GENERALIZED ISORADIAL CIRCLE PATTERNS

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Abstract. This work is about introducing generalized isoradial circle patterns. These are circle patterns for which the centers of the circles are the points of intersection of a circle pattern themselves. This generalizes the class of isoradial circle patterns. We thereby restrict ourselves to the case of circle patterns with square grid combinatorics and describe our circle patterns by their points of intersection, i.e. maps of the form $f : \mathbb{Z}^2 \to \mathbb{R}^2$ which we call nets.

On the way we introduce discrete systems and their multi-dimensional consistency and study various characterizations of circular and conical nets in higher dimensions which can be immediately transferred to generalized isoradial nets.

We prove the statement that a net is generalized isoradial if and only if its edges are parallel to the corresponding edges of an isoradial net.

The net of the centers of a generalized isoradial circle pattern is generalized isoradial itself which gives rise to an iteration process. Studying generalized isoradial nets on the torus is a case suited for simulating this process on a whole net while only dealing with finitely many points. The initial conditions used in the simulations always led to nets which converged to an isoradial net. We prove the convergence for rectangular nets on the torus for which the process reduces to a one-dimensional process of averaging points on a circle. It is shown by counter-example that the iteration process does not have an averaging character in general though.

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German Abstract. Diese Arbeit führt eine mögliche Verallgemeinerung der Kreismuster mit konstantem Radius ein (*generalized isoradial circle patterns*). Als charakterisierende Eigenschaft wird hierbei gewählt, dass die Zentren der Kreise erneut ein Kreismuster definieren. Wir beschränken uns auf den Fall von Kreismustern mit $\mathbb{Z}^2$-Kombinatorik und beschreiben diese vornehmlich durch die Schnittpunkte ihrer Kreise, also Abbildungen der Form $f : \mathbb{Z}^2 \to \mathbb{R}^2$, die wir als Netze bezeichnen.

Einleitend führen wir den Begriff des diskreten Systems und dessen multidimensionaler Kosistenz ein und untersuchen verschiedene Kriterien zur Charakterisierung von zirkulären und konischen Netzen in höheren Dimensionen, die wir daraufhin auf unsere verallgemeinerten isoradialen Netze anwenden können.

Weiterhin beweisen wir, dass ein Netz genau dann verallgemeinert isoradial ist, wenn seine Kanten parallel zu den Kanten eines isoradialen Netzes sind.

Das Netz der Kreiszentren eines verallgemeinerten isoradialen Netzes ist verallgemeinert isoradial, so dass wir diesen Prozess weiter iterieren können. Verallgemeinerte isoradiale Netze auf dem Torus lassen sich durch eine endliche Anzahl von Punkten komplett beschreiben und dienen damit als geeignete Grundlage für die Simulation dieses Prozesses. Mit allen verwendeten Anfangsbedingungen in den durchgeführten Simulationen war die Konvergenz gegen ein isoradiales Netz festzustellen. Wir beweisen die Konvergenz für den Fall von rechteckigen Netzen auf dem Torus, für die sich der Iterationsprozess auf das eindimensionale Problem der Mittelung von endlich vielen Punkten auf einem Kreis zurückführen lässt. Dass der Iterationsprozess jedoch im Allgemeinen keinen Mittelungsprozess darstellt, zeigen wir an Hand eines Gegenbeispiels.

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und eigenhändig sowie ohne unerlaubte fremde Hilfe und ausschließlich unter Verwendung der aufgeführten Quellen und Hilfsmittel angefertigt habe.

Berlin, den ....

Contents

1. Introduction

We introduce circle patterns, give the definition for generalized isoradial circle patterns and outline the path taken before coming back their study.

1.1. **Circle Patterns.** Let $\mathcal{G}$ be a *graph* which is a strongly regular polytopal cell-decomposition of a surface. Let $\mathcal{G}^*$ be its *dual graph* generated by choosing one point for each face of the original cell-decomposition $\mathcal{G}$. Connect each pair of vertices of $\mathcal{G}^*$ by an edge if and only if the corresponding faces in $\mathcal{G}$ are adjacent, i.e. separated by an edge. The vertices of $\mathcal{G}$ become the faces of $\mathcal{G}^*$ in such a way that the map which identifies the elements of $\mathcal{G}$ with the elements of $\mathcal{G}^*$ preserves adjacency. We also have $\mathcal{G}^{**} = \mathcal{G}$.

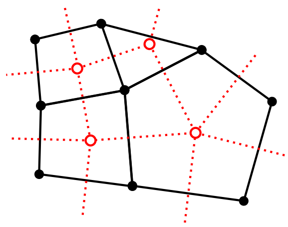

Figure 1.1. Part of a graph $\mathcal{G}$ (black) and its dual graph $\mathcal{G}^*$ (red).

Although the graph $\mathcal{G}$ actually consists of vertices, edges and faces, we write $V \in \mathcal{G}$ in the sense of V being a vertex of the graph $\mathcal{G}$. We also identify the faces of $\mathcal{G}$ with the vertices of $\mathcal{G}^*$ so that we can write $F \in \mathcal{G}^*$ to determine F to mean a face of $\mathcal{G}$.

A planar *circle pattern* is a map $C : \mathcal{G}^* \to \{\text{circles in } \mathbb{R}^2\}$ mapping faces of $\mathcal{G}$ to circles in the Euclidean plane such that each set of circles corresponding to faces adjacent to one vertex $V \in \mathcal{G}$ intersect in one point.

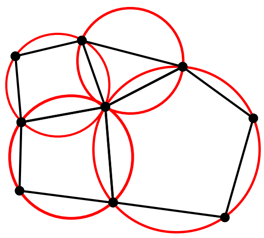

Figure 1.2. Part of a circle pattern (red circles) with corresponding circular net (black intersection points).

We will describe circle patterns by these intersection points as a map $f : \mathcal{G} \to \mathbb{R}^2$ which we call a *circular net* since each set of points corresponding to vertices adjacent to one face have to lie on a circle.
These two descriptions are dual to each other in the sense that they are equivalent and live on graphs which are dual to each other.

Let

$$\mathrm{CN}\,(\mathcal{G}) \coloneqq \{f : \mathcal{G} \to \mathbb{R}^2 \mid f \text{ circular net}\} \tag{1.1}$$

denote the set of all circular nets.
Given a circular net $f \in \mathrm{CN}\,(\mathcal{G})$ the centers of its circles are naturally assigned to $\mathcal{G}^*$ like the circles themselves. So we get a map $c(f) : \mathcal{G}^* \to \mathbb{R}^2$ which we call the *central*

net of f. This makes c a map on $\mathrm{CN}\,(\mathcal{G})$ into the set $\mathrm{N}(\mathcal{G}^*) \coloneqq \{f : \mathcal{G}^* \to \mathbb{R}^2 \mid f \text{ map}\}$ of nets on $\mathcal{G}^*$

$$c : \mathrm{CN}\,(\mathcal{G}) \to \mathrm{N}(\mathcal{G}^*) \tag{1.2}$$

1.2. **Generalized Isoradial Circle Patterns.** A circular net is called *isoradial* if all circles of its circle pattern have the same radius. For an isoradial net f we immediately see that its central net $c(f)$ defines an isoradial net itself.

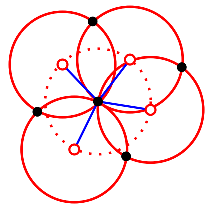

FIGURE 1.3. Vertex of an isoradial net. The distances (blue lines) of the centers of the adjacent circles to the vertex are equal.

Proposition 1.1 (centers of an isoradial circle pattern)**.** *The central net $c : \mathcal{G}^* \to \mathbb{R}^2$ of an isoradial net $f : \mathcal{G} \to \mathbb{R}^2$ is an isoradial net with same radius and central net f.*

Proof. Consider a vertex $V \in \mathcal{G}$ and the centers of the circles intersecting in $f(V)$. Then they all have the same distance to $f(V)$ since f is isoradial. See Figure 1.3. □

In particular, the central net of an isoradial net is a circular net itself which we will make our defining property for generalized isoradial circle patterns.

Definition 1.1 (generalized isoradial circle pattern)**.** Let $f : \mathcal{G} \to \mathbb{R}^2$ be a circular net. Then f is called *generalized isoradial* if its central net $c : \mathcal{G}^* \to \mathbb{R}^2$ is circular. Let $\mathrm{GI}\,(\mathcal{G})$ denote the set of all generalized isoradial nets on $\mathcal{G}$.

So the set of generalized isoradial nets is the preimage of the set of circular nets on the dual graph with respect to the map c, i.e.

$$\mathrm{GI}\,(\mathcal{G}) = c^{-1}\big(\mathrm{CN}\,(\mathcal{G}^*)\big) \tag{1.3}$$

We will see that the central net of a generalized isoradial net is itself generalized isoradial, i.e. its central map is again circular. This central net of the central net is again defined on the original graph $\mathcal{G}$ making the composition of the map c with itself a map on $\mathrm{GI}\,(\mathcal{G})$

$$c^2 : \mathrm{GI}\,(\mathcal{G}) \to \mathrm{GI}\,(\mathcal{G}) \tag{1.4}$$

which we can iterate further on. The isoradial nets are exactly the elements of $\mathrm{GI}\,(\mathcal{G})$ which are invariant with respect to c^2.
We will see that one is tempted to state the conjecture that iteration of this map for suitable initial generalized isoradial nets $f \in \mathrm{GI}\,(\mathcal{G})$ makes the radii of the circle pattern become equally distributed, i.e. the sequence $\big(c^{2n}(f)\big)_{n \in \mathbb{N}_0}$ converges to an isoradial net.

Giving emphasis to the vertices and edges of the map $f : \mathcal{G} \to \mathbb{R}^2$ rather than the circles by defining its edges in $\mathbb{R}^2$ to be the line-segments connecting neighboring points we end up with a polytopal cell-decomposition in $\mathbb{R}^2$ which is a (not necessarily embedded) realization of the graph $\mathcal{G}$ where each polygon has a circumcircle. We mainly restrict ourselves to circular patterns with square grid combinatorics, i.e. $\mathcal{G} = \mathbb{Z}^2$ making all the polytopes quadrilaterals. The conditions of circularity of the points and circularity of the centers can be imposed in the form of conditions on the angles between the introduced edges of the map $f : \mathbb{Z}^2 \to \mathbb{R}^2$. We will also see that the generalized isoradial nets $\mathrm{GI}\left(\mathbb{Z}^2\right)$ are exactly the circle patterns which are parallel to an isoradial net.[1]

Viewing circular nets with square grid combinatorics $f : \mathbb{Z}^2 \to \mathbb{R}^2$ as a special case of *circular nets* $f : \mathbb{Z}^m \to \mathbb{R}^N$ (where each four points adjacent to the same face are circular) we will explore the condition of circularity in the form of the symmetry with respect to certain reflections in this general setting.
In addition we will introduce *conical nets* in $\mathbb{R}^3$ which possess the same kind of symmetry as circular nets before eventually returning to the study of the main subject of generalized isoradial nets in the plane which we will see to incorporate properties of both these classes of nets.

Since this line of investigation belongs to the setting of *discrete differential geometry* we will first introduce the notion of a *discrete system* and its *multi-dimensional consistency* in the preliminaries.

[1]In the sense of parallel edges.

2. Preliminaries

The main part of this work deals with circle patterns in the plane with square grid combinatorics, which we describe by their points of intersection, i.e. maps of the form $f : \mathbb{Z}^2 \to \mathbb{R}^2$ which we call *nets*. Generalized isoradial nets have common properties with circular nets and conical nets which we will introduce in a more general multi-dimensional setting making them special cases of Q-nets and Q*-nets respectively.
Regarding the question of what kind of initial data is needed in the different settings and dimensions to describe one of these nets uniquely it is suitable to introduce the notion of discrete systems and their consistency which is done loosely following [BS09].

Notation 2.1. Let $m \in \mathbb{N}$, $f : \mathbb{Z}^m \to X$ a map to some set X.
For $i = 1, \dots, m$ we define translation operators

$$(\tau_i f)(u) := f(u + e_i) \quad \text{for } u \in \mathbb{Z}^m \tag{2.1}$$

where $e_i \in \mathbb{Z}^m$ denotes the i-th canonical basis vector. If X has a group structure we also define difference operators

$$(\delta_i f)(u) := (\tau_i f)(u) - f(u) \quad \text{for } u \in \mathbb{Z}^m. \tag{2.2}$$

We often write $f_i := \tau_i f$, $f_{ij} := \tau_i \tau_j f$ and use the symbols f, f_i, f_{ij} for the maps $\mathbb{Z}^m \to X$ as well as for a specific point $f(u)$, $f_i(u)$, $f_{ij}(u)$ for $u \in \mathbb{Z}^m$ if the meaning follows from the context.
We denote the coordinate d-planes and elementary d-dimensional cells of $\mathbb{Z}^m$ by

$$\begin{aligned} \mathcal{B}_{i_1 \dots i_d} &:= \{u \in \mathbb{Z}^m \mid u_i = 0 \text{ for } i \neq i_1, \dots, i_d\} \\ \mathcal{C}_{i_1 \dots i_d}(u) &:= \{u + \varepsilon_1 e_{i_1} + \dots + \varepsilon_d e_{i_d} \mid \varepsilon_1, \dots \varepsilon_d \in \{0, 1\}\} \text{ for } u \in \mathbb{Z}^m \end{aligned} \tag{2.3}$$

The m-dimensional cells $\mathcal{C}_{1,\dots,m}$ can be identified with elements of the dual[2] lattice $(\mathbb{Z}^m)^*$ which can be seen as $\mathbb{Z}^m$ translated by $\frac{1}{2}$ in all coordinate directions and therefore is isomorphic to $\mathbb{Z}^m$ itself.

$$(\mathbb{Z}^m)^* = \{u + \sum_{i=1}^{m} \tfrac{1}{2} e_i \mid u \in \mathbb{Z}^m\} \cong \mathbb{Z}^m \tag{2.4}$$

2.1. Discrete Systems.

Let $m \in \mathbb{N}$, X some set, called the *phase space*.

By a discrete system with X-valued fields on the vertices of $\mathbb{Z}^m$ we understand a set of admissible initial values $f_0 : U \to X$, $U \subset \mathbb{Z}^m$ and a set of rules which determine how to propagate these values throughout $\mathbb{Z}^m$, such that for any admissible initial values f_0 the values of all other fields are uniquely determined by these rules, i.e. there exists a unique map

$$f : \mathbb{Z}^m \to X \tag{2.5}$$

such that $f|_U = f_0$ and the values of f comply with the rules. f is then called the *solution* of the discrete system for the initial data f_0.

Remark 2.1. The notion of a *discrete system* can also be introduced on more general cell-decompositions and with fields on any elementary sub-cells (e.g. fields on edges instead of vertices).

Instead we will specialize the notion even more for our purposes and generate the global behavior of a discrete system by imposing rules locally on the elementary cells.

[2] Dual in the sense of *dual cell decomposition*.

Definition 2.1 (discrete mD-system)**.** A *discrete mD-system*[3] is a set of rules such that there is one rule[4] regarding all 2^m fields adjacent to each elementary m-dimensional cell $\mathcal{C} \in (\mathbb{Z}^m)^*$ such that for any given $2^m - 1$ values the 2^m-th value is uniquely determined.

So on each cell $\mathcal{C}$ the set of rules determines 2^m *maps of propagation*

$$\varphi^{\mathcal{C}}_{\pm 1,\dots,\pm m} : X^{2^m-1} \to X. \tag{2.6}$$

and for any suitable "$(m-1)$-dimensional" subset $\mathcal{B} \subset \mathbb{Z}^m$ (such as the union of the coordinate hyperplanes $\mathcal{B}_{i_1 \dots \hat{i}_k \dots i_m}$ or any translate of them[5]) any admissible initial values can be propagated through the whole lattice of $\mathbb{Z}^m$ using those maps.

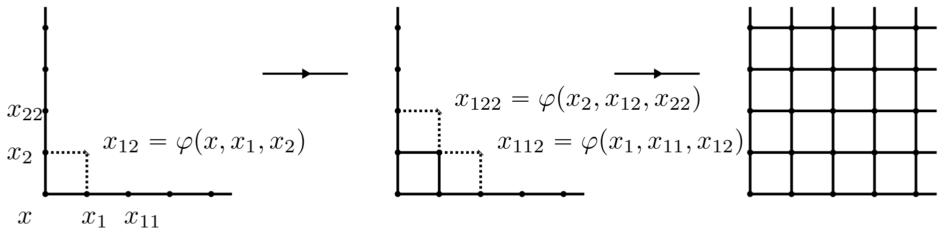

FIGURE 2.1. Propagation of some initial values of a discrete system on the coordinate lines of $\mathbb{Z}^2$ through the first quadrant with the same propagation map $\varphi = \varphi_{12}$ on all faces.

Remark 2.2 (compatibility conditions). The local solvability condition can also be stated in the following way. Any two sets of 2^m values adjacent to one elementary cell which agree in $2^m - 1$ values and satisfy the rule also agree in the 2^m-th value. This locally insures the uniqueness of the solution in a way that does not depend on where we choose our initial $2^m - 1$ values. When using the maps of propagation (2.6) this implies certain *compatibility conditions* for those maps (see also Example 2.1). On the global scale[6] this means that two solutions f_1 and f_2 satisfying the same set of local rules but coming from initial values on different subsets $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathbb{Z}^m$ are identical if they agree on $\mathcal{B}_1$ or $\mathcal{B}_2$ or any other suitable "$(m-1)$-dimensional" subset of $\mathbb{Z}^m$.[7] Since it does not matter where we take the initial values we excluded them from Definition 2.1 and implicitly understand them to be values on any suitable "$(m-1)$-dimensional" subset of $\mathbb{Z}^m$.[8]

In the following we restrict ourselves to the case of one and the same rule for each $(m-1)$-dimensional elementary cell of the lattice $\mathbb{Z}^m$.

Remark 2.3. We additionally get translational invariance of the solutions in this case, i.e. any initial data which are translates of each other give us solutions which are translates of each other.

Example 2.1 (quad equation)**.** Consider an equation

$$Q(x, x_1, x_2, x_{12}) = 0 \tag{2.7}$$

over some field X which can be solved for any of its four variables. Imposing this *quad-equation* onto all elementary quadrilaterals of $\mathbb{Z}^2$ induces a 2D-system in the sense of Definition 2.1.

[3]A more precise terminology would be "a local discrete m-dimensional system on $\mathbb{Z}^m$".

[4]Rule in the sense of a condition.

[5]Where $\hat{i}_k$ means that the index i_k is to be omitted.

[6]The transition from local to global is possible since we are dealing with the topology of $\mathbb{Z}^m$.

[7]Of course, if we want to get the same solution from initial values given at different positions, the values at these positions have to be different in general.

[8]Where suitable means that it ensures the existence and uniqueness of a solution.

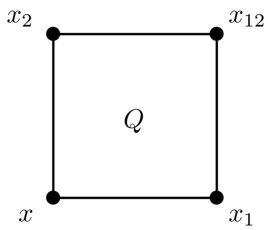

FIGURE 2.2. Elementary quadrilateral of $\mathbb{Z}^2$ with fields x, x_1, x_2, x_{12} on the vertices and equation $Q(x, x_1, x_2, x_{12}) = 0$ on the face.

It gives rise to four maps $\varphi_{12}, \varphi_{-12}, \varphi_{1-2}, \varphi_{-1-2} : X^3 \to X$ such that for four values $x, x_1, x_2, x_{12} \in X$ (2.7) is equivalent to satisfying one of the following equations

$$\begin{aligned} x &= \varphi_{-1-2}(x_{12}, x_2, x_1) \\ x_1 &= \varphi_{1-2}(x_2, x_{12}, x) \\ x_2 &= \varphi_{-12}(x_1, x, x_{12}) \\ x_{12} &= \varphi_{12}(x, x_1, x_2). \end{aligned} \tag{2.8}$$

So given any initial values $f|_{\mathcal{B}_1} \in X^{\mathcal{B}_1}$, $f|_{\mathcal{B}_2} \in X^{\mathcal{B}_2}$ on the coordinate axis we can use the map φ_{12} to propagate them through the first quadrant, φ_{-12} to propagate them through the second quadrant and so on until we get values $f \in X^{(\mathbb{Z}^2)}$ for all fields throughout the whole lattice satisfying

$$Q(f(u), \tau_1 f(u), \tau_2 f(u), \tau_{12} f(u)) = 0 \text{ for } u \in \mathbb{Z}^2. \tag{2.9}$$

The solvability for each variable, i.e. uniqueness of the solutions automatically imposes the compatibility conditions on our maps mentioned in Remark 2.2. E.g. for any $x, x_1, x_2, x_{12} \in X$ we have

$$x_{12} = \varphi_{12}(x, x_1, x_2) \Leftrightarrow x = \varphi_{-1-2}(x_{12}, x_2, x_1), \tag{2.10}$$

so

$$\varphi_{-1-2}(\varphi_{12}(x, x_1, x_2), x_2, x_1) = x \quad \text{for } x, x_1, x_2 \in X. \tag{2.11}$$

If we additionally assume our equation to be symmetric with respect to any permutation of the variables the propagation in any direction becomes the same, i.e. we only have one map $\varphi = \varphi_{12} = \varphi_{-12} = \varphi_{1-2} = \varphi_{-1-2}$ and end up with the situation depicted in Figure 2.1.

