Discrete z^N

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Overview

- Introduction, discrete conformal maps
- Second constraint from monodromy problem
- Discrete z^γ, $0 < \gamma < 2$
- Discrete z^N, $N \in \mathbb{N}$

Discrete conformal map

Definition

$f : \mathbb{Z}^2 \to \mathbb{C}$ discrete conformal

$$:\Leftrightarrow \quad q(f, f_1, f_{12}, f_2) = \frac{f - f_1}{f_1 - f_{12}} \frac{f_{12} - f_2}{f_2 - f} = -1 \tag{1}$$

for all elementary quadrilaterals (f, f_1, f_{12}, f_2)

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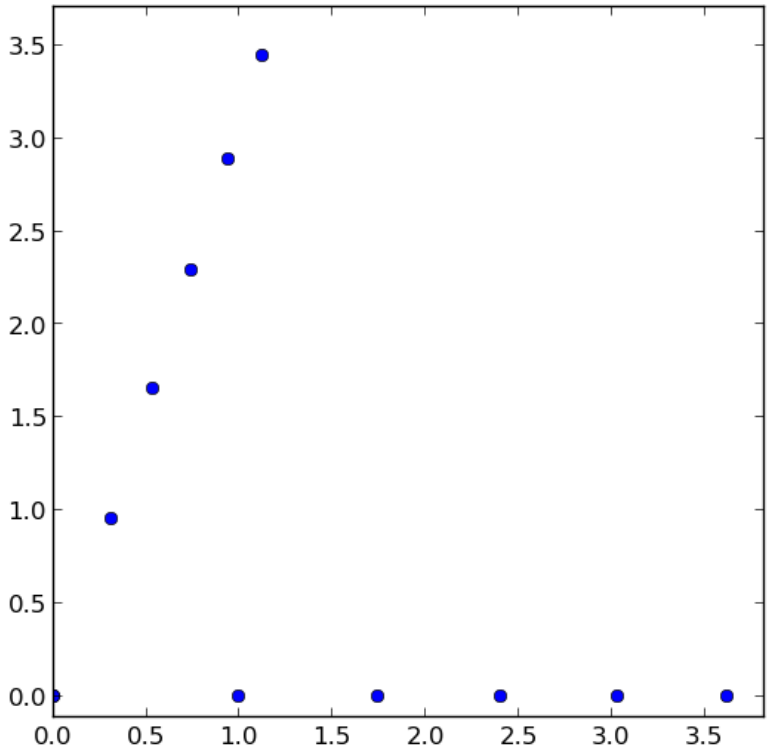
- Möbius invariant
- smooth limit: $-1 = q \to \frac{f_x^2}{f_y^2}$

First try to obtain discrete z^γ

Initial data on coordinate axes from continuous case:

$$f(m,0) = m^\gamma, \quad f(0,n) = (in)^\gamma \quad \text{for } m,n \geqslant 0$$



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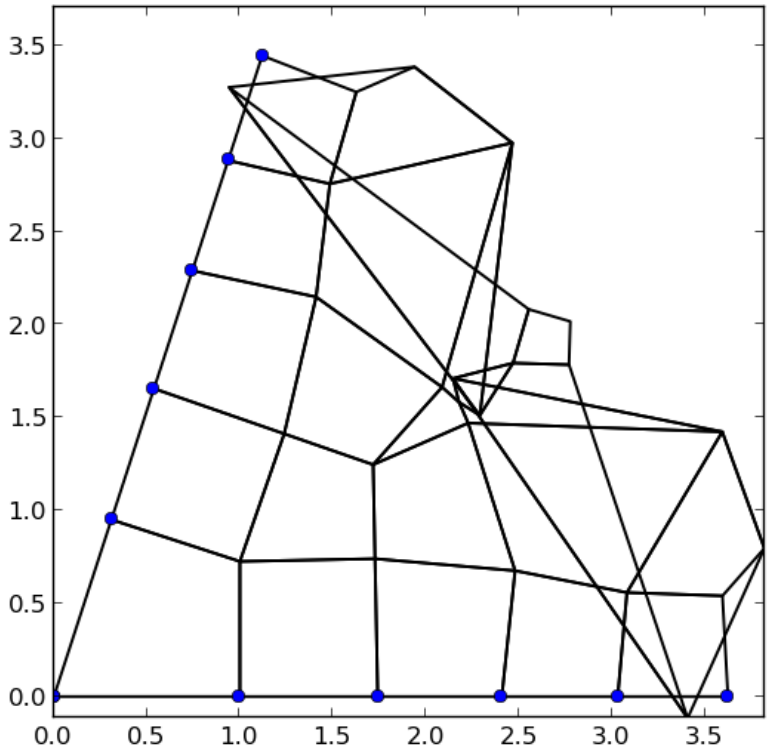

