (x,t) satisfying one of the conditions of (3.43) is open and nonempty. Consequently, the set of points where Pontryagin's principle fails has a positive Lebesgue measure, which is not possible. $\square$

Remark 3.16. *The coercivity assumption* (3.15) *can only hold, if there is a constant $\delta_0 > 0$ such that $C_Q^Y(x,t) \geq \delta_0$ holds for a.a. $(x,t) \in Q$ and $C_T^Y(x) \geq \delta_0$ holds for a.a. $x \in \Omega$. Our theory is worked out for this case.*

If C_T^Y does not obey this assumption, for instance if $C_T^Y = 0$, then the main results must be modified in an obvious way. The reader will confirm that then the theory remains valid under the following changes: The norm squares of $\eta_v(T)$, $\eta_{u-\bar{u}}(T)$ in (3.15), (3.16), (3.37), *and* (3.38) *are to be deleted. Analogously, the norm squares of $y_u(T) - \bar{y}(T)$ must be deleted in* (3.36) *and* (3.41).

4. Applications to the Stability Analysis with Respect to Perturbations. Here, we exemplarily show by three case studies how the second order sufficient optimality conditions can be invoked as main assumption for proving the stability of optimal solutions with respect to perturbations of our optimal control problem. We consider perturbations of the desired state functions and also the case where the Tikhonov regularization parameter ν tends to zero. This can be viewed as a perturbation of the reference parameter $\nu = 0$.

4.1. Perturbations of the desired state functions.

4.1.1. Case $\nu > 0$. Assume that a family of perturbed desired state functions y_Q^ε, $z_Q^\varepsilon \in L^p(Q)$, $p > 5/2$, and y_T^ε, $z_T^\varepsilon \in L^\infty(\Omega)$, $\varepsilon > 0$, is given such that

$$\max\{\|y_Q^\varepsilon - y_Q\|_{L^2(Q)}, \|z_Q^\varepsilon - z_Q\|_{L^2(Q)}, \|y_T^\varepsilon - y_T\|_{L^2(\Omega)}, \|z_T^\varepsilon - z_T\|_{L^2(\Omega)}\} \le C\varepsilon \quad \forall \varepsilon > 0 \tag{4.1}$$

is satisfied. We also assume that $\{y_T^\varepsilon, z_T^\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(\Omega)$. Associated with these perturbed data, we define the perturbed objective functionals

$$F_\nu^\varepsilon(u) = \frac{1}{2} \int_Q \{C_Q^Y(y_u - y_Q^\varepsilon)^2 + C_Q^Z(z_u - z_Q^\varepsilon)^2\}\, dx\, dt$$

$$+ \frac{1}{2} \int_\Omega \{C_T^Y(y_u(\cdot, T) - y_T^\varepsilon)^2 + C_T^Z(z_u(\cdot, T) - z_T^\varepsilon)^2\}\, dx + \frac{\nu}{2} \int_Q u^2\, dx\, dt,$$

$$J_\nu^\varepsilon(u) = F_\nu^\varepsilon(u) + \kappa\, j(u)$$

and the family of perturbed optimal control problems

$$(\mathrm{P}_\nu^\varepsilon) \left\{ \begin{array}{cc} \mathrm{Min} & J_\nu^\varepsilon(u). \\ u \in \mathcal{U}_{\mathrm{ad}} & \end{array} \right.$$

Prior to proving the next result, for convenience we introduce the notation

$$F(u) := F_0(u), \quad F^\varepsilon(u) := F_0^\varepsilon(u).$$

These functions contain only the parts of J and J^ε that depend on the state functions. The Tikhonov regularization term and $\kappa\, j$ are separated from them.

THEOREM 4.1. *If $\{\bar{u}_\nu^\varepsilon\}_\varepsilon$ is any sequence of optimal controls of problems $(\mathrm{P}_\nu^\varepsilon)$ that converges weakly in $L^2(Q)$ to some $\bar{u}_\nu$, then $\bar{u}_\nu$ is optimal for (P_ν) and*

$$\lim_{\varepsilon \to 0} \|\bar{u}_\nu^\varepsilon - \bar{u}_\nu\|_{L^p(Q)} = 0 \tag{4.2}$$

holds for all $p \in [1, \infty)$. Conversely, if $\bar{u}_\nu$ is a strict locally optimal control of (P_ν), then there exists a sequence $\{\bar{u}_\nu^\varepsilon\}_\varepsilon$ of locally optimal controls of $(\mathrm{P}_\nu^\varepsilon)$ converging to $\bar{u}_\nu$. This sequence obeys (4.2). Furthermore, there exists $\rho > 0$ such that every $\bar{u}_\nu^\varepsilon$ affords a global minimum to J_ν^ε in $\mathcal{U}_{\mathrm{ad}} \cap \bar{B}_\rho(\bar{u}_\nu)$, where $B_\rho(\bar{u}_\nu)$ denotes the $L^2(Q)$-ball.

Proof. Let us skip the subscript ν in the proof for convenience. We write $\bar{u} := \bar{u}_\nu$ and $\bar{u}_\varepsilon := \bar{u}_\nu^\varepsilon$. From $\bar{u}_\varepsilon \rightharpoonup \bar{u}$ we get $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$ and, in a standard way, that $\bar{u}$ is optimal for (P_ν). It remains to show the strong convergence.

Applying a result of [11], we know that $\bar{y}_\varepsilon - \bar{y}$ is bounded in some Hölder space $C^\lambda(\bar{Q})$, $\lambda \in (0, 1)$. Therefore we obtain by compact embedding in $C(\bar{Q})$ that a subsequence $\bar{y}_\varepsilon := y_{\bar{u}_\varepsilon}$ converges strongly in $L^\infty(Q)$ towards $\bar{y}$ as $\varepsilon \to 0$. Since the same holds for all subsequences, we even have $\|\bar{y}_\varepsilon - \bar{y}\|_{L^\infty(Q)} \to 0$ as $\varepsilon \to 0$ for the whole sequence. From (1.1), we also deduced that $\|\bar{z}_\varepsilon - \bar{z}\|_{L^\infty(Q)} \to 0$ as $\varepsilon \to 0$.

Now we show the strong convergence $\bar{u}_\varepsilon \to \bar{u}$ in $L^2(Q)$. Writing $\|\cdot\| := \|\cdot\|_{L^2(Q)}$ for short, we find

$$J(\bar{u}) \le \liminf_{\varepsilon \to 0} J(\bar{u}_\varepsilon)$$

$$\le \liminf_{\varepsilon \to 0} \left(F(\bar{u}_\varepsilon) + \frac{\nu}{2}\|\bar{u}_\varepsilon\|^2 + \kappa j(\bar{u}_\varepsilon) \right) = \liminf_{\varepsilon \to 0} \left(F^\varepsilon(\bar{u}_\varepsilon) + \frac{\nu}{2}\|\bar{u}_\varepsilon\|^2 + \kappa j(\bar{u}_\varepsilon) \right)$$

$$\le \limsup_{\varepsilon \to 0} \left(F^\varepsilon(\bar{u}_\varepsilon) + \frac{\nu}{2}\|\bar{u}_\varepsilon\|^2 + \kappa j(\bar{u}_\varepsilon) \right) \le \limsup_{\varepsilon \to 0} \left(F^\varepsilon(\bar{u}) + \frac{\nu}{2}\|\bar{u}\|^2 + \kappa j(\bar{u}) \right)$$

$$= F(\bar{u}) + \frac{\nu}{2}\|\bar{u}\|^2 + \kappa j(\bar{u}) = J(\bar{u}).$$

Obviously, this chain of inequalities implies the convergence

$$\frac{\nu}{2}\|\bar{u}_\varepsilon\|^2 + \kappa j(\bar{u}_\varepsilon) \to \frac{\nu}{2}\|\bar{u}\|^2 + \kappa j(\bar{u}), \ \varepsilon \to 0.$$

As in (3.11), we infer that $\|\bar{u}_\varepsilon\|_{L^2(Q)} \to \|\bar{u}\|_{L^2(Q)}$. Along with the weak convergence, this leads to $\bar{u}_\varepsilon \to \bar{u}$ in $L^2(Q)$. By boundedness of $\mathcal{U}_{\mathrm{ad}}$ in $L^\infty(Q)$, we obtain this convergence also in $L^p(Q)$ for all $p \in [1, \infty)$.

