SECOND ORDER ANALYSIS FOR OPTIMAL CONTROL PROBLEMS: IMPROVING RESULTS EXPECTED FROM ABSTRACT THEORY *

EDUARDO CASAS† AND FREDI TRÖLTZSCH‡

Abstract. An abstract optimization problem of minimizing a functional on a convex subset of a Banach space is considered. We discuss natural assumptions on the functional that permit to establish sufficient second-order optimality conditions with minimal gap with respect to the associated necessary ones. Though the two-norm discrepancy is taken into account, the obtained results exhibit the same formulation than the classical ones known from finite-dimensional optimization. We demonstrate that these assumptions are fulfilled in particular by important optimal control problems for partial differential equations. We prove that, in contrast to a widespread common belief, the standard second-order conditions formulated for these control problems imply strict local optimality of the controls not only in the sense of L^∞, but also in that of L^2.

AMS subject classifications. 90C48, 49K20, 35J61, 35K58

Key words. Optimal control, semilinear partial differential equation, second order optimality conditions, quadratic growth condition, two-norm discrepancy

1. Introduction.

It is well known that second order optimality conditions are an important tool in the numerical analysis of optimization problems. They are essential in proving superlinear or quadratic convergence of numerical algorithms, in deriving error estimates for the numerical discretization of infinite-dimensional optimization problems or just for the proof of local uniqueness of optimal solutions. Although there is an extensive literature on second order optimality conditions, there are still some gaps arising in applications to problems posed in function spaces.

In this paper, we address some specific questions of second order analysis for optimization problems in Banach spaces. We present some new abstract results on local stability of second order condition and discuss their application to optimal control problems of partial differential equations.

A study of the existing theory of first order optimality conditions reveals that the situation for finite-dimensional problems is very close to the infinite-dimensional one. However, there are big differences when we look at sufficient second order conditions. Let us mention some of these differences.

Consider a differentiable functional $J : U \longrightarrow \mathbb{R}$, where U is a Banach space. If $\bar{u}$ is a local minimum of J, then we know that $J'(\bar{u}) = 0$. This is a necessary condition. If J is not convex, we have to invoke a sufficient condition and should study the second derivative. In the finite-dimensional case, say $U = \mathbb{R}^n$, the first order optimality condition $J'(\bar{u}) = 0$ and the second order condition $J''(\bar{u})v^2 > 0$ for

*The first author was supported by Spanish Ministerio de Ciencia e Innovación under projects MTM2008-04206 and "Ingenio Mathematica (i-MATH)" No. CSD2006-00032 (Consolider Ingenio 2010). The second author was also supported by the DFG Research Center MATHEON "Mathematics for key technologies" Berlin.

†Departmento de Matemática Aplicada y Ciencias de la Computación, E.T.S.I. Industriales y de Telecomunicación, Universidad de Cantabria, 39005 Santander, Spain (eduardo.casas@unican.es).

‡Institut für Mathematik, Technische Universität Berlin, D-10623 Berlin, Germany (troeltzsch@math.tu-berlin.de).

every $v \in U \setminus \{0\}$ imply that $\bar{u}$ is a strict local minimum of J. This second order condition says that the quadratic form $v \to J''(\bar{u})v^2$ is positive definite in $\mathbb{R}^n$, which is equivalent to the strict positivity of the smallest eigenvalue λ_m of the associated symmetric matrix. Moreover $J''(\bar{u})v^2 \geq \lambda_m \|v\|^2$ for every $v \in \mathbb{R}^n$.

However, if U is an infinite-dimensional space, then the condition $J''(\bar{u})v^2 > 0$ is not equivalent to $J''(\bar{u})v^2 \geq \lambda_m \|v\|^2$ for some $\lambda_m > 0$. Is one of the two conditions sufficient for local optimality? The answer is well known since long time and it is documented extensively in literature: the first condition is not sufficient while the second one, together with the first order optimality condition, implies strict local optimality of $\bar{u}$ in the right setting. Let us discard the first (weaker) condition by an example.

Example 1.1. *Consider the optimization problem*

$$(\mathrm{Ex}_1) \quad \min_{u \in L^\infty(0,1)} J(u) = \int_0^1 [tu^2(t) - u^3(t)]\, dt.$$

The function $\bar{u}(t) \equiv 0$ *satisfies the first-order necessary condition* $J'(\bar{u}) = 0$ *and*

$$J''(\bar{u})v^2 = \int_0^1 2tv^2(t)\, dt > 0 \quad \forall v \in L^\infty(0,1) \setminus \{0\}.$$

However, $\bar{u}$ *is not a local minimum of* (Ex_1)*. Indeed, if we define*

$$u_k(t) = \begin{cases} 2t & \text{if } t \in (0, \dfrac{1}{k}), \\ 0 & \text{otherwise,} \end{cases}$$

then it holds $J(u_k) = -\frac{1}{k^4} < J(\bar{u})$*, and* $\|u_k - \bar{u}\|_{L^\infty(0,1)} = \frac{2}{k}$.

Now the question seems to be answered – the second and stronger condition should be sufficient for optimality. The next example shows that also this is not true in general.

Example 1.2. *We discuss the optimization problem*

$$(\mathrm{Ex}_2) \quad \min_{u \in L^2(0,1)} J(u) = \int_0^1 \sin(u(t))\, dt.$$

Obviously, $\bar{u}(t) \equiv -\pi/2$ *is a global solution. Some fast but formal computations lead to*

$$\begin{aligned} J'(\bar{u})v &= \int_0^1 \cos(\bar{u}(t))v(t)\, dt = 0 \quad \text{and} \\ J''(\bar{u})v^2 &= -\int_0^1 \sin(\bar{u}(t))v^2(t)\, dt = \int_0^1 v^2(t)\, dt = \|v\|^2_{L^2(0,1)} \ \forall v \in L^2(0,1). \end{aligned}$$

If the second, stronger condition were sufficient for local optimality, $\bar{u}$ *would be strict local minimum of* (Ex_2)*. However, this is not true. Indeed, for every* $0 < \varepsilon < 1$*, the functions*

$$u_\varepsilon(t) = \begin{cases} -\dfrac{\pi}{2} & \text{if } t \in [0, 1-\varepsilon], \\[2ex] +\dfrac{3\pi}{2} & \text{if } t \in (1-\varepsilon, 1], \end{cases}$$

are also global solutions of (Ex_2)*, with* $J(\bar u) = J(u_\varepsilon)$ *and* $\|\bar u - u_\varepsilon\|_{L^2(0,1)} = 2\pi\sqrt{\varepsilon}$*. Therefore, infinitely many different global solutions of* (Ex_2) *are contained in any* L^2*-neighborhood of* $\bar u$ *and* $\bar u$ *is not a strict solution.*

The reader will easily confirm that this property holds for any solution $\hat u$ *of the problem. What is wrong?*

The reason is that J is not of class C^2 in $L^2(0,1)$, our fast computations was too careless. Therefore we cannot apply the abstract theorem on sufficient conditions for local optimality in $L^2(0,1)$. On the other hand, J is of class C^2 in $L^\infty(0,1)$ and the derivatives computed above are correct in $L^\infty(0,1)$. However, the inequality $J''(\bar u)v^2 \ge \delta\|v\|^2_{L^\infty(0,1)}$ does not hold for any $\delta > 0$.

This phenomenon is called the *two-norm discrepancy*: the functional J is twice differentiable with respect to one norm, but the inequality $J''(\bar u)v^2 \ge \delta\|v\|^2$ holds in a weaker norm in which J is not twice differentiable; see, for instance, [13]. This situation arises frequently in infinite-dimensional problems but it does not happen for finite-dimensions because all the norms are equivalent in this case. The classical theorem on second order optimality conditions can easily be modified to deal with the two norm-discrepancy.

Theorem 1.3. *Let* U *be a vector space endowed with two norms,* $\| \, \|_\infty$ *and* $\| \, \|_2$*, such that* $J : (U, \| \, \|_\infty) \mapsto \mathbb{R}$ *is of class* C^2 *in a* $(U, \| \, \|_\infty)$*-neighborhood* $\mathcal{A} \subset U$ *of* $\bar u$ *and assume that the following properties hold*

$$J'(\bar u) = 0 \quad \text{and} \quad \exists\delta > 0 \text{ such that } J''(\bar u)v^2 \ge \delta\|v\|_2^2 \ \forall v \in U, \tag{1.1}$$

and there exists some $\varepsilon > 0$ *such that* $\bar B_\infty(\bar u;\varepsilon) \subset \mathcal{A}$ *and*

$$|J''(\bar u)v^2 - J''(u)v^2| \le \frac{\delta}{2}\|v\|_2^2 \ \forall v \in U \text{ if } \|u - \bar u\|_\infty \le \varepsilon. \tag{1.2}$$

Then there holds

$$\frac{\delta}{4}\|u - \bar u\|_2^2 + J(\bar u) \le J(u) \quad \text{if} \quad \|u - \bar u\|_\infty \le \varepsilon \tag{1.3}$$

so that $\bar u$ *is a strictly locally optimal with respect to the norm* $\| \cdot \|_\infty$*.*

In the above theorem and hereafter $B_\infty(\bar u;\varepsilon)$ (respectively, $B_2(\bar u;\varepsilon)$) will denote the ball of radius ε and centered at $\bar u$ with respect to the norm $\|\cdot\|_\infty$ (respectively, $\|\cdot\|_2$).

The proof of this theorem is quite elementary. To our knowledge, Ioffe [13] was the first who proved a result of this type by using two norms in the context of optimal control for ordinary differential equations. We refer also to the discussion of the two-norm discrepancy by Malanowski [15] and Maurer [16]. In the context of PDE constrained optimization, the proof of Theorem 1.3 can be found e.g. in [10] or [19, Thm. 4.29].

