Geometry 2
Discrete Differential Geometry

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Contents

1 Curves in projective geometry

First, we introduce the notions of regularity, tangent lines, and osculating plane for curves in the Euclidean space $\mathbb{R}^n$. Then we consider their lift and corresponding definitions in the projective space $\mathbb{RP}^n$, and check in which sense these notions are projectively well-defined and invariant. We assume $n \geqslant 2$.

We start by recalling the definition of a curve in $\mathbb{R}^n$.

Definition 1.1. Let $I \subset \mathbb{R}$ be an interval. Then a smooth map

$$\gamma : I \to \mathbb{R}^n$$

is called a *(smooth parametrized) curve* in $\mathbb{R}^n$.

We usually denote the curve parameter by t and the derivatives with respect to t by

$$\gamma(t), \dot{\gamma}(t), \ddot{\gamma}(t), \ldots$$

We will mostly deal with regular curves, that is curves for which the first derivative does not vanish.

Definition 1.2. A curve $\gamma : I \to \mathbb{R}^n$ is called *regular* if

$$\dot{\gamma}(t) \neq 0 \qquad \text{for all } t \in I.$$

For a regular curve the tangent line is well-defined at every point of the curve.

Definition 1.3. Let $\gamma : I \to \mathbb{R}^n$ be a regular curve. Then the line

$$T(t) := \{\gamma(t) + \alpha\dot{\gamma}(t) \mid \alpha \in \mathbb{R}\}$$

is called the *tangent line* of γ at $t \in I$.

The tangent line is the line that best approximates the curve at some point up to first order. Similarly, the osculating plane is the plane that best approximates the curve at some point up to second order.

Definition 1.4. Let $\gamma : I \to \mathbb{R}^n$ be a regular curve. If additionally $\dot{\gamma}(t)$ and $\ddot{\gamma}(t)$ are linearly independent, then the plane

$$\{\gamma(t) + \alpha\dot{\gamma}(t) + \beta\ddot{\gamma}(t) \mid \alpha, \beta \in \mathbb{R}\}$$

is called the *osculating plane* of γ at $t \in I$.

We can lift a curve $\gamma : I \to \mathbb{R}^n$ to the projective space $\mathbb{RP}^n$ by

$$[\hat{\gamma}] : I \to \mathbb{RP}^n, \quad \hat{\gamma}(t) := \begin{pmatrix} \gamma(t) \\ 1 \end{pmatrix}$$

If γ is regular, then

$$\dot{\hat{\gamma}}(t) = \begin{pmatrix} \dot{\gamma}(t) \\ 0 \end{pmatrix} \neq 0$$

describes a point at infinity on the lift of the tangent line

$$\mathrm{P}\left\{\alpha_1\hat{\gamma}(t) + \alpha_2\dot{\hat{\gamma}}(t) \;\middle|\; \alpha_1, \alpha_2 \in \mathbb{R}\right\} = [\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)].$$

Immediately the following questions arise:

- What happens if we choose different representative vectors for the lift of the curve?
- Are the point $[\dot{\hat{\gamma}}(t)]$ and the tangent line well-defined by the curve?

Let us answer these questions in a more general setup.

Definition 1.5. Let $I \subset \mathbb{R}$ be an interval and $\hat{\gamma} : I \to \mathbb{R}^{n+1}$ a smooth map. Then

$$[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n, \qquad t \mapsto [\hat{\gamma}(t)]$$

is called a *(smooth parametrized) curve* in $\mathbb{R}\mathrm{P}^n$.

Consider a curve $[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n$ and a smooth function

$$\lambda : I \to \mathbb{R}\backslash\{0\}.$$

Then $\hat{\gamma}$ and $\tilde{\gamma} := \lambda\hat{\gamma}$ define the same curve in $\mathbb{R}\mathrm{P}^n$,

$$[\hat{\gamma}(t)] = [\tilde{\gamma}(t)] \qquad \text{for all } t \in I.$$

But the first derivative changes in the following way

$$\dot{\tilde{\gamma}}(t) = \dot{\lambda}(t)\hat{\gamma}(t) + \lambda(t)\dot{\hat{\gamma}}(t).$$

Thus, in general, $[\dot{\tilde{\gamma}}(t)] \neq [\dot{\hat{\gamma}}(t)]$. However, the point $[\dot{\tilde{\gamma}}(t)]$ still lies in the span $[\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)]$, and we have

$$[\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)] = [\tilde{\gamma}(t)] \vee [\dot{\tilde{\gamma}}(t)]$$

In particular, $[\hat{\gamma}(t)] = [\dot{\hat{\gamma}}(t)]$ if and only if $[\tilde{\gamma}(t)] = [\dot{\tilde{\gamma}}(t)]$. Thus, the following definition of regularity is independent of the choice of representative vectors. Furthermore, in affine coordinates, it coincides with the corresponding definition for curves in $\mathbb{R}^n$.