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How to obtain immersed discrete conformal z^γ?

Isomonodromic solutions

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Let $f : \mathbb{Z}^2 \to \mathbb{C}$,

$$U(\lambda) = \begin{pmatrix} 1 & f - f_1 \\ \frac{\lambda}{f_1 - f} & 1 \end{pmatrix}, \quad V(\lambda) = \begin{pmatrix} 1 & f - f_2 \\ \frac{\lambda}{f - f_2} & 1 \end{pmatrix},$$

$$A(\lambda) = \frac{B}{1+\lambda} + \frac{C}{1-\lambda} + \frac{D}{\lambda} : \mathbb{Z}^2 \to \mathrm{gl}(2, \mathbb{C}).$$

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$$\beta f^2 + \gamma f + \delta = 2(m - \phi)\frac{(f_1 - f)(f - f_{-1})}{(f_1 - f_{-1})} + 2(n - \theta)\frac{(f_2 - f)(f - f_{-2})}{(f_2 - f_{-2})}.$$

Discrete z^γ, $0 < \gamma < 2$

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Definition

For $0 < \gamma < 2$ define $Z^\gamma : \mathbb{Z}_+^2 \to \mathbb{C}$ as the solution of (1) and (2) with initial conditions:

$$Z^\gamma(0,0) = 0, \quad Z^\gamma(1,0) = 1, \quad Z^\gamma(0,1) = e^{i\frac{\pi}{2}\gamma}$$