Since we have only one equation for all faces of $\mathbb{Z}^2$ we can easily add one dimension and impose this equation onto all faces of the 3-dimensional lattice $\mathbb{Z}^3$.

To answer the question whether this induces a 3D-system consider an elemen-

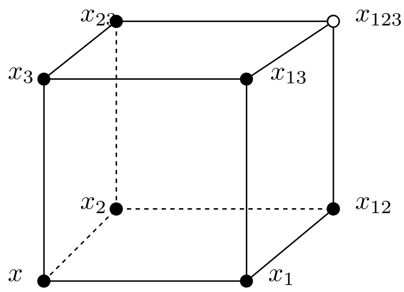

FIGURE 2.3. 3D-consistency of a 2D-system. The value for x_{123} has to be uniquely determined by the seven initial values.

tary cube $\mathcal{C}_{123}(u)$ of $\mathbb{Z}^3$. If we now start with four initial values x, x_1, x_2, x_3 as in Figure 2.3 we can calculate in a first step the values $x_{12} = \varphi(x, x_1, x_2)$, $x_{23} = \varphi(x, x_2, x_3)$, $x_{13} = \varphi(x, x_1, x_3)$ using the 2D-system and end up with 7 admissible initial values for our local 3D-system.
In a second step we now have 3 possibilities using our 2D-system to calculate the value of x_{123}, namely $\varphi(x_1, x_{12}, x_{13})$, $\varphi(x_2, x_{12}, x_{23})$, $\varphi(x_3, x_{13}, x_{23})$. To get a well-defined 3D-system these values should coincide and we end up with a 3D-consistency condition for our 2D-system:

$$\begin{aligned} \varphi(x_1, &\varphi(x, x_1, x_2), \varphi(x, x_1, x_3)) \\ &= \varphi(x_2, \varphi(x, x_1, x_2), \varphi(x, x_2, x_3)) \\ &= \varphi(x_3, \varphi(x, x_1, x_3), \varphi(x, x_2, x_3)) \end{aligned} \tag{2.12}$$

We don't have to stop here but rather go on asking the question about 4D-consistency of our new attained 3D-system as well. But this is automatically fulfilled as Theorem 2.1 states.

Definition 2.2 (consistency)**.** Impose an mD-system onto the m-dimensional cells of $\mathbb{Z}^{m+1}$. If this induces an $(m+1)$D-system for any admissible initial values[9] the mD-system is called $(m+1)$D-consistent.

Remark 2.4. As stated in Example 2.1 the $(m+1)$D-consistency can be checked by verifying that using any admissible initial data of the mD-system on an $(m+1)$-dimensional elementary cell produces the same values independent of along which path of m-dimensional elementary cells we propagate these values using the mD-system.

Theorem 2.1 (multi-dimensional consistency)**.** *An mD-system which is $(m+1)$D-consistent is also nD-consistent for all $n \geqslant m$.*

Proof. Proof by induction, see [BS09]. The proof there is given for $m = 3$. □

If an mD-system is multi-dimensional consistent it induces an nD-system for $n > m$ where the admissible initial data on the $(n-1)$-dimensional hyperplanes can be any solutions of the $(n-1)$D-system induced by the original mD-system.
Alternatively one can always use admissible values for the original mD-system on the m-dimensional coordinate planes as initial values for the nD-system since the consistency allows us to propagate them first to all $(n-1)$-dimensional coordinate planes.

Remark 2.5. The multi-dimensional consistency of an mD-system leads to permutability claims about its discrete transformations governed by the induced $(m+1)$D-system on $\mathbb{Z}^m \times \{0,1\} \subset \mathbb{Z}^{m+1}$. See [BS09].

2.2. **Q-nets.** We now introduce Q-nets as an example of how geometric constraints on a discrete net $f : \mathbb{Z}^m \to \mathbb{R}^N$ give rise to a discrete system. The notion of Q-nets actually belongs to projective geometry so we formulate it for $\mathbb{RP}^N$.

Definition 2.3 (Q-net)**.** Let $m \geqslant 2$, $N \geqslant 3$, $f : \mathbb{Z}^m \to \mathbb{RP}^N$.
Then f is called an m-dimensional *Q-net* (discrete conjugate net) in $\mathbb{RP}^N$ if for each $u \in \mathbb{Z}^m$ its elementary quadrilaterals (f, f_i, f_j, f_{ij}), $i, j = 1, \dots, m$, $i \neq j$ are coplanar.

[9] $2^{m+1} - 1$ initial values on an $(m+1)$-dimensional elementary cell are called admissible in this context if each 2^m or $2^m - 1$ values adjacent to an m-dimensional sub-cell are either a solution of the mD-system or admissible initial values for it.

In affine coordinates[10] this means $f_{ij} \in f + \operatorname{span}\{f_i, f_j\}$ or equivalently[11]

$$\delta_i \delta_j f \in \operatorname{span}\{\delta_i f, \delta_j f\}, \tag{2.13}$$

i.e.

$$\delta_i \delta_j f = c_{ji} \delta_i f + c_{ij} \delta_j f \tag{2.14}$$

for some $c_{ij}, c_{ji} \in \mathbb{R}$.

2.2.1. *Q-surfaces.* For $m = 2$ the Q-net condition by itself does not induce a 2D-system for initial data $f|_{\mathcal{B}_1},\ f|_{\mathcal{B}_2}$ on the vertices of the coordinate lines. Given three points of a face there are still two degrees of freedom to determine a fourth point on the plane spanned by them.
So, for example, additionally prescribing c_{12} and c_{21} in (2.14) on all faces of $\mathbb{Z}^2$ determines the whole Q-surface $f : \mathbb{Z}^2 \to \mathbb{R}^N$ uniquely.

Remark 2.6. Consider a parametrized continuous surface $f : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \mapsto f(x, y)$ with Gauss-map $N : \mathbb{R}^2 \to \mathbb{S}^2$. Then[12]

$$\begin{aligned} f \text{ conjugate line parametrization} &\Leftrightarrow \langle f_x, \mathrm{d}N(f_y)\rangle = 0 \\ &\qquad \text{i.e. the second fundamental form is diagonal} \\ &\Leftrightarrow \langle f_{xy}, N\rangle = 0 \\ &\Leftrightarrow f_{xy} \in \operatorname{span}\{f_x, f_y\} \end{aligned}$$

So condition (2.14) resembles the condition for conjugate line parametrizations in the case of discrete surfaces.

2.2.2. *Q*-nets.*

Definition 2.4 (Q*-net)**.** A net $f : \mathbb{Z}^m \to \{\text{planes in } \mathbb{RP}^3\}$ is called Q*-net if each four planes adjacent to one face intersect in one point.

Example 2.2. The planes of a Q-surface $f : \mathbb{Z}^2 \to \mathbb{RP}^3$ can be assigned to the faces of $\mathbb{Z}^2$ or equivalently to the vertices of the dual lattice $(\mathbb{Z}^2)^*$. This gives rise to a map $f^* : (\mathbb{Z}^2)^* \to (\mathbb{RP}^3)^*$ where planes of $\mathbb{RP}^3$ are identified with points in the dual space $(\mathbb{RP}^3)^*$. The statement of four planes in $\mathbb{RP}^3$ meeting in one point is dual to the statement of four points in $(\mathbb{RP}^3)^*$ lying on a plane. So the Q*-net f^* is a Q-net in $(\mathbb{RP}^3)^*$.

2.2.3. *Basic 3D-system and consistency.*

Theorem 2.2. *The system governing Q-nets is a discrete 3D-system.*[13]

Proof. Consider an elementary cube of $\mathbb{Z}^3$ with seven admissible initial values f, f_i, f_{ij}. The Q-net condition implies that f_{123} has to lie on the three planes $\tau_i \Pi_{jk} := f_i + \operatorname{span}\{\delta_j f_i, \delta_k f_i\}$. Since three planes in $\mathbb{RP}^3$ always intersect in one point, f_{123} is uniquely determined. □

Remark 2.7. We silently assumed the data to be in general position to avoid degenerate cases where some of the planes coincide. We always assume this kind of *generic* data in the following without mentioning it explicitly.

---

[10]We denote the affine coordinates $g \in \mathbb{R}^N$ such that $(g, 1)$ are some homogeneous coordinates for f by the same symbol f here.

[11]Note that $\delta_i \delta_j = f_{ij} - f_i - f_j - f \neq f_{ij} - f$.

[12]Where $f_x = \frac{\partial f}{\partial x}$ denotes the partial derivative of f with respect to x and $\mathrm{d}N$ the differential of N.

[13]i.e. admissible initial data on the coordinate planes determines a whole Q-net uniquely.

Remark 2.8. The case of $N = 2$ where everything lies in one plane has been excluded from our definition of Q-nets since it obviously does not lead to a 3D-system. Studying properties of projections of Q-nets in higher dimensions to the plane allows to find alternative characterizing properties of Q-nets which are also applicable to the plane, see [BS09].

Theorem 2.3. *The 3D-system governing Q-nets is 4D-consistent.*

Proof. See [BS09]. □

So from Theorem 2.1 follows that it is also mD-consistent for $m \geqslant 4$.

Remark 2.9. 3-dimensional Q-nets $f : \mathbb{Z}^3 \to \mathbb{R}^N$ can be viewed as a family of Q-surfaces where each two consecutive surfaces are connected by an *F-transform*, i.e. a transformation for which each two neighboring points and its images are coplanar. In this interpretation the statement that Q-nets build a 3D-system is a statement about the uniqueness of such transformations and the 4D-consistency a statement about their permutability.

3. Circular nets

The introduction of circular nets follows [BS09] until we introduce our concept of defining the angles of a quadrilateral.

Definition 3.1 (Circular nets)**.** Let $m \geqslant 2$, $N \geqslant 2$, $f : \mathbb{Z}^m \to \mathbb{R}^N$.
Then f is called an m-dimensional discrete *circular net* in $\mathbb{R}^N$ if all its elementary quadrilaterals (f, f_i, f_{ij}, f_j), $i, j = 1, \ldots, m$, $i \neq j$ are circular.

Notation 3.1. Denote by $C : (\mathbb{Z}^2)^* \to \{\text{circles in } \mathbb{R}^N\}$ the circumcircles of f and by $c : (\mathbb{Z}^2)^* \to \mathbb{R}^N$ their midpoints, where each circle and midpoint is associated to an elementary quad of $\mathbb{Z}^m$, i.e. a point of $(\mathbb{Z}^m)^*$. We call c the *central net* of f or its circular centers.

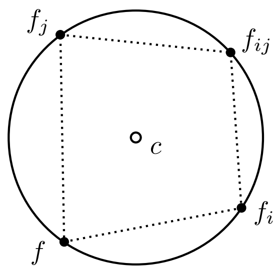

Figure 3.1. Elementary quadrilateral of a circular net with circumcircle.

Remark 3.1. Four points lying on a circle are planar. So in the case $N \geqslant 3$ circular nets are special Q-nets. The case of $N = 2$ is the case of circle patterns in the plane.

Remark 3.2. This definition belongs to Möbius geometry. So one can use $\mathbb{R}^N \cup \{\infty\}$ instead of $\mathbb{R}^N$.
Using the projective model of Möbius geometry, i.e. the quadric $\mathrm{P}(L^{N+1})$ which is the projectivized light cone in the projectivized Lorentz-space $\mathrm{P}(\mathbb{R}^{N+1,1})$, one can treat circular nets as Q-nets restricted to a quadric. See [BS09].

3.1. **Circular surfaces.** For $m = 2$ consider three points of an elementary quad of a circular net. Then there is one degree of freedom when choosing the fourth point on the circle passing through the three points. So the circularity condition reduces the degrees of freedom in comparison to the planarity condition of Q-nets by one. For initial values $f|_{\mathcal{B}_1}$, $f|_{\mathcal{B}_2}$ on the coordinate lines one can, for example, prescribe real values for the cross-ratio on each face using Proposition 3.1 for determining a whole net.
The cross-ratio of four points z_1, z_2, z_3, z_4 in the complex plane which is identified with the affine 2-dimensional subspace of the quad in $\mathbb{R}^N$ is understood to be

$$\mathrm{cr}(z_1, z_2, z_3, z_4) := \frac{z_1 - z_2}{z_2 - z_3} \frac{z_3 - z_4}{z_4 - z_1} \tag{3.1}$$

Proposition 3.1. *Let $z_1, z_2, z_3, z_4 \in \mathbb{C} \cong \mathbb{R}^2$ be four distinct points in the plane. Then*

$$(z_1, z_2, z_3, z_4) \text{ circular} \Leftrightarrow q := cr(z_1, z_2, z_3, z_4) \in \mathbb{R}\backslash\{0, 1\}$$

Moreover if (z_1, z_2, z_3, z_4) is circular, then

$$(z_1, z_2, z_3, z_4) \text{ embedded}^{14} \Leftrightarrow q < 0$$

[14] Meaning that z_1, z_2, z_3, z_4 are in cyclic order on the circle or equivalently that z_1 and z_3 are separated by z_2 and z_4.

Proof. Map z_2, z_3, z_4 by the Möbius transformation $m := \mathrm{cr}(\bullet, z_2, z_3, z_4)$ to $0, 1, \infty$. So $q = m(z_1)$. Since z_1 lies on the circle through z_2, z_3 and z_4, $q = m(z_1)$ lies on the line through $0, 1, \infty$, i.e. $q \in \mathbb{R}$.
The rest follows from comparing the order of the points on the circle to its images on the real line. □

Consider the case $N = 3$ of a discrete circular surface in $\mathbb{R}^3$. For each circle there is a unique line through its center orthogonal to the plane of the circle. Denote those lines by $l : (\mathbb{Z}^2)^* \to \{\text{lines in } \mathbb{R}^3\}$. They can be interpreted as the normal field of our discrete surface and possess the following property.

Proposition 3.2. *l is a line congruence, i.e. any two neighboring lines intersect.*

Proof. For any $u \in (\mathbb{Z}^2)^*$ and direction $i = 1, 2$ consider the two neighboring lines l and l_i. The corresponding circles C and C_i intersect in two points and therefore lie on a 2-dimensional sphere. So l and l_i both pass through the center of the sphere. □

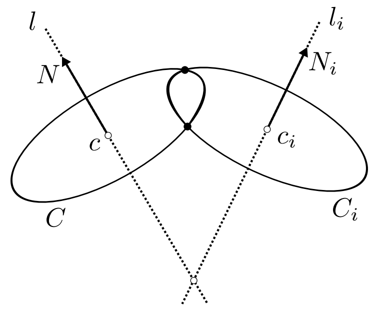

FIGURE 3.2. Two adjacent circles of a circular net. The normal lines intersect in a focal point.

Remark 3.3. These lines also exist for $N > 3$. In this case they are characterized by going through the center of the 2-sphere and the center of the circle, rather than the orthogonality condition.

Denoting by $N : (\mathbb{Z}^2)^* \to \mathbb{S}^2$ some chosen normal vectors on l we can reformulate Proposition 3.2 as

$$\delta_i N \in \operatorname{span}\{N, \delta_i c\}. \tag{3.2}$$

Remark 3.4. Consider a conjugate line parametrization of a continuous surface $f : \mathbb{R}^2 \to \mathbb{R}^3$, with Gauss-map $N : \mathbb{R}^2 \to \mathbb{R}^3$. Then

$$f \text{ curvature line parametrization} \Leftrightarrow \langle f_x, f_y \rangle = 0$$
$$\text{i.e. the first fundamental form is diagonal}$$

Since f_x and f_y are conjugated, i.e. $\langle N_x, f_y \rangle = \langle N_y, f_x \rangle = 0$ we even get

$$\begin{aligned} \langle f_x, f_y \rangle = 0 &\Leftrightarrow N_x \in \operatorname{span}\{N, f_x\} \\ &\Leftrightarrow N_y \in \operatorname{span}\{N, f_y\} \end{aligned}$$

So while the planarity condition of a Q-net resembles the property of a conjugate line parametrization, the circularity condition resembles the additional property of a curvature line parametrization, i.e. being orthogonal.

Going along a strip of a circular net from face to face crossing opposite edges corresponds to going along a curvature line. The radius r of the sphere containing

two adjacent circles can be interpreted as the discrete principal curvature $\frac{1}{r}$ in that direction which is naturally assigned to the edges one crosses along the strip.
The centers of the spheres can be interpreted as *discrete focal points* of that direction. Joining all focal points for one direction into a new net gives rise to a discrete focal net (as described in [PW08]) which can easily seen to be a Q-net.

3.2. **Basic 3D-system and consistency.** Circular nets are Q-nets and we already know that for any seven points f, f_i, f_{ij} on an elementary cube of $\mathbb{Z}^3$ which fulfill the Q-net condition, the eighth point f_{123} is uniquely determined as the intersection of three planes. Now the question is whether the circularity condition is compatible with this.

Theorem 3.3. *The system governing circular nets is a discrete 3D-system.*

Proof. Suppose $N \geqslant 3$ at first.
Let f, f_i, f_{ij}, $i, j = 1, 2, 3$, $i \neq j$ be seven points in $\mathbb{R}^N$ such that each of the three elementary quadrilaterals (f, f_i, f_{ij}, f_j) is circular. We have to show that the three circles $\tau_i C_{jk}$ through f_i, f_{ij}, f_{ik} intersect.
The four points f, f_i lie on a 2-dimensional sphere. This sphere contains the circles through each three of those points so in particular the three points f_{ij}. So all considered points and circles lie on a sphere which can be stereographically projected to a plane.
For $N = 2$ we start at this point.
Now mapping the point f to ∞ by a Möbius-transformation[15] makes the claim equivalent to Miquel's theorem.
See, for example, [BS09]. □

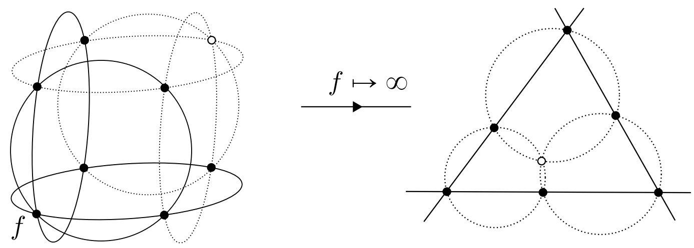

FIGURE 3.3. Elementary cube of a circular net. Map f to ∞ and use Miquel's theorem to see that circular nets build a 3D-system.

For $N \geqslant 3$ and generic data the eighth point of an elementary cell determined in the proof is in particular the intersection point of the three planes the circles $\tau_i C_{jk}$ lie in, i.e. the point determined by the 3D-system governing Q-nets.
So having circular initial data $f|_{\mathcal{B}_{ij}}$ on the 2-dimensional coordinate planes of $\mathbb{Z}^m$ we can use the system governing Q-nets to propagate it through the whole lattice, while Theorem 3.4 assures that the circularity condition is preserved.
So the multidimensional consistency of Q-nets implies the multidimensional consistency of circular nets for $N \geqslant 3$.
In the case of $N = 2$ the plane can be stereographically projected to a 2-sphere lying in $\mathbb{R}^3$. The circles in the plane become circles on the sphere. So in particular the net becomes a Q-net in $\mathbb{R}^3$ and the consistency follows from the consistency for $N = 3$.

Theorem 3.4. *The 3D-system governing circular nets is multi-dimensional consistent.*

[15] In the case of $N \geqslant 3$ this is equivalent to directly projecting stereographically with f as north-pole.

3.3. **Concerning angles.** For any of the following sections involving angles one has to be careful which angles to take, especially when dealing with non-embedded and clockwise oriented quads.

For three points $z_1, z_2, z_3 \in \mathbb{C}$ in the plane we define an angle

$$\sphericalangle(z_1, z_2, z_3) \coloneqq \arccos \frac{\langle z_1 - z_2, z_3 - z_2 \rangle}{|z_1 - z_2||z_3 - z_2|} \in (0, \pi) \tag{3.3}$$

Now for two distinct lines l_1, l_2 in the plane intersecting at a point we have to decide between two possible choices. We denote by $\sphericalangle(l_1, l_2)$ the following angle: Take any point on l_1 and go to the left around the point of intersection until you first meet l_2.

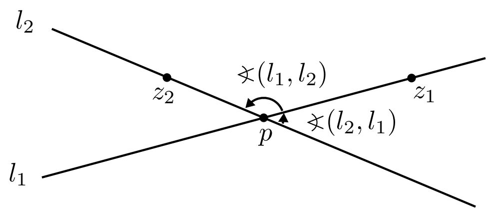

FIGURE 3.4. Convention for angles between two lines.

We state this in the following formal way.