Let us now prove the second part of the theorem and assume that $\bar{u}$ is a strict local solution to (P_ν). The controls $\bar{u}_\varepsilon$ are defined as (global) solutions to the auxiliary problems

$$\min J_\nu^\varepsilon(u), \quad u \in \mathcal{U}_{\mathrm{ad}} \cap \overline{B}_\rho(\bar{u}),$$

where $\rho > 0$ is taken sufficiently small. The existence of such controls $\bar{u}_\varepsilon$ follows by standard arguments. Thanks to boundedness and closedness of $\mathcal{U}_{\mathrm{ad}}$, we can assume that $\bar{u}_\varepsilon \rightharpoonup \tilde{u} \in \mathcal{U}_{\mathrm{ad}} \cap \overline{B}_\rho(\bar{u})$, possibly after selecting a subsequence. Now we apply the first result of the theorem in $\mathcal{U}_{\mathrm{ad}} \cap \overline{B}_\rho(\bar{u})$ and obtain that $\tilde{u}$ is globally optimal in $\mathcal{U}_{\mathrm{ad}} \cap \overline{B}_\rho(\bar{u})$, the convergence to $\tilde{u}$ is strong, and (4.2) holds. Therefore, since $\bar{u}$ is a strict local solution, $\tilde{u} = \bar{u}$ must hold provided that ρ was taken sufficiently small. This is true for any subsequence of $\bar{u}_\varepsilon$ and hence the convergence result refers to the whole sequence $\bar{u}_\varepsilon$. $\square$

THEOREM 4.2 (Lipschitz stability for $\nu > 0$). *Assume that $\nu > 0$ and $\bar{u}_\nu$ is a locally optimal control of (P_ν) that satisfies the second order sufficient optimality condition (3.5). Assume further that (4.1) holds. Let $(\bar{u}_\nu^\varepsilon)$ be a sequence of locally optimal controls of $(\mathrm{P}_\nu^\varepsilon)$ that converges to $\bar{u}_\nu$ in $L^2(Q)$ as $\varepsilon \to 0$ and enjoys the properties established in Theorem 4.1. Then, there are constants $C_L > 0$ and $\varepsilon_0 > 0$ such that*

$$\|\bar{u}_\nu^\varepsilon - \bar{u}_\nu\|_{L^2(Q)} \leq C_L\,\varepsilon \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{4.3}$$

Proof. Since $\bar{u}_\nu^\varepsilon \to \bar{u}_\nu$ as $\varepsilon \to 0$, there exists $\varepsilon_0 > 0$ such that the quadratic growth condition (3.6) holds and $\bar{u}_\nu^\varepsilon$ is a global minimum of J_ν^ε in $\mathcal{U}_{\mathrm{ad}} \cap B_{\varepsilon_0}(\bar{u}_\nu)$. Then, we can argue as follows:

$$\begin{aligned}
J_\nu^\varepsilon(\bar{u}_\nu) \geq J_\nu^\varepsilon(\bar{u}_\nu^\varepsilon) &= F^\varepsilon(\bar{u}_\nu^\varepsilon) + \frac{\nu}{2}\|\bar{u}_\nu^\varepsilon\|_{L^2(Q)}^2 + \kappa j(\bar{u}_\nu^\varepsilon) \\
&= F(\bar{u}_\nu^\varepsilon) + \frac{\nu}{2}\|\bar{u}_\nu^\varepsilon\|_{L^2(Q)}^2 + \kappa j(\bar{u}_\varepsilon) + F^\varepsilon(\bar{u}_\nu^\varepsilon) - F(\bar{u}_\nu^\varepsilon) \\
&= J_\nu(\bar{u}_\varepsilon) + F^\varepsilon(\bar{u}_\nu^\varepsilon) - F(\bar{u}_\nu^\varepsilon) \\
&\geq J_\nu(\bar{u}_\nu) + \frac{\delta}{2}\|\bar{u}_\nu^\varepsilon - \bar{u}_\nu\|_{L^2(Q)}^2 + F^\varepsilon(\bar{u}_\nu^\varepsilon) - F(\bar{u}_\nu^\varepsilon).
\end{aligned}$$

After re-arranging, the last inequality admits the form

$$(F^\varepsilon(\bar{u}_\nu) - F(\bar{u}_\nu)) - (F^\varepsilon(\bar{u}_\nu^\varepsilon) - F(\bar{u}_\nu^\varepsilon)) \geq \frac{\delta}{2}\|\bar{u}_\nu^\varepsilon - \bar{u}_\nu\|_{L^2(Q)}^2. \tag{4.4}$$

Notice that the same terms $\frac{\nu}{2}\|\bar{u}_\nu\|_{L^2(Q)}^2$ and $\kappa j(\bar{u})$ are included in $J_\varepsilon(\bar{u}_\nu)$ and $J(\bar{u}_\nu)$ and are hence cancelled. Inserting the concrete expressions for F^ε and F, the left-hand side of this inequality can be simplified considerably. For instance, by $a_1^2 - a_2^2 =$

$(a_1 - a_2)(a_1 + a_2)$ we obtain for the terms related to C_Q^Y:

$$\frac{1}{2} \int_Q C_Q^Y ((\bar{y}_\nu - y_Q^\varepsilon)^2 - (\bar{y}_\nu - y_Q)^2) \, dx \, dt - \frac{1}{2} \int_Q C_Q^Y ((\bar{y}_\nu^\varepsilon - y_Q^\varepsilon)^2 - (\bar{y}_\nu^\varepsilon - y_Q)^2) \, dx \, dt$$

$$= \frac{1}{2} \int_Q C_Q^Y \left((y_Q - y_Q^\varepsilon)(2\bar{y}_\nu - y_Q - y_Q^\varepsilon) - (y_Q - y_Q^\varepsilon)(2\bar{y}_\nu^\varepsilon - y_Q - y_Q^\varepsilon) \right) \, dx \, dt$$

$$= \int_Q C_Q^Y (y_Q - y_Q^\varepsilon)(\bar{y}_\nu - \bar{y}_\nu^\varepsilon) \, dx \, dt \le \|C_Q^Y\|_{L^\infty(Q)} \|y_Q - y_Q^\varepsilon\|_{L^2(Q)} \|\bar{y}_\nu - \bar{y}_\nu^\varepsilon\|_{L^2(Q)}$$

$$\le C\,\varepsilon\, \|\bar{y}_\nu - \bar{y}_\nu^\varepsilon\|_{L^2(Q)} \le C'\,\varepsilon\, \|\bar{u}_\nu - \bar{u}_\nu^\varepsilon\|_{L^2(Q)}.$$

In the same way, all other integrals appearing in (4.4) are first simplified and next estimated by the Cauchy-Schwarz inequality. Now the result follows immediately from these estimates and (4.4). ☐

4.1.2. Case $\nu = 0$. Now we consider the problem (P_ν) for $\nu = 0$ and we write for short $(P) := (P_0)$. Let us prove the following result analogous to Theorem 4.1.

THEOREM 4.3. *If $\{\bar{u}^\varepsilon\}_\varepsilon$ is any sequence of optimal controls of problems (P^ε) that converges weakly in $L^2(Q)$ to some $\bar{u}$, then $\bar{u}$ is optimal for (P) and*

$$\lim_{\varepsilon \to 0} \|\bar{y}^\varepsilon - \bar{y}\|_{L^\infty(Q)} = 0. \tag{4.5}$$

Conversely, if $\bar{u}$ is a strict locally optimal control of (P), then there exists a sequence $\{\bar{u}^\varepsilon\}_\varepsilon$ of locally optimal controls of (P^ε) converging weakly to $\bar{u}$. Moreover, (4.5) holds. Furthermore, there exists $\rho > 0$ such that every $\bar{u}^\varepsilon$ affords a global minimum to J^ε with respect to the elements $u \in \mathcal{U}_{ad}$ such that $\|y_u - \bar{y}\|_\infty \le \rho$.