Theorem 1.3 can be applied to Example 1.2 to deduce that $\bar u$ is a strict local minimum in the sense of $L^\infty(0,1)$. Strict local optimality of $\bar u$ means that $J(u) > J(\bar u)$ holds for all admissible u out of a certain neighborhood of $\bar u$. This does not yet exclude that $\bar u$ is possibly an accumulation point of locally optimal solutions.

If the two-norm discrepancy occurs in an optimal control problem, we consider two norms, namely the L^∞-norm for differentiation and the L^2-norm for expressing the coercivity of J''. Then local optimality should hold only in the stronger L^∞ sense.

However, we will prove in this paper that for standard optimal control problems for distributed parameter systems, where the control appears linearly in the state equation, the sufficient second order condition implies also strict local optimality in the L^2 sense. Even more, we can find an L^2-neighborhood of this local minimum where local uniqueness holds. This means that there does not exist any other stationary point of (P) that neighborhood. Let us underline this even more: in many cases with two-norm discrepancy, results expected to hold only in an L^∞-neighborhood around the local solution, are even true in an L^2-neighborhood.

The plan of this work is as follows. In §2 we will formulate an abstract optimization problem and fix the assumptions that lead to the results mentioned above. In §3 and §4 we will apply the abstract results to elliptic control problems with Neumann and Dirichlet boundary controls, respectively. Finally, in §5 a distributed parabolic control problem is considered. We do not need the restrictions on the dimension of the spatial domain, which are usually required in these cases.

2. An abstract optimization problem in Banach spaces. Let U_∞ and U_2 be Banach and Hilbert spaces, respectively, endowed with the norms $\|\cdot\|_\infty$ and $\|\cdot\|_2$. We assume that $U_\infty \subset U_2$ with continuous embedding; in particular, the choice $U_\infty = U_2$ is possible. A nonempty convex subset $\mathcal{K} \subset U_\infty$ is given, and $\mathcal{A} \subset U_\infty$ is an open set covering $\mathcal{K}$. Moreover, an objective function $J : \mathcal{A} \longrightarrow \mathbb{R}$ is given. We consider the abstract optimization problem

$$\text{(P)} \qquad \min_{u\in\mathcal{K}} J(u).$$

The next well known result expresses the first order optimality conditions in form of a variational inequality.

THEOREM 2.1. *If $\bar u$ is a local solution of* (P) *and J is differentiable at $\bar u$, both in the sense of U_∞, then*

$$J'(\bar u)(u - \bar u) \ge 0 \quad \forall u \in \mathcal{K}. \tag{2.1}$$

We say that $\bar u$ is a *local solution* of (P) in the sense of U_∞, if $J(\bar u) \le J(u)$ holds for all $u \in \mathcal{K} \cap \{u \in U_\infty : \|u - \bar u\|_\infty < \varepsilon\}$ with some $\varepsilon > 0$. If the strong inequality $J(\bar u) < J(u)$ is satisfied in this set for all $u \ne \bar u$, then this solution is called a *strict local solution.* Notice that any local solution of (P) in the U_2 sense is also a local solution in the U_∞ sense. Therefore, (2.1) holds also for local solutions of (P) in the U_2 sense.

The rest of this section is devoted to the study of the necessary and sufficient second order optimality conditions for problem (P). Throughout the section all notions of differentiability of J are to be understood in the sense of U_∞. We fix an element $\bar u$ of $\mathcal{K}$ and require the following assumptions on (P).

(A1) The functional $J : \mathcal{A} \longrightarrow \mathbb{R}$ is of class C^2. Furthermore, for every $u \in \mathcal{K}$ there exist continuous extensions

$$J'(u) \in \mathcal{L}(U_2, \mathbb{R}) \quad \textit{and} \quad J''(u) \in \mathcal{B}(U_2, \mathbb{R}). \tag{2.2}$$

(A2) For any sequence $\{(u_k, v_k)\}_{k=1}^{\infty} \subset \mathcal{K} \times U_2$ *with* $\|u_k - \bar{u}\|_2 \to 0$ *and* $v_k \rightharpoonup v$ *weakly in* U_2,

$$J'(\bar{u})v = \lim_{k\to\infty} J'(u_k)v_k, \tag{2.3}$$

$$J''(\bar{u})v^2 \leq \liminf_{k\to\infty} J''(u_k)v_k^2, \tag{2.4}$$

$$\textit{if } v = 0, \ \textit{then} \quad \Lambda \liminf_{k\to\infty} \|v_k\|_2^2 \leq \liminf_{k\to\infty} J''(u_k)v_k^2, \tag{2.5}$$

hold for some $\Lambda > 0$.

The reader might have the impression that Assumptions *(A1)* and *(A2)*, mainly *(A2)*, are too strong. However, we will see in the next sections that they are fulfilled by many optimal control problems.

Associated with $\bar{u}$, we define the sets

$$\begin{aligned} S_{\bar{u}} &= \{v \in U_\infty : v = \lambda(u - \bar{u}) \text{ for some } \lambda > 0 \text{ and } u \in \mathcal{K}\}, \\ C_{\bar{u}} &= cl_2(S_{\bar{u}}) \cap \{v \in U_2 : J'(\bar{u})v = 0\} \\ D_{\bar{u}} &= \{v \in S_{\bar{u}} : J'(\bar{u})v = 0\}, \end{aligned} \tag{2.6}$$

where $cl_2(S_{\bar{u}})$ denotes the closure of $S_{\bar{u}}$ in U_2. The set $S_{\bar{u}}$ is called the cone of feasible directions and $C_{\bar{u}}$ is said the critical cone.

Now, we formulate the necessary second order optimality conditions under a regularity assumption stated in the next theorem; we refer to [2, §3.2] or [9] for the proof.

THEOREM 2.2. *Let* $\bar{u}$ *be a local solution of* (P) *in* U_∞. *Assume that (A1) and the regularity condition* $C_{\bar{u}} = cl_2(D_{\bar{u}})$ *are satisfied. Then* $J''(\bar{u})v^2 \geq 0$ *holds for all* $v \in C_{\bar{u}}$.

Let us mention that the regularity assumption of the above theorem is equivalent to the notion of polyhedricity of $\mathcal{K}$; see [1] or [2, §3.2].

Finally we prove a theorem on sufficient second order optimality conditions. Its novelty is that the obtained quadratic growth condition holds in a U_2-neighborhood of $\bar{u}$ rather than only in a U_∞-neighborhood.

THEOREM 2.3. *Suppose that assumptions (A1) and (A2) hold. Let* $\bar{u} \in \mathcal{K}$ *satisfy* (2.1) *and*

$$J''(\bar{u})v^2 > 0 \quad \forall v \in C_{\bar{u}} \setminus \{0\}. \tag{2.7}$$

Then, there exist $\varepsilon > 0$ *and* $\delta > 0$ *such that*

$$J(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|_2^2 \leq J(u) \quad \forall u \in \mathcal{K} \cap B_2(\bar{u}; \varepsilon). \tag{2.8}$$

Proof. We argue by contradiction and assume that for any positive integer k there exists $u_k \in \mathcal{K}$ such that

$$\|u_k - \bar{u}\|_2 < \frac{1}{k} \quad \text{and} \quad J(\bar{u}) + \frac{1}{2k}\|u_k - \bar{u}\|_2^2 > J(u_k). \tag{2.9}$$

Setting $\rho_k = \|u_k - \bar{u}\|_2$ and $v_k = (u_k - \bar{u})/\rho_k$, we can assume that $v_k \rightharpoonup v$ in U_2; if necessary, we select a subsequence. Let us prove that $v \in C_{\bar{u}}$. From assumption *(A1)* and (2.1) we deduce

$$J'(\bar u)v = \lim_{k\to\infty} J'(\bar u)v_k = \lim_{k\to\infty} \frac{1}{\rho_k} J'(\bar u)(u_k - \bar u) \geq 0.$$

We also derive the converse inequality. Due to the definition of v_k and (2.3), we have for some $\theta_k \in (0,1)$

$$\frac{J(\bar u + \rho_k v_k) - J(\bar u)}{\rho_k} = \frac{J(u_k) - J(\bar u)}{\rho_k} = J'(\bar u + \theta_k(u_k - \bar u))v_k \to J'(\bar u)v.$$

Hence, (2.9) leads to

$$J'(\bar u)v = \lim_{k\to\infty} \frac{J(\bar u + \rho_k v_k) - J(\bar u)}{\rho_k} = \lim_{k\to\infty} \frac{J(u_k) - J(\bar u)}{\rho_k} \leq$$

$$\leq \lim_{k\to\infty} \frac{1}{2k}\|u_k - \bar u\|_2 \leq \lim_{k\to\infty} \frac{1}{2k^2} = 0.$$

Thus it holds $J'(\bar u)v = 0$.

Next, we prove that $v \in cl_2(S_{\bar u})$. From $v_k = (u_k - \bar u)/\rho_k$ and $u_k \in \mathcal{K}$, we conclude $v_k \in S_{\bar u} \subset cl_2(S_{\bar u})$. The set $cl_2(S_{\bar u})$ is closed and convex in U_2, hence $v \in cl_2(S_{\bar u})$. Thus, we obtain $v \in C_{\bar u}$.