Definition 3.2 (angle between lines)**.** Let l_1, l_2 be two distinct lines in the plane intersecting at $p \in \mathbb{C}$. Let $z_1 \in l_1$ and $z_2 \in l_2$ such that $\operatorname{Im} \frac{z_2}{z_1} > 0$. Then

$$\sphericalangle(l_1, l_2) \coloneqq \sphericalangle(z_1, p, z_2) \tag{3.4}$$

It immediately follows that $\sphericalangle(l_1, l_2) \in (0, \pi)$ and for any lines b_1 and b_2 perpendicular to l_1 and l_2 respectively we get

$$\sphericalangle(l_1, l_2) = \sphericalangle(b_1, b_2) \tag{3.5}$$

$$\sphericalangle(l_1, l_2) + \sphericalangle(b_2, b_1) = \pi \tag{3.6}$$

In fact rotating l_1 and l_2 by any common angle will yield these results.

Remark 3.5. For line segments $[z_1, z_2] \subset \mathbb{C}$ we define the angles by the lines containing the segments. Especially for two adjacent edges $[z_1, z_2]$ and $[z_2, z_3]$ this means that in general $\sphericalangle([z_1, z_2], [z_2, z_3]) \neq \sphericalangle(z_1, z_2, z_3)$ but they are either equal or supplementary.

We now introduce a convention on how we want to label angles in quadrilaterals and planar vertex stars.

Notation 3.2 (quad-angles). For a quadrilateral with vertices $z_1, z_2, z_3, z_4 \in \mathbb{C}$ we denote its four *quad-angles* by $\alpha, \beta, \gamma, \delta$ and define them to be the angles between the lines $l_i \coloneqq (z_i z_{i+1})$ through its edges, i.e.

$$\begin{aligned} \alpha &= \measuredangle(l_1, l_4) \quad & \beta &= \measuredangle(l_2, l_1) \\ \gamma &= \measuredangle(l_3, l_2) \quad & \delta &= \measuredangle(l_4, l_3) \end{aligned} \tag{3.7}$$

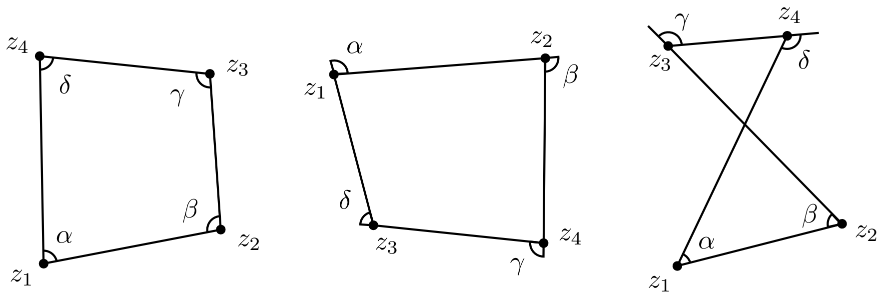

FIGURE 3.5. Convention for measuring angles in a quadrilateral as the angles between the lines containing their edges. For an embedded quad of positive and negative orientation on the left and a non-embedded quad on the right.

Note that enumerating the vertices induces an orientation for the quad. The two possible orientations carry supplementary angles. We call an embedded quad with a clockwise orientation *flipped*.

Notation 3.3 (vertex-angles). Let $w, w_1, w_2, w_3, w_4 \in \mathbb{C}$ be a planar vertex star consisting of the edges $[w, w_i]$ which are contained in the lines $b_i \coloneqq (ww_i)$. Then we denote its *vertex-angles* by $\alpha, \beta, \gamma, \delta$ and define them to be the angles between the lines $b_i \coloneqq (ww_i)$, i.e.

$$\begin{aligned} \alpha &= \measuredangle(b_2, b_3) \quad & \beta &= \measuredangle(b_3, b_4) \\ \gamma &= \measuredangle(b_4, b_1) \quad & \delta &= \measuredangle(b_1, b_2) \end{aligned} \tag{3.8}$$

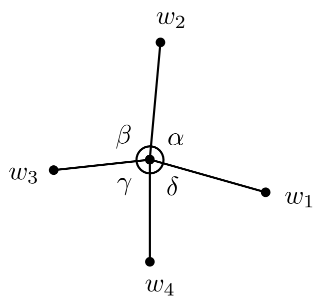

FIGURE 3.6. Convention for labeling angles around a vertex star.

We will later adapt this convention for labeling all angles in a discrete net $f : \mathbb{Z}^2 \to \mathbb{R}^2$ by indexing $\alpha, \beta, \gamma, \delta$ by the quadrilaterals $u \in (\mathbb{Z}^2)^*$ or vertices $v \in \mathbb{Z}^2$.

If dealing with a vertex star inside a quadrilateral as in the next chapter we will equip the angles of the vertex star with tildes.

3.4. **Reflection and angle criteria.** Beside the cross-ratio criterion of Proposition 3.1 we now state two additional criteria for the circularity of planar quadrilaterals which can be used to characterize circular nets.

Proposition 3.5 (Reflection criterion for circular quads)**.** *Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ be four points in the plane.*
Then z_1, z_2, z_3, z_4 lie on a circle if and only if there are four lines b_i intersecting in one point,

$$\{c\} := \bigcap_{i=1}^{4} b_i \neq \varnothing, \tag{3.9}$$

such that for the reflections R_i in the lines b_i

$$z_i = R_{i-1}(z_{i-1}) \qquad \textit{for } i = 1, \ldots, 4.^{16} \tag{3.10}$$

In this case b_i is the bisector of the segment $[z_i, z_{i+1}]$, c the center of the circle and

$$R_4 \circ R_3 \circ R_2 \circ R_1 = id \tag{3.11}$$

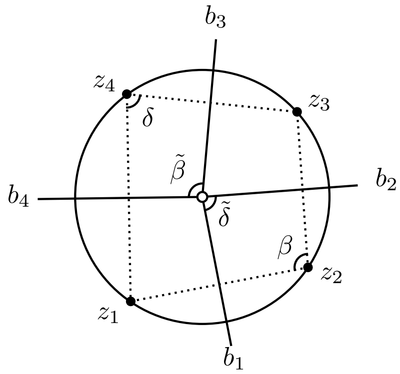

FIGURE 3.7. The vertices of a circular quad are symmetric with respect to reflections in the bisectors of its edges.

Proof. "$\Rightarrow$" Choose b_i to be the bisectors of $[z_i, z_{i+1}]$ for $i = 1, \ldots, 4$ respectively. Then they all go through the center c of the circle and the corresponding reflections R_i fulfill (3.10).
"$\Leftarrow$" A reflection R_i leaves the distance to all points on the line b_i invariant. Since all the lines intersect in one point c (3.10) implies

$$\|z_i - c\| = \|z_j - c\| \qquad \text{for} \quad i, j = 1, \ldots, 4$$

and that b_i bisects $[z_i, z_{i+1}]$.

We still have to prove (3.11).
The composition of two successive reflections is a rotation around c by the double angle between the reflection lines. W.l.o.g $c = 0$. Let $\tilde{\delta} := \measuredangle(b_1, b_2)$, $\tilde{\beta} := \measuredangle(b_3, b_4)$ be the angles between the bisectors. Then

$$\begin{aligned} R_2 \circ R_1 &: z \mapsto e^{2i\tilde{\delta}} z \\ R_4 \circ R_3 &: z \mapsto e^{2i\tilde{\beta}} z.^{17} \end{aligned} \tag{3.12}$$

We also know

$$z_1 = R_4 \circ R_3 \circ R_2 \circ R_1(z_1) = e^{2i(\tilde{\beta}+\tilde{\delta})} z_1$$

[16] Where indices are to be taken modulo 4.

So

$$e^{2i(\tilde{\beta}+\tilde{\delta})} = 1 \tag{3.13}$$

and therefore $R_4 \circ R_3 \circ R_2 \circ R_1 = \mathrm{id}$.
We additionally see that $\tilde{\beta} + \tilde{\delta} = \pi$ since $\tilde{\beta}, \tilde{\delta} \in (0, \pi)$. □

Remark 3.6. The reflection principle can easily be generalized to an arbitrary number of points on a circle.

Corollary 3.6. *Let $x_1, x_2, x_3, x_4 \in \mathbb{R}^N$ be four points on a circle. Then the reflections R_i in the hyperplanes bisecting the edges $[z_i, z_{i+1}]$ fulfill*

$$R_4 \circ R_3 \circ R_2 \circ R_1 = id \tag{3.14}$$

Given a circular net f we assign its bisecting hyperplanes to the edges of $\mathbb{Z}^m$. The corresponding reflections adjacent to one vertex fulfill (3.14). We call this the *symmetry net of f* which motivates the following general definition.

Definition 3.3 (symmetry net)**.** Let $\mathrm{E}(\mathbb{Z}^m)$ be the set of unoriented edges of $\mathbb{Z}^m$. A map $b : \mathrm{E}(\mathbb{Z}^m) \to \{\text{hyperplanes in } \mathbb{R}^N\}$ is called a *symmetry net* if and only if for each four edges the corresponding hyperplanes b_1, b_2, b_3, b_4 intersect in an $(N-2)$-dimensional affine subspace and the four reflections R_i in b_i fulfill (3.14).

Remark 3.7. Interpreted as a Q*-net $b^* : E(\mathbb{Z}^2) \to (\mathbb{RP}^N)^*$ the incidence condition dual to b_1, b_2, b_3, b_4 intersecting in a $(N-2)$-dimensional subspace becomes $b_1^*, b_2^*, b_3^*, b_4^*$ lying on a line.

Using Definition 3.3 and the higher-dimensional implications of Proposition 3.5 we can now formulate a criterion for circular nets in the following way.

Proposition 3.7 (reflection criterion for circular nets)**.** *Let $f : \mathbb{Z}^m \to \mathbb{R}^N$ be a net.*
Then f is circular if and only if it possesses a reflection net, i.e. there is a reflection net $b : E(\mathbb{Z}^m) \to \{\text{hyperplanes in } \mathbb{R}^N\}$ such that each two adjacent vertices of f are symmetric with respect to the reflection corresponding to the edge joining them.

$$R_{(u,u+e_i)}(f(u)) = \tau_i f(u) \qquad \text{for } u \in \mathbb{Z}^m, i = 1, \dots, 4 \tag{3.15}$$

A circular net uniquely defines a corresponding reflection net.
Given a reflection net on the other hand, we get an N-parameter family of corresponding circular nets by choosing one point $x \in \mathbb{R}^N$ and successively reflecting it throughout the lattice $\mathbb{Z}^m$. The incidence condition of four adjacent hyperplanes intersecting in an $(N-2)$-dimensional subspace assures that the corresponding reflections can be restricted to the 2-dimensional orthogonal complements of the intersection while Proposition 3.5 together with (3.14) assures that each four points adjacent to one face lie on a circle as well as that the construction does not conclude in any contradictions.

Proposition 3.8 (angle-criterion for the reflection condition)**.** *Four hyperplanes b_1, b_2, b_3, b_4 intersecting in an $(N-2)$-dimensional affine subspace fulfill (3.14) if and only if the angles $\tilde{\delta} = \sphericalangle(b_1, b_2)$ and $\tilde{\beta} = \sphericalangle(b_3, b_4)$ fulfill*

$$\tilde{\beta} + \tilde{\delta} = \pi \tag{3.16}$$

Notation 3.4. For two hyperplanes b_1 and b_2 intersecting in an $(N-2)$-dimensional affine subspace U we define their angle $\sphericalangle(b_1, b_2)$ to be the angle between the lines $b_1 \cap U^\perp$ and $b_2 \cap U^\perp$ for some 2-dimensional plane $U^\perp$ orthogonal to U. This definition is independent of the choice of $U^\perp$.

Proof. Reducing the problem to any plane orthogonal to the $(N-2)$-dimensional intersection we see that we have already proven this angle-condition in the proof of Proposition 3.5. $\square$

So Proposition 3.8 immediately gives an angle criterion for symmetry nets and additionally for circular nets as can be seen in the next proposition.

Proposition 3.9 (angle criterion for circular quads). *A quadrilateral with vertices $z_1, z_2, z_3, z_4 \in \mathbb{C}$ and quad-angles $\alpha, \beta, \gamma, \delta$ is circular if and only if opposite angles are supplementary, i.e*

$$\alpha + \gamma = \pi \tag{3.17}$$

or equivalently

$$\beta + \delta = \pi \tag{3.18}$$

Proof. "$\Rightarrow$" Let z_1, z_2, z_3, z_4 lie on a circle. Let $\tilde{\delta} = \measuredangle(b_1, b_2)$ and $\tilde{\beta} = \measuredangle(b_3, b_4)$ denote the angles between the orthogonal bisectors b_i as in Proposition 3.5, see Figure 3.4. So $\beta + \tilde{\delta} = \delta + \tilde{\beta} = \pi$. From (3.13) we get $\tilde{\beta} + \tilde{\delta} = \pi$. and therefore also for the supplementary angles $\beta + \delta = \pi$.
"$\Leftarrow$" Define b_1, b_2 to be the orthogonal bisecting lines of $[z_1, z_2]$, $[z_2, z_3]$ respectively. b_1 and b_2 meet in some point c. W.l.o.g. $c = 0$. Now let b_3 and b_4 be the lines through the origin perpendicular to $(z_3 z_4)$ and $(z_4 z_1)$ respectively. So

$$\begin{aligned} \tilde{\delta} &\coloneqq \measuredangle(b_1, b_2) = \beta - \pi \\ \tilde{\beta} &\coloneqq \measuredangle(b_3, b_4) = \delta - \pi \end{aligned} \tag{3.19}$$

Consider the reflections R_i in b_i, $i = 1, \ldots, 4$. Then

$$\begin{aligned} \beta + \delta = \pi \Rightarrow \tilde{\beta} + \tilde{\delta} &= \pi \\ &\Rightarrow R_4 \circ R_3 \circ R_2 \circ R_1 = id \\ &\Leftrightarrow R_3 \circ R_1 = R_4 \circ R_1 \end{aligned} \tag{3.20}$$

since the R_i are involutions. We get

$$\begin{aligned} (z_3 z_4) \ni R_3(z_3) &= R_3 \circ R_2(z_2) \\ &= R_4 \circ R_1(z_2) = R_4(z_1) \in (z_4 z_1). \end{aligned} \tag{3.21}$$

So $R_4(z_1) = (z_3 z_4) \cap (z_4 z_1) = z_4$ and therefore $z_i = R_{i-1}(z_{i-1})$ for all $i = 1, \ldots, 4$. From Proposition 3.5 follows that z_1, z_2, z_3, z_4 lie on a circle. $\square$

Remark 3.8. This proof being based on the symmetry criterion makes it immediately obvious which angles to use in the various cases of embedded and non-embedded quads. The symmetry criterion helps identifying them.

4. Conical nets

We will now consider another class of nets which possesses the same kind of symmetry. Indeed given a symmetry net in $\mathbb{R}^3$ as defined in Definition 3.3 we can take a plane and assign it to any vertex of $\mathbb{Z}^m$. We then use the reflections to propagate this plane throughout the whole lattice constructing a Q*-net. We will see that this procedure actually yields a conical net as defined below.
If we now take any point in one of the planes and reflect it throughout the lattice we get a circular net lying on the conical net. Both share the same symmetry net and together define a contact element net.

The concept of introducing and characterizing conical nets is adapted and put together from [BS09], [PW08] and [WWL07].

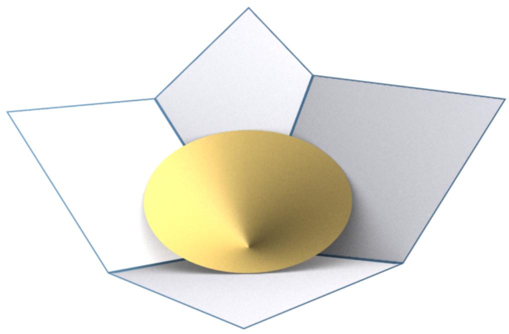

Figure 4.1. Conical vertex. Four planes tangent to a cone.

Definition 4.1 (conical net)**.** A net $f : \mathbb{Z}^m \to \{\text{planes in } \mathbb{R}^3\}$ is called conical if each four planes adjacent to one face of $\mathbb{Z}^m$ are in contact to a cone of revolution.

Remark 4.1. Four planes in contact to a cone intersect in one point. So conical nets are a special class of Q*-nets.

Proposition 4.1. *Let P_1, P_2, P_3, P_4 be four planes in $\mathbb{R}^3$ intersecting in one common point. Then they are tangent to a common cone if and only if they are tangent to a common sphere.*

Remark 4.2. Unlike the cone the sphere is not unique but one element of a one-parameter family of spheres with centers on the axis of the cone.

Proof. "$\Rightarrow$" Each of the planes touches the cone K in a line $l_i \coloneqq P_i \cap K$. Let S be a sphere with center on the axis of the cone touching the cone in a circle C. Then the line l_i is tangent to the sphere S and the plane P_i touches the sphere S in the point $l_i \cap C$.

"$\Leftarrow$" Let S be the sphere with center s tangent to each of the four planes. Let x be the point of intersection of the four planes and x_i the touching point of the plane P_i and the sphere S. Define the lines $l_i \coloneqq (xx_i)$. Since they all go through the point x and are all tangent to the sphere S, they all lie on the cone of revolution defined by rotating one of these lines around the axis (xs). $\square$

So for a Q*-net the condition of tangency to a cone at each face is equivalent to the tangency to a sphere.

4.1. **Reflection and angle criteria.** Let P_1, P_2 be two planes tangent to a sphere S with center s. Then the symmetry plane transforming these two planes into each other by reflection is the plane going through the line $P_1 \cap P_2$ and the center of the sphere s.

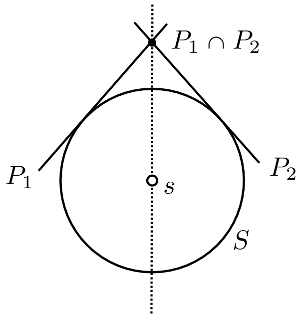

FIGURE 4.2. Two planes P_1 and P_2 tangent to a sphere S and their symmetry plane. View orthogonal to their line of intersection $P_1 \cap P_2$.

Considering four planes P_1, P_2, P_3, P_4 with a common point x and tangent to S all those symmetry planes contain the points x and s. They therefore share a common line (xs) which is the axis of the tangent cone.

Proposition 4.2 (reflection criterion for conical nets)**.** *Let $f : \mathbb{Z}^m \to \{planes\ in\ \mathbb{R}^3\}$ be some net of planes.*
Then f is conical if and only if it possesses a reflection net, i.e. there is a symmetry net $b : E(\mathbb{Z}^m) \to \{planes\ in\ \mathbb{R}^3\}$ such that each two adjacent planes of f are symmetric with respect to the reflection corresponding to the edge joining them.

Proof. We only have to prove this locally. So let P_1, P_2, P_3, P_4 be four planes in $\mathbb{R}^3$ adjacent to one face of $\mathbb{Z}^m$.

"$\Rightarrow$" has already been shown above.

"$\Leftarrow$" Let b_1, b_2, b_3, b_4 be four planes intersecting in a line a with corresponding reflections R_i satisfying $P_i = R_{i-1}(P_{i-1})$.[18]. This implies that the four planes have a common point x lying on a. Prescribing the axis of revolution a and tangency to the plane P_1 uniquely defines a cone K with its tip at x. Reflecting P_1 in a plane containing the axis a preserves tangency to that cone. So all four planes are tangent to K. □

Let P_1, P_2, P_3, P_4 be four planes in contact to a cone K. Intersecting everything with a plane Π orthogonal to the cone axis results in four lines $l_i \coloneqq P_i \cap \Pi$ tangent to a circle $C \coloneqq K \cap \Pi$.[19]
We see that the reflection condition for conical nets can be seen as a 3-dimensional version of the reflection condition on this 2-dimensional setting which is similar to Proposition 3.5 but concerning quadrilaterals with an incircle rather than a circumcircle.

[18]As we know this already implies the reflection identity (3.11)
[19]Assuming Π does not contain the tip of the cone.

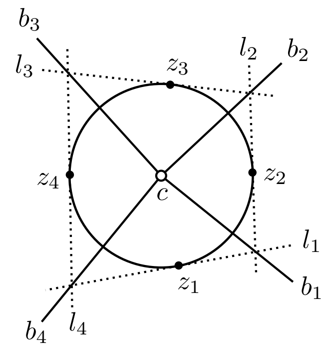

FIGURE 4.3. Four lines l_i tangent to a circle, i.e. a tangential quadrilateral of a circle. b_i are the symmetry lines of the edges, z_i the touching points on the incircle.

Proposition 4.3 (reflection criterion for tangential quadrilateral of a circle)**.** *Four lines l_1, l_2, l_3, l_4 in the plane are tangent to a circle if and only if there are four lines b_1, b_2, b_3, b_4 intersecting in one point*

$$\{c\} := \bigcap_{i=1}^{4} b_i \neq \varnothing \tag{4.1}$$

such that for the reflections R_i in b_i

$$l_i = R_{i-1}(l_{i-1}) \qquad \text{for } i = 1, \ldots, 4. \tag{4.2}$$

In this case each b_i is the angular bisector of the angle $\measuredangle(l_i, l_{i+1})$ or its supplementary angle $\measuredangle(l_{i+1}, l_i)$, c is the center of the circle and

$$R_4 \circ R_3 \circ R_2 \circ R_1 = id \tag{4.3}$$

Remark 4.3. In the 2-dimensional cut of Proposition 4.3 the symmetry lines b_i are angular bisectors of the lines l_i and l_{i+1}. In the 3-dimensional picture of Proposition 4.2 the symmetry planes b_i fulfill the same property of being angular bisectors of the planes P_i and P_{i+1}.