Proof. The first part of the theorem can be proved arguing as in the proof of Theorem 4.1. For the second part, we take $\rho > 0$ such that $\bar{u}$ is the minimizer of the functional J in the convex and closed set

$$K_\rho = \{u \in \mathcal{U}_{ad} : \|y_u - \bar{y}\|_\infty \le \rho\}.$$

Now we take $\bar{u}^\epsilon$ as a global minimizer of J^ε in K_ρ. Arguing again as in the proof of Theorem 4.1, we deduce that $\{\bar{u}^\varepsilon\}_\varepsilon$ converges weakly to $\bar{u}$ in $L^p(Q)$ and (4.5) holds. ☐

THEOREM 4.4 (Lipschitz stability for $\nu = 0$). *Let $\bar{u}$ be a locally optimal control of (P) that satisfies the second order sufficient optimality condition (3.37) and let $\{\bar{u}_\varepsilon\}$ be a sequence of locally optimal controls of (P^ε) that converges weakly to $\bar{u}$ in $L^p(Q)$ as $\varepsilon \to 0$ with the properties established in Theorem 4.3. Denote the associated states by $(\bar{y}, \bar{z})$ and $(\bar{y}_\varepsilon, \bar{z}_\varepsilon)$, respectively. Then there exists $C > 0$ such that*

$$\|\bar{y}_\varepsilon - \bar{y}\|_{L^2(Q)} + \|\bar{z}_\varepsilon - \bar{z}\|_{L^2(Q)} + \|\bar{y}_\varepsilon(T) - \bar{y}(T)\|_{L^2(\Omega)} + \|\bar{z}_\varepsilon(T) - \bar{z}(T)\|_{L^2(\Omega)} \le C\,\varepsilon. \tag{4.6}$$

Proof. Since $\|\bar{y}_\varepsilon - \bar{y}\|_{L^\infty(Q)} \to 0$ as $\varepsilon \to 0$, for all sufficiently small $\varepsilon > 0$, $\bar{y}_\varepsilon$ belongs to a neighborhood of $\bar{y}$, where the quadratic growth condition (3.41) holds by Corollary 3.14. Thanks to this growth condition, we can argue as follows:

$$J_\varepsilon(\bar{u}) \ge J_\varepsilon(\bar{u}_\varepsilon) = F^\varepsilon(\bar{u}_\varepsilon) + \kappa j(\bar{u}_\varepsilon) = F(\bar{u}_\varepsilon) + \kappa j(\bar{u}_\varepsilon) + F^\varepsilon(\bar{u}_\varepsilon) - F(\bar{u}_\varepsilon)$$

$$= J(\bar{u}_\varepsilon) + F^\varepsilon(\bar{u}_\varepsilon) - F(\bar{u}_\varepsilon)$$

$$\ge J(\bar{u}) + \frac{\delta}{2}\{\|\bar{y}_\varepsilon - \bar{y}\|_{L^2(Q)}^2 + \|\bar{y}_\varepsilon(T) - \bar{y}(T)\|_{L^2(\Omega)}^2\} + F^\varepsilon(\bar{u}_\varepsilon) - F(\bar{u}_\varepsilon).$$

After re-arranging, the last inequality admits the form

$$F^\varepsilon(\bar u) - F(\bar u) - (F^\varepsilon(\bar u_\varepsilon) - F(\bar u_\varepsilon)) \geq \frac{\delta}{2}\{\|\bar y_\varepsilon - \bar y\|_{L^2(Q)}^2 + \|\bar y_\varepsilon(T) - \bar y(T)\|_{L^2(\Omega)}^2\}, \quad (4.7)$$

Arguing as in the proof of the previous theorem we get

$$\frac{1}{2}\int_Q C_Q^Y((\bar y - y_Q^\varepsilon)^2 - (\bar y - y_Q)^2)\,dx\,dt - \frac{1}{2}\int_Q C_Q^Y((\bar y_\varepsilon - y_Q^\varepsilon)^2 - (\bar y_\varepsilon - y_Q)^2)\,dx\,dt$$

$$\leq C\,\varepsilon\,\|\bar y - \bar y_\varepsilon\|_{L^2(Q)}.$$

In the same way, all other integrals appearing in (4.7) are first simplified and next estimated by the Cauchy-Schwarz inequality. Finally, we find

$$c_1\,\varepsilon\,\{\|\bar y - \bar y_\varepsilon\|_{L^2(Q)} + \|\bar z - \bar z_\varepsilon\|_{L^2(Q)} + \|\bar y(T) - \bar y_\varepsilon(T)\|_{L^2(\Omega)} + \|\bar z(T) - \bar z_\varepsilon(T)\|_{L^2(\Omega)}\}$$

$$\geq \frac{\delta}{2}\{\|\bar y_\varepsilon - \bar y\|_{L^2(Q)}^2 + \|\bar y_\varepsilon(T) - \bar y(T)\|_{L^2(\Omega)}^2\}$$

$$\geq c_2\,\{\|\bar y_\varepsilon - \bar y\|_{L^2(Q)}^2 + \|\bar y_\varepsilon(T) - \bar y(T)\|_{L^2(\Omega)}^2 + \|\bar z_\varepsilon - \bar z\|_{L^2(Q)}^2 + \|\bar z_\varepsilon(T) - \bar z(T)\|_{L^2(\Omega)}^2\}$$

$$\geq c\,\{\|\bar y_\varepsilon - \bar y\|_{L^2(Q)} + \|\bar y_\varepsilon(T) - \bar y(T)\|_{L^2(\Omega)} + \|\bar z_\varepsilon - \bar z\|_{L^2(Q)} + \|\bar z_\varepsilon(T) - \bar z(T)\|_{L^2(\Omega)}\}^2.$$

Here, we have exploited the estimate

$$\|\bar z_\varepsilon - \bar z\|_{L^2(Q)} = \|K_0(\bar y_\varepsilon - \bar y)\|_{L^2(Q)} \leq C\|\bar y_\varepsilon - \bar y\|_{L^2(Q)}$$

and an associated inequality for the time $t = T$ (cf. the proof of Lemma 3.8) to get the second inequality and the equivalence of all norms in $\mathbb{R}^4$ to obtain the third one. Now the result follows immediately. □

4.2. Tikhonov parameter tending to zero. In this section, we investigate the behavior of a sequence of optimal controls $\{\bar u_\nu\}_{\nu>0}$ of (P_ν) and the corresponding states $\{(\bar y_\nu, \bar z_\nu)\}_{\nu>0}$ as $\nu \to 0$. Since $\mathcal{U}_{ad}$ is bounded in $L^\infty(Q)$, any sequence of solutions of (P_ν) contains subsequences converging weakly* in $L^\infty(Q)$. Below, we will deduce consequences of this convergence.