Invoking again (2.9) and (2.1) we get by a Taylor expansion

$$\begin{aligned}\frac{\rho_k^2}{2k} &= \frac{1}{2k}\|u_k - \bar u\|_2^2 > J(u_k) - J(\bar u) = J(\bar u + \rho_k v_k) - J(\bar u)\\ &= \rho_k J'(\bar u)v_k + \frac{\rho_k^2}{2} J''(\bar u + \theta_k\rho_k v_k)v_k^2 \geq \frac{\rho_k^2}{2} J''(\bar u + \theta_k \rho_k v_k)v_k^2.\end{aligned}$$

Therefore, it holds

$$J''(\bar u + \theta_k\rho_k v_k)v_k^2 < \frac{1}{k}.$$

Using first (2.7) and then (2.4), the above inequality leads to

$$0 \leq J''(\bar u)v^2 \leq \liminf_{k\to\infty} J''(\bar u + \theta_k\rho_k v_k)v_k^2 \leq \limsup_{k\to\infty} J''(\bar u + \theta_k\rho_k v_k)v_k^2 \leq \limsup_{k\to\infty} \frac{1}{k} = 0,$$

so that $J''(\bar u)v^2 = 0$. From (2.7), it follows $v = 0$. Finally, using (2.5) and the fact that $\|v_k\|_2 = 1$, we get the contradiction as follows

$$0 < \Lambda = \Lambda \liminf_{k\to\infty} \|v_k\|_2^2 \leq \liminf_{k\to\infty} J''(\bar u + \theta_k\rho_k v_k)v_k^2 = 0.$$

$\square$

Remark 2.4. *The main novelty in the proof is the use of the assumptions* (2.4) *and* (2.5). *They generalize the requirement of other papers that $J''(\bar u)$ is a so-called Legendre form. In former contributions to the subject it was not known that assumption* (2.4) *can be deduced in the context of control theory by an application of Egorov's theorem. Therefore, they needed the U_∞-convergence of the sequence $\{u_k\}_{k=1}^\infty$. In our approach, a generalization of the Legendre quadratic form hypothesis was necessary to* *achieve the final contradiction in the precedent proof;* (2.5) *was developed in this way. We refer to the first author's paper [5]. For the use of Legendre forms, the reader is referred to Hestenes [12] and Ioffe and Tihomirov [14].*

To explain some more specific difficulties related to second-order conditions, we consider also a modified version of example (Ex_2).

Example 2.5. *We discuss the optimization problem*

$$(\mathrm{Ex}_3) \qquad \min_{u \in L^2(0,1)} J(u) = \int_0^1 \{(u(t) + \frac{\pi}{2})^2 + \sin(u(t))\}\, dt.$$

Obviously, $\bar{u}(t) \equiv -\pi/2$ *is the unique global solution of this problem.* J *is a* C^2 *functional in* $L^\infty(0,1)$ *and*

$$J'(\bar{u})v = \int_0^1 \{2(\bar{u}(t) + \frac{\pi}{2}) + \cos(\bar{u}(t))\} v(t)\, dt = 0 \quad \textit{and}$$
$$J''(\bar{u})v^2 = \int_0^1 \{2 - \sin(\bar{u}(t))\} v^2(t)\, dt = 3 \int_0^1 v^2(t)\, dt = 3\, \|v\|^2_{L^2(0,1)} \ \forall v \in L^2(0,1).$$

From Theorem 1.3 we can only deduce that $\bar{u}$ is a strict local minimum in the sense of $L^\infty(0,1)$. However, it is easy to check that the assumptions (2.3)–(2.5) are fulfilled; thus Theorem 2.3 implies that $\bar{u}$ is a strict local minimum in the sense of $L^2(0,1)$. The crucial point is once again that J is not twice Fréchet differentiable in $L^2(0,1)$; it is only twice Fréchet differentiable in $L^\infty(0,1)$.

Even more, though there exists a continuous extension $J''(u) \in \mathcal{B}(L^2(0,1),\mathbb{R})$ for every $u \in L^\infty(0,1)$, the continuity property $J''(u_k) \to J''(\bar{u})$ in $\mathcal{B}(L^2(0,1),\mathbb{R})$ does not hold for every sequence $\{u_k\}_{k=1}^\infty$ that is bounded in $L^\infty(0,1)$ and converges strongly in $L^2(0,1)$ to $\bar{u}$. Indeed, it suffices to consider u_k and v_k defined by

$$u_k(t) = \begin{cases} 0 & \text{if } t \in [0, \frac{1}{k}) \\ -\frac{\pi}{2} & \text{if } t \in [\frac{1}{k}, 1] \end{cases}$$
$$v_k(t) = \begin{cases} \sqrt{k} & \text{if } t \in [0, \frac{1}{k}) \\ 0 & \text{if } t \in [\frac{1}{k}, 1]. \end{cases}$$

We have $u_k \to \bar{u}$ strongly in $L^2(0,1)$ and $\|v_k\|_{L^2(0,1)} = 1$, but it holds

$$[J''(u_k) - J''(\bar{u})]v_k^2 = \int_0^{1/k} v_k^2(t)\, dt = 1.$$

This proves the lack of the continuity property $J''(u_k) \to J''(\bar{u})$ in $\mathcal{B}(L^2(0,1),\mathbb{R})$.

However, if this property would be satisfied, then the assumptions (2.2)–(2.5) can be simplified, as one of the referees suggested: We can substitute them by

(A2')

(i) For any sequence $\{u_k\}_{k=1}^\infty \subset \mathcal{K}$ converging strongly in U_2 to u, the convergence properties $J'(u_k) \to J'(\bar{u})$ in $\mathcal{L}(U_2,\mathbb{R})$ and $J''(u_k) \to J''(\bar{u})$ in $\mathcal{B}(U_2,\mathbb{R})$ hold.

(ii) $J''(\bar{u}) : U_2 \times U_2 \longrightarrow \mathbb{R}$ is a Legendre form.

Theorem 2.3 holds under the assumptions *(A1)* and *(A2')*. Indeed, it is obvious that *(A1)* and *(A2')* imply (2.3) and (2.4). Therefore, the proof is the same except for the final contradiction that can be obtained as follows. First, using the notation of the former proof, we observe that *(A2')-(i)* implies

$$\lim_{k\to\infty} J''(\bar u)v_k^2 = \lim_{k\to\infty} J''(\bar u + \theta_k\rho_k v_k)v_k^2 = 0.$$

Since $J''(\bar u)$ is a Legendre form, we deduce that $v_k \to 0$ strongly in U_2. This contradicts the fact that $\|v_k\|_{U_2} = 1$ holds for all k.

Unfortunately, the assumption *(A2')-(i)* is too restrictive. It does not hold for the simple problem (Ex$_3$) and it also fails for optimal control problems with highly nonlinear terms in the cost functional or in the state equation. The abstract framework given by assumptions *(A1)* and *(A2)* has a wider range of applications.

As an important consequence of Theorem 2.3, we are able to show local uniqueness of stationary points in the sense of L^2. Recall that $\tilde u \in \mathcal{K}$ is said to be a stationary point if

$$J'(\tilde u)(u - \tilde u) \ge 0 \quad \text{for all} \quad u \in \mathcal{K}.$$

COROLLARY 2.6. *Under the assumptions of Theorem 2.3, there exists a ball $B_2(\bar u;\varepsilon)$ such that there is no stationary point $\tilde u \in B_2(\bar u;\varepsilon)\cap\mathcal{K}$ different from $\bar u$.*

Proof. We prove the assertion by contradiction. Assume that there exists a sequence $\{u_k\}_{k=1}^\infty \subset \mathcal{K}$ such that $u_k \to \bar u$ in U_2, $u_k \ne \bar u$ for all k and $J'(u_k)(u-u_k) \ge 0$ for every $u \in \mathcal{K}$. Then, using the quadratic growth condition (2.8) and performing a Taylor expansion of $J(\bar u)$ around u_k, we get

$$\begin{aligned}
&J(u_k) \ge J(\bar u) + \frac{\delta}{2}\|u_k - \bar u\|^2\\
&= J(u_k) + J'(u_k)(\bar u - u_k) + \frac{1}{2}J''(\hat u_k)(\bar u - u_k)^2 + \frac{\delta}{2}\|u_k - \bar u\|^2\\
&\ge J(u_k) + \frac{1}{2}J''(\hat u_k)(\bar u - u_k)^2 + \frac{\delta}{2}\|u_k - \bar u\|^2,
\end{aligned}$$

for some $\hat u_k \in [u_k, \bar u]$. Setting $v_k = (u_k - \bar u)/\|u_k - \bar u\|_2$, we deduce from the inequality above

$$J''(\hat u_k)v_k^2 + \delta \le 0.$$

Selecting a subsequence, if necessary, we can assume that $v_k \rightharpoonup v$ in U_2. Invoking (2.4) we obtain

$$J''(\bar u)v^2 + \delta \le 0.$$

If we are able to show $v \in C_{\bar u}$, then the above inequality contradicts (2.7) and the proof is complete. Let us prove this. Obviously, v_k belongs to $S_{\bar u}$ for every k, hence we have $v \in cl_2(S_{\bar u})$, since $cl_2(S_{\bar u})$ is convex and closed in U_2. Let us check that $J'(\bar u)v = 0$. From (2.1) we get $J'(\bar u)v_k \ge 0$. Therefore, the inequality $J'(\bar u)v \ge 0$ follows from (2.3). On the other hand, $J'(u_k)v_k \le 0$ follows from the definition of u_k. Invoking again (2.3), we obtain $J'(\bar u)v \le 0$, which completes the proof. ∎

Assumption (2.7) has another consequence that was known up to now only in an L^∞-neighborhood of $\bar{u}$. The result expresses some alternative formulation of second-order sufficient conditions that is useful for applications in the numerical analysis.