Having seen that the reflection planes of a conical net contain the axes of the cones we immediately deduct that each two neighboring axes intersect. Indeed, each axis of a pair of neighboring axes belongs to a quadruple of planes where both quadruples have two planes in common, i.e. they share one symmetry plane. So both axes are contained in this symmetry plane and therefore generically intersect in one point.

Let us first associate the axis of the cones to the faces of the lattice giving rise to a map $a : (\mathbb{Z}^m)^* \to \{\text{lines in } \mathbb{R}^3\}$. Then we can formulate this claim in the following way.

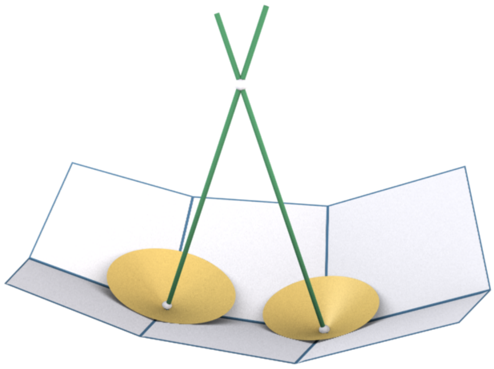

FIGURE 4.4. Symmetry planes of conical nets contain the cone axes. So neighboring axes intersect.

Proposition 4.4. *a is a line congruence, cf. Proposition 3.2.*

Proof. Beside the consideration above one can see this in the following way. Taking a circular net lying on the planes of the conical net as described in Section 4.3 one sees that the lines l from Proposition 3.2 coincide with the lines a and the latter therefore also build a line congruence. □

Remark 4.4. So for $m = 2$ conical nets $f : \mathbb{Z}^2 \to \{\text{planes in } \mathbb{R}^3\}$ can also be interpreted as discrete curvature line parametrizations, cf. Remark 3.4.

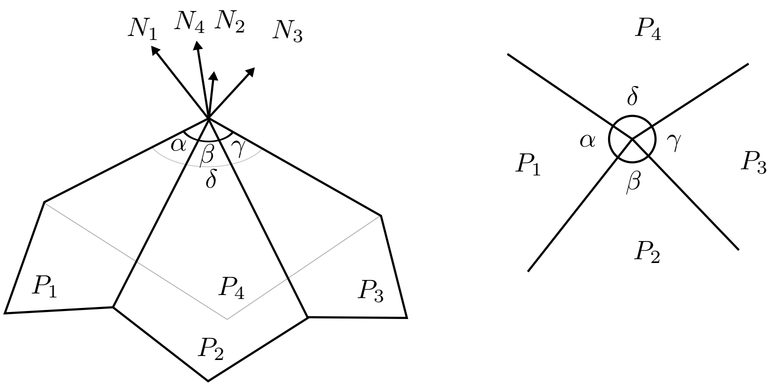

FIGURE 4.5. Four planes P_1, P_2, P_3, P_4 of a Q*-net with vertex-angles $\alpha, \beta, \gamma, \delta$. And view from the top.

Notation 4.1 (vertex-angles of a Q*-net). Let P_1, P_2, P_3, P_4 be four adjacent planes in $\mathbb{R}^3$ of a Q*-net, i.e. intersecting in one common point. Then we define the corresponding *vertex-angles* $\alpha, \beta, \gamma, \delta$ at this point by

$$\begin{aligned} \alpha &= \sphericalangle(P_4 \cap P_1, P_1 \cap P_2) & \beta &= \sphericalangle(P_1 \cap P_2, P_2 \cap P_3) \\ \gamma &= \sphericalangle(P_2 \cap P_3, P_3 \cap P_4) & \delta &= \sphericalangle(P_3 \cap P_4, P_4 \cap P_1) \end{aligned} \tag{4.4}$$

where the angles between the lines are measured according to Notation 3.3 on the common plane with orientation determined by the Gauss-map, i.e. a normal vector on each plane.

Remark 4.5. Note that in general $\alpha + \beta + \gamma + \delta \neq 2\pi$. The difference is one way of measuring the discrete curvature of the surface.

Imagining a unit sphere $\mathbb{S}^2$ centered at a vertex[20] of a Q*-net the vertex-angles become the lengths of the edges of a spherical quadrilateral with vertices defined by the lines of intersection of the planes.
Each plane is separated into four regions by the lines of intersection with its two neighboring planes. Since the defined vertex-angles occur in two of those quadrants there are actually two spherical quadrilaterals defined which are reflections of each other in the center of the sphere. We choose one of them arbitrarily[21] and call the corresponding quadrants which contain the edges *angle-quadrants*, cf. Figure 4.5.

We assume the normals to be chosen consistently, i.e. such that they define an orientation on the union of the four angle quadrants and only consider the case that the defined spherical quadrilateral is embedded. We call such Q*-nets *admissible*.[22]

Remark 4.6. Note that the spherical quadrilateral defined by the vertex-angles can have any possible shape a spherical quadrilateral can have. And due to our definition of the vertex-angles the shape depends not only on the geometric arrangement of the planes but also on their enumeration.

Definition 4.2. We call a vertex of a Q*-net *elliptic* if the spherical quadrilateral defined by its vertex-angles is embedded, its vertices are contained in one hemisphere and no vertex is contained in the spherical triangle formed by the other three vertices.

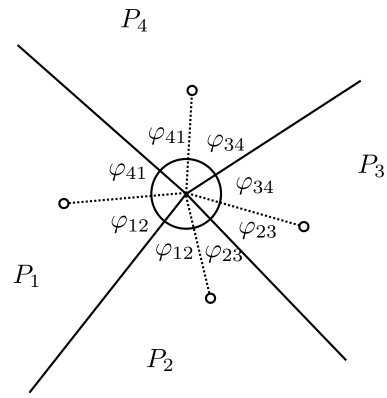

FIGURE 4.6. Conical vertex with touching points to a sphere. Each two angles between a line of intersection and its two neighboring touching points are identical.

Now consider an elliptic vertex which is conical. Then the cone touches the planes in the angle-quadrants. So we can take a sphere touching each of the planes in a point contained in the angle-quadrants as depicted in Figure 4.6 from above. The symmetry plane for each two neighboring planes is going through the center of the sphere ensuring that each of the two angles φ_{ij} in Figure 4.6 are the same. So

$$\alpha + \gamma = \varphi_{41} + \varphi_{12} + \varphi_{23} + \varphi_{34} = \beta + \delta \tag{4.5}$$

[20]A vertex now means the point of intersection of four neighboring planes, which is actually a point naturally associated to a face of $\mathbb{Z}^m$.

[21]Or such that the edges of the spherical quadrilateral actually lie on the surface defined by the conical net if possible.

[22]This is the case in which the Q*-net locally at each vertex (not necessarily at each quad) defines an embedded surface.

For all other types of vertices of an admissible Q*-net the angle-criterion can be derived from the elliptic case using the following claim.[23]

Claim 4.5. *Let P_1, P_2, P_3, P_4 be four planes of a non-elliptic vertex of an admissible Q*-net, i.e. its vertex-angles $\alpha, \beta, \gamma, \delta$ build an embedded spherical quadrilateral. Then it can be made elliptic by a renumeration of the planes while preserving the angle-balance, i.e.*

$$\alpha + \gamma = \beta + \delta \tag{4.6}$$

holds if and only if it holds for the renumerated planes.

Remark 4.7. This is essentially the claim from Theorem 4 in [WWL07].

We illustrate this by an example.

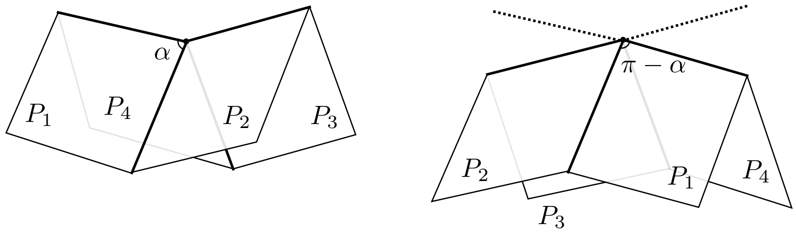

FIGURE 4.7. A non-elliptic vertex of a Q*-net made elliptic by renumerating the planes.

Example 4.1. Consider the vertex depicted in Figure 4.7.
Labeling the planes as in the picture results in angle-quadrants as they are depicted on the left side of the figure and in this case a non-elliptic vertex.[24] Taking the angles contained in the quadrants of the planes depicted on the right side of the figure leads to an elliptic vertex. The angle α on P_1 becomes the supplementary angle on the neighboring quadrant of P_1.

Thus renumerating the planes

$$(P_1, P_2, P_3, P_4) \to (P_2, P_1, P_4, P_3) \tag{4.7}$$

leads to the new angles

$$(\alpha, \beta, \gamma, \delta) \to (\pi - \beta, \pi - \alpha, \pi - \delta, \pi - \gamma) \tag{4.8}$$

which define an elliptic vertex while the truth-value of (4.6) is invariant under the renumeration.

Proposition 4.6 (angle criterion for conical nets)**.** *Let P_1, P_2, P_3, P_4 be four planes of an admissible Q*-net with vertex-angles $\alpha, \beta, \gamma, \delta$, i.e. the corresponding spherical quadrilateral is embedded.*
Then P_1, P_2, P_3, P_4 are in contact to a cone if and only if

$$\alpha + \gamma = \beta + \delta \tag{4.9}$$

Proof. The "$\Rightarrow$"-direction has already been proven above. Nonetheless both directions can be done by reformulating this into the following fundamental statement on spherical quadrilaterals which is proven in [WWL07]. ☐

Proposition 4.7 (incircle of elliptic spherical quadrilaterals)**.** *An elliptic spherical quadrilateral with side-lengths $\alpha, \beta, \gamma, \delta$ has an incircle if and only if*

$$\alpha + \gamma = \beta + \delta \tag{4.10}$$

[23] And in the non-admissible case the angle-criterion actually does not apply!

[24] The lines of intersection of the planes do not intersect one common hemisphere.

4.2. **On the Gauss-map and spherical duality.** Considering the unit normal vectors of the planes instead of the line congruence a we get a Gauss-map for f defined on the vertices of $\mathbb{Z}^m$ for which Proposition 4.2 implies a circular symmetry.

For a Q*-net $f : \mathbb{Z}^2 \to \{\text{planes in } \mathbb{R}^3\}$ we choose its Gauss-map

$$N : \mathbb{Z}^m \to \mathbb{S}^2, \quad u \mapsto \text{normal of the plane } f(u) \tag{4.11}$$

such that it orients neighboring planes consistently. This is at least possible locally in the admissible case.

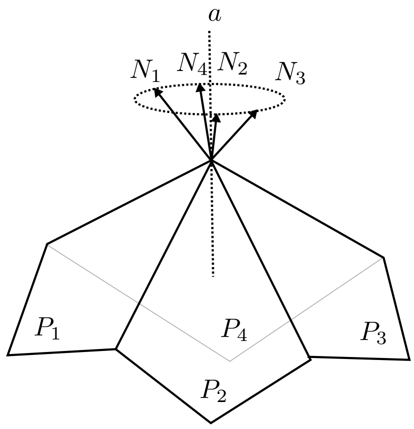

FIGURE 4.8. Four planes P_1, P_2, P_3, P_4 of a conical vertex. Their normal vectors N_1, N_2, N_3, N_4 lie on a circle.

Proposition 4.8 (conical nets have circular Gauss-map)**.** *Let $f : \mathbb{Z}^m \to \{\text{planes in } \mathbb{R}^3\}$ be a Q*-net and $N : \mathbb{Z}^m \to \mathbb{S}^2$ be its Gauss-map.*
Then f is conical if and only if N is circular.

Proof. Consider four adjacent planes of the Q*-net f. They meet in one point. Imagine the normal vectors to the planes being based in this point.

"$\Rightarrow$" The normal vectors inherit the symmetry of the planes[25] and are therefore circular according to Proposition 3.5.

"$\Leftarrow$" The planes inherit the symmetry of the normal vectors and are therefore conical according to Proposition 4.2. □

Consider the vertex angle β generated by the three planes P_1, P_2, P_3 with normal vectors N_1, N_2, N_3, lines of intersection $l_1 := P_1 \cap P_2$, $l_2 := P_2 \cap P_3$ and point of intersection $x := P_1 \cap P_2 \cap P_3$. Imagine x to be the origin and the normal vectors based in this point. They build the vertices of a spherical triangle on the unit sphere $\mathbb{S}^2$. The angle β^* at the vertex N_2, i.e. the angle between the two spherical lines $l_1^* := \operatorname{span}\{N_1, N_2\} \cap \mathbb{S}^2$ and $l_2^* := \operatorname{span}\{N_2, N_3\} \cap \mathbb{S}^2$ is supplementary to the angle β.

$$\beta + \beta^* = \pi \tag{4.12}$$

Adding a fourth plane through the point of intersection the corresponding four normals define a spherical quadrilateral of which the angles are supplementary to the angles of the Q*-net vertex.

This is actually the polar spherical quadrilateral to the one defined by the vertex-angles. So the correspondence between the vertex-angles and the spherical angles of the normal vectors is just the correspondence between the side lengths of a spherical quadrilateral and the angles of its polar quadrilateral.

[25]Due to the their consistent orientation.

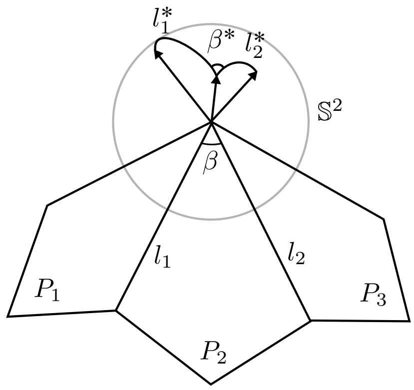

FIGURE 4.9. The vertex angle β of a Q*-net can be converted to a spherical angle β^* of the Gauss-map which is supplementary.

For a conical net we have just seen that these normal vectors form circular quads. The angle-criterion is invariant with respect to substituting all angles by their supplementary angles. So instead of proving the angle-criterion for conical vertices by the spherical incircle criterion we can just as well use its polar version on circumcircles of spherical quadrilaterals which is nothing but the statement about the normal vectors being circular.

Proposition 4.9 (Angle criterion for circular quadrilaterals in spherical geometry)**.** *A spherical quadrilateral with vertices $N_1, N_2, N_3, N_4 \in \mathbb{S}^2$ and quad-angles $\alpha, \beta, \gamma, \delta$ on $\mathbb{S}^2$ is circular if and only if*

$$\alpha + \gamma = \beta + \delta \tag{4.13}$$

4.3. **The connection to circular nets.**

Proposition 4.10. *For any circular net $f : \mathbb{Z}^m \to \mathbb{R}^3$ there is a two-parameter family of conical nets $P : \mathbb{Z}^m \to \{\text{planes in } \mathbb{R}^3\}$ such that $f(u) \in P(u)$ for all $u \in \mathbb{Z}^m$.*
Conversely, for any conical net there is a two-parameter family of circular nets lying on it.

Proof. Given a circular net f choose one normal vector $N \in \mathbb{S}^2$ at some vertex $u \in \mathbb{Z}^m$ which defines a plane through the point $f(u)$. Propagate this plane throughout $\mathbb{Z}^m$ using the symmetry net of the circular net.

Given a conical net P choose one point $x \in P$ at some vertex $u \in \mathbb{Z}^m$ and propagate it throughout $\mathbb{Z}^m$ using the symmetry net of the conical net. □

Remark 4.8. A point and a plane through the point define a contact element containing the family of all spheres through this point and tangent to the plane. This makes a pair of a circular and appropriate conical net a *contact element net* and subject to Lie-geometry. See [BS09].

From this connection between conical and circular nets we can transfer the consistency of circular nets to conical nets.

Proposition 4.11 (basic 3D-system and 4D-consistency)**.** *The system governing conical nets is a discrete 3D-system which is 4D-consistent.*

Proof. $m = 3$: Any admissible initial data for a conical net $P : \mathbb{Z}^m \to \{\text{planes in } \mathbb{R}^3\}$ on the coordinate planes can be replaced by a set of initial data for a circular net by choosing one point in one plane and applying the reflections of the conical sub-nets on the coordinate planes. Circular nets build a 3D-system. So the initial data for the circular net can be consistently propagated throughout $\mathbb{Z}^3$. The corresponding conical net is uniquely determined by one plane of the original initial data due to the symmetry net of the circular net. The planes of the original initial data obviously coincide with the planes of the constructed conical net.

The same principle can be applied to proof the 4D-consistency. □

4.4. **Conical nets in the plane.** Ideas to get similar classes of nets in other dimensions than $N = 3$ could include the following.

- Changing the dimension of the ambient space but keep using 2-planes, i.e. $f : \mathbb{Z}^m \to \{\text{2-planes in } \mathbb{R}^N\}$ where each four neighboring planes intersect in one point and are in contact to a 2-sphere.
- Using Q*-nets in higher dimensions and adapting the dimensions of the incidence conditions accordingly i.e. $f : \mathbb{Z}^m \to \{\text{hyper-planes in } \mathbb{R}^N\}$ where each four neighboring hyper-planes intersect in a $(N-3)$-dimensional affine subspace and are in contact to a hyper-sphere.

Obviously both ideas are not applicable to the plane.

Let us start with a vertex of a conical net in $\mathbb{R}^3$ but lay emphasis on the points of intersection rather than the planes. Consider one intersection point x and the adjacent intersection points x_1, x_2, x_3, x_4. The vertex-angles in this case are the angles between the edges $\omega_i = \sphericalangle([xx_i], [xx_{i+1}])$. If we now start to flatten the vertex by letting the cone angle going to π while keeping its symmetry planes the angles ω_i go to the angles between the reflection planes, satisfying

$$\omega_1 + \omega_3 = \omega_2 + \omega_4 = \pi \tag{4.14}$$

We will see that generalized isoradial nets satisfy this condition at every vertex. This way they can be interpreted as a degenerate case of conical nets with constant cone opening angle π.

So we have seen that the angle-criterion for conical nets is applicable in the plane. But it also becomes the same as the angle-criterion for symmetry-nets.

Definition 4.3 (planar conical vertex stars)**.** We call a planar vertex star x, x_1, x_2, x_3, $x_4 \in \mathbb{R}^2$ *conical* if opposite angles are supplementary.

5. Generalized isoradial circle patterns

Coming back to circle patterns in the plane with square grid combinatorics we let them be represented by its intersection points, i.e. by circular nets $f : \mathbb{Z}^2 \to \mathbb{R}^2$ in the plane.

$$\mathrm{CN}\left(\mathbb{Z}^2\right) = \{f : \mathbb{Z}^2 \to \mathbb{R}^2 \mid f \text{ circular}\} \tag{5.1}$$

We have defined the map c on $\mathrm{CN}\left(\mathbb{Z}^2\right)$

$$c : \mathrm{CN}\left(\mathbb{Z}^2\right) \to \{f : (\mathbb{Z}^2)^* \to \mathbb{R}^2 \mid f \text{ net}\} \tag{5.2}$$

which assigns to each circular net its central net, i.e. the net of circular centers. The generalized isoradial nets are the circular nets with circular central net.

$$\mathrm{GI}\left(\mathbb{Z}^2\right) = c^{-1}\left(\mathrm{CN}\left((\mathbb{Z}^2)^*\right)\right) \tag{5.3}$$

5.1. **Characterization and angles.**

Proposition 5.1. *Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be a circular net. Then*

$$c(f) \text{ circular net} \Leftrightarrow f \text{ symmetry net} \tag{5.4}$$

Remark 5.1. $f : \mathbb{Z}^2 \to \mathbb{R}^2$ being a symmetry net as defined in Definition 3.3 is to be understood for the edges of f. Each four edges adjacent to one vertex automatically intersect in one point. Going once around a vertex the composition of the reflections in the edges has to be the identity. Equivalently, this means that opposite angles are supplementary.

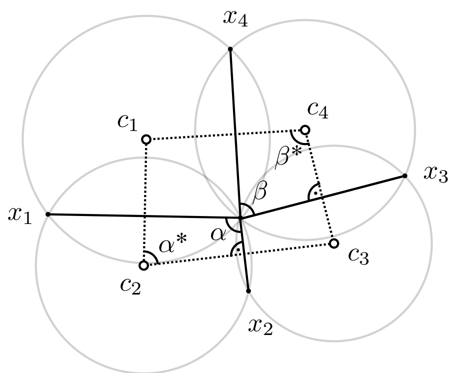

Figure 5.1. Vertex-star of a circular net. The edges of the net and the corresponding edges of its central net are orthogonal. Opposite angles are therefore supplementary.