THEOREM 4.5. *Let $\{\bar u_\nu\}_{\nu>0}$ be a sequence of global solutions of (P_ν) such that $\bar u_\nu \rightharpoonup \bar u$ in $L^2(Q)$ for $\nu \downarrow 0$. Then $\bar u$ is a global solution of (P) and $\|\bar u_\nu - \bar u\|_{L^2(Q)} \to 0$. Moreover, the following identity holds*

$$\|\bar u\|_{L^2(Q)} = \min\{\|u\|_{L^2(Q)} : \ u \text{ is a global solution of } (P)\}. \quad (4.8)$$

Proof. First we observe that the boundedness of $\{\bar u_\nu\}_{\nu>0}$ in $L^\infty(Q)$ and the weak convergence $\bar u_\nu \rightharpoonup \bar u$ for $\nu \downarrow 0$ in $L^2(Q)$ implies that $\bar u_\nu \rightharpoonup \bar u$ for $\nu \downarrow 0$ in $L^p(Q)$ for any $1 \leq p < \infty$. Moreover, $\bar u \in \mathcal{U}_{ad}$ holds. Let $(\bar y_\nu, \bar z_\nu) = G(\bar u_\nu)$ and $(\bar y, \bar z) = G(\bar u)$. From the equation satisfied by $\bar z_\nu$, we deduce

$$\bar z_\nu(x,t) = e^{-\beta t} z_0(x,t) + \int_0^t e^{-\beta(t-s)}(\gamma \bar y_\nu(x,s) - \delta)\,ds. \quad (4.9)$$

We have a similar representation for $\bar z$ in terms of $\bar y$. Using (3.17) and (3.19) and the compactness of the embedding $W(0,T) \subset L^2(Q)$, it is easy to pass to the limit in the state equation (1.1) satisfied by $(\bar y_\nu, \bar z_\nu)$ and to confirm that $(\bar y_\nu, \bar z_\nu) \rightharpoonup (\bar y, \bar z)$ in $W(0,T)$. By the continuity of the embedding $W(0,T) \hookrightarrow C([0,T], L^2(\Omega))$, the

convexity of the cost functional with respect to (y, z, u), and using that $\bar{u}_\nu$ is a solution of (P_ν) it follows

$$J(\bar{u}) \leq \liminf_{\nu \to 0} J(\bar{u}_\nu) \leq \liminf_{\nu \to 0} J_\nu(\bar{u}_\nu) \leq \lim_{\nu \to 0} J_\nu(u) = J(u) \quad \forall u \in \mathcal{U}_{\mathrm{ad}}.$$

This implies that $\bar{u}$ is a solution of (P). Let us prove that $\{\bar{u}_\nu\}_{\nu > 0}$ converges strongly to $\bar{u}$. Since $\bar{u}_\nu$ and $\bar{u}$ are solutions of (P_ν) and (P), respectively, we obtain

$$J(\bar{u}_\nu) + \frac{\nu}{2}\|\bar{u}_\nu\|^2_{L^2(Q)} = J_\nu(\bar{u}_\nu) \leq J_\nu(\bar{u}) = J(\bar{u}) + \frac{\nu}{2}\|\bar{u}\|^2_{L^2(Q)} \leq J(\bar{u}_\nu) + \frac{\nu}{2}\|\bar{u}\|^2_{L^2(Q)},$$

which implies that $\|\bar{u}_\nu\|^2_{L^2(Q)} \leq \|\bar{u}\|^2_{L^2(Q)}$ for every $\nu > 0$. From here, we infer

$$\|\bar{u}\|_{L^2(Q)} \leq \liminf_{\nu \to 0} \|\bar{u}_\nu\|_{L^2(Q)} \leq \limsup_{\nu \to 0} \|\bar{u}_\nu\|_{L^2(Q)} \leq \|\bar{u}\|_{L^2(Q)}.$$

Thus, we have $\|\bar{u}_\nu\|_{L^2(Q)} \to \|\bar{u}\|_{L^2(Q)}$, which leads to the strong convergence $\bar{u}_\nu \to \bar{u}$ in $L^2(Q)$. Finally, arguing as above, for any (global) solution u of (P) we have that $\|\bar{u}_\nu\|_{L^2(Q)} \leq \|u\|_{L^2(Q)} \ \forall \nu > 0$. Hence,

$$\|\bar{u}\|_{L^2(Q)} = \lim_{\nu \to 0} \|\bar{u}_\nu\|_{L^2(Q)} \leq \|u\|_{L^2(Q)},$$

which implies (4.8). ☐

Remark 4.6. *Except for the strong convergence $\bar{u}_\nu \to \bar{u}$ in $L^2(Q)$, the rest of the statements in the above theorem can be deduced from Danskin's Theorem [10, Thm. 1]. However, to apply this result, we would have to verify the hypotheses of the theorem. Moreover, our proof above is short. To make our paper self-contained, we preferred to present a complete proof without using Danskin's Theorem.*

Now, we formulate a converse result, namely that strict local solutions to (P) can be approximated by local solutions of (P_ν). This is an analogon of Theorem 4.1, therefore we omit the proof, because it is almost identical to that of Theorem 4.1.

THEOREM 4.7. *Let $\bar{u}$ be a strict local solution of (P). Then, there exist $\rho > 0$, $\bar{\nu} > 0$ and a sequence $\{\bar{u}_\nu\}_{0 < \nu \leq \bar{\nu}}$ of local solutions of (P_ν) such that $\bar{u}_\nu \to \bar{u}$ in $L^2(Q)$ and every $\bar{u}_\nu$ affords a global minimum to J_ν in $\mathbb{K}_\rho = \mathcal{U}_{\mathrm{ad}} \cap \bar{B}_\rho(\bar{u})$.*

Let us mention the following property. If $\bar{u}$ is as in Theorem 4.5 and $\tilde{u}$ is a global solution of (P) with $\|\tilde{u}\|_{L^2(Q)} > \|\bar{u}\|_{L^2(Q)}$, then (4.8) implies that there is no sequence $\{\bar{u}_\nu\}_{\nu > 0}$ of global solutions of problems (P_ν) converging to $\tilde{u}$. However, Theorem 4.7 proves that $\tilde{u}$ can be approximated by local solutions of problems (P_ν).

We know the convergence of global (local) solutions $\bar{u}_\nu$ of (P_ν) to global (local) solutions $\bar{u}$ of (P). But, we are interested to determine the rate of convergence. The next theorem provides such a rate for the associated states. It is applicable, if the coercivity condition is fulfilled in the form (3.15) (based on $C_{\bar{u}}^\tau$) or in the form (3.37) (based on $E_{\bar{u}}^\tau$). Notice that $y_u \neq y_v$ implies $u \neq v$. Therefore, the strong quadratic growth condition (3.41) ensures in particular that $\bar{u}$ is a strict local solution.

THEOREM 4.8 (Hölder rate of convergence as $\nu \downarrow 0$). *Let $\bar{u}$ and $\{\bar{u}_\nu\}_{0 < \nu \leq \bar{\nu}}$ be as in Theorem 4.7. Let us assume that one of the conditions (3.15) or (3.37) is satisfied. Then, the following identities hold*

$$
\begin{aligned}
&\lim_{\nu \to 0} \frac{1}{\sqrt{\nu}} \left\{ \|\bar{y}_\nu - \bar{y}\|_{L^2(Q)} + \|\bar{z}_\nu - \bar{z}\|_{L^2(Q)} \right\} = 0, \\
&\lim_{\nu \to 0} \frac{1}{\sqrt{\nu}} \left\{ \|\bar{y}_\nu(T) - \bar{y}(T)\|_{L^2(\Omega)} + \|\bar{z}_\nu(T) - \bar{z}(T)\|_{L^2(\Omega)} \right\} = 0,
\end{aligned}
\tag{4.10}
$$

where $(\bar{y}_\nu, \bar{z}_\nu) = G(\bar{u}_\nu)$ and $(\bar{y}, \bar{z}) = G(\bar{u})$.