Definition 1.6. A curve $[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n$ is called *regular* if

$$[\hat{\gamma}(t)] \neq [\dot{\hat{\gamma}}(t)] \qquad \text{for all } t \in I.$$

For a regular curve the span $[\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)]$ is a line, which again is independent of the choice of representative vectors. Indeed, in this case, by choice of the function λ, the point $[\dot{\tilde{\gamma}}(t)]$ can be mapped to any point on this line except $[\hat{\gamma}(t)] = [\tilde{\gamma}(t)]$. In affine coordinates, this line coincides with the tangent line as defined for curves in $\mathbb{R}^n$.

Definition 1.7. Let $[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n$ be a regular curve. Then the line

$$T(t) := [\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)]$$

is called the *tangent line* of $[\hat{\gamma}]$ at $t \in I$.

Similarly, for higher derivatives, in general, $[\ddot{\tilde{\gamma}}(t)] \neq [\ddot{\hat{\gamma}}(t)]$, but

$$[\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)] \vee [\ddot{\hat{\gamma}}(t)] = [\tilde{\gamma}(t)] \vee [\dot{\tilde{\gamma}}(t)] \vee [\ddot{\tilde{\gamma}}(t)],$$

and thus, this plane is independent of the choice of representative vectors. In affine coordinates, it coincides with the osculating plane as defined for curves in $\mathbb{R}^n$.

Definition 1.8. Let $[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n$ be a regular curve. If additionally $[\ddot{\hat{\gamma}}(t)] \notin [\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)]$, then the plane

$$[\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)] \vee [\ddot{\hat{\gamma}}(t)]$$

is called the *osculating plane* of $[\hat{\gamma}]$ at $t \in I$.

Thus, we have found that regularity, tangent line, and osculating plane are well-defined for a curve in $\mathbb{R}\mathrm{P}^n$ in the sense that their definition is independent of the choice of representative vectors.

Proposition 1.1. *For a curve $[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n$, regularity, tangent line, and osculating plane are independent of the choice of representative vectors*

$$\hat{\gamma}(t) \mapsto \lambda(t)\hat{\gamma}(t)$$

with a smooth non-vanishing function λ.

Next, we investigate how these properties of a curve depend on the parametrization. Thus, let $I, \tilde{I} \subset \mathbb{R}$ be two intervals, $[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n$ a curve, and

$$\varphi : \tilde{I} \to I$$

a smooth bijective map. Then, $\tilde{\gamma} := \hat{\gamma} \circ \varphi$ defines a reparametrization

$$[\tilde{\gamma}] : \tilde{I} \mapsto \mathbb{R}\mathrm{P}^n, \qquad s \mapsto [\hat{\gamma} \circ \varphi(s)].$$

Its derivative

$$\dot{\tilde{\gamma}}(s) = (\dot{\hat{\gamma}} \circ \varphi)(s)\, \varphi'(s)$$

defines the same point

$$[\dot{\tilde{\gamma}}(s)] = [\dot{\hat{\gamma}} \circ \varphi(s)].$$

Thus, regularity, the tangent line

$$[\tilde{\gamma}(s)] \vee [\dot{\tilde{\gamma}}(s)] = [\hat{\gamma} \circ \varphi(s)] \vee [\dot{\hat{\gamma}} \circ \varphi(s)],$$

and similarly the osculating plane are invariant under reparametrization.

Proposition 1.2. *For a curve $[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n$, regularity, tangent line, and osculating plane are invariant under reparametrization*

$$\hat{\gamma}(t) \mapsto \hat{\gamma} \circ \varphi(s)$$

with a smooth bijective function φ.