THEOREM 2.7. *Under the assumptions of Theorem 2.3, there exist a ball $B_2(\bar{u};\varepsilon)$ in U_2 and numbers $\nu > 0$ and $\tau > 0$ such that*

$$J''(u)v^2 \geq \frac{\nu}{2}\|v\|_2^2 \quad \forall\, v \in E_{\bar{u}}^\tau \quad \textit{and} \quad \forall\, u \in \mathcal{K} \cap B_2(\bar{u};\varepsilon), \tag{2.10}$$

where

$$E_{\bar{u}}^\tau = \{v \in cl_2(S_{\bar{u}}) : |J'(\bar{u})v| \leq \tau\|v\|_2\}.$$

Proof. We argue again by contradiction. Assume the existence of a sequence $\{(u_k, v_k)\}_{k=1}^\infty \subset \mathcal{K} \times U_2$ such that $\|u_k - \bar{u}\|_2 \to 0$, $J''(\bar{u}_k)v_k^2 \leq \frac{1}{k}\|v_k\|_2^2$ and $v_k \in C_{\bar{u}}^{1/k}$ for every k. Renaming $v_k/\|v_k\|_2$ by v_k, we still have that $v_k \in C_{\bar{u}}^{1/k}$. Now, selecting a subsequence, if necessary, we obtain an element $v \in U_2$ such that $v_k \rightharpoonup v$ in U_2. Furthermore, $|J'(\bar{u})v_k| \leq \frac{1}{k}$ and $J''(u_k)v_k^2 \leq \frac{1}{k}$ hold for all k. Therefore, (2.2) and (2.4) imply $J'(\bar{u})v = 0$ and $J''(\bar{u})v^2 \leq 0$. It is also clear that $v \in cl_2(S_{\bar{u}})$. It follows $v \in C_{\bar{u}}$ and, as a consequence of (2.7), $v = 0$. Finally, taking into account that $\|v_k\|_2 = 1$, we obtain the contradiction from (2.5):

$$0 < \Lambda = \Lambda \liminf_{k\to\infty} \|v_k\|_2^2 \leq \liminf_{k\to\infty} J''(u_k)v_k^2 \leq 0.$$

$\square$

Remark 2.8. *Suppose that the Assumptions (A1) and (A2) are changed as follows:*

1. *In (A1), the relations* (2.2) *are required to hold for every $u \in \mathcal{A}$;*
2. *In (A2), all properties are required for all sequences $\{u_k\}_{k=1}^\infty$ converging to $\bar{u}$ and belonging to $\mathcal{A}$ instead of $\mathcal{K}$.*

Then (2.10) *holds for every element $u \in \mathcal{A} \cap B_2(\bar{u};\varepsilon)$. The same proof remains valid just by changing $\mathcal{K}$ by $\mathcal{A}$.*

This extension to the open set $\mathcal{A}$ can be important in cases, where the sequence $\{u_k\}$ cannot be required to be in $\mathcal{K}$. For instance, this might be interesting for numerical discretizations.

In the sequel, we demonstrate the applicability of our results to PDE constrained optimal control problems.

3. Application I. An elliptic Neumann control problem. In this section we study the optimal control problem

$$(\mathrm{P}_1) \qquad \min_{u \in \mathcal{K}} J(u),$$

where

$$J(u) = \int_\Omega L(x, y_u(x))\,dx + \int_\Gamma l(x, y_u(x), u(x))\,d\sigma(x), \tag{3.1}$$

$$\mathcal{K} = \{u \in L^\infty(\Gamma) : \alpha \leq u(x) \leq \beta \ \text{ for a.a. } x \in \Gamma\},$$

where $-\infty < \alpha < \beta < +\infty$, and y_u is the solution of the following Neumann problem

$$\begin{cases} -\Delta y + f(y) = 0 & \text{in } \Omega, \\ \partial_\nu y = u & \text{on } \Gamma. \end{cases} \tag{3.2}$$

Hereafter $\nu(x)$ denotes the unit outward normal vector to Γ at the point x and $\partial_\nu y$ is the normal derivative of y. We impose the following assumptions on the functions and parameters appearing in the control problem (P$_1$).

Assumption (N1): Ω is an open, bounded and connected subset of $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary Γ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function of class C^2 such that $f'(t) \geq c_o > 0$ for all $t \in \mathbb{R}$. The reader is referred to [7] for more general non-linear terms in the state equation.

Assumption (N2): We assume that $L : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ and $l : \Gamma \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are Carathéodory functions of class C^2 with respect to the second variable for L and with respect to the second and third variables for l with $L(\cdot, 0) \in L^1(\Omega)$, $l(\cdot, 0, 0) \in L^1(\Gamma)$. For every $M > 0$ there exist functions $\psi_M \in L^{\bar p}(\Omega)$, $\bar p > n/2$, and $\phi_M \in L^{\bar q}(\Gamma)$, $\bar q > n - 1$, and a constant $C_M > 0$ such that

$$\begin{cases} \left| \dfrac{\partial^j L}{\partial y^j}(x, y) \right| \leq \psi_M(x), \quad \text{with } j = 1, 2, \\ \left| \dfrac{\partial^j l}{\partial y^j}(x, y, u) \right| \leq \phi_M(x), \quad \text{with } j = 1, 2, \\ \left| \dfrac{\partial^{i+j} l}{\partial u^i \partial y^j}(x, y, u) \right| \leq C_M, \;\; 1 \leq i + j \leq 2 \text{ and } i \geq 1 \end{cases}$$

are satisfied for a.a. $x \in \Omega$ and every $u, y \in \mathbb{R}$, with $|y| \leq M$ and $|u| \leq M$.

Moreover, for every $\varepsilon > 0$ there exists $\eta > 0$ such that for a.a. $x \in \Omega$ and all $u_i, y_i \in \mathbb{R}$, with $i = 1, 2$,

$$\begin{cases} |y_2 - y_1| \leq \eta \Rightarrow \left| \dfrac{\partial^2 L}{\partial y^2}(x, y_2) - \dfrac{\partial^2 L}{\partial y^2}(x, y_1) \right| \leq \varepsilon, \\ |u_2 - u_1| + |y_2 - y_1| \leq \eta \Rightarrow \left| D^2_{(y,u)} l(x, y_2, u_2) - D^2_{(y,u)} l(x, y_1, u_1) \right| \leq \varepsilon. \end{cases}$$

Here $D^2_{(y,u)} l(x, y, u)$ denotes the Hessian matrix of l with respect to the variables (y, u).

We also assume the Legendre-Clebsch type condition

$$\exists \Lambda > 0 \text{ such that } \frac{\partial^2 l}{\partial u^2}(x, y, u) \geq \Lambda \;\text{ for a.a. } x \in \Gamma \text{ and } \forall y, u \in \mathbb{R}. \tag{3.3}$$

We should mention that the frequently used function $L(x, y) = \frac{1}{2}(y - y_d(x))^2$ satisfies Assumption *(N2)* if $y_d \in L^{\bar p}(\Omega)$.

On the state equation (2.1), the following result is known.

Theorem 3.1. *Under the Assumption (N1), for every $u \in L^{\bar q}(\Gamma)$ the equation (3.2) has a unique solution $y_u \in H^1(\Omega) \cap C(\bar\Omega)$. The mapping $G : L^{\bar q}(\Gamma) \longrightarrow H^1(\Omega) \cap C(\bar\Omega)$, defined by $G(u) = y_u$, is of class C^2. For elements u, v, v_1 and v_2 of $L^{\bar q}(\Gamma)$,* *the functions* $z_v = G'(u)v$ *and* $z_{v_1 v_2} = G''(u)(v_1, v_2)$ *are the solutions of the problems*

$$\begin{cases} Az + f'(y_u)z = 0 & \text{in } \Omega, \\ \partial_{\nu_A} z = v & \text{on } \Gamma, \end{cases} \tag{3.4}$$

and

$$\begin{cases} Az + f'(y_u)z + f''(y_u)z_{v_1}z_{v_2} = 0 & \text{in } \Omega, \\ \partial_{\nu_A} z = 0 & \text{on } \Gamma, \end{cases} \tag{3.5}$$

respectively.

The proof of the existence and uniqueness of a solution y_u in $H^1(\Omega) \cap L^\infty(\Omega)$ is standard; see, for instance, [3]. For the continuity of y_u, the reader is referred to [11] or [18]. Let us show for convenience the differentiability of G. We set

$$V = \{y \in H^1(\Omega) : \Delta y \in L^{\bar{p}}(\Omega) \text{ and } \partial_\nu y \in L^{\bar{q}}(\Gamma)\}.$$

It is known that, given $y \in W^{1,r}(\Omega)$ such that $\Delta y \in L^r(\Omega)$, $1 < r < +\infty$, one can define $\partial_\nu y \in W^{-1/r,r}(\Gamma)$, see [6]. Therefore, V is well defined for $r = \min\{\bar{p}, 2\}$. Endowed with the graph norm, V is a Banach space. Moreover, we deduce from [11] or [18] that V is embedded in $C(\bar{\Omega})$. Now, we consider

$$F : V \times L^{\bar{q}}(\Gamma) \longrightarrow L^{\bar{p}}(\Omega) \times L^{\bar{q}}(\Gamma), \quad F(y, u) = (-\Delta y + f(y), \partial_\nu y - u).$$

It is easy to check that F is C^2, $F(y_u, u) = (0, 0)$ for every $u \in L^{\bar{q}}(\Gamma)$ and

$$\frac{\partial F}{\partial y}(y_u, u) : V \longrightarrow L^{\bar{p}}(\Omega) \times L^{\bar{q}}(\Gamma), \quad \frac{\partial F}{\partial y}(y_u, u)z = (-\Delta z + f'(y_u)z, \partial_\nu z)$$

defines an isomorphism. Now the implicit function theorem yields that G is of class C^2 and (3.4) and (3.5) are fulfilled.