Proof. Using (3.36) or (3.41) and the fact that $J_\nu(\bar{u}_\nu) \leq J_\nu(\bar{u})$, we get

$$J(\bar{u}) + \frac{\delta}{2} \left\{ \|y_\nu - \bar{y}\|_{L^2(Q)}^2 + \|y_\nu(T) - \bar{y}(T)\|_{L^2(\Omega)}^2 \right\} + \frac{\nu}{2} \|\bar{u}_\nu\|_{L^2(Q)}^2$$
$$\leq J(\bar{u}_\nu) + \frac{\nu}{2} \|\bar{u}_\nu\|_{L^2(Q)}^2 = J_\nu(\bar{u}_\nu) \leq J_\nu(\bar{u}) = J(\bar{u}) + \frac{\nu}{2} \|\bar{u}\|_{L^2(Q)}^2.$$

From here, we deduce that $\|\bar{u}_\nu\|_{L^2(Q)} \leq \|\bar{u}\|_{L^2(Q)}$ and furthermore

$$\frac{\delta}{2} \left\{ \|y_\nu - \bar{y}\|_{L^2(Q)}^2 + \|y_\nu(T) - \bar{y}(T)\|_{L^2(\Omega)}^2 \right\}$$
$$\leq \frac{\nu}{2} \left(\|\bar{u}\|_{L^2(Q)} - \|\bar{u}_\nu\|_{L^2(Q)} \right) \leq \nu \|\bar{u}\|_{L^2(Q)} \|\bar{u} - \bar{u}_\nu\|_{L^2(Q)}.$$

Hence, the inequality

$$\frac{1}{\sqrt{\nu}} \left\{ \|y_\nu - \bar{y}\|_{L^2(Q)} + \|y_\nu(T) - \bar{y}(T)\|_{L^2(\Omega)} \right\} \leq \left(\frac{2}{\delta} \|\bar{u}\|_{L^2(Q)} \|\bar{u} - \bar{u}_\nu\|_{L^2(Q)} \right)^{1/2}$$

is fulfilled, where the right hand side converges to zero. The proof is finished by recalling that

$$\|\bar{z}_\nu - \bar{z}\|_{L^2(Q)} + \|\bar{z}_\nu(T) - \bar{z}(T)\|_{L^2(\Omega)} \leq C \left\{ \|y_\nu - \bar{y}\|_{L^2(Q)} + \|y_\nu(T) - \bar{y}(T)\|_{L^2(\Omega)} \right\}.$$

☐

We should mention that this estimate is fairly pessimistic. All of our numerical tests below show that the convergence has the order ν, i.e. numerically we see a Lipschitz rather than a Hölder estimate. Though this is not a proof, there is a theoretical explanation for this. It is known from similar discussions for problems with linear elliptic equations that a Lipschitz estimate for $\nu \to 0$ is true provided that the adjoint state function for $\nu = 0$ satisfies a certain assumption on the behavior in its zeros, cf. [19]. This assumption is often fulfilled and the result of Lipschitz stability seems to be satisfied also in our case of a nonlinear system of parabolic equations. However, an associated discussion would go beyond the scope of our paper.

5. Numerical results. The goal of this section is to confirm the convergence of $\bar{y}_\nu$ against $\bar{y}$ for $\nu \to 0$. This needs a very precise computation of optimal solutions and therefore we concentrate on the spatial 1D case. In 1D, the Schlögl model develops traveling wave fronts, while the FitzHugh-Nagumo system exhibits traveling pulses as typical solutions. For 2D examples, we refer to the various examples in our former paper [8]. For instance, we showed the control of moving spiral waves.

We apply a semi-smooth Newton method since it allows to determine solutions that satisfy the first order necessary optimality conditions with high accuracy. This method requires the solution of forward-backward parabolic systems that can be efficiently solved in 1D but would be very demanding in 2D.

Let us briefly sketch our numerical approach. The optimal solutions must obey the projection-formulas

$$\bar{u}_\nu(x,t) = \text{Proj}_{[a,b]} \left(-\frac{1}{\nu} [\bar{\varphi}_\nu(x,t) + \kappa \bar{\lambda}_\nu(x,t)] \right)$$

$$\bar{\lambda}_\nu(x,t) = \text{Proj}_{[-1,+1]} \left(-\frac{1}{\kappa} \bar{\varphi}_\nu(x,t) \right)$$

for the optimal control and the associated the sub-gradient. We define active and inactive sets for the control u depending on the adjoint state φ_u:

$$
\begin{aligned}
\mathcal{A}_a &:= \{(x,t) \in Q \mid -\varphi_u(x,t) + \kappa < \nu\, a \text{ a.e. in } Q\}, \\
\mathcal{I}_- &:= \{(x,t) \in Q \mid \nu\, a \leq -\varphi_u(x,t) + \kappa \leq 0 \text{ a.e. in } Q\}, \\
\mathcal{A}_0 &:= \{(x,t) \in Q \mid |\varphi_u(x,t)| < \kappa \text{ a.e. in } Q\}, \\
\mathcal{I}_+ &:= \{(x,t) \in Q \mid 0 \leq -\varphi_u(x,t) - \kappa \leq \nu\, b \text{ a.e. in } Q\}, \\
\mathcal{A}_b &:= \{(x,t) \in Q \mid \nu\, b < -\varphi_u(x,t) - \kappa \text{ a.e. in } Q\}.
\end{aligned}
$$

Almost everywhere on $\mathcal{A}_a$, $\mathcal{A}_0$, and $\mathcal{A}_b$, the optimal control must be equal to a, 0, and b, respectively. Almost everywhere in $\mathcal{I}_-$ resp. $\mathcal{I}_+$, the equation $u = \frac{1}{\nu}[-\varphi_u + \kappa]$ resp. $u = \frac{1}{\nu}[-\varphi_u - \kappa]$ must hold true.

By these sets, the system of necessary optimality conditions can be expressed in the form

$$
0 = \mathcal{F}(y, z, \varphi, \psi, u) := \begin{pmatrix} \dfrac{\partial}{\partial t} y - \Delta y + R(y) + \alpha\, y - u \\[2mm] \dfrac{\partial}{\partial t} z + \beta\, x - \gamma\, y \\[2mm] -\dfrac{\partial}{\partial t}\varphi - \Delta\varphi + R'(y)\varphi - \gamma\,\psi - c_Q^Y\,(y - y_Q) \\[2mm] -\dfrac{\partial}{\partial t}\psi + \beta\psi + \alpha\,\varphi - c_Q^Z\,(z - z_Q) \\[2mm] u - \chi_{\mathcal{A}_a}\, a - \chi_{\mathcal{A}_b}\, b + \dfrac{1}{\nu}\left(\chi_{\mathcal{I}_+}(\varphi + \kappa) + \chi_{\mathcal{I}_-}(\varphi - \kappa)\right) \end{pmatrix}
$$

where χ_k, $k \in \{\mathcal{A}_a, \mathcal{I}_-, \mathcal{A}_0, \mathcal{I}_+, \mathcal{A}_b\}$, denotes the indicator function of the associated sets. This optimality system, along with the associated initial and boundary conditions, is solved by the semi-smooth Newton method. The shorten the presentation, we do not explain the steps of this standard method and refer to [18], where the method is presented in the context of sparse controls for elliptic equations. The method is terminated, if a suitable norm $\|\mathcal{F}\|$ is sufficiently small.

To avoid certain oscillation effects of the control iterates, we did not take the whole Newton step and added the computed Newton direction $(d_y, d_z, d_\varphi, d_\psi, d_u)$ with some step-size s to the last iterate $(y_k, z_k, \varphi_k, \psi_k, u_k)$. We invoked a modified Armijo-Goldstein rule, since the standard one slowed down the algorithm considerably:

We fix $m \in \mathbb{N}$ and determine step-sizes $\{s_1, \ldots, s_m\}$ by $s_1 := 1$ and $s_l := \left(\frac{1}{2} + \mathrm{rand}\left(0, \frac{1}{5}\right)\right) s_{l-1}$, $l \in \{2, \ldots, m\}$. Here, $\mathrm{rand}\left(0, \frac{1}{5}\right)$ denotes a random value between 0 and $\frac{1}{5}$. Out of this set, the step-size s was taken that minimizes the norm of $\mathcal{F}(y_k + s\, d_y, z_k + s\, d_z, \varphi_k + s\, d_\varphi, \psi_k + s\, d_\psi, u_k + s\, d_u)$. Fairly small numbers $m \in \{10, \ldots, 25\}$ were sufficient for making the method fast.

To have a sufficiently accurate initial iterate for the semi-smooth Newton method, we applied a few iterations of the nonlinear CG method that we also used in [8]. Moreover, the CG method was our method of choice in the case $\nu = 0$.