Proof. Consider a planar vertex star x, x_1, x_2, x_3, x_4. Let R_i be the reflection in the corresponding edge $[x, x_i]$ for $i = 1, \ldots, 4$. Let c_i be the center of the circle through x, x_{i-1}, x_i. Then $[c_i, c_{i+1}]$ is orthogonal to $[x, x_i]$. So the angles

$$\begin{aligned} \alpha &= \measuredangle([x, x_1], [x, x_2]) \text{ and} \\ \alpha^* &= \measuredangle([c_1, c_2], [c_2, c_3]) \end{aligned} \tag{5.5}$$

as well as the angles

$$\begin{aligned} \gamma &= \measuredangle([x, x_3], [x, x_4]) \text{ and} \\ \gamma^* &= \measuredangle([c_3, c_4], [c_4, c_1]) \end{aligned} \tag{5.6}$$

are supplementary.

$$\alpha + \alpha^* = \gamma + \gamma^* = \pi \tag{5.7}$$

So

$$\alpha + \gamma = \pi \Leftrightarrow \alpha^* + \gamma^* = \pi \tag{5.8}$$

□

So we see that the angle-condition by which we defined conical vertex stars in Definition 4.3 means that the four circular centers c_1, c_2, c_3 and c_4 lie on a circle.

We get the following characterizations of generalized isoradial circle patterns.

Proposition 5.2 (Characterization of generalized isoradial circle patterns)**.** *Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be some net. Then*

$$\begin{aligned} & f \text{ generalized isoradial circle pattern} \\ \Leftrightarrow\ & f \text{ circular net and } c(f) \text{ circular net} \\ \Leftrightarrow\ & f \text{ circular net and reflection net} \\ \Leftrightarrow\ & \text{each quad of } f \text{ is circular and each vertex is conical} \\ \Leftrightarrow\ & \text{opposite angles at each face and at each vertex of } f \text{ are supplementary} \end{aligned} \tag{5.9}$$

From the angle condition at faces and vertices we see that the angles between the edges of a generalized isoradial circle pattern recur along the diagonals of the net.

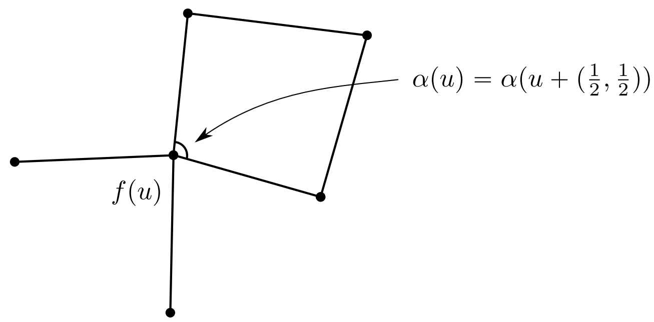

FIGURE 5.2. Referencing the angles of the net f by vertices $u \in \mathbb{Z}^2$ or quads $\left(u + \left(\frac{1}{2}, \frac{1}{2}\right)\right) \in (\mathbb{Z}^2)^*$.

For stating this we use our Notation 3.2 for *quad-angles* and Notation 3.3 for vertex-angles from Section 3.

Extending the notion of those angles $\alpha, \beta, \gamma, \delta$ of a net $f : \mathbb{Z}^2 \to \mathbb{R}^2$ to the whole lattice $\mathbb{Z}^2$ we get four maps

$$\alpha, \beta, \gamma, \delta : \mathbb{Z}^2 \to (0, \pi) \tag{5.10}$$

for the vertex-angles and four maps

$$\alpha, \beta, \gamma, \delta : (\mathbb{Z}^2)^* \to (0, \pi) \tag{5.11}$$

for the quad-angles which are connected by

$$\begin{aligned} \alpha(u) &= \alpha\left(u + \left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right) \\ \beta(u) &= \beta\left(u + \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)\right) \\ \gamma(u) &= \gamma\left(u + \left(-\tfrac{1}{2}, -\tfrac{1}{2}\right)\right) \\ \delta(u) &= \delta\left(u + \left(\tfrac{1}{2}, -\tfrac{1}{2}\right)\right) \end{aligned} \tag{5.12}$$

for $u \in \mathbb{Z}^2$.

The combined map $\mathbb{Z}^2 \cup (\mathbb{Z}^2)^* \to (0, \pi)$ can be referenced via the vertex or the quad and we just call it the *angles* of the net.

Proposition 5.3 (Angles along the diagonals). *Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be a generalized isoradial net and α, β its angles.*[26] *Then at every vertex $u \in \mathbb{Z}^2$*

$$\tau_1\tau_2\alpha = \alpha \qquad \text{and} \qquad \tau_{-1}\tau_2\beta = \beta \tag{5.13}$$

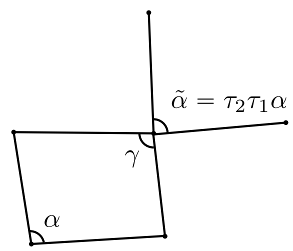

FIGURE 5.3. The quad-angles along the diagonals of the net of a generalized isoradial circle pattern stay the same. $\alpha = \tilde{\alpha}$.

Proof. Let $u \in \mathbb{Z}^2$, $\alpha := \alpha(u)$, $\tilde{\alpha} := \tau_2\tau_1\alpha(u) = \alpha(u + e_1 + e_2)$, $\gamma := \gamma(u)$. The circularity of the quad implies

$$\alpha + \gamma = \pi \tag{5.14}$$

and the symmetry property of the vertex

$$\gamma + \tilde{\alpha} = \pi \tag{5.15}$$

So $\alpha = \tilde{\alpha}$.
Same for β. □

5.2. **The central extension.**

Definition 5.1 (central extension). Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be a circular net and $c(f) : (\mathbb{Z}^2)^* \to \mathbb{R}^2$ its central net. Then we define the central extension of f on the *double* $D := \mathbb{Z}^2 \cup (\mathbb{Z}^2)^*$ by

$$f^\diamond : D \to \mathbb{R}^2, \quad u \mapsto \begin{cases} f(u) & \text{for } u \in \mathbb{Z}^2 \\ c(f)(u) & \text{for } u \in (\mathbb{Z}^2)^* \end{cases} \tag{5.16}$$

Making D a graph we define its unoriented edges by

$$E(D) := \{(u, v) \in \mathbb{Z}^2 \times (\mathbb{Z}^2)^* \mid \text{vertex } u \text{ and face } v \text{ are adjacent in } \mathbb{Z}^2\} \tag{5.17}$$

The resulting faces correspond to the edges of $\mathbb{Z}^2$ or its dual

$$F(D) \cong E(\mathbb{Z}^2) \cong E((\mathbb{Z}^2)^*) \tag{5.18}$$

This makes D a bipartite quad-graph.

The corresponding edges and faces of $f^\diamond$ in $\mathbb{R}^2$ are defined by the line segments and quadrilaterals given by the image points.

Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be circular. Then the quads of its central extension $f^\diamond$ are kites, see Figure 5.4.

[26] Note that the information about the other angles γ and δ is redundant since they are supplementary to α and β respectively.

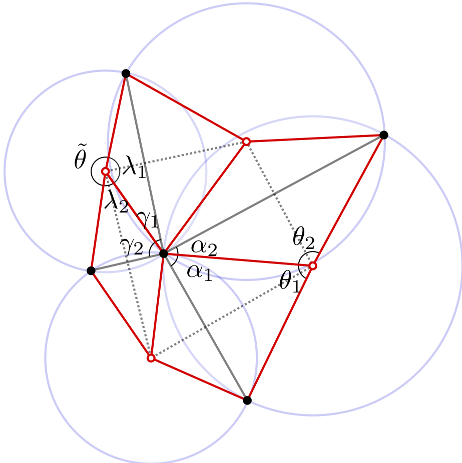

FIGURE 5.4. For the central extension at a conical vertex of circular net we get $\tilde{\theta} = \theta \coloneqq \theta_1 + \theta_2$.

Now let f be generalized isoradial and consider one of its conical vertex stars with angles as depicted in Figure 5.4.

$$\frac{\theta_1}{2} + \alpha_1 = \frac{\theta_2}{2} + \alpha_2 = \frac{\lambda_1}{2} + \gamma_1 = \frac{\lambda_1}{2} + \gamma_1 \tag{5.19}$$

implies

$$\begin{aligned} \alpha &\coloneqq \alpha_1 + \alpha_2 = \pi - \frac{\theta_1}{2} - \frac{\theta_2}{2} \\ \gamma &\coloneqq \gamma_1 + \gamma_2 = \pi - \frac{\lambda_1}{2} - \frac{\lambda_2}{2} \end{aligned} \tag{5.20}$$

So the conicality of the vertex gives us

$$\begin{aligned} \alpha + \gamma = \pi &\Leftrightarrow \left(\frac{\theta_1}{2} + \frac{\theta_2}{2}\right) + \left(\frac{\lambda_1}{2} + \frac{\lambda_2}{2}\right) = \pi \\ &\Leftrightarrow \theta_1 + \theta_2 + \lambda_1 + \lambda_2 = \pi \\ &\Leftrightarrow \theta = \tilde{\theta} \end{aligned} \tag{5.21}$$

with $\theta \coloneqq \theta_1 + \theta_2$. So the angle

$$\theta : (\mathbb{Z}^2)^* \to (0, 2\pi), \quad u \mapsto \measuredangle\left(f\left(u + (\tfrac{1}{2}, \tfrac{1}{2})\right), c(f)(u), f\left(u - (\tfrac{1}{2}, \tfrac{1}{2})\right)\right) \tag{5.22}$$

which is the sum of two adjacent quad-angles at a white (circular center) vertex is constant along the direction $(-1, 1)$ as shown schematically in Figure 5.5.

Since $\theta = 2(\pi - \alpha) = 2\gamma$ this corresponds to the property of α and γ being constant along the diagonals of the original net f.

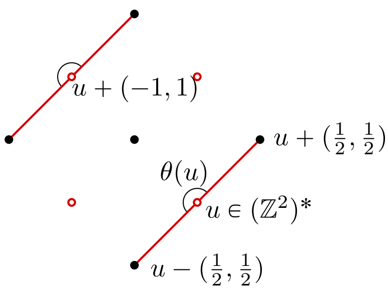

FIGURE 5.5. The angle θ schematically drawn on the double D.

Of course one can define analogously a sum of quad-angles at the white vertices which is constant along the direction $(1,1)$.

Finally we summarize this by stating the following characterization of generalized isoradial nets in terms of its central extension.

Proposition 5.4 (characterization in terms of the central extension)**.** *Let $f^\diamond : D \to \mathbb{R}^2$ be a net on the double D.*
Then $f^\diamond$ is the central extension of a generalized isoradial net if and only if its quads are kites and $\theta : (\mathbb{Z}^2)^* \to (0, 2\pi)$ is invariant with respect to translation along the direction $(-1,1)$.

5.3. **Initial data.** Given initial points along the coordinate lines, where the vertex at the origin is conical, are not sufficient to determine the whole generalized isoradial net.
We consider the propagation process locally. Let $f, f_1, f_2 \in \mathbb{R}^2$ be three given points. We try to determine f_{12}. f, f_1, f_2 determine a circle on which f_{12} has to lie but we need further information to determine its position.
Suppose we additionally have the points f_{11} and f_{1-2}. The angle α^* between $[f_{1-2}, f_1]$ and $[f_1, f]$ determines a direction by using the supplementary angle $\alpha = \pi - \alpha^*$ as the angle between $[f_{11}, f_1]$ and $[f_1, f_{12}]$. Now f_{12} can be determined by the intersection of a line with a circle.

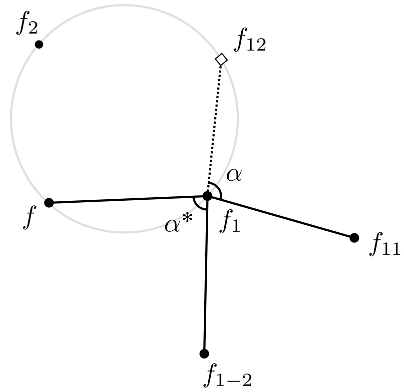

FIGURE 5.6. Constructing the vertex f_{12} of the net of a generalized isoradial circle pattern from f, f_1, f_2 and neighboring circles, given direction at f_1 or additional vertices f_{11} and f_{1-2}.

So given initial points on the coordinate lines we need additional access to points of one row below. This additional row can be given by

(a) prescribing the circumcircles along one strip adjacent to one of the coordinate lines. Then the additional points are given by the points of intersection.
(b) prescribing the points directly. But then we have to make sure that emerging quads are circular.
(c) prescribing the angles α along one coordinate line. Then we can successively build up the rows by taking the circle through three points and intersect it with the direction given by the angle.

Using the local propagation method described above we can construct a complete generalized isoradial circle pattern from any of those data without contradiction as long as the emerging lines and circles always intersect in two points, i.e. any two neighboring circles intersect in two points, i.e. no two vertices coincide along the way. We call this the *generic* case and always assume it to be given.

Proposition 5.5 (initial data for generalized isoradial nets). *Given*

(1) f along the coordinate lines $\mathcal{B}_1$ and $\mathcal{B}_2$ where the vertex at the origin has to be conical.

and one of the following

(2a) circumcircles along one strip of faces adjacent to a coordinate line

(2b) f along $\mathcal{B}_1 + e_2$ such that at each vertex $u \in \mathcal{B}_1$ the four points f, f_1, f_2, f_{12} are circular

(2c) α along $\mathcal{B}_1$ where the angle at the origin has to correspond with the angle given by the points around this vertex

as initial data on $\mathbb{Z}^2$, there generically exists a unique generalized isoradial net f restricting to this data.

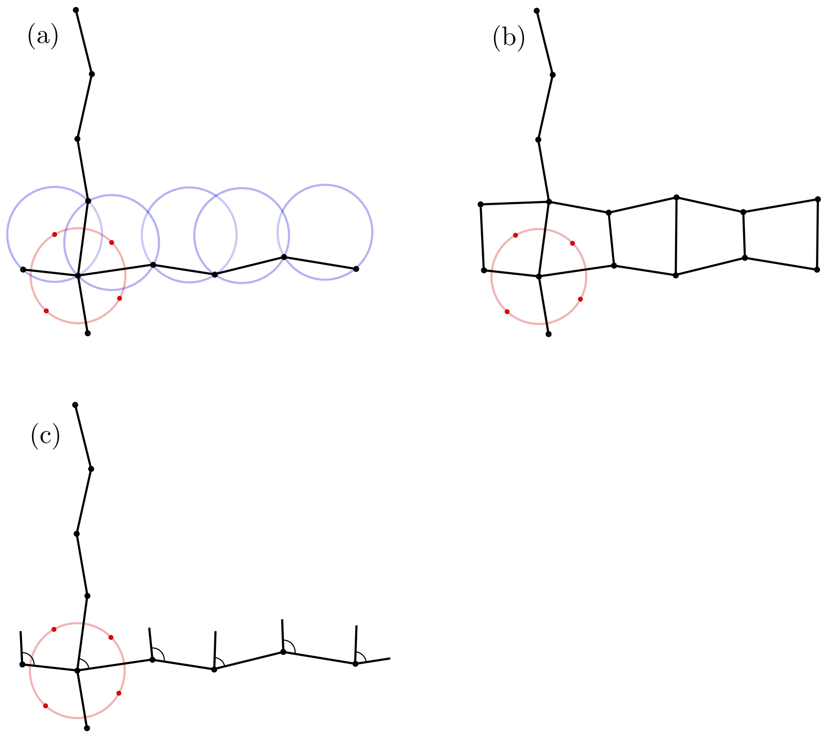

FIGURE 5.7. Initial data for constructing a generalized isoradial net. Points along the coordinate lines where the origin has to be conical plus (a) one strip of circumcircles or (b) one additional strip of circular points or (c) one strip of angles.

5.4. **Examples.**

Example 5.1 (Rectangular nets)**.** A net $f : \mathbb{Z}^2 \to \mathbb{R}^2$ with all rectangular faces only consists of straight lines, i.e. after a suitable rotation $f = (x, y)$ fulfills

$$\tau_2 x = x \qquad \text{and} \qquad \tau_1 y = y \tag{5.23}$$

All angles are $\frac{\pi}{2}$, so all faces circular and all vertices conical.

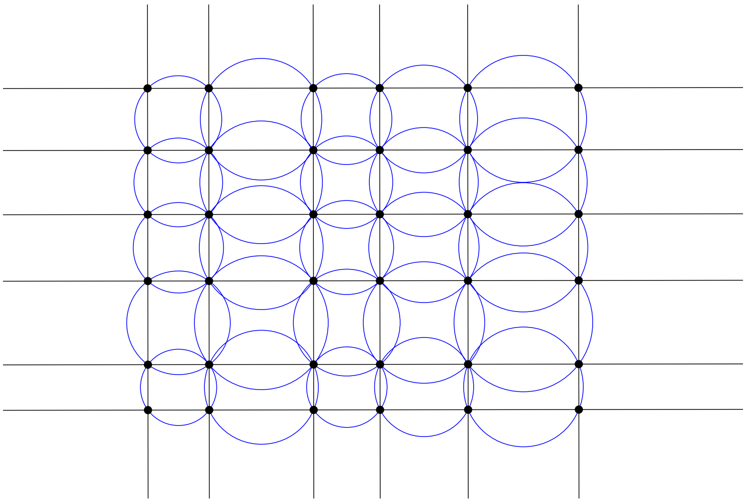

FIGURE 5.8. Rectangular nets are generalized isoradial.

Example 5.2 (Isoradial nets)**.** Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be circular with constant circumcircle radius. As already seen in Proposition 1.1 of the introduction, $c(f)$ is an isoradial net itself, so f generalized isoradial.

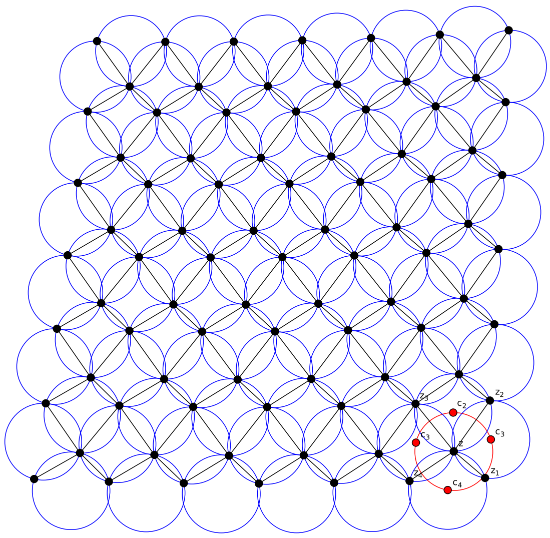

FIGURE 5.9. Isoradial net. This is actually also a regular circle pattern from a conical vertex, cf. Example 5.4

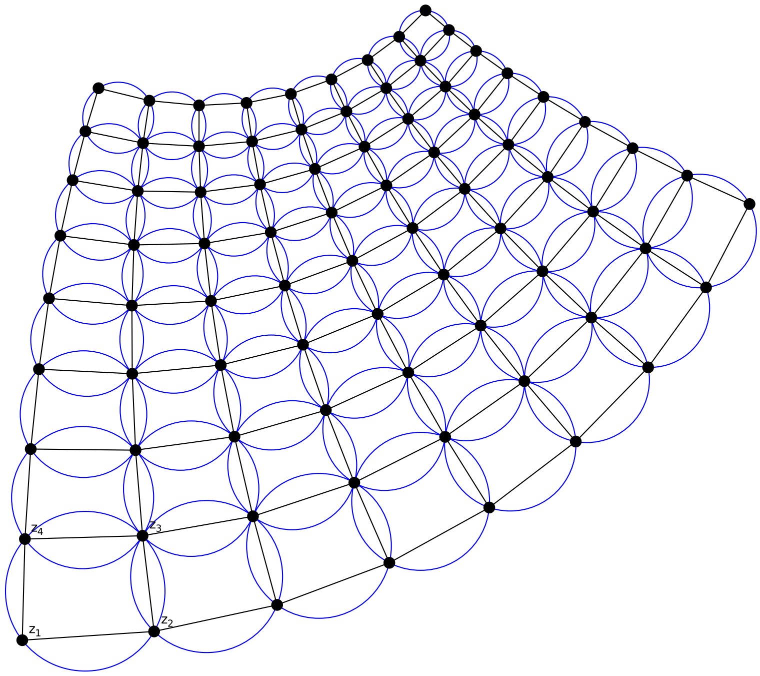

FIGURE 5.10. Generalized isoradial circle pattern from similarity transformations of one circular quad.