In view of this theorem, the chain rule applies to show the following result:

Theorem 3.2. *Assuming (N1) and (N2), then the mapping* $J : L^\infty(\Gamma) \longrightarrow \mathbb{R}$, *defined by* (3.1), *is of class* C^2. *For all* u, v, v_1 *and* v_2 *of* $L^\infty(\Gamma)$ *we have*

$$J'(u)v = \int_\Gamma \left(\varphi_u + \frac{\partial l}{\partial u}(x, y_u, u) \right) v \, d\sigma \tag{3.6}$$

$$\begin{aligned} J''(u)(v_1, v_2) &= \int_\Omega \left(\frac{\partial^2 L}{\partial y^2}(x, y_u) - \varphi_u f''(y_u) \right) z_{v_1} z_{v_2} \, dx \\ &+ \int_\Gamma \left(\frac{\partial^2 l}{\partial y^2}(x, y_u, u) z_{v_1} z_{v_2} + \frac{\partial^2 l}{\partial y \partial u}(x, y_u, u)(v_1 z_{v_2} + v_2 z_{v_1}) \right) d\sigma \\ &+ \int_\Gamma \frac{\partial^2 l}{\partial u^2}(x, y_u, u) v_1 v_2 \, d\sigma, \end{aligned} \tag{3.7}$$

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$, *and* $\varphi_u \in H^1(\Omega) \cap C(\bar{\Omega})$ *is the solution of*

$$\begin{cases} -\Delta \varphi + f'(y_u)\varphi = \dfrac{\partial L}{\partial y}(x, y_u) & \text{in } \Omega, \\ \partial_\nu \varphi = \dfrac{\partial l}{\partial y}(x, y_u, u) & \text{on } \Gamma. \end{cases} \tag{3.8}$$

Remark 3.3. *From the above expressions for $J'(u)$ and $J''(u)$ and Assumption (N2) we deduce that $J'(u)$ and $J''(u)$ can be extended to linear and bilinear forms, respectively, on $L^2(\Gamma)$.*

Indeed, since $u \in L^\infty(\Gamma)$, then set $y_u \in H^1(\Omega) \cap C(\bar\Omega)$. In particular, there exists $M > 0$ such that $||u||_\infty \le M$ and $\|y_u\|_\infty \le M$ holds. Moreover, (3.8) *yields*

$$\|\varphi_u\|_{H^1(\Omega)} + \|\varphi_u\|_{C(\bar\Omega)} \le C(\|\psi_M\|_{L^{\bar p}(\Omega)} + \|\phi_M\|_{L^{\bar q}(\Gamma)}),$$

where C is independent of M, y and u. We also know

$$\|z_v\|_{H^1(\Omega)} \le C\|v\|_{L^2(\Gamma)}.$$

All these estimates ensure the existence of two constants $M_1 > 0$ and $M_2 > 0$ such that for every $v, v_1, v_2 \in L^2(\Gamma)$

$$|J'(u)v| \le M_1\|v\|_{L^2(\Gamma)} \quad \textit{and} \quad |J''(u)(v_1, v_2)| \le M_2\|v_1\|_{L^2(\Gamma)}\|v_2\|_{L^2(\Gamma)},$$

Furthermore, the constants M_i can be taken the same for every u belonging to a bounded set of $L^\infty(\Gamma)$.

We now demonstrate that Theorem 2.3 can be applied to the problem (P_1). To this end, we set $U_2 = L^2(\Gamma)$ and $U_\infty = L^\infty(\Gamma)$. Theorem 3.2 shows that $J : U_\infty \longrightarrow \mathbb{R}$ is of class C^2. By Remark 3.3, also (2.2) holds. Let us prove (2.3)-(2.5).

PROPOSITION 3.4. *Let $\{(u_k, v_k\}_{k=1}^\infty \subset \mathcal{K} \times L^2(\Gamma)$ such that $u_k \to u$ strongly in $L^2(\Gamma)$ and $v_k \rightharpoonup v$ weakly in $L^2(\Gamma)$. Then* (2.3)-(2.5) *are satisfied.*

Proof. The convergence of $\{u_k\}_{k=1}^\infty$ in $L^2(\Gamma)$, along with the boundedness in $L^\infty(\Gamma)$ implies that $u_k \to u$ in $L^q(\Gamma)$ for every $1 \le q < +\infty$. In particular this is true for $\bar q$. Therefore, invoking Theorem 3.1 we get $y_{u_k} = G(u_k) \to G(u) = y_u$ strongly in $H^1(\Omega) \cap C(\bar\Omega)$. Using this fact in (3.8), we deduce with the help of the assumption *(N1)* that $\varphi_{u_k} \to \varphi_u$ in $H^1(\Omega) \cap C(\bar\Omega)$. From (3.4) we also know that $z_{v_k} = G'(u_k)v_k \to G'(u)v$ strongly in $H^1(\Omega)$. In view of these convergence properties and the Assumptions *(N1)–(N2)*, we easily obtain (2.3). Now we consider the expression for $J''(u_k)v_k^2$ and observe that it is easy to pass to the limit in all the integral terms except in the last one. To confirm (2.4) we apply Lemma 3.5 stated below for $X = \Gamma$, $\mu = \sigma$ and

$$0 < \Lambda \le g_k(x) = \frac{\partial^2 l}{\partial u^2}(x, y_{u_k}(x), u_k(x)) \to g(x) = \frac{\partial^2 l}{\partial u^2}(x, y_u(x), u(x)) \text{ in } L^1(\Gamma).$$

Then we deduce from (3.9)

$$\liminf_{k\to\infty} \int_\Gamma \frac{\partial^2 l}{\partial u^2}(x, y_{u_k}, u_k)v_k^2\, d\sigma \ge \int_\Gamma \frac{\partial^2 l}{\partial u^2}(x, y_u, u)v^2\, d\sigma.$$

Together with the previous comments, this confirms (2.4).

Let us prove (2.5). Since $v = 0$, then all the integral terms of $J''(u_k)v_k^2$ tend to zero, except the last one. To get (2.5), we use (3.3) and find

$$\Lambda \liminf_{k\to 0} \|v_k\|_{L^2(\Gamma)}^2 \le \liminf_{k\to 0} \int_\Gamma \frac{\partial^2 l}{\partial u^2}(x, y_{u_k}, u_k)v_k^2\, d\sigma = \liminf_{k\to 0} J''(u_k)v_k^2.$$

□

LEMMA 3.5. *Let* (X, Σ, μ) *be a measure space with* $\mu(X) < +\infty$. *Suppose that* $\{g_k\}_{k=1}^{\infty} \subset L^{\infty}(X)$ *and* $\{v_k\}_{k=1}^{\infty} \subset L^2(X)$ *satisfy the assumptions*

- $g_k \geq 0$ *a.e. in* X, $\{g_k\}_{k=1}^{\infty}$ *is bounded in* $L^{\infty}(X)$ *and* $g_k \to g$ *in* $L^1(X)$.
- $v_k \rightharpoonup v$ *in* $L^2(X)$.

Then there holds the inequality

$$\int_X g(x)v^2(x)\,d\mu(x) \leq \liminf_{k\to\infty} \int_X g_k(x)v_k^2(x)\,d\mu(x). \tag{3.9}$$

Proof. Since $\{g_k\}_{k=1}^{\infty}$ is bounded in $L^{\infty}(X)$, it holds $g \in L^{\infty}(X)$. Denote the lower limit in (3.9) by λ. Then there exists a subsequence of functions, denoted in the same way, such that the integrals of the right hand side of (3.9) converge to λ. Again, we can select a new subsequence of this one such that $g_k(x) \to g(x)$ *a.e.* in X. Let $\varepsilon > 0$ be arbitrary. By Egorov's theorem there exists a measurable set $K_\varepsilon \subset X$ such that $\mu(X \setminus K_\varepsilon) < \varepsilon$ and $\|g - g_k\|_{L^{\infty}(K_\varepsilon)} \to 0$ as $k \to \infty$. Then we have

$$\begin{aligned}
&\liminf_{k\to\infty} \int_X g_k(x)v_k^2(x)\,d\mu(x) \geq \liminf_{k\to\infty} \int_{K_\varepsilon} g_k(x)v_k^2(x)\,d\mu(x) \\
&\geq \liminf_{k\to\infty} \int_{K_\varepsilon} [g_k(x) - g(x)]v_k^2(x)\,d\mu(x) + \liminf_{k\to\infty} \int_{K_\varepsilon} g(x)v_k^2(x)\,d\mu(x) \\
&= \liminf_{k\to\infty} \int_{K_\varepsilon} g(x)v_k^2(x)\,d\mu(x) \geq \int_{K_\varepsilon} g(x)v^2(x)\,d\mu(x).
\end{aligned}$$

Finally, passing to the limit $\varepsilon \to 0$ we get (3.9) □

After having verified all the necessary assumptions, we are justified to apply Theorems 2.2 and 2.3 to the problem (P_1). Given $\bar{u} \in \mathcal{K}$, we see that the cone of critical directions $C_{\bar{u}}$ defined in §2 can be expressed for the problem (P_1) in the form

$$C_{\bar{u}} = \{v \in L^2(\Gamma) : v(x) = \begin{cases} \geq 0 & \text{if } \bar{u}(x) = \alpha \\ \leq 0 & \text{if } \bar{u}(x) = \beta \\ \ \ 0 & \text{if } \bar{d}(x) \neq 0 \end{cases} \text{a.e. in } \Gamma\},$$

where

$$\bar{d}(x) = \bar{\varphi}(x) + \frac{\partial l}{\partial u}(x, \bar{y}(x), \bar{u}(x))$$

and $\bar{y} = y_{\bar{u}}$ and $\bar{\varphi} = \varphi_{\bar{u}}$ denote the state and adjoint state associated to $\bar{u}$, respectively.

Let us check this claim. We have to prove that the defined cone $C_{\bar{u}}$ coincides with the set $\{v \in cl_2(S_{\bar{u}}) : J'(\bar{u})v = 0\}$ denoted by $E_{\bar{u}}$ for a while. We recall the following well known property of the optimal control, see e.g. [19, Lemma 2.26],

$$\begin{cases} \bar{d}(x) > 0 & \Rightarrow \quad \bar{u}(x) = \alpha, \\ \bar{d}(x) < 0 & \Rightarrow \quad \bar{u}(x) = \beta. \end{cases}$$

If $v \in E_{\bar{u}}$, then there exists a sequence $\{v_k\}_{k=1}^{\infty} \subset S_{\bar{u}}$ such that $v_k \to v$ in $L^2(\Gamma)$. By the definition of $S_{\bar{u}}$, it is obvious that $v_k(x) \geq 0$ whenever $\bar{u}(x) = \alpha$ and $v_k(x) \leq 0$ if

$\bar u(x) = \beta$. The set of elements of $L^2(\Gamma)$ enjoying this property is closed, consequently v also has this property. Therefore, from the above property of $\bar u$ we get

$$0 = J'(\bar u)v = \int_\Gamma \bar d(x)v(x)\,d\sigma(x) = \int_\Gamma |\bar d(x)||v(x)|\,d\sigma(x),$$

which implies that $v(x) = 0$ if $\bar d(x) \neq 0$, thus $v \in C_{\bar u}$ and hence $E_{\bar u} \subset C_{\bar u}$.