To set up an optimality indicator for u, we define the function

$$
\mathcal{R}_u(x,t) := \begin{cases} \max\left(0, -\bar{\varphi}_u(x,t) - \kappa_u\,\bar{\lambda}(x,t)\right), & \text{if } \bar{u}(x,t) = a \\[2mm] |\bar{\varphi}_u(x,t) + \kappa_u\,\bar{\lambda}(x,t)|, & \text{if } a < \bar{u}(x,t) < b \\[2mm] \max\left(0, \bar{\varphi}_u(x,t) + \kappa_u\,\bar{\lambda}(x,t)\right), & \text{if } \bar{u}(x,t) = b. \end{cases}
$$

As optimality indicator, we used

$$
\Theta := |Q|^{-1}\|\mathcal{R}_u\|_{L^2(Q)}, \tag{5.1}
$$

where $|Q|$ is the volume of Q. Our optimization algorithms are terminated for sufficiently small Θ.

5.1. Example 1 (Schlögl model). At first, we consider the Schlögl model and fix $\alpha = \beta = \gamma = \delta = 0$, $c_T^Z = 0$, $c_Q^Z = 0$, $c_T^Y = 0$, and $c_Q^Y = 1$, i.e. we are tracking some y_Q in Q. Starting from a naturally developed uncontrolled wave profile y_{nat} shown in Fig. 5.1 (left) at $t = 0$, the target is to approach a piecewise linear shape defined by

$$y_Q(x, t) = \begin{cases} 1, & 0 \le x \le 9.5, \\ 10.5 - x, & 9.5 < x \le 10.5, \\ 0, & 10.5 < x \le 20, \end{cases}$$

hence the desired front y_Q is a stationary wave front of edged shape. As non-linearity, we fix R by

$$R(y) = y(y - 0.25)(y - 1).$$

For $\Omega = (0, 20)$, partitioned by 201 equidistant nodes, the initial value of our desired wave front satisfies $y_0(10) = \frac{1}{2}$. Fig. 5.1 displays the natural development of this wave profile as well as the desired state y_Q for the observed time horizon $(0, 10)$.
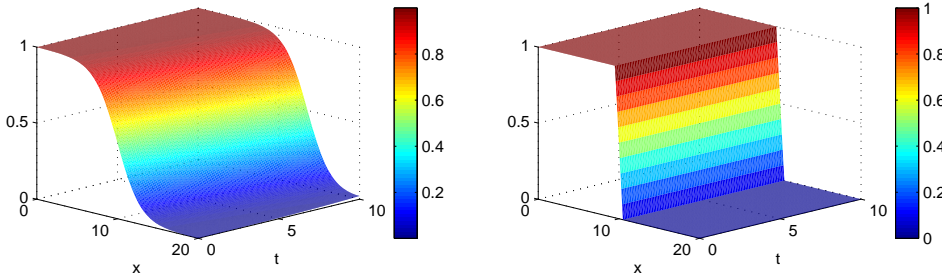

FIGURE 5.1. *Example 1, natural development y_{nat} of the wave profile for $u = 0$ (left) and the desired state y_Q (right).*

As sparse parameter, we set $\kappa = 1e - 2$ and fix the control bounds $a = -1$ and $b = 1$. For $\nu = 1e - 9$ the optimal control is displayed in Fig. 5.2 together with its associated state, adjoint state and the related sets $\mathcal{A}_a$, $\mathcal{A}_b$, $\mathcal{A}_0$ that should coincide with the active/inactive sets of $\bar{u}$. As in the other examples, they show a very good coincidence.

Let us explain why this form of the optimal control can be expected. In the unconstrained case, we insert the desired function y_Q in the left-hand side of the 1D Schlögl model,

$$\frac{\partial}{\partial t} y_Q - \frac{\partial^2}{\partial x^2} y_Q + R(y_Q) = 0 + \delta_{9.5} - \delta_{10.5} + R(y_Q) =: \hat{u}.$$

The result is a control such that the objective functional is zero for $\nu = 0$. Notice that the first derivative of y_Q is a function of Heaviside type with jumps at 9.5 and 10.5, hence the Dirac Delta functions $\delta_{9.5}$ and $-\delta_{10.5}$ are obtained as derivatives. The "best" unconstrained control $\hat{u}$ is a measure and not a measurable function.
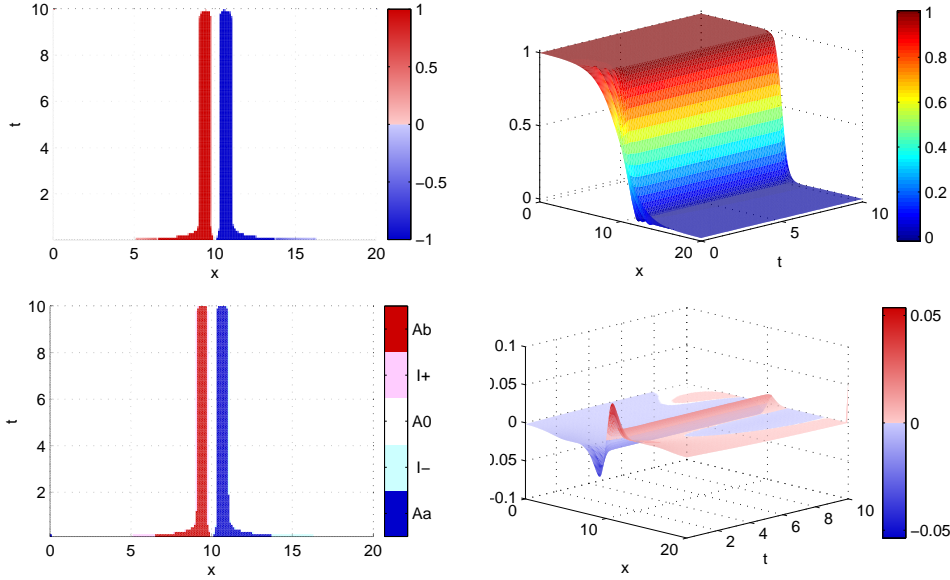
FIGURE 5.2. *Example 1, optimal control $\bar{u}$ (top left), state $\bar{y}$ (top right), adjoint state $\bar{\varphi}$ (bottom right) and associated active/inactive sets (bottom left) for $\nu = 1e - 9$.*

With bounded and measurable control functions taking values in $[a, b]$, the desired state y_Q cannot be reached. However, the optimal controls will approximate the measure defined above. This explains why $\bar{u}$ is of bang-bang-bang type, cf. Fig. 5.2.

To compute the optimal solution for $\nu = 0$, we used the non-linear CG-method and took the optimal control for $\nu = 1e-9$ as initial iterate. However, the CG method did not improve the initial iterate; the optimality indicator $\Theta = 6.21e - 13$ from (5.1) was very small. For this reason, we took $\bar{y} := \bar{y}_{1e-9}$ as reference solution to determine the order of convergence as $\nu \downarrow 0$. In both the $L^2(Q)$- and $L^\infty(Q)$-norm, $\|\bar{y}_\nu - \bar{y}\|$ appears to decay linearly for $\nu \downarrow 0$, cf. Fig. 5.3.
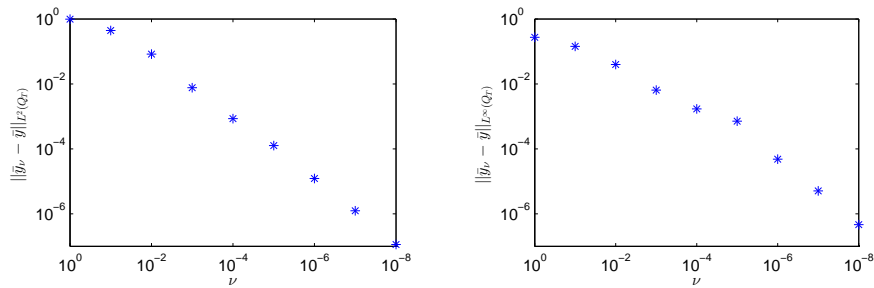