Example 5.3 (Regular circle pattern from a circular quadrilateral)**.** Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ be the vertices of a circular quadrilateral. Consider the two affine transformations

$$\begin{aligned} m_1(z) &= s_1 z + t_1, \quad s_1 = \frac{z_2 - z_3}{z_1 - z_4}, t_1 = \frac{z_1 z_3 - z_2 z_4}{z_1 - z_4} \\ m_2(z) &= s_2 z + t_2, \quad s_2 = \frac{z_4 - z_3}{z_1 - z_2}, t_2 = \frac{z_1 z_3 - z_4 z_2}{z_1 - z_2} \end{aligned} \tag{5.24}$$

m_1 mapping $z_1 \mapsto z_2$, $z_4 \mapsto z_3$ and m_2 mapping $z_1 \mapsto z_4$, $z_2 \mapsto z_3$. Such two affine transformations commute

$$\begin{aligned} [m_1, m_2] = 0 &\Leftrightarrow (m_1 \circ m_2)(z) = (m_2 \circ m_1)(z) \\ &\Leftrightarrow s_1(s_2 z + t_2) + t_1 = s_2(s_1 z + t_1) + t_2 \text{ for all } z \in \mathbb{C} \\ &\Leftrightarrow s_1 t_2 + t_1 = s_2 t_1 + t_2 \end{aligned} \tag{5.25}$$

since we have

(5.26)

$$s_1 t_2 + t_1 - (s_2 t_1 + t_2) = \frac{z_1 z_3 - z_2 z_4}{(z_1 - z_4)(z_1 - z_2)} \left(z_2 - z_3 + z_1 - z_2 - (z_4 - z_3 + z_1 - z_4)\right) = 0$$

So applying these transformations repetitively to the quadrilateral yields a (possibly not embedded) tiling of the plane with similar quadrilaterals. The commutativity of m_1 and m_2 ensures that everything fits together, i.e. no contradictions arise.

Since opposite angles of the original quadrilateral are supplementary so are opposite angles of the similar image quads. So all quads are circular. On the other hand opposite angles at each vertex are the same as opposite angles at each face. So the arising net is generalized isoradial.

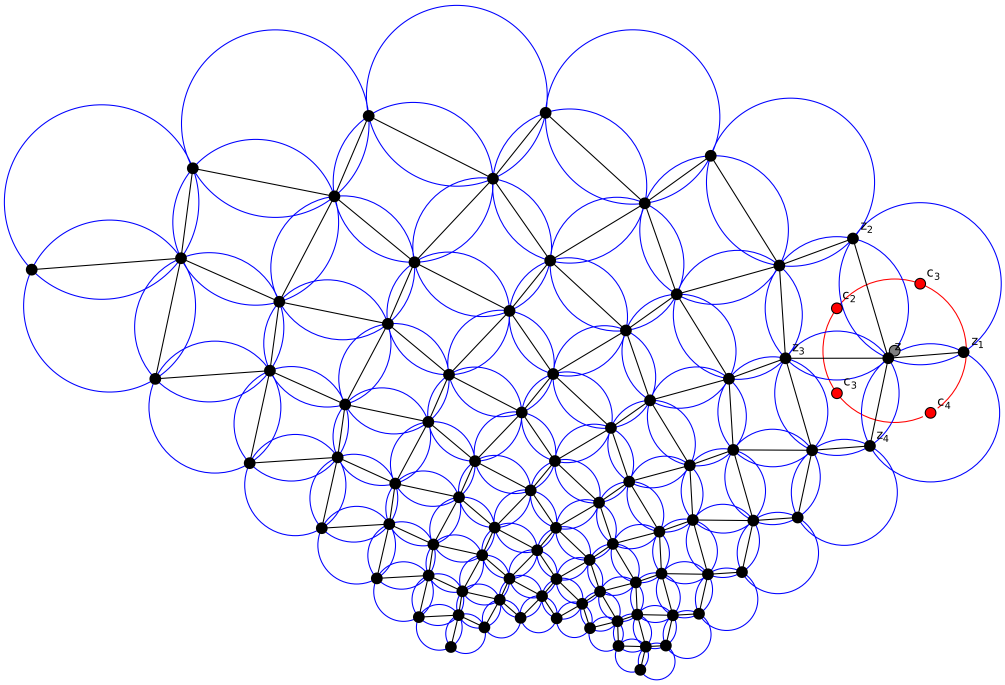

Figure 5.11. Generalized isoradial circle pattern from one conical vertex star.

Example 5.4 (Regular circle pattern from conical vertex star)**.** In a similar manner we can start with a conical vertex star which can be constructed by taking four circles intersecting in one point with circular centers.

- Take a circle C which shall be the circle on which the centers lie.
- Choose four points $c_1, c_2, c_3, c_4 \in C$ which shall be the centers of the circles.
- Choose one point z which shall be the central vertex.
- Draw four circles C_1, C_2, C_3, C_4 with centers c_1, c_2, c_3, c_4 each going through z.
- Denote the second intersection points of neighboring circles in a cyclic order by z_1, z_2, z_3, z_4.

Now the planar vertex star z, z_1, z_2, z_3, z_4 is conical.

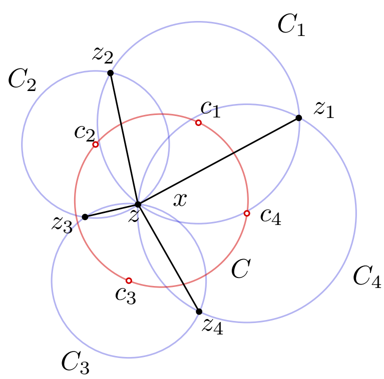

Figure 5.12. How to construct a generalized isoradial vertex. Start with the circle on which the circular centers lie.

Define two affine transformations literally as in Example 5.3.[27] Applying the affine transformations to the circles we get a tiling which consists of two quads up to similarity. The fourth point of each quad is automatically determined by intersection of the circles. Since the affine transformations commute no contradiction occurs and we get a generalized isoradial circle pattern again.

5.5. **Parallelism to isoradial nets.** Parallel translating any edge of a quadrilateral does not change the quad-angles. It is possible to make an embedded quadrilateral non-embedded and to change its orientation, but the angles measured according to our convention will not change.

Definition 5.2 (Combescure transform)**.** Let $f, \tilde{f} : \mathbb{Z}^m \to \mathbb{R}^N$ be any two nets. Then f and $\tilde{f}$ are called *Combescure transforms* of each other or *parallel* if each two corresponding edges $[f, f_i]$ and $[\tilde{f}, \tilde{f}_i]$, $i = 1, \ldots, m$ are parallel.

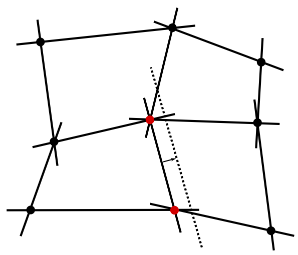

FIGURE 5.13. The lines $l : \mathrm{E}(\mathbb{Z}^2) \to \{\text{lines in } \mathbb{R}^2\}$ through the edges of a net $f : \mathbb{Z}^2 \to \mathbb{R}^2$. Parallel translating one of the lines does not constitute a net anymore.

Since all angles of a Combescure transform are the same as in the original net a Combescure transform of an generalized isoradial net $f : \mathbb{Z}^2 \to \mathbb{R}^2$ is generalized isoradial. So by starting for example with an isoradial circle pattern we can construct a generalized isoradial circle pattern by Combescure transformation. Whether such transformations exist will be answered by the following consideration.

We lay emphasis on the lines $l : \mathrm{E}(\mathbb{Z}^2) \to \{\text{lines in } \mathbb{R}^2\}$ going through the edges of f. If we parallel translate one line of l it does not constitute a net f any more since each four lines corresponding to four edges which have a vertex in common have to intersect in one point. By moving the rest of the lines accordingly we can repair this property getting a Combescure transform of f by only manipulating "half" of the edges of f.

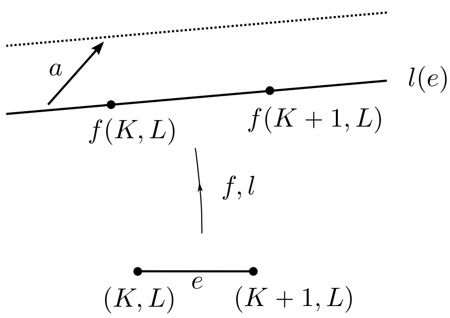

FIGURE 5.14. Parallel translation of one line $l(e)$ of the net f with notations from Lemma 5.6

[27]Note that the points z_i have another meaning as in Example 5.3.

Lemma 5.6 (Strip-wise manipulation of a net)**.** *Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be a net, $l : E(\mathbb{Z}^2) \to \{\text{lines in } \mathbb{R}^2\}$ the lines through its edges. Let $(K, L) \in \mathbb{Z}^2$, $e := ((K, L), (K + 1, L)) \in E(\mathbb{Z}^2)$ an edge.*
Then for any translate $a \in \mathbb{R}^2$ there is a Combescure transform $\tilde{f}$ of f with lines $\tilde{l} : E(\mathbb{Z}^2) \to \{\text{lines in } \mathbb{R}^2\}$ such that

$$\tilde{l}(e) = l(e) + a \tag{5.27}$$

$$\tilde{f}(u) = f(u) \text{ for all } u \in \{(k, l) \in \mathbb{Z}^2 \mid k \in \mathbb{Z}, l < L\} \tag{5.28}$$

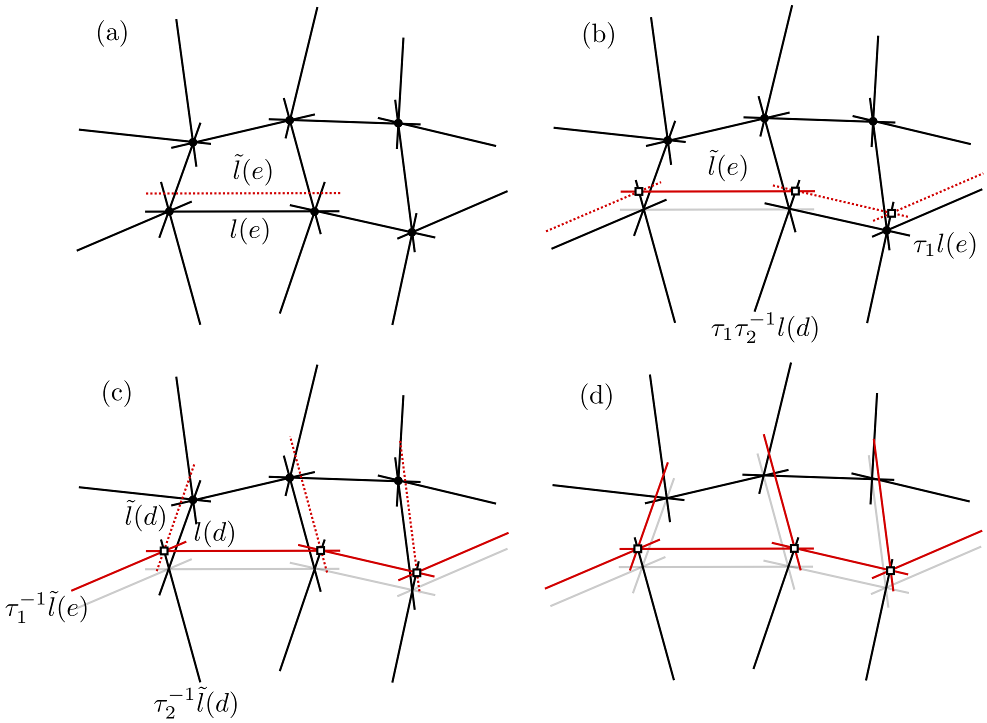

FIGURE 5.15. Strip-wise manipulation of parallel translations: (a) Parallel translate one given line. (b) Adjust the horizontal lines along this strip. (c) Adjust the vertical lines along this strip. (d) Result of manipulations along one strip. Go on like this along all strips above this strip until the intersections of the lines constitute a Combescure transform of the original net.

Proof. Denote $d := ((K, L), (K, L + 1)) \in \mathrm{E}(\mathbb{Z}^2)$. The Combescure transform can be obtained applying the following procedure of stripwise manipulation. You can follow this in Figure 5.15

- Translate $l(e)$ to $\tilde{l}(e) = l(e) + a$.
- Translate $\tau_1 l(e)$ to go through the intersection point of $\tilde{l}(e)$ and $\tau_1(\tau_2)^{-1}l(d)$ and continue in this manner along the strip $\{(K + k, L) \mid k \in \mathbb{Z}\}$.
- Translate $l(d)$ to go through the point of intersection of $\tilde{l}(e)$, $(\tau_1)^{-1}\tilde{l}(e)$ and $\tilde{l}(d) = (\tau_2)^{-1}l(d)$ and continue in this manner along the strip $\{(K + k, L) \mid k \in \mathbb{Z}\}$.

Having done this we go up one strip to $\{(K + k, L + 1) \mid k \in \mathbb{Z}\}$ substituting $e \to e + (0, 1)$ and $d \to d + (0, 1)$ and start over again. This time the translation a for $l(e + (0, 1))$ is not prescribed and can be chosen arbitrary, e.g. zero. So we see that this Combescure transform is not unique. Applying the procedure successively to all strips $\{(K+k, L+l) \mid k \in \mathbb{Z}\}$ for $l \geqslant 0$ each four adjacent new lines of $\tilde{l}$ intersect in one point, therefore defining a new net $\tilde{f} : \mathbb{Z}^2 \to \mathbb{R}^2$ which is parallel to f. □

Remark 5.2. The direction of the edge e (vertical or horizontal) can of course be chosen arbitrarily. Same whether the top or bottom (left or right respectively) half space of the edge stays invariant.

Together with the following Lemma we will be prepared to proof the fact that not only can we construct a generalized isoradial net from an isoradial net by Combescure transformation but that conversely, also every generalized isoradial net is parallel to an isoradial net.

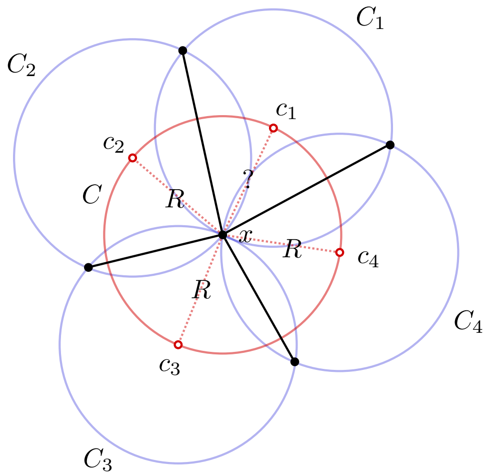

FIGURE 5.16. C_2, C_3, C_4 being of the same radius R implies for a conical vertex-star that the center of C coincides with x. So the radius of C_1 has to be also R.

Lemma 5.7. *Let $x, x_1, x_2, x_3, x_4 \in \mathbb{R}^2$ be a planar conical vertex star. Let C_i be the circle through x, x_i, x_{i+1} for $i = 1, \ldots, 4$.*
If three of the four circles are of the same radius $R > 0$ then so is the fourth.

Proof. For $i = 1, \ldots, 4$ let c_i denote the radius of C_i. W.l.o.g. let C_2, C_3, C_4 be of radius R. Since the vertex star is conical c_1, c_2, c_3, c_4 lie on a circle C. We see that the distance of c_2, c_3 and c_4 to x coincides Since the radius R of C_2, C_3, C_4. So C is a circle with radius R and its center coincides with the point x. Since c_1 is also lying on C this means that the distance of x and c_1 is R which is also the radius of C_1. □

Theorem 5.8 (Generalized isoradial nets as Combescure transforms of isoradial nets)**.** *Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be a net.*
Then f is a generalized isoradial circle pattern if and only if it is Combescure transform of an isoradial circle pattern.

Proof. Only one direction remains to show.
"$\Rightarrow$" Let $q : (\mathbb{Z}^2)^* \to \{\text{quadrilaterals of } f\}$ be the net of quadrilaterals of f. Choose any $u \in (\mathbb{Z}^2)^*$. Let R be the radius of the circumcircle of $q(u)$. Consider the quadrilateral to its right $\tau_1 q(u)$. By translating its right edge we can change the radius of its circumcircle to any value we like. So apply the corresponding Combescure

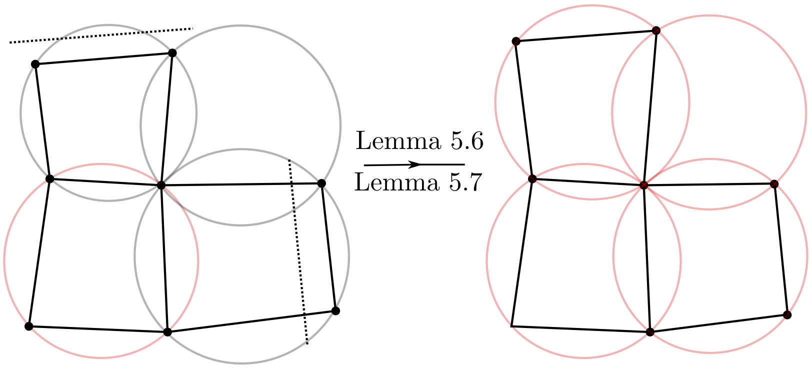

FIGURE 5.17. Use stripwise Combescure transformations to make generalized isoradial nets isoradial.

transformation of the kind from Lemma 5.6 such that the radius of its circumcircle is R and all points to the left stay invariant.
For $\tau_2 q(u)$ we translate its top edge such that the radius of its circumcircle becomes R and all points below stay invariant.
Analogously for $(\tau_1)^{-1}q(u)$ and $(\tau_2)^{-1}q(u)$. Since each of the four Combescure transformations only act on one half space they do not change any of the other three quadrilaterals we manipulate. So after applying them each of the four quadrilaterals $\tau_1 q(u), \tau_2 q(u), (\tau_1)^{-1}q(u), (\tau_2)^{-1}q(u)$ has radius R as $q(u)$. Now Lemma 5.7 implies that $\tau_1\tau_2 q(u), (\tau_1)^{-1}\tau_2 q(u), (\tau_1)^{-1}(\tau_2)^{-1}q(u), \tau_1(\tau_2)^{-1}q(u)$ also have radius R.
We can go on like this making all the radii equal along the coordinate lines $u + \mathcal{B}_1$, $u + \mathcal{B}_2$ of $(\mathbb{Z}^2)^*$ using Combescure transformations which each do not interfere with those parts of the axes which we have already made isoradial. Lemma 5.7 makes sure that after this all other quads of the net are of the same radius. □

5.6. **Iteration of circular centers.** Proving Proposition 5.1 we have already seen the following

Proposition 5.9. *Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be generalized isoradial.*
The vertex-angles (quad-angles) of $c(f)$ are supplementary to the opposite lying quad-angles (vertex-angles) of f.

Proof. Follows from the orthogonality of corresponding edges of f and $c(f)$. □

From this we immediately see what happens to the angles of the net of a generalized isoradial circle pattern under the map c.

Corollary 5.10. *Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be generalized isoradial.*
Then $c(f)$ is generalized isoradial and $c(c(f))$ is generalized isoradial with the same angles as f.

So restricting the map c to generalized isoradial circle patterns yields

$$c : \mathrm{GI}\left(\mathbb{Z}^2\right) \to \mathrm{GI}\left((\mathbb{Z}^2)^*\right) \tag{5.29}$$

and

$$c^2 : \mathrm{GI}\left(\mathbb{Z}^2\right) \to \mathrm{GI}\left(\mathbb{Z}^2\right) \tag{5.30}$$

where c^2 preserves all the nets angles.[28] So we can iterate c^2 to get a sequence of generalized isoradial nets.

[28]Note that this means the angles between edges, not the intersection angles of the circles.

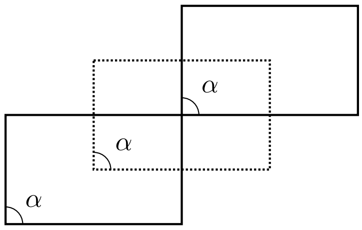

FIGURE 5.18. Schematic image of two quads of a generalized isoradial net and one quad of its central extension. The three angles depicted are equal.

Actually the angle α even stays invariant with respect to c if we identify corresponding diagonals of $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^*$ as shown in Figure 5.18.

In terms of the angle θ of the central extension $f^\diamond$ defined in Section 5.2 this looks like depicted in Figure 5.19

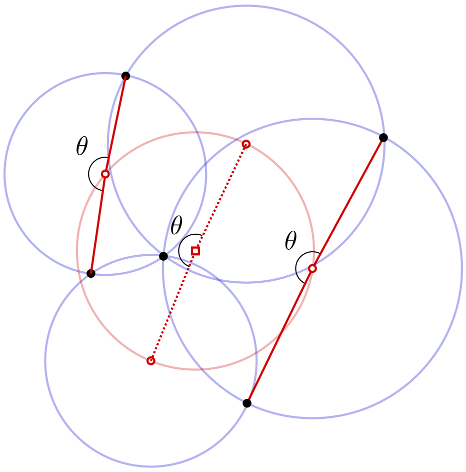

FIGURE 5.19. The angle θ of the central extension upon iteration.

Getting back to the properties of iteration we see that the only nets invariant with respect to this process are the isoradial nets.

Proposition 5.11. *Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be generalized isoradial. Then*

$$c^2(f) = f \Leftrightarrow f \text{ isoradial} \tag{5.31}$$

Proof. "$\Leftarrow$" Follows from Proposition 1.1.
"$\Rightarrow$" $c^2(f) = f$ means that the points of f are the centers of the circumcircles of $c(f)$. So the radii of four circles of f adjacent to one circle of $c(f)$ equal the radius of this one circle of $c(f)$. Each neighboring circles of f are of the same radius and therefore f isoradial. $\square$

We will now investigate the local behavior of this process. If you construct a planar vertex star of a generalized isoradial circle pattern as in Example 5.4 in a more or less "regular" way the radius of the central circle C will be less than the maximal and greater than the minimal radius of the surrounding circles C_1, C_2, C_3, C_4.