Now, we prove the converse inclusion. We will even get more, because we prove that the regularity assumption of Theorem 2.2 holds: $C_{\bar u} \subset cl_2(D_{\bar u}) \subset E_{\bar u}$. Given any element $v \in C_{\bar u}$, for every positive integer k, we define

$$v_k(x) = \begin{cases} 0 & \text{if } \alpha < \bar u(x) < \alpha + \dfrac{1}{k} \ \text{ or } \ \beta - \dfrac{1}{k} < \bar u(x) < \beta, \\ \mathbb{P}_{[-k,+k]}(v(x)) & \text{otherwise.} \end{cases}$$

Above, $\mathbb{P}_{[-k,+k]}$ denotes the pointwise projection on the interval $[-k,+k]$. It is obvious that $\alpha \le u_k(x) = \bar u(x) + \rho_k v_k(x) \le \beta$ for every $\rho_k = \min\{1/k^2, (\beta-\alpha)/k\}$ and *a.a.* $x \in \Gamma$. Hence, $v_k = (u_k - \bar u)/\rho_k \in S_{\bar u}$. Furthermore, $|v_k(x)| \le |v(x)|$, which implies that $v_k \to v$ strongly in $L^2(\Gamma)$ and $|v_k(x)\bar d(x)| \le |v(x)\bar d(x)| = 0$ a.e. in Γ. Thus, we have that every v_k belongs to $D_{\bar u}$, which leads to $v \in cl_2(D_{\bar u})$, as desired.

Corollary 3.6. *Let the Assumption (N1) be satisfied and suppose that $\bar u$ is a local minimum of* (P_1) *in the $L^\infty(\Gamma)$ sense. Then there holds $J'(\bar u)(u - \bar u) \ge 0$ for all $u \in \mathcal{K}$ and $J''(\bar u)v^2 \ge 0$ $\forall v \in C_{\bar u}$. Conversely, if $\bar u \in \mathcal{K}$ obeys*

$$J'(\bar u)(u - \bar u) \ge 0 \quad \forall u \in \mathcal{K}, \tag{3.10}$$

$$J''(\bar u)v^2 > 0 \quad \forall v \in C_{\bar u} \setminus \{0\}, \tag{3.11}$$

then there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$J(\bar u) + \frac{\delta}{2}\|u - \bar u\|^2_{L^2(\Gamma)} \le J(u) \quad \forall u \in \mathcal{K} \cap B_2(\bar u;\varepsilon). \tag{3.12}$$

Corollary 3.7. *Under the assumption (N1) and (N2), there exists a ball $B_2(\bar u;\varepsilon)$ in $L^2(\Gamma)$ such that there is no other stationary point in $B_2(\bar u;\varepsilon) \cap \mathcal{K}$ than $\bar u$. Moreover, there exist numbers $\nu > 0$ and $\tau > 0$ such that*

$$J''(u)v^2 \ge \frac{\nu}{2}\|v\|^2_{L^2(\Gamma)} \quad \forall\, v \in C^\tau_{\bar u} \quad \textit{and} \quad \forall\, u \in \mathcal{A} \cap B_2(\bar u;\varepsilon),$$

where $\mathcal{A}$ is a bounded open subset of $L^\infty(\Gamma)$ containing $\mathcal{K}$ and

$$C^\tau_{\bar u} = \{v \in L^2(\Gamma) : v(x) = \begin{cases} \ge 0 & \textit{if } \bar u(x) = \alpha \\ \le 0 & \textit{if } \bar u(x) = \beta \\ 0 & \textit{if } |\bar d(x)| > \tau \end{cases} \quad \textit{a.e. in } \Gamma\}.$$

Observe that the above cone $C^\tau_{\bar u}$ is not equal to the cone $E^\tau_{\bar u}$ defined in Theorem 2.7. However, if $v \in C^\tau_{\bar u}$, then

$$|J'(\bar u)v| = \int_\Gamma |\bar d(x)v(x)|\,dx \le \tau \int_{\{x:|\bar d(x)| \le \tau\}} |v(x)|\,dx \le \tau\sqrt{|\Gamma|}\|v\|_{L^2(\Gamma)}.$$

Thus, we have that $C^\tau_{\bar u} \subset E^{\tau_\Gamma}_{\bar u}$, with $\tau_\Gamma = \tau\sqrt{|\Gamma|}$. Hence, Theorem 2.7 can be applied.

Let us underline that the mapping G is only differentiable in $L^q(\Gamma)$ for $q > n-1$. For all $n \geq 3$, G is not differentiable in $L^2(\Gamma)$. Even if the objective functional J were quadratic, the classical theory of second order conditions would only assure local optimality in the sense of $L^\infty(\Gamma)$. The general nonlinear cost functional J is only differentiable in $L^\infty(\Gamma)$. Hence, for any dimension n, the classical theory of second order conditions would only assure the local optimality of $\bar{u}$ in the $L^\infty(\Gamma)$ sense. In contrast to this, our result guarantees local optimality in the sense of $L^2(\Gamma)$. Let us recall a well known fact. Since $\mathcal{K}$ is bounded in $L^\infty(\Gamma)$, then $\bar{u}$ is a (strict) local solution of (P$_1$) in the sense of $L^2(\Gamma)$ if and only if it is a (strict) local solution of (P$_1$) in the sense of $L^r(\Gamma)$ for all $1 \leq r \leq \infty$.

4. Application II. An elliptic Dirichlet control problem. In this section, we assume that $\Omega \subset \mathbb{R}^n$ is an open domain whose boundary Γ is of class $C^{1,1}$. In this domain we formulate the following control problem

$$(\mathrm{P}_2) \qquad \min_{u \in \mathcal{K}} J(u),$$

with

$$J(u) = \frac{1}{2} \int_\Omega |y_u(x) - y_d(x)|^2 \, dx \; + \frac{\Lambda}{2} \int_\Gamma u^2(x) \, dx, \tag{4.1}$$

$$\mathcal{K} = \{u \in L^\infty(\Gamma) : \alpha \leq u(x) \leq \beta \ \text{ for a.a. } x \in \Gamma\},$$

where $-\infty < \alpha < \beta < +\infty$ and y_u is the solution of the state equation

$$\begin{cases} -\Delta y + f(y) = 0 & \text{in } \Omega, \\ \qquad\quad y = u & \text{on } \Gamma. \end{cases} \tag{4.2}$$

The following hypotheses are assumed about the functions involved in the control problem (P$_2$).

Assumption (D1): We assume that $y_d \in L^{\bar{p}}(\Omega)$, with $\bar{p} \geq 2$ and $\bar{p} > n/2$, and $\Lambda > 0$.

Assumption (D2): The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is of class C^2 and $f'(t) \geq 0$ for all $t \in \mathbb{R}$.

As usual, we will say that an element $y_u \in L^\infty(\Omega)$ is a solution of (2.1) if

$$\int_\Omega -\Delta w \, y \, dx + \int_\Omega f(y) w \, dx + \int_\Gamma u \partial_\nu w \, dx = 0 \quad \forall w \in H^2(\Omega) \cap H_0^1(\Omega), \tag{4.3}$$

The problem (P$_2$) was studied by Casas and Raymond [8]. The reader is referred to this paper for a more general formulation of the problem concerning the cost functional and the non-linear term of the state equation, as well as and for the proof of the following results.

Theorem 4.1. *For every $u \in L^\infty(\Gamma)$ the state equation (4.2) has a unique solution $y_u \in L^\infty(\Omega) \cap H^{1/2}(\Omega)$. Moreover the following Lipschitz properties hold*

$$\begin{aligned} &\|y_u - y_v\|_{L^\infty(\Omega)} \leq \|u - v\|_{L^\infty(\Gamma)} \\ &\|y_u - y_v\|_{H^{1/2}(\Omega)} \leq C\|u - v\|_{L^2(\Gamma)} \quad \forall u, v \in L^\infty(\Gamma). \end{aligned} \tag{4.4}$$

Finally if $u_n \rightharpoonup u$ weakly* in $L^\infty(\Gamma)$, then $y_{u_n} \to y_u$ strongly in $L^r(\Omega)$ for all $r < +\infty$.

Using this result, it is easy to prove that (P_2) has at least one solution.