Discrete z^γ, $0 < \gamma < 2$

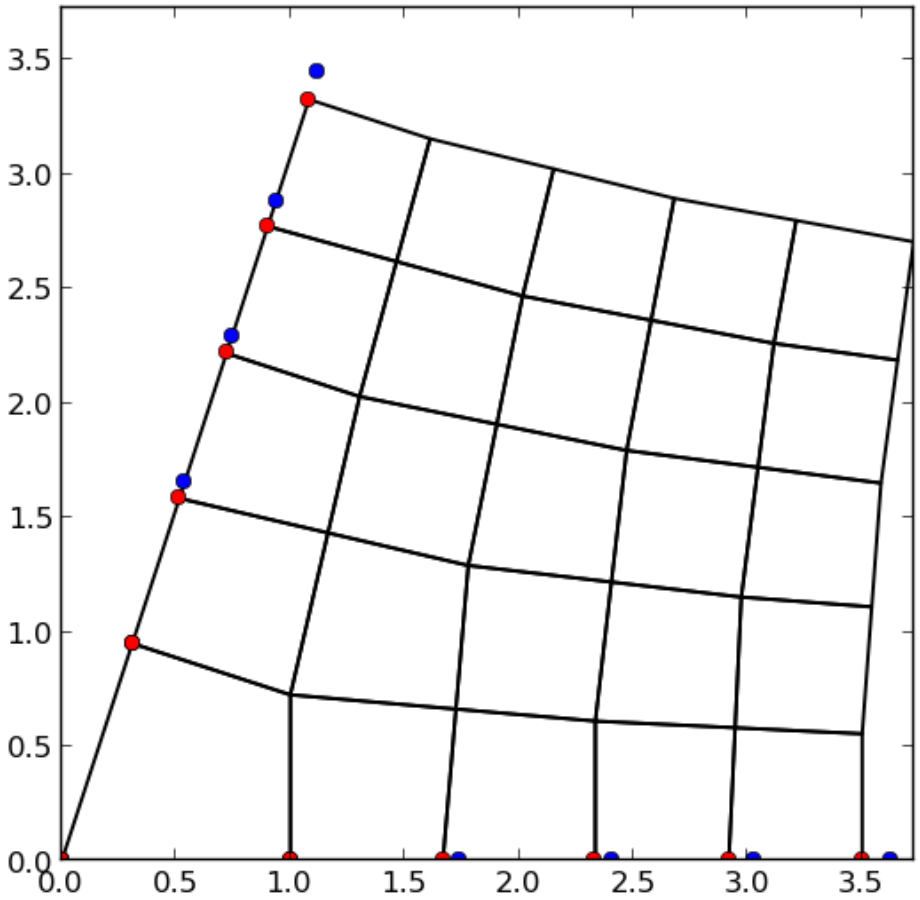

Figure: $\gamma = 0.8$

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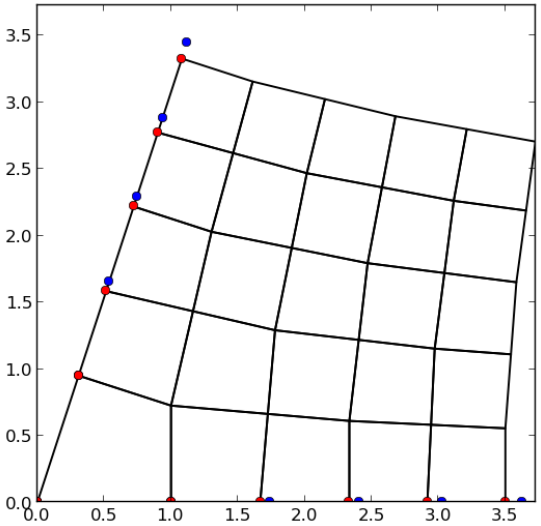

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Theorem (Agafonov, Bobenko)

Discrete Z^γ, $0 < \gamma < 2$ is immersed.

Discrete z^N, $N \in \mathbb{N}$

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For $N \in \mathbb{N}$ define $Z^N : \mathbb{Z}_+^2 \to \mathbb{C}$ as the solution of (1) and (2) with initial conditions:

$$Z^N(m,n) = 0 \quad \text{for } n+m < N,$$
$$Z^N(N,0) = 1,$$
$$Z^N(N-1,1) = i\frac{2^{N-1}\Gamma^2(\frac{N}{2})}{\pi\Gamma(N)} = \begin{cases} i\frac{(N-2)!!}{(N-1)!!} & \text{for } N \text{ odd} \\ i\frac{2}{\pi}\frac{(N-2)!!}{(N-1)!!} & \text{for } N \text{ even} \end{cases}$$

Discrete z^N on the real line

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- "integral property": $x_{2k}^{(N+1)} = \sum_{l=0}^{k} x_{2l}^{(N)}$

(2) at the origin of discrete z^N

Proposition (equidistant property at the origin)

Let $m + n = N$, $f_{-1} = f_{-2} = 0$, $|f| = |f_1 - f| =: r$.
Then

$$(2) \Rightarrow |f_2 - f| = r$$

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↳ (1) $\Rightarrow$ orthogonal circle pattern

Radii at the origin

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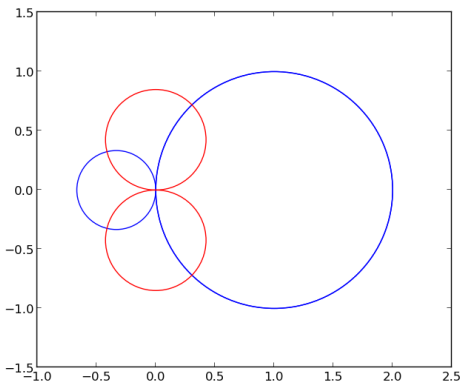