This supports the idea that through this averaging process iterating the sequence

$\left(c^{2n}(f)\right)_{n\in\mathbb{N}_0}$ will make the radii more and more equal, eventually converging to an isoradial net.
But in general c does not convey in a local averaging of radii as you can see in the following example.

Example 5.5 (counter-example against local averaging)**.** In the construction of Example 5.4 z, c_1, c_2, c_3, c_4 can be chosen as in Figure 5.20. The radius of C is greater than all radii of the C_i and the resulting vertex star is admissible in the sense that it is suitable to be part of a whole general isoradial net.

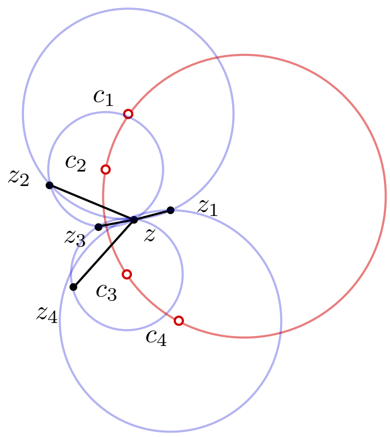

FIGURE 5.20. Construction of a generalized isoradial vertex star with circles all smaller than the circle of the centers.

On the other hand the sequence $\left(c^{2n}(f)\right)_{n\in\mathbb{N}}$ will also not necessarily converge if the radii of f are unbounded as the following example shows.

Example 5.6 (non-converging sequence of unbounded radii)**.** Consider the following rectangular net.

$$f(u) \coloneqq (x(k,l), y(k,l)) \coloneqq \left(2^k, l\right) \text{ for } u = (k,l) \in \mathbb{Z}^2 \tag{5.32}$$

The center of the circumcircles coincides with the point of intersection of the diagonals for each of the rectangular quads. So the y-coordinate stays invariant with respect to c^2, i.e the height of all rectangles stays 1.
Denote the sequence of x-coordinates of $\left(c^{2n}(f)\right)_{n\in\mathbb{N}_0}$ by $(x_n)_{n\in\mathbb{N}_0}$. So

$$x_0(k) = x(k,l) = 2^k. \tag{5.33}$$

The x-coordinate of midpoints of the quadrilaterals in the vertical strip to the left of $u = (k,l)$ is $\frac{2^{k-1}+2^k}{2}$. The x-coordinate of midpoints to right is $\frac{2^k+2^{k+1}}{2}$. So

$$x_1(k) = \frac{2^{k-2} + 2^{k-1} + 2^{k-1} + 2^k}{2} = 9 \cdot 2^{k-3}. \tag{5.34}$$

For the next step we get

$$x_2(k) = 9^2 \cdot 2^{k-6} \tag{5.35}$$

and going on like this

$$x_n(k) = 9^n \cdot 2^{k-3n} = \left(\frac{9}{8}\right)^n 2^k \tag{5.36}$$

which can be verified by induction. So the width of the rectangles to the right of (k,l) is

$$\Delta_n(k) := |x_n(k+1) - x_n(k)| = \left(\frac{9}{8}\right)^n (2^{k+1} - 2^k) = \left(\frac{9}{8}\right)^n 2^k \tag{5.37}$$

and therefore the radius of its circumcircle

$$r_n(k) := \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\Delta_n(k)}{2}\right)^2} = \frac{1}{2}\sqrt{1 + \left(\frac{9}{8}\right)^n 2^k} \tag{5.38}$$

which diverges for $n \to \infty$.

To ensure convergence the radii should at least be bounded. We will study the stronger restriction of doubly periodic generalized isoradial nets, i.e. generalized isoradial nets on the torus.

5.7. **On the torus.** For simulations this is a natural case to consider since we can describe the whole net by finitely many points making us able to actually compute the iteration process for a given initial generalized isoradial net in a simulation.

Definition 5.3 (generalized isoradial net on the torus)**.** Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$, $M, N \in \mathbb{N}$, $a, b \in \mathbb{R}^2$ with $\det(a,b) \neq 0$.
Then f is called an $M \times N$ *generalized isoradial net with periods* a *and* b if

$$f(m + kM, n + lN) = f(m,n) + ka + lb \qquad \text{for } m,n,k,l \in \mathbb{Z} \tag{5.39}$$

Remark 5.3. Defining

$$\begin{aligned} \mathbb{Z}^2_{MN} &:= (\mathbb{Z}/M\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z}) \\ \mathbb{R}^2_{ab} &:= \mathbb{R}^2/\{x \mapsto x + a, x \mapsto x + b\} \end{aligned} \tag{5.40}$$

the periodic map $f : \mathbb{Z}^2 \to \mathbb{R}^2$ induces a map $\tilde{f} : \mathbb{Z}^2_{MN} \to \mathbb{R}^2_{ab}$ from the discrete torus to the smooth torus. But the information on how two adjacent points are connected gets lots since there is more than one straight line connecting two points on $\mathbb{R}^2_{ab}$

Let us address the question of existence. The condition of periodicity is of course a major restriction to our net. The restriction weighs heaviest if M and N do not have a common divisor. In this case the periodicity ensures that the number of possible angles in the net drops to two (and the two supplementary angles).

Proposition 5.12 (angles for incommensurable M and N)**.** *Let f be an $M \times N$ generalized isoradial net with periods a and b.*
If M and N have no common divisor then the corresponding angles of each two quads of f are equal, i.e. $\alpha, \beta = const.$

Proof. Let $\alpha, \beta : \mathbb{Z}^2 \to \mathbb{R}^2$ be the angles of f. α stays constant along the diagonals from bottom-left to top-right. Because of the periodicity the diagonal starting at $(0,0) \in \mathbb{Z}^2_{MN}$ is represented on the torus by

$$D := \{(k \bmod M, k \bmod N) \mid k \in \mathbb{Z}\} \tag{5.41}$$

Since M and N have no common divisor each point $(m,n) \in \mathbb{Z}^2_{MN}$ can be reached by a suitable choice of k. Indeed, let $l \in \mathbb{Z}$ such that $(m + lM) \bmod N = n$. Then for $k := m + lM$ we have $(k \bmod M, k \bmod N) = (m,n)$. So

$$D = \mathbb{Z}^2_{MN} \tag{5.42}$$

Same for the other diagonal. $\square$

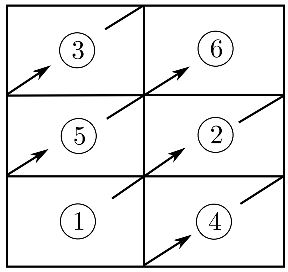

FIGURE 5.21. Schematic picture of a net on a 2×3-torus. Going from bottom-left to top-right along the diagonals starting at quad 1 we cross all quads of the net, eventually returning to quad 1.

We will only consider the case $a \perp b$, so w.l.o.g $a \parallel e_1$ and $b \parallel e_2$. In this case there always exist periodic generalized isoradial nets in the form of rectangular nets. But from Proposition 5.12 we see that at least in the embedded case there can exist more only if M and N have a common divisor.

Proposition 5.13. *Let f be an $M \times N$ generalized isoradial net with periods $a \perp b$ for which no two quads overlap.*
Then if M and N have no common divisor, all quads of f are rectangular.

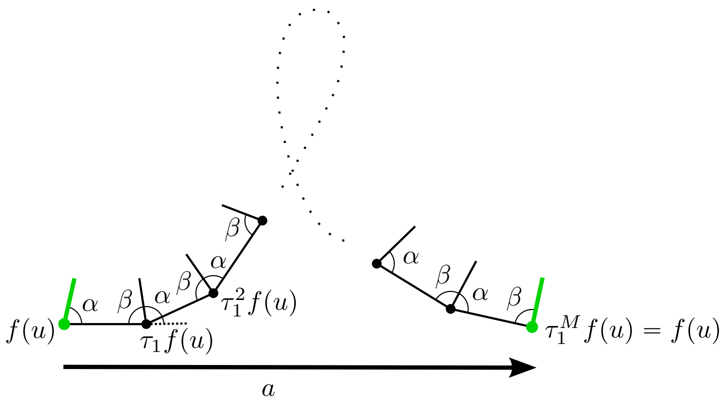

FIGURE 5.22. A closed path of edges on the torus along direction 1.

Proof. From Proposition 5.12 we know that all corresponding angles of each two quads are identical.
For any vertex $u \in \mathbb{Z}^2$ and any direction $i \in \{1, 2\}$ consider the closed path

$$[f, \tau_i f] \cup [\tau_i f, \tau_i^2 f] \cup \cdots \cup [\tau^{K_i - 1}, f] \quad \text{where } K_1 = M \text{ and } K_2 = N \tag{5.43}$$

The paths along direction 1 consist of M segments. The angle between two adjacent segments is $\alpha + \beta$. Since the quads do not flip along the way, a necessary condition for those paths to be closed is

$$M(\pi - \alpha - \beta) = 2\pi m \quad \text{for some } m \in \mathbb{Z} \tag{5.44}$$

cf. Figure 5.22.
For the paths along direction 2 we get the analogue statement for the angles α and $\delta = \pi - \beta$

$$N\left(\pi - \alpha - (\pi - \beta)\right) = N(\beta - \alpha) = 2\pi n \quad \text{for some } n \in \mathbb{Z} \tag{5.45}$$

So possible values for the angles are included in

$$\begin{aligned}\alpha &= \pi\left(-\frac{n}{N}-\frac{m}{M}+\frac{1}{2}\right)\\ \beta &= \pi\left(\frac{n}{N}-\frac{m}{M}+\frac{1}{2}\right)\end{aligned} \tag{5.46}$$

with $m, n \in \mathbb{Z}$.
The condition of no two quads overlapping further implies that the paths have no self-intersections, i.e. $n = m = 0$ and therefore

$$\alpha = \beta = \frac{\pi}{2} \tag{5.47}$$

□

On the other hand, starting with a path of say one self intersection on the torus, it is hard to imagine a way to extend it to a closed generalized isoradial net on the whole torus. This brings us to the following conjecture.

Conjecture 5.14. *Any $M \times N$ generalized isoradial net f with periods $a \perp b$ where M and N have no common divisor is rectangular.*

So the simplest case in which non-rectangular nets can occur is $M = N = 2$.

Example 5.7 ($M = N = 2$)**.** Let $a \parallel e_1$, $b \parallel e_2$ some periods. Choose $x \in (0, |a|)$ and $\varepsilon \in \left(-\frac{|a|-x}{2}, \frac{x}{2}\right)$. Define the four points of our 2×2-net to be

$$\begin{aligned} f(0,0) &:= (0,0)\\ f(1,0) &:= (x,0)\\ f(0,1) &:= \left(\varepsilon, \tfrac{|b|}{2}\right)\\ f(1,1) &:= \left(x-\varepsilon, \tfrac{|b|}{2}\right)\end{aligned} \tag{5.48}$$

All four trapezoids of this net are symmetric and have the same angles, cf. Figure 5.23, so it is generalized isoradial.[29]

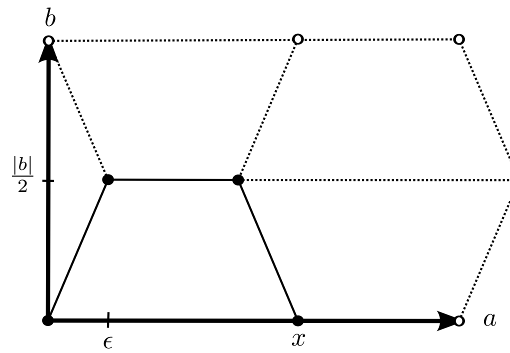

FIGURE 5.23. A symmetric trapezoid with height equal to half the length of the b-period constitutes in a generalized isoradial net on the 2×2-torus

Since generating generalized isoradial nets on a 2×2-torus with the methods of Section 5.8 did not result in any other nets in our simulations we state the following conjecture.

[29] $\varepsilon \in \mathbb{R}$ would also be possible, allowing non-embedded quads.

Conjecture 5.15. *Any 2×2-generalized isoradial net f with periods $a \perp b$ is of the trapez-form described in Example 5.7 or rectangular.*[30]

To generate more examples on larger tori one can use variational methods minimizing a suitable energy as described in Section 5.8.

5.8. **Circular and conical energies.** Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be a net (not necessarily periodic) and let $\alpha, \beta, \gamma, \delta : \mathbb{Z}^2 \cup (\mathbb{Z}^2)^* \to \mathbb{R}^2$ be its angles.
For any quad $u \in (\mathbb{Z}^2)^*$ its "circularity" can be measured by the term

$$(\alpha(u) + \gamma(u) - \pi)^2 \tag{5.49}$$

It is non-negative and 0 if and only if the quad is circular. It is also smooth in the coordinates of the vertices of the quad as long as the angles stay in $(0, \pi)$. Due to the definition of the quad-angles in Notation 3.2 the angles jump from 0 to π and from π to 0 when rotating one edge while holding the other as seen in Figure 5.24.

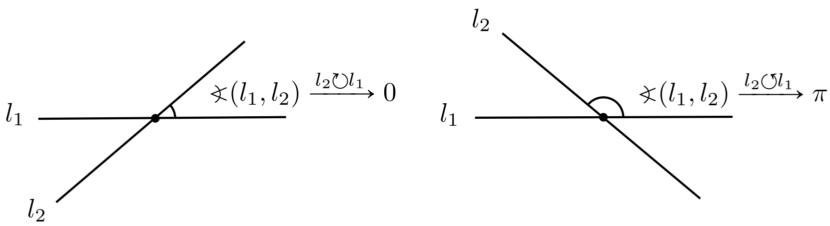

FIGURE 5.24. The defined angles are not continuous at the point where the two lines become identical.

In the same way we can measure the "conicality" of any vertex $v \in \mathbb{Z}^2$ by

$$(\alpha(v) + \gamma(v) - \pi)^2 \tag{5.50}$$

Definition 5.4 (Circular and conical energy)**.** Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be a net. Then we define its *circular energy* by

$$E_{circ}(f) := \sum_{u \in (\mathbb{Z}^2)^*} (\alpha(u) + \gamma(u) - \pi)^2 \tag{5.51}$$

and its *conical energy* by

$$E_{con}(f) := \sum_{v \in \mathbb{Z}^2} (\alpha(v) + \gamma(v) - \pi)^2 \tag{5.52}$$

So we can obtain more generalized isoradial circle pattern on the torus by minimizing

$$E(f) := E_{circ}(f) + E_{con}(f) \tag{5.53}$$

interpreting this as a function on all the coordinates of all the vertices of f.

Remark 5.4. The introduced energies are smooth functions on the coordinates as long as all the quadrilaterals stay embedded and non-flipped.
So in order to efficiently minimize E by using its gradient one has to ensure this condition.

Calculation of the gradient can be found in Section B, some sample results of minimizing these energies in Section A.

[30]Up to interchanging the roles of a and b.

5.9. **Convergence of rectangular nets.** In our simulations all generalized isoradial circle patterns we were able to generate on the torus converged to an isoradial circle pattern under iteration of the map c^2, cf. Section A, which leads us to the following conjecture.

Conjecture 5.16 (convergence of generalized isoradial nets on the torus)**.** *Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be a $M \times N$ generalized isoradial net with periods $a \perp b$. Then, under certain regularity conditions on f, the sequence $\left(c^{2n}(f)\right)_{n\in\mathbb{N}}$ converges to an isoradial net.*

For the special case of rectangular nets a proof of convergence will be given in the following.

Proposition 5.17 (convergence of rectangular nets)**.** *Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be a $M \times N$ rectangular net with periods $a \perp b$ and no flips, i.e. the vertical and horizontal lines of the net are enumerated as they are ordered geometrically.*
Then $\left(c^{2n}(f)\right)_{n\in\mathbb{N}}$ converges to an rectangular net with all identical rectangles, i.e. in particular an isoradial net.

Proof. W.l.o.g let the lines of the rectangular net be parallel to the x and y-axes, i.e. $a \parallel e_1$ and $b \parallel e_2$.

Consider the sequence $\left(c^{n}(f)\right)_{n\in\mathbb{N}}$ instead of $\left(c^{2n}(f)\right)_{n\in\mathbb{N}}$. We first note that all nets in this sequence are rectangular and we therefore only have to consider the x-coordinates of the vertical lines and the y-coordinates of the horizontal lines.
The positions of the horizontal lines after one step does not depend on the position of the vertical lines, i.e. the x and y-coordinates vary independently and can be treated separately. Since both x and y are interchangeable this reduces the problem to an one-dimensional problem.

We show that the distance between neighboring lines becomes equally distributed under iteration therefore making all the rectangles identical.

Identifying the torus lattice $\mathbb{Z}^2_{MN}$ of each even step with $(\mathbb{Z}^2_{MN})^*$ of the subsequent odd step by moving it up and to the right

$$\mathbb{Z}^2_{MN} + \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \cong (\mathbb{Z}^2_{MN})^* \tag{5.54}$$

and identifying each lattice $(\mathbb{Z}^2_{MN})^*$ of each odd step with $\mathbb{Z}^2_{MN}$ of the subsequent even step in the same way by moving it up and to the right

$$(\mathbb{Z}^2_{MN})^* + \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \cong \mathbb{Z}^2_{MN} \tag{5.55}$$

makes each point walking diagonally through the torus lattice while iterating the sequence. Since we only want to show the equal distribution of the distances between points this does not matter to us.

Let $X^{(n)} = (x_1^{(n)}, \dots, x_M^{(n)})$ be the vector of x-coordinates of vertical lines in the n-th step resulting from this identification.
Then

$$x_k^{(n+1)} = \frac{x_k^{(n)} + x_{k+1}^{(n)}}{2} \mod |a| \tag{5.56}$$

for $k \in 1, \dots, M-1$ and

$$x_M^{(n+1)} = \frac{x_M^{(n)} + x_1^{(n)} + |a|}{2} \mod |a| \tag{5.57}$$

due to the identification at the boundary. This is nothing but an averaging process of M points on a circle.

Defining the distances

$$l_k^{(n)} := x_{k+1}^{(n)} - x_k^{(n)} \tag{5.58}$$

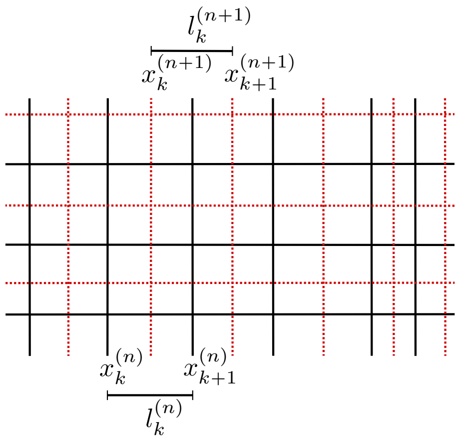

FIGURE 5.25. One step of the sequence of central nets for a rectangular net on the torus

the iteration formula for those distances becomes

$$l_k^{(n+1)} = \frac{l_{k-1}^{(n)} + l_k^{(n)}}{2} \tag{5.59}$$

for $k \in 1, \ldots, M$, indices taken modulo M. These are all non-negative since no flipped lines were allowed.

We have to show that these distances become equally distributed reducing the claim to the following lemma. $\square$

Lemma 5.18 (averaging on a circle)**.** *Let $X = (x_1, \ldots, x_M)$ be M points on a circle $\mathbb{R}/|a|\mathbb{R}$ of length $|a|$, such that neighboring points on the circle are actually successive points in the enumeration X.*
Then they become equally distributed under the process which arithmetically averages each two neighboring points upon each step.

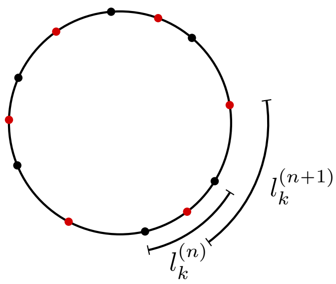

FIGURE 5.26. One step of the averaging process on a circle.

Proof. Let $t^{(n)} = (l_1^{(n)}, \ldots, l_M^{(n)})$ be the distances of neighboring points after the n-th step. Then

$$t^{(n+1)} = At^{(n)} = A^n t \quad \text{with } A = \begin{pmatrix} \frac{1}{2} & & & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & & \\ & \ddots & \ddots & \\ & & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{5.60}$$

for $n \in \mathbb{N}$.
A is a stochastic matrix since columns (and even rows) add up to one. Additionally

each entry of A^k is positive for $k \in \mathbb{N}$ big enough.
So from Theorem C.1 follows that $A^n t$ converges to the eigenvector of A with corresponding eigenvalue 1 and length $L := |t| = |a|$. Since $(1, \ldots, 1)$ is obviously an eigenvector of A with eigenvalue 1 we get

$$t^{(n)} \to \left(\tfrac{L}{M}, \ldots, \tfrac{L}{M}\right) \qquad \text{for } n \to \infty \tag{5.61}$$

i.e. the points become equally distributed. □

6. Circular conical nets

The considerations in Section 4.4 suggest that in the spirit of investigating generalized isoradial circle patterns we also consider nets in $\mathbb{R}^3$ which are circular and conical at the same time with constant cone opening angle.[31]

In [PW08] it is shown that the nets $f : \mathbb{Z}^2 \to \mathbb{R}^3$ which possess a Combescure transform which is both of constant vertex distance and of constant face distance is circular and conical of constant cone opening angle. They also show that these nets are exactly the nets which possess a Combescure transform on the sphere with circurmcircles of constant radius. A similar property to that of generalized isoradial circle patterns being exactly the Combescure transforms of isoradial circle patterns.