THEOREM 4.2. *The mapping* $G : L^\infty(\Gamma) \longrightarrow L^\infty(\Omega) \cap H^{1/2}(\Omega)$ *defined by* $G(u) = y_u$ *is of class* C^2*. Moreover, for all* $u, v \in L^\infty(\Gamma)$*,* $z_v = G'(u)v$ *is the solution of*

$$\begin{cases} -\Delta z_v + f'(y_u) z_v = 0 & \text{in } \Omega, \\ z_v = v & \text{on } \Gamma. \end{cases} \tag{4.5}$$

For every $v_1, v_2 \in L^\infty(\Omega)$*,* $z_{v_1 v_2} = G''(u)v_1 v_2$ *is the solution of*

$$\begin{cases} -\Delta z_{v_1 v_2} + f'(y_u) z_{v_1 v_2} + f''(y_u) z_{v_1} z_{v_2} = 0 & \text{in } \Omega, \\ z_{v_1 v_2} = 0 & \text{on } \Gamma, \end{cases} \tag{4.6}$$

where $z_{v_i} = G'(u)v_i$*,* $i = 1, 2$*.*

THEOREM 4.3. *The functional* $J : L^\infty(\Gamma) \to \mathbb{R}$ *is of class* C^2*. For every* $u, v, v_1, v_2 \in L^\infty(\Gamma)$

$$J'(u)v = \int_\Gamma (\Lambda u - \partial_\nu \varphi_u)\, v\, dx \tag{4.7}$$

and

$$J''(u)v_1 v_2 = \int_\Omega [1 + \varphi_u f''(y_u)]\, z_{v_1} z_{v_2}\, dx + \Lambda \int_\Gamma v_1 v_2\, dx, \tag{4.8}$$

where $z_{v_i} = G'(u)v_i$*,* $i = 1, 2$*,* $y_u = G(u)$*, and the adjoint state* $\varphi_u \in H^2(\Omega) \cap L^\infty(\Omega)$ *is the unique solution of the problem*

$$\begin{cases} -\Delta \varphi + f'(y_u)\varphi = y_u - y_d & \text{in } \Omega, \\ \varphi = 0 & \text{on } \Gamma. \end{cases} \tag{4.9}$$

By the preceding results, the reader can easily check that assumptions (2.2)-(2.5) are satisfied. The most delicate point is certainly the proof of (2.4). To this end, the reader should observe that the boundedness of $\{u_k\}_{k=1}^\infty$ in $L^\infty(\Gamma)$ and the strong convergence $u_k \to \bar{u}$ in $L^2(\Gamma)$ imply the boundedness of $\{y_{u_k}\}_{k=1}^\infty$ in $L^\infty(\Omega)$ and the strong convergence $y_{u_k} \to \bar{y}$ in $L^q(\Omega)$ for every $1 \le q < \infty$. On the other hand, the weak convergence $v_k \rightharpoonup v$ in $L^2(\Gamma)$ implies that $z_{v_k} \rightharpoonup z_v$ weakly in $H^{1/2}(\Omega)$. The compactness of the embedding $H^{1/2}(\Omega) \subset L^r(\Omega)$ for every $1 \le r < 2n/(n-1)$ implies that $z_{v_k}^2 \to z_v^2$ strongly in $L^r(\Omega)$ for some $r > 1$. Finally, it follows immediately that $\varphi_{u_k} \to \bar{\varphi}$ strongly in $L^\infty(\Omega)$. Having in mind these facts and taking into account the expression of J'' given by (4.8), it is easy to pass to the limit and to prove (2.4).

Now, given an optimal control $\bar{u}$ with associated adjoint state $\bar{\varphi}$, we define $\bar{d} = \Lambda \bar{u} - \partial_\nu \bar{\varphi}$. Then, the critical cone $C_{\bar{u}}$ is defined as for the problem (P_1) and the analogous versions of corollaries 3.6 and 3.7 hold for the problem (P_2). The reader should notice that the mapping $G(u) = y_u$ is not differentiable, even probably not well defined in $L^q(\Gamma)$ for any $q < \infty$. Therefore, the use of $L^\infty(\Gamma)$ as control space is crucial. Once again, the classical theory of second order conditions is improved by assuring the strict local optimality of $\bar{u}$ in the sense of $L^2(\Gamma)$ under the standard second order optimality conditions.

5. Application III. A parabolic distributed control problem. Now we consider the distributed control problem

$$(\mathrm{P}_3) \qquad \min_{u\in\mathcal{K}} J(u),$$

with

$$J(u) = \int_0^T \int_\Omega L(x,t,y_u(x,t),u(x,t))\,dxdt \tag{5.1}$$

$$\mathcal{K} = \{u \in L^\infty(\Omega_T) : \alpha \le u(x,t) \le \beta \ \text{ for a.a. } (x,t) \in \Omega_T\},$$

where $\Omega_T = \Omega \times (0,T)$ and y_u is the solution of the state equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + f(x,t,y) = u & \text{in } \Omega_T, \\ \hfill y = 0 & \text{on } \Sigma_T, \\ \hfill y(x,0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{5.2}$$

Here, $\Sigma_T = \Gamma \times (0,T)$. We impose the following assumptions on the functions and parameters appearing in the control problem (P_3).

Assumption (P1): $y_0 \in C_0(\Omega)$ and the function $f : \Omega_T \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the second variable and satisfies the conditions

$$\begin{cases} \exists \psi_0 \in L^{\hat p}([0,T], L^{\hat q}(\Omega)) \text{ and } C_1 > 0 \text{ such that} \\ f(x,t,y)y \ge \psi_0(x,t) - C_1 y^2 \ \ \forall (x,t,y) \in \Omega_T \times \mathbb{R}, \end{cases}$$

$$\begin{cases} f(\cdot,\cdot,0) \in L^{\hat p}([0,T], L^{\hat q}(\Omega)) \text{ and } \forall M > 0\ \exists C_M > 0 \text{ such that} \\ \left|\dfrac{\partial^j f}{\partial y^j}(x,t,y)\right| \le C_M \ \forall (x,t) \in \Omega_T, |y| \le M, j = 1,2, \end{cases}$$

$$\forall \varepsilon > 0\ \exists \eta > 0 \text{ such that } |y_2 - y_1| \le \eta \Rightarrow \left|\frac{\partial^2 f}{\partial y^2}(x,t,y_2) - \frac{\partial^2 f}{\partial y^2}(x,t,y_1)\right| \le \varepsilon,$$

where $\hat q, \hat p \in [1,+\infty]$ and $\dfrac{1}{\hat p} + \dfrac{n}{2\hat q} < 1$.

Assumption (P2): We require $-\infty < \alpha < \beta < +\infty$. Moreover, $L : \Omega_T \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the last two variables and $L(\cdot,\cdot,0,0)$ belongs to $L^1(\Omega_T)$. For every $M > 0$, there exist a function $\psi_M \in L^{\hat p}([0,T], L^{\hat q}(\Omega))$ and a constant $C_M > 0$ such that

$$\begin{cases} \left|\dfrac{\partial L}{\partial y}(x,t,y,u)\right| \le \psi_M(x,t), \\ \left|\dfrac{\partial^{i+j} L}{\partial u^i \partial y^j}(x,y,u)\right| \le C_M, \ 1 \le i + j \le 2, \end{cases}$$

are satisfied for a.a. $(x,t) \in \Omega_T$ and every $u, y \in \mathbb{R}$, with $|y| \le M$ and $|u| \le M$.

For every $\varepsilon > 0$ there exists $\eta > 0$ such that

$$|u_2 - u_1| + |y_2 - y_1| \le \eta \Rightarrow \left|D^2_{(y,u)}L(x,t,y_2,u_2) - D^2_{(y,u)}L(x,t,y_1,u_1)\right| \le \varepsilon,$$

for a.a. $(x,t) \in \Omega_T$ and all $u_i, y_i \in \mathbb{R}$, with $i = 1,2$. Here $D^2_{(y,u)}L(x,t,y,u)$ denotes the Hessian matrix of L with respect to the variables (y,u).

We also assume the Legendre-Clebsch type condition

$$\exists \Lambda > 0 \text{ such that } \frac{\partial^2 L}{\partial u^2}(x,t,y,u) \geq \Lambda \quad \text{for a.a. } (x,t) \in \Omega_T \text{ and } \forall y,u \in \mathbb{R}. \tag{5.3}$$

Then the following parabolic counterpart to the theorems 3.1 and 3.2 holds true.

THEOREM 5.1. *Under the assumption (P1), for all $u \in L^{\hat{p}}([0,T], L^{\hat{q}}(\Omega))$ the equation* (5.2) *has a unique solution $y_u \in L^2([0,T], H^1_0(\Omega)) \cap C(\bar{\Omega}_T)$. The mapping $G : L^{\hat{p}}([0,T], L^{\hat{q}}(\Omega)) \longrightarrow L^2([0,T], H^1_0(\Omega)) \cap C(\bar{\Omega}_T)$ defined by $G(u) = y_u$ is of class C^2. For all elements u, v, v_1 and v_2 of $L^{\hat{p}}([0,T], L^{\hat{q}}(\Omega))$, the functions $z_v = G'(u)v$ and $z_{v_1v_2} = G''(u)(v_1, v_2)$ are the solutions of the problems*

$$\left\{\begin{aligned} \frac{\partial z}{\partial t} - \Delta z + \frac{\partial f}{\partial y}(x,t,y_u)z = v & \quad \text{in } \Omega_T, \\ z = 0 & \quad \text{on } \Sigma_T, \\ z(x,0) = 0 & \quad \text{in } \Omega, \end{aligned}\right. \tag{5.4}$$

and

$$\left\{\begin{aligned} \frac{\partial z}{\partial t} - \Delta z + \frac{\partial f}{\partial y}(x,t,y_u)z + \frac{\partial^2 f}{\partial y^2}(x,t,y_u)z_{v_1}z_{v_2} = 0 & \quad \text{in } \Omega_T, \\ z = 0 & \quad \text{on } \Sigma_T, \\ z(x,0) = 0 & \quad \text{in } \Omega, \end{aligned}\right. \tag{5.5}$$

respectively.

The reader is referred to [4] for the proof of the existence of a unique solution in $L^2([0,T], H^1_0(\Omega)) \cap C(\bar{\Omega}_T)$; see also [19, Theorem 5.5]. For the proof of the differentiability we can proceed analogously to the proof of Theorem 3.1. We set

$$V = \{y \in L^2([0,T], H^1_0(\Omega)) \cap C(\bar{\Omega}_T) : \frac{\partial y}{\partial t} - \Delta y \in L^{\hat{p}}([0,T], L^{\hat{q}}(\Omega))\},$$

endowed with the graph norm. Defining

$$F : V \times L^{\hat{p}}([0,T], L^{\hat{q}}(\Omega)) \longrightarrow L^{\hat{p}}([0,T], L^{\hat{q}}(\Omega)) \times C(\bar{\Omega}),$$

$$F(y,u) = (\frac{\partial y}{\partial t} - \Delta y + f(x,t,y) - u, y(\cdot,0) - y_0),$$

we can apply again the implicit function theorem to deduce (5.4) and (5.5).