Figure: $N = 4$

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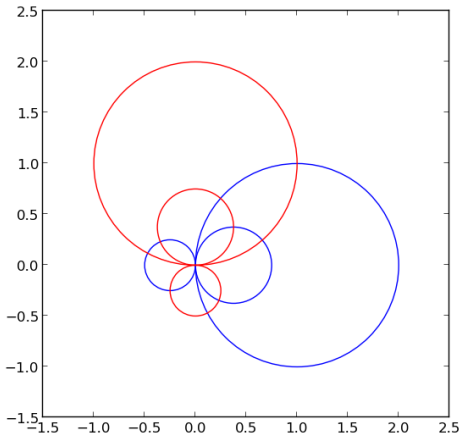

Figure: $N = 5$

Initial conditions for odd N from symmetry

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$$\vdots$$

$$r_N = \frac{(N-1)(N-3)\ldots 4 \cdot 2}{(N-2)(N-4)\ldots 3 \cdot 1} r_1 = \frac{(N-1)!!}{(N-2)!!} r_1$$

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$$r_N = 1 \quad \Rightarrow \quad Z^N(N-1,1) = i r_1 = i \frac{(N-2)!!}{(N-1)!!}$$

Initial conditions for all N from asymptotics

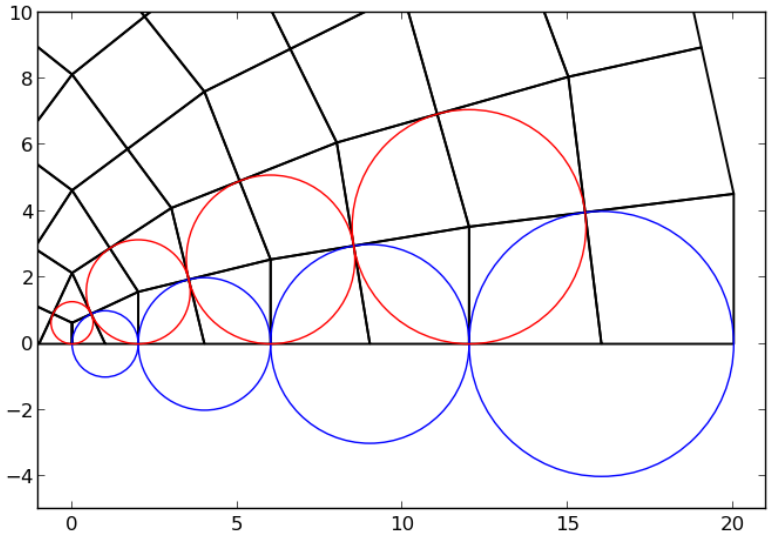

Figure: $N = 2$

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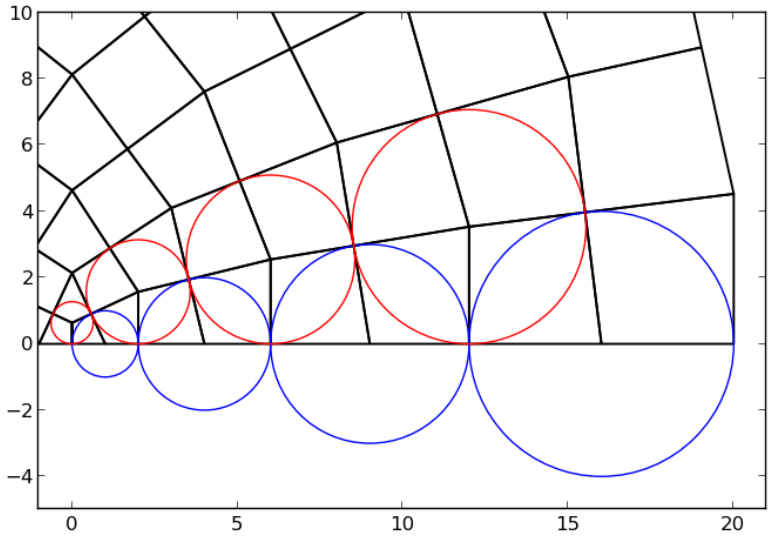

Figure: $N = 2$

$$\frac{r_k}{R_k} \to 1 \quad \Rightarrow \quad Z^N(N-1,1) \text{ for all } N \in \mathbb{N}$$

References