FIGURE 5.3. *Example 1, $\|\bar{y}_\nu - \bar{y}\|_{L^2(Q)}$ (left) and $\|\bar{y}_\nu - \bar{y}\|_{L^\infty(Q)}$ (right) for $\nu \downarrow 0$.*

5.2. Example 2 (FitzHugh-Nagumo system). Here we consider the 1D FitzHugh-Nagumo system for $\Omega = (0, 75)$, $T = 10$, $\alpha = 1$, $\beta = 0$, $\gamma = 0.33$, $\delta = -0.429$, and $R(y) = y(y - \sqrt{3})(y + \sqrt{3})$. We take a natural impulse as initial value; this is a snapshot of an (instationary) impulse that develops in the uncontrolled case as a typical solution of the 1D FitzHugh-Nagumo system. For $\kappa = 0.1$, $a = -10$, and

$b = 10$, the goal is to double the velocity of the natural impulse in the y-component of the solution (y, z). To this aim, we fix $c_T^Y = c_T^Z = c_Q^Z = 0$ and $c_Q^Y = 1$. The natural development of the initial impulse and the desired trajectory are shown in Fig. 5.4.
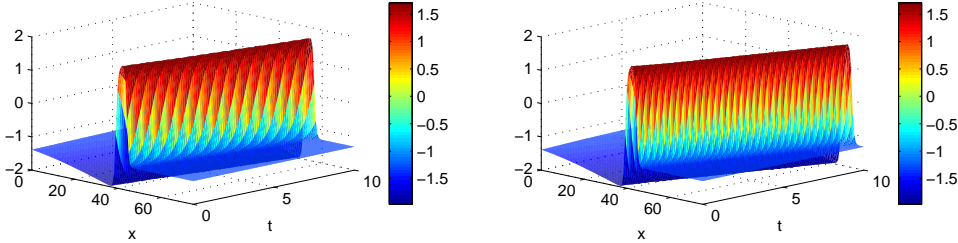

FIGURE 5.4. *Example 2, natural development y_{nat} of the impulse for $u = 0$ (left) and the desired state y_Q (right).*

Fig. 5.5 displays the optimal control and the adjoint state for $\nu = 1e - 10$. Graphically, the optimal state does not differ from the desired state displayed in Fig. 5.4. Moreover, also here the active sets of the optimal control and the sets $\mathcal{I}_+$, $\mathcal{A}_0$, $\mathcal{I}_-$ coincide. We skip these figures not to exceed the paper. The calculated control does not touch the bounds $a = -10$ or $b = 10$. Its maximal absolute value is 8.89.

As in Example 1, taking this result as initial control for the case $\nu = 0$, the CG-method did not further decrease the objective function; the value $\Theta = 9.81e - 13$ of the optimality indicator has already been very small.
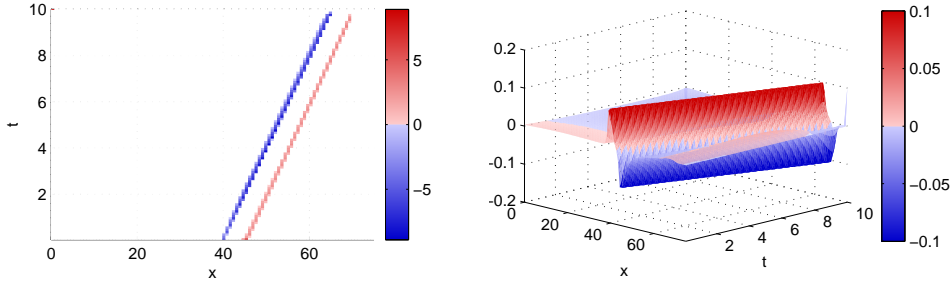

FIGURE 5.5. *Example 2, optimal control $\bar{u}$ (left) and adjoint state $\bar{\varphi}$ (right) for $\nu = 1e - 10$.*

Also in this example, the decay of $\|\bar{y}_\nu - \bar{y}\|$ in the $L^2(Q)$- and $L^\infty(Q)$-norm seems to be almost linear. The deviation of the last value for $\nu = 1e - 9$ is due to numerical limitations. For very small $\nu > 0$, the accuracy cannot expected to get any better. Notice that the term $1/\nu$ appears in the optimality system. We dispense with an associated graphical representation as in Fig. 5.3, as the decay looks similarly.

Let us emphasize that the values of $\|\bar{y}_\nu - y_Q\|_{L^\infty(Q)}$ and $\|\bar{y}_\nu - y_Q\|_{L^2(Q)}$ from Tab. 5.1 are quite large. Since the sparse optimal control acts very localized, not all features of the desired trajectory can be achieved. As Fig. 5.6 displays, the profile of the impulse is not conserved perfectly, but the main characteristic is. Therefore, we believe the result is close to global optimality.

5.3. Example 3 (FitzHugh-Nagumo system with explicitly known optimal solution). In the preceding example, for $\nu = 0$ or ν very close to 0, the figures

TABLE 5.1
Example 2, $\|\bar{y}_\nu - y_Q\|_{L^2(Q)}$ and $\|\bar{y}_\nu - \bar{y}\|_{L^\infty(Q)}$.

| ν	$\|\bar{y}_\nu - y_Q\|_{L^2(Q)}$	$\|\bar{y}_\nu - y_Q\|_{L^\infty(Q)}$	$\|\bar{y}_\nu - \bar{y}\|_{L^2(Q)}$	$\|\bar{y}_\nu - \bar{y}\|_{L^\infty(Q)}$
1	7.95036	2.86134	7.18160	2.26412
1e-1	3.22170	1.28987	1.52707	7.70512e-1
1e-2	2.94165	0.91893	4.96245e-1	3.28877e-1
1e-3	2.98672	0.73515	1.32929e-1	1.66305e-1
1e-4	3.00047	0.71178	2.14528e-2	4.08718e-2
1e-5	3.00258	0.70999	2.43604e-3	5.02801e-3
1e-6	3.00281	0.70981	2.47567e-4	5.15669e-4
1e-7	3.00284	0.70979	2.51592e-5	5.17744e-5
1e-8	3.00284	0.70979	4.82772e-6	5.25352e-6
1e-9	3.00284	0.70979	4.13440e-6	6.00743e-7
1e-10	3.00284	0.70979	0	0
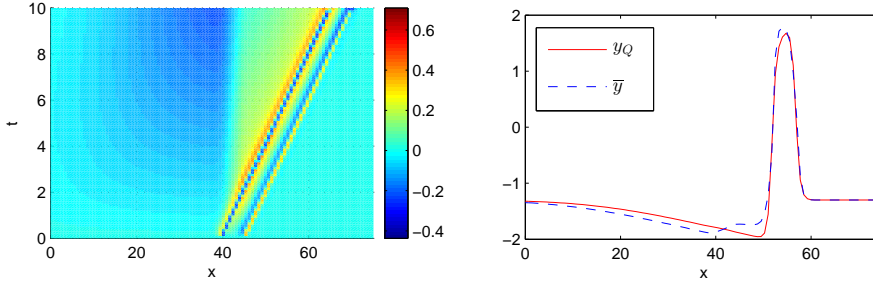
FIGURE 5.6. *Example 2, difference $\bar{y}-y_Q$ in the space-time domain Q (left) and $\bar{y}$ in comparison to y_Q at $t = 5$ (right) for $\nu = 1e - 10$.*

revealed areas with non-empty interior, where the optimal control took nonzero values strictly between a and b. According to the analysis, the absolute value of the adjoint state should be equal to κ in these regions. However, this is a numerical result for a finite dimensional approximation. We can not be certain about the structure of the real optimal solution. There are two possibilities. The exact set $\{(x,t) \in Q \mid |\bar{\varphi}(x,t)| = \kappa\}$ might have positive measure as the computation seems to indicate. On the other hand, this might be a numerical artifact and the measure of the set is zero. In that case, the optimal control should touch the bounds ± 10. Perhaps, our numerical resolution is not sufficiently fine to see this. To show that also the first explanation might hold true, we construct an associated example as follows:

Let $\Omega = (0,50)$, $T = 10$, and define $\bar{\varphi}(x,t) := \mathcal{E}(x)\,\mathcal{T}(t)$, where $\mathcal{E} \in C^2(\bar{\Omega})$ and $\mathcal{T} \in C^1[0,T]$. Under these conditions, we first construct an adjoint state $\bar{\varphi}$ that is equal to $-\kappa$ in a subset Q_κ of Q with positive measure. Next, we fix an associated control $\bar{u}$ of our choice which is equal to zero in $Q \setminus Q_\kappa$ but non-negative in Q_κ.