The central extension[32] of an isoradial net $f : \mathbb{Z}^2 \to \mathbb{R}^3$ is a rhombic net with planar vertex stars. Such nets have been studied in [Wun51] as a suitable discretization of surfaces with constant negative Gaussian curvature.

The other way round does a rhombic net with planar vertex stars define a pair of isoradial nets by taking its two diagonal nets which build a subclass of circular-conical nets with constant cone opening angle.

---

[31] We described in Section 4.4 how conical nets in the plane can be viewed as degenerated conical nets in $\mathbb{R}^3$ with constant cone opening angle π.

[32] As defined in Section 5.2.

Appendix A. Sample simulations on the torus

Starting with a square grid and perturbing each vertex a little bit gives a suitable initial condition to minimize the energy making flips and non-embedded quads rather unprobable.

We generated generalized isoradial circle patterns on various sized tori minimizing the energy using the Broyden-Fletcher-Goldfarb-Shanno algorithm with the gradient calculated in Section B.

In the following you can see some sample results. The periods for an $M \times N$ torus are always chosen to be $a = (M, 0)$ and $b = (0, N)$.

On the left the initial data is shown and on the right the resulting generalized isoradial net. The dots indicate the vertices of the net contained in one fundamental domain of the torus while the unmarked vertices are copies obtained by the identification.

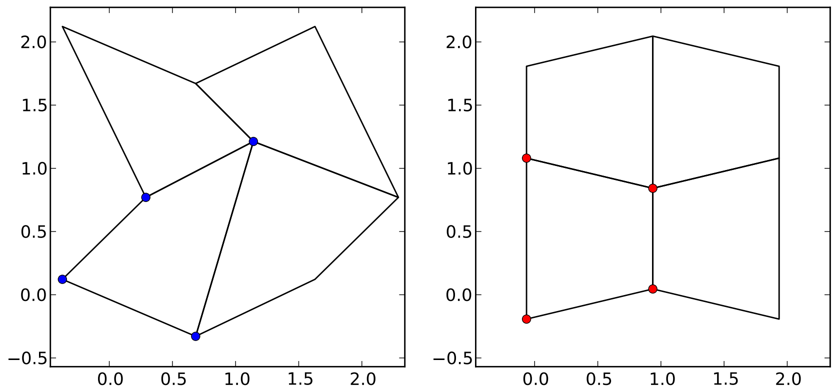

Figure A.1. Minimizing on a 2×2-torus. The result is a symmetric trapezoid as described in Conjecture 5.15.

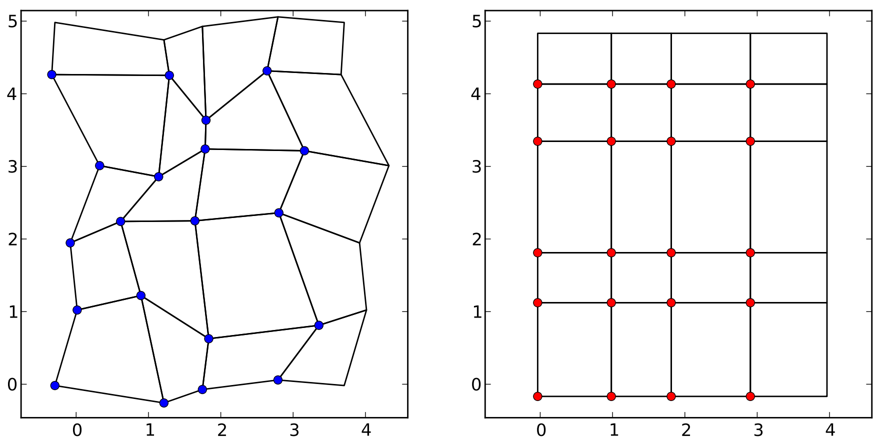

Figure A.2. Minimizing on a 4×5-torus. The result is a rectangular net as described in Proposition 5.13.

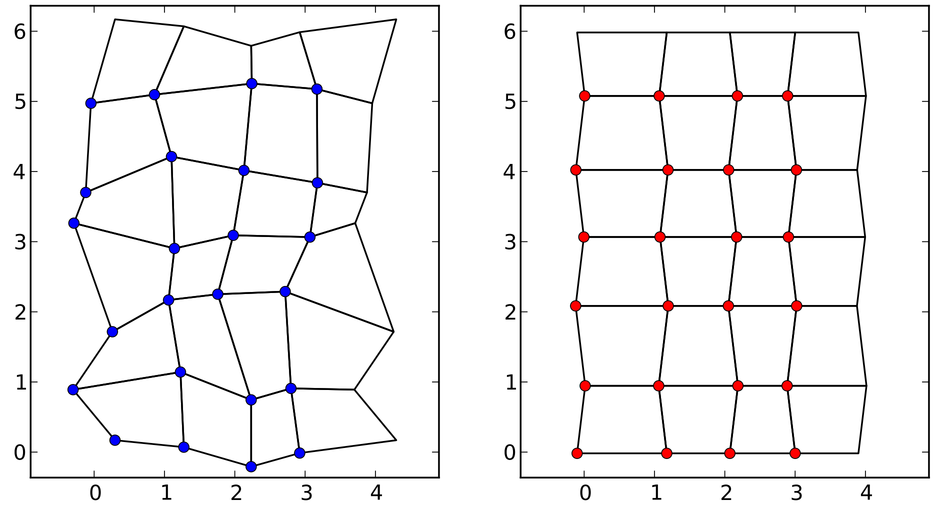

FIGURE A.3. Minimizing on a 4×6-torus. There are only four different diagonals and therefore only four different angles (plus supplementary angles) in this net as on the 2×2-torus, making the result very similar.

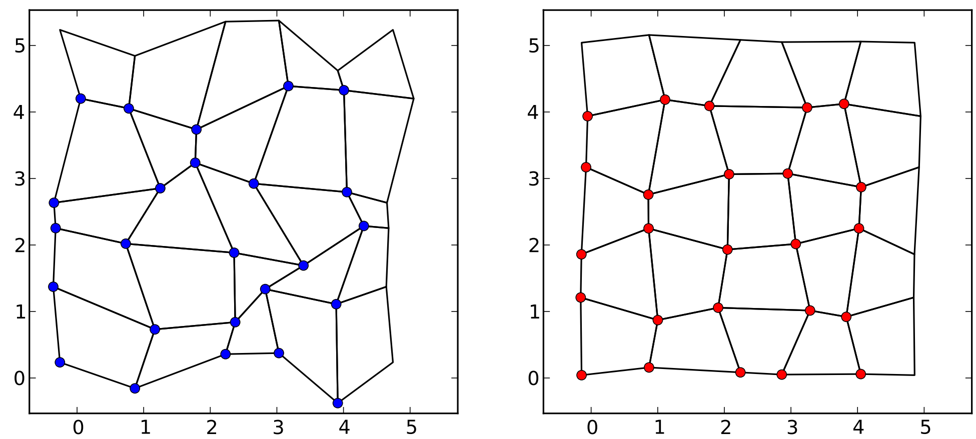

FIGURE A.4. Minimizing on a 5×5-torus. The patterns become richer on square-tori.

The following figures show sample simulations of the iteration process described in Section 5.6. In the top left the initial generalized isoradial net is shown generated on an $M \times N$-torus as in the examples above. Below you can see the resulting net after 20 steps of iteration. On the right side from top to bottom are shown

- total circular (blue) and conical (green) energy
- the average radius of the circumcircles
- the mean square error of the radii of the circumcircles

all plotted against the iteration step.
You can see that in the examples the radii become equally distributed while the circular and conical energies do not rise significantly, i.e. the nets stay generalized isoradial during the iteration process.

This behavior was observed with all initial data generated which led to an iteration process in which the energies of the net did not explode. Note though, that the method by which the initial nets are created leads to fairly "regular" nets with all embedded quads which are not far from being isoradial themselves if the minimizing process of the energies converges.

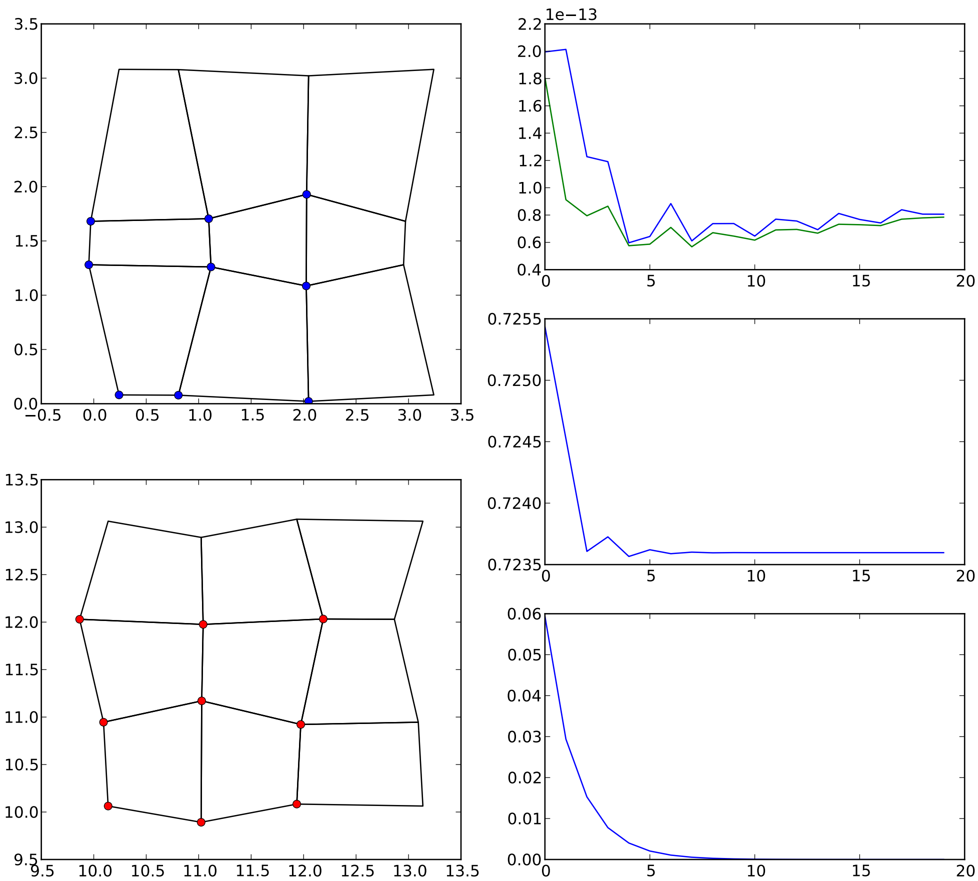

FIGURE A.5. Iteration on a 3×3-torus.

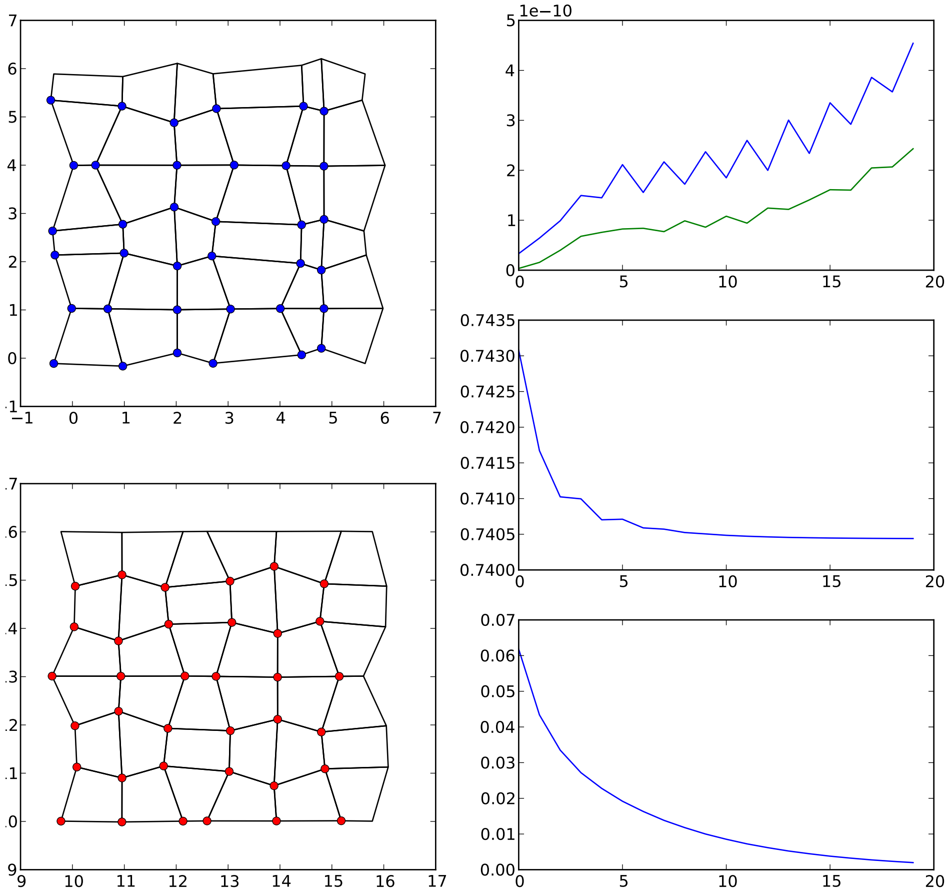

FIGURE A.6. Iteration on a 6×6-torus.

For running the simulations with various initial conditions yourself you can download the source code from [Tec13].

`https://gitlab.discretization.de/public/projects/techter/ddgtools`

Appendix B. Gradient of the circular and conical energy

Let $x, x_1, x_{12}, x_2 \in \mathbb{R}^2$ be the vertices of an embedded, non-flipped quadrilateral in cyclic order. In this case the angle $\sphericalangle(x_2, x, x_1)$ at the vertex of x is given by

$$\sphericalangle(x_2, x, x_1) = \arccos \underbrace{\frac{\langle x_1 - x, x_2 - x\rangle}{\|x_1 - x\| \, \|x_2 - x\|}}_{:= h(x_2, x, x_1)} \in (0, \pi) \tag{B.1}$$

Let us calculate the gradient.[33]

(B.2)

$$\begin{aligned}
\nabla_{x_1} h(x_2, x, x_1) &= \frac{1}{\|x_1 - x\| \, \|x_2 - x\|} \left((x_2 - x) - \frac{\langle x_1 - x, x_2 - x\rangle}{\|x_1 - x\|^2} (x_1 - x) \right) \\
&= -\frac{\cos \sphericalangle}{l_1^2} (x_1 - x) + \frac{1}{l_1 l_2} (x_2 - x) \\
\nabla_{x_2} h(x_2, x, x_1) &= \frac{1}{\|x_1 - x\| \, \|x_2 - x\|} \left((x_1 - x) - \frac{\langle x_1 - x, x_2 - x\rangle}{\|x_2 - x\|^2} (x_1 - x) \right) \\
&= \frac{1}{l_1 l_2} (x_1 - x) - \frac{\cos \sphericalangle}{l_1^2} (x_2 - x) \\
\nabla_x h(x_2, x, x_1) &= \frac{1}{\|x_1 - x\| \, \|x_2 - x\|} \left(\left(\frac{\langle x_1 - x, x_2 - x\rangle}{\|x_1 - x\|^2} - 1 \right) (x_1 - x) + \left(\frac{\langle x_1 - x, x_2 - x\rangle}{\|x_2 - x\|^2} - 1 \right) (x_2 - x) \right) \\
&= \left(\frac{\cos \sphericalangle}{l_1^2} - \frac{1}{l_1 l_2} \right) (x_1 - x) + \left(\frac{\cos \sphericalangle}{l_2^2} - \frac{1}{l_1 l_2} \right) (x_2 - x)
\end{aligned}$$

where $l_i := \|x_i - x\|$, $i = 1, 2$.[34].

Now

$$\nabla \sphericalangle(x_2, x, x_1) = -\frac{1}{\sqrt{1 - h^2}} (\nabla_x h, \nabla_{x1} h, \nabla_{x_2} h) = -\frac{1}{\sin \alpha} (\nabla_x h, \nabla_{x1} h, \nabla_{x_2} h) \tag{B.3}$$

Note that $\nabla_{x_i} \sphericalangle(x_2, x, x_1) \perp (x_i - x)$.

Finally the gradient of the circular energy for the whole net f with $x = f(v)$, $x_1 = \tau_1 f(v)$,... for $v \in \mathbb{Z}^2$

(B.4)

$$\nabla E_{circ}(f) = \sum_{v \in \mathbb{Z}^2} 2 \left(\alpha(x_2, x, x_1) + \gamma(x_1, x_{12}, x_2) - \pi \right) \left(\nabla \alpha(x_2, x, x_1) + \nabla \gamma(x_1, x_{12}, x_2) \right)$$

Be careful to add the right components of the gradient. There are four summands which contribute to one component (corresponding to the four quads adjacent to the point $x = f(v)$)

(B.5)

$$\begin{aligned}
\nabla_x E_{circ} = {} & 2 \left(\alpha + \gamma_{12} - \pi \right) \nabla_x \alpha(x_2, x, x_1) + 2 \left(\alpha_{-1-2} + \gamma - \pi \right) \nabla_x \gamma(x_{-2}, x, x_{-1}) \\
& + 2 \left(\alpha_{-1} + \gamma_2 - \pi \right) \left(\nabla_x \alpha_{-1}(x_{-1-2}, x_{-1}, x) + \nabla_x \gamma_2(x, x_2, x_{-1-2}) \right) \\
& + 2 \left(\alpha_{-2} + \gamma_1 - \pi \right) \left(\nabla_x \alpha_{-2}(x, x_{-2}, x_{1-2}) + \nabla_x \gamma_1(x_{1-2}, x_1, x) \right)
\end{aligned}$$

For the gradient of the conical energy we get

(B.6)

$$\nabla E_{con}(f) = \sum_{v \in \mathbb{Z}^2} 2 \left(\alpha(x_2, x, x_1) + \gamma(x_{-2}, x, x_{-1}) - \pi \right) \left(\nabla \alpha(x_2, x, x_1) + \nabla \gamma(x_{-2}, x, x_{-1}) \right)$$

---

[33] $\frac{\partial}{\partial a^i} \left(\frac{\langle a, b\rangle}{\|a\| \|b\|} \right) = \frac{1}{\|a\| \|b\|} \left(b^i - \frac{\langle a, b\rangle}{\|a\|^2} a^i \right)$ for $i = 1, 2$, $a = (a^1, a^2), b = (b^1, b^2) \in \mathbb{R}^2$.

[34] We express the coefficients in term of the angle itself since it is most times available when minimizing and simplifies the expression.

There are five summands contributing to one component (corresponding to the five points of vertex star with center $x = f(v)$.

(B.7)

$$\begin{aligned}
\nabla_x E_{con} &= 2\,(\alpha + \gamma - \pi)\,(\nabla_x \alpha(x_2, x, x_1) + \nabla_x \gamma(x_{-2}, x, x_{-1})) \\
&\quad + 2\,(\alpha_1 + \gamma_1 - \pi)\,\nabla_x \gamma_1(x_{1-2}, x_1, x) + 2\,(\alpha_2 + \gamma_2 - \pi)\,\nabla_x \gamma_2(x, x_2, x_{-12}) \\
&\quad + 2\,(\alpha_{-1} + \gamma_{-1} - \pi)\,\nabla_x \alpha_{-1}(x, x_{-1}, x_{-12}) + 2\,(\alpha_{-2} + \gamma_{-2} - \pi)\,\nabla_x \alpha_{-2}(x, x_{-2}, x_{1-2})
\end{aligned}$$

Appendix C. Ergodic theorem for Markov chains

Theorem C.1. *Let $M \in \mathbb{N}$, $A = (a_{ij})_{i,j=1,\dots,M} \in \mathbb{R}^{M\times M}$ be a stochastic matrix, i.e. $a_{ij} > 0$ for $i,j = 1,\dots,M$ and*

$$\sum_{i=1}^{M} a_{ij} = 1 \tag{C.1}$$

for $j = 1,\dots,M$.
Denote $\left(a_{ij}^{(k)}\right)_{i,j=1,\dots,M} \coloneqq A^k$.
If for every $i,j = 1,\dots,M$ there is a $k \in \mathbb{N}$ such that $a_{ij}^{(k)} > 0$, then for any $t \in \mathbb{R}^M$ with non-negative entries

$$\lim_{n\to\infty} A^n t = b \tag{C.2}$$

where $b \in \mathbb{R}^M$ is the only vector with non-negative entries satisfying

$$Ab = b \tag{C.3}$$

and $|b| = |t|$.

Remark C.1. We want our state vectors to be column vectors getting multiplied by the stochastic matrix from the left upon each step. So, we use the more unusual convention of stochastic matrices with columns adding up to one instead of rows.

Proof. See for example [Geo00]. □

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