THEOREM 5.2. *Assuming (P1) and (P2), the functional $J : L^\infty(\Omega_T) \longrightarrow \mathbb{R}$, defined by* (5.1)*, is of class C^2. For all u, v, v_1 and v_2 of $L^\infty(\Omega_T)$ we have*

$$J'(u)v = \int_0^T \int_\Omega \left(\varphi_u + \frac{\partial L}{\partial u}(x,t,y_u,u)\right) v \, dxdt \tag{5.6}$$

$$\begin{aligned} J''(u)(v_1,v_2) = & \int_0^T \int_\Omega \left(\frac{\partial^2 L}{\partial y^2}(x,t,y_u,u) - \varphi_u \frac{\partial^2 f}{\partial y^2}(x,t,y_u)\right) z_{v_1} z_{v_2} \, dxdt \\ & + \int_0^T \int_\Omega \left(\frac{\partial^2 L}{\partial y \partial u}(x,t,y_u,u)(v_1 z_{v_2} + v_2 z_{v_1}) + \frac{\partial^2 L}{\partial u^2}(x,t,y_u,u) v_1 v_2\right) dxdt \end{aligned} \tag{5.7}$$

where $z_{v_i} = G'(u)v_i$, $i = 1,2$, *and* $\varphi_u \in L^2([0,T], H_0^1(\Omega)) \cap C(\bar\Omega_T)$ *is the solution of*

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} - \Delta\varphi + \dfrac{\partial f}{\partial y}(x,t,y_u)\varphi = \dfrac{\partial L}{\partial y}(x,t,y_u,u) & \text{in } \Omega_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{5.8}$$

Now, we verify that problem (P$_3$) satisfies the assumptions of Theorem 2.3 with $U_\infty = L^\infty(\Omega_T)$ and $U_2 = L^2(\Omega_T)$. To confirm (2.2), we argue as we did for problem (P$_1$) ; see Remark 3.3. Let us verify the second assumption.

PROPOSITION 5.3. *Let* $\{(u_k, v_k\}_{k=1}^\infty \subset \mathcal{K} \times L^2(\Omega_T)$ *such that* $u_k \to u$ *strongly in* $L^2(\Omega_T)$ *and* $v_k \rightharpoonup v$ *weakly in* $L^2(\Omega_T)$. *Then* (2.3)-(2.5) *are satisfied.*

Proof. We follow the steps of the proof of Proposition 3.4. First, we mention that the convergence of $\{u_k\}_{k=1}^\infty$ in $L^2(\Omega_T)$ along with the boundedness in $L^\infty(\Omega_T)$ imply that $u_k \to u$ in $L^{\hat p}([0,T], L^{\hat q}(\Omega))$. Applying Theorem 5.1 we get that $y_{u_k} = G(u_k) \to G(u) = y_u$ strongly in $L^2([0,T], H_0^1(\Omega)) \cap C(\bar\Omega_T)$. Invoking the assumption *(P1)*, we obtain from (5.5) that $\varphi_{u_k} \to \varphi_u$ in $L^2([0,T], H_0^1(\Omega)) \cap C(\bar\Omega_T)$. Now (5.4) implies that $z_{v_k} = G'(u_k)v_k \to G'(u)v$ strongly in $L^2([0,T], H_0^1(\Omega)) \cap C(\bar\Omega_T)$. These convergence properties and the Assumptions *(P1)* and *(P2)* yield (2.3). We now proceed as in the proof of Proposition 3.4. The only delicate term for passing to the limit is the last one in the expression for $J''(u_k)v_k^2$. Inequality (2.4) follows from Lemma 3.5, where we set $X = \Omega_T$, μ is the Lebesgue measure in Ω_T, and

$$0 < \Lambda \le g_k(x,t) \to g(x,t) \quad \text{in} \quad L^1(\Omega_T),$$

with

$$g_k(x,t) = \frac{\partial^2 L}{\partial u^2}(x,t,y_{u_k}(x,t),u_k(x,t)) \quad \text{and} \quad g(x,t) = \frac{\partial^2 L}{\partial u^2}(x,t,y_u(x,t),u(x,t)).$$

Then we deduce from (3.9)

$$\liminf_{k\to\infty} \int_0^T \int_\Omega \frac{\partial^2 L}{\partial u^2}(x,t,y_{u_k},u_k)v_k^2\,dxdt \ge \int_0^T \int_\Omega \frac{\partial^2 L}{\partial u^2}(x,t,y_u,u)v^2\,dxdt,$$

which together with the previous comments prove (2.4).

Let us prove (2.5). Assuming that $v = 0$, thanks to (5.3), we deal with the last term of $J''(u_k)v_k^2$ by

$$\Lambda \liminf_{k\to 0} \|v_k\|^2_{L^2(\Omega_T)} \le \liminf_{k\to 0} \int_0^T \int_\Omega \frac{\partial^2 L}{\partial u^2}(x,t,y_{u_k},u_k)v_k^2\,dxdt = \liminf_{k\to 0} J''(u_k)v_k^2.$$

∎

Now we can apply Theorem 2.3 to the problem (P$_3$). For given $\bar u \in \mathcal{K}$, the cone of critical directions $C_{\bar u}$ defined in §2 admits for (P$_1$) the form

$$C_{\bar u} = \{v \in L^2(\Omega_T) : v(x,t) = \begin{cases} \ge 0 & \text{if } \bar u(x,t) = \alpha \\ \le 0 & \text{if } \bar u(x,t) = \beta \\ 0 & \text{if } \bar d(x,t) \ne 0 \end{cases} \quad \text{a.e. in } \Omega_T\},$$

where

$$\bar{d}(x,t) = \bar{\varphi}(x,t) + \frac{\partial L}{\partial u}(x,t,\bar{y}(x,t),\bar{u}(x,t)).$$

Here, $\bar{y} = y_{\bar{u}}$ and $\bar{\varphi} = \varphi_{\bar{u}}$ are the state and adjoint state associated with $\bar{u}$. As for the elliptic control problem (P_1), the cone of critical directions $C_{\bar{u}}$ coincides with the one defined in §2 and the regularity condition of Theorem 2.2 holds. Therefore, analogous corollaries to 3.6 and 3.7 hold for the control problem (P_3).

Once again G is not differentiable in $L^2(\Omega_T)$ for all $n > 1$ and, because of its general form, the cost functional J is only differentiable in $L^\infty(\Omega_T)$. Therefore, the classical theory of second order conditions would only assure the local optimality of $\bar{u}$ in the $L^\infty(\Omega_T)$ sense. However, our result ensures local optimality in the sense of $L^2(\Omega_T)$ in all cases.

REFERENCES

[1] J.F. Bonnans. Second-order analysis for control constrained optimal control problems of semilinear elliptic systems. *Appl. Math. Optim.*, 38:303–325, 1998.

[2] F. Bonnans and A. Shapiro. *Perturbation Analysis of Optimization Problems.* Springer-Verlag, 2000.

[3] E. Casas. Boundary control of semilinear elliptic equations with pointwise state constraints. *SIAM J. Control Optim.*, 31(4):993–1006, 1993.

[4] E. Casas. Pontryagin's principle for state-constrained boundary control problems of semilinear parabolic equations. *SIAM J. Control Optim.*, 35(4):1297–1327, 1997.

[5] E. Casas. Using piecewise linear functions in the numerical approximation of semilinear elliptic control problems. *Adv. Comp. Math.*, 16:137–153, 2007.

[6] E. Casas and L.A. Fernández. A Green's formula for quasilinear elliptic operators. *J. Math. Anal. Appl.*, 142(1):62–72, 1989.

[7] E. Casas and M. Mateos. Second order optimality conditions for semilinear elliptic control problems with finitely many state constraints. *SIAM J. Control Optim.*, 45(5):1586–1611, 2006.

[8] E. Casas and J.-P. Raymond. Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations. *SIAM J. Control Optim.*, 45(5):1586–1611, 2006.

[9] E. Casas and F. Tröltzsch. A general theorem on error estimates with application to elliptic optimal control problems. *To appear in Comput. Optim. Appl.*, 2011.

[10] E. Casas, F. Tröltzsch and A. Unger. Second order sufficient optimality conditions for some state-constrained control problems of semilinear elliptic equations. *SIAM J. Control Optim.*, 38(5):1369–1391, 2000.

[11] R. Haller-Dintelmann, C. Meyer, J. Rehberg, and A. Schiela. Hölder continuity and optimal control for nonsmooth elliptic domains. *Appl. Math. Optim.*, 60:397–428, 2009.

[12] M. R. Hestenes. Applications of the theory of quadratic forms in Hilbert space to the calculus of variations. *Pacific J. Math.*, 1:525–581, 1951.

[13] A. D. Ioffe. Necessary and sufficient conditions for a local minimum. III. A reduction theorem and first order conditions. *SIAM J. Control Optim.*, 17(2):266–288, 1979.

[14] A. D. Ioffe and V. M. Tihomirov. *Theory of extremal problems.* North-Holland Publishing Co., Amsterdam, 1979.

[15] K. Malanowski. Two norm approach in stability analysis of optimization and optimal control problems. In *World Congress of Nonlinear Analysts '92, Vol. I–IV (Tampa, FL, 1992)*, pages 2557–2568. de Gruyter, Berlin, 1996.

[16] H. Maurer. First and second order sufficient optimality conditions in mathematical programming and optimal control. *Math. Programming Study*, 14:163–177, 1981.

[17] H. Maurer and J. Zowe. First- and second-order conditions in infinite-dimensional programming problems. *Math.Programming*, 16:98–110, 1979.

[18] G. Stampacchia. Problemi al contorno ellittici con dati discontinui dotati di soluzioni Hölderiane. *Ann. Mat. Pura Appl.*, 51:1–38, 1960.

[19] F. Tröltzsch. *Optimal Control of Partial Differential Equations*, volume 112 of *Graduate Studies in Mathematics.* American Mathematical Society, Philadelphia, 2010.