Setting $c_T^Z = c_Q^Z = c_T^Y = 0$ and $c_Q^Y = 1$ once again, for fixed parameters the second adjoint state $\bar{\psi}$ is obtained directly from $\bar{\varphi}$ by (1.8). Moreover, since $\bar{u}$ is also fixed, $\bar{y}$ can be calculated numerically from the state equation. Along with $\bar{\varphi}$, $\bar{\psi}$, and $\bar{y}$, the adjoint equation leads to the desired trajectory y_Q. With this approach, we know an exact stationary point for $\nu = 0$.

Let $\alpha = 1$, $\beta = 0$, $\gamma = \frac{1}{2}$, $\delta = -\frac{1}{4}$ and the non-linearity $R(y) = y(y - 2)(y + 1)$ be

given. The constant initial values $y_0 \equiv -\frac{1}{2}$ and $z_0 \equiv -\frac{5}{8}$ are stable for this setting. With $\kappa = 1e - 2$, we define $f_\kappa(x) := -8\,\kappa\,(x^3 - x^4)$, $g_\kappa(x) := \kappa - f_\kappa(1 - x) = -\kappa\,(1 - 8x + 24x^2 - 24x^3 + 8x^4)$ and

$$
\mathcal{E}(x) := \begin{cases} f_\kappa((x - 10)/5), & \text{if } x \in [10, 12.5] \\ g_\kappa((x - 10)/5), & \text{if } x \in (12.5, 15] \\ -\kappa, & \text{if } x \in (15, 35) \\ g_\kappa((40 - x)/5), & \text{if } x \in [35, 37.5) \\ f_\kappa((40 - x)/5), & \text{if } x \in [37.5, 40] \\ 0, & \text{else}, \end{cases}
$$

$$
\mathcal{T}(t) := \begin{cases} (\cos(2\,\pi\,(t - 0.5)) + 1)\,/2, & \text{if } t \in [1, 1.5] \cup [3.5, 4] \\ 1, & \text{if } t \in (1.5, 3.5) \\ 0, & \text{else} \end{cases}
$$

to obtain $\bar{\varphi}$. For the control, we define $\bar{u}(x, t) := \max(0, (1 - (t - 2.5)^2) * \bar{u}_0(x))$, where $\bar{u}_0(x) := \max(0, 1 - ((x - 25)/10)^2)$. This analytically defined optimal control along with its associated state and adjoint state are shown in figure 5.7.
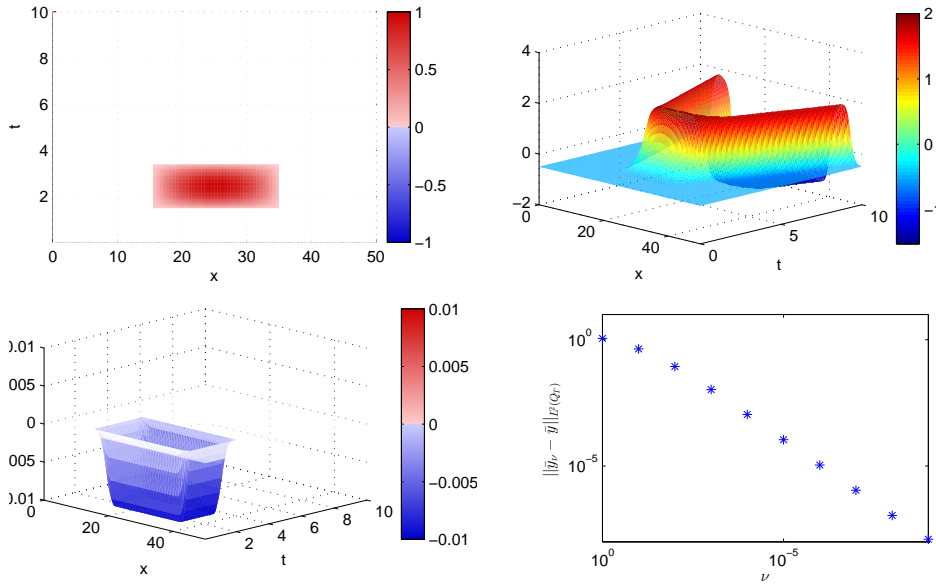

FIGURE 5.7. Example 3, analytically constructed optimal control $\bar{u}$ (top left), state $\bar{y}$ (top right), adjoint state $\bar{\varphi}$ (left) and decay of $\|\bar{y}_\nu - \bar{y}\|_{L^2(Q)}$ for $\nu \downarrow 0$ (bottom right).

This approach has the advantage that we have a well defined stationary point for $\nu = 0$ and can confirm the convergence against this point. Also in this example, we observe a linear decay of $\|\bar{y}_\nu - \bar{y}\|$. For the norm of $L^2(Q)$ this is displayed in Fig. 5.7, for the norm of $L^\infty(Q)$ this looks similar.

Remark: Since $\bar{u}$ only takes values in $[0, 1]$ and our approach does not depend on the bounds a and b, $\bar{u}$ satisfies the first order necessary optimality conditions for any $a \leq 0$ and $b \geq 1$. In particular, this works for $a < 0$ and $b > 1$ so that the control does not touch the bounds. Let us mention that our example is not completely analytic.

TABLE 5.2
Example 3, $\|\bar{y}_\nu - y_Q\|_{L^2(Q)}$ and $\|\bar{y}_\nu - \bar{y}\|_{L^\infty(Q)}$.

ν	$\|\bar{y}_\nu - y_Q\|_{L^2(Q)}$	$\|\bar{y}_\nu - y_Q\|_{L^\infty(Q)}$	$\|\bar{y}_\nu - \bar{y}\|_{L^2(Q)}$	$\|\bar{y}_\nu - \bar{y}\|_{L^\infty(Q)}$
1	1.25458	0.32876	1.13356	2.86116e-1
1e-1	0.59012	0.18273	4.31176e-1	1.30223e-1
1e-2	0.29383	0.09338	8.71793e-2	3.26041e-2
1e-3	0.24118	0.07169	1.06175e-2	4.29191e-3
1e-4	0.23582	0.07172	1.09357e-3	4.45709e-4
1e-5	0.23529	0.07173	1.09709e-4	4.47483e-5
1e-6	0.23523	0.07173	1.09745e-5	4.47662e-6
1e-7	0.23523	0.07173	1.09749e-6	4.47680e-7
1e-8	0.23523	0.07173	1.09750e-7	4.47629e-8
1e-9	0.23523	0.07173	1.26651e-8	4.53634e-9
0	0.23523	0.07173	0	0

The states $\bar{y}$ and y_Q are numerically obtained. We emphasize that the first order necessary optimality conditions are satisfied but that global optimality of $\bar{u}$ cannot be deduced from this. Even the local optimality cannot be assured. For this purpose, we would have to check second order sufficient optimality conditions. This cannot be done numerically. However, even with the quite high penalization from the sparse term, the computed state $\bar{y}$ is very close to the desired y_Q. Most likely, the consistence of the first order optimality conditions together with the quite small objective value indicate global optimality of $\bar{u}$.

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