Finally, how do these properties change under projective transformations? Let $[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n$ be a curve, and

$$F \in \mathrm{GL}(n+1, \mathbb{R}),$$

i.e.,

$$f := [F] \in \mathrm{PGL}(n+1, \mathbb{R})$$

is a projective transformation. Then $\tilde{\gamma} := F\hat{\gamma}$ defines the transformed curve

$$[\tilde{\gamma}] : I \to \mathbb{R}\mathrm{P}^n, \qquad t \mapsto f([\hat{\gamma}(t)]) = [F\hat{\gamma}(t)]$$

Its derivative
$$\dot{\tilde{\gamma}}(t) = F\dot{\hat{\gamma}}(t)$$
defines a point, which is transformed by the same projective transformation
$$[\dot{\tilde{\gamma}}(t)] = [F\dot{\hat{\gamma}}(t)] = f([\dot{\hat{\gamma}}(t)])$$
Thus, regularity, the tangent line
$$[\tilde{\gamma}(t)] \vee [\dot{\tilde{\gamma}}(t)] = f\left([\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)]\right),$$
and similarly the osculating plane are invariant under projective transformations.

Proposition 1.3. *For a curve $[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n$, regularity, tangent line, and osculating plane are invariant under projective transformations*
$$\hat{\gamma}(t) \mapsto F\hat{\gamma}(t)$$
with $F \in \mathrm{GL}(n+1, \mathbb{R})$.

Discrete curves

As a discrete version of Definition 1.5, we could consider maps of the form $I \to \mathbb{R}\mathrm{P}^n$ with some interval $I \subset \mathbb{Z}$. For simplicity, to avoid treating boundary cases, we mostly use maps defined on all of $\mathbb{Z}$.

Definition 1.9. A map
$$\gamma : \mathbb{Z} \to \mathbb{R}\mathrm{P}^n$$
is called a *discrete curve* in $\mathbb{R}\mathrm{P}^n$. We denote its *vertices* by
$$\gamma_k := \gamma(k) \qquad \text{for } k \in \mathbb{Z}.$$

We can define regularity, tangent line, and osculating plane in ways very similar to the smooth case.

Definition 1.10. Let $\gamma : \mathbb{Z} \to \mathbb{R}\mathrm{P}^n$ be a discrete curve.

(i) γ is called regular if any two successive points γ_k, γ_{k+1} are distinct.

(ii) If γ is regular, then the line
$$T_k := \gamma_k \vee \gamma_{k+1}$$
is called the *(edge) tangent line* at the edge $(k, k+1)$.

(iii) If γ is regular and additionally $\gamma_{k-1}, \gamma_k, \gamma_{k+1}$ are not collinear, then the plane
$$\gamma_{k-1} \vee \gamma_k \vee \gamma_{k+1}$$
is called the *(vertex) osculating plane* at k.

Note how in the discrete case regularity, tangent line, and osculating plane are immediately seen to be independent of the choice of representative vectors and invariant under projective transformations. Reparametrization, on the other hand, is not a well-defined concept for discrete curves.

2 Surfaces in projective geometry

We now turn to parametrized surfaces. Again first in $\mathbb{R}^n$, before we lift them to $\mathbb{RP}^n$. We assume $n \geqslant 3$.

Definition 2.1. Let $U \subset \mathbb{R}^n$ be an open rectangle. Then a smooth map

$$x : U \to \mathbb{R}^n, \qquad (u, v) \mapsto x(u, v)$$

is called a *(smooth parametrized) surface (patch)* in $\mathbb{R}^n$.

The curves

$$u \mapsto x(u, v = \text{const}), \qquad v \mapsto x(u = \text{const}, v)$$

are called *parameter lines* of x.

We usually denote the two parameters by u and v, and the partial derivatives with respect to u and v by

$$x_u := \frac{\partial x}{\partial u}, \quad x_v := \frac{\partial x}{\partial v}.$$

For surfaces regularity is defined by the linear independence of the first partial derivatives.

Definition 2.2. A surface $x : U \to \mathbb{R}^n$ is called *regular* if $x_u(u, v)$ and $x_v(u, v)$ are linearly independent at every point $(u, v) \in U$.

For a regular surface the parameter lines are regular curves, and the tangent plane is well-defined at every point. It is the plane that best approximates the surface at some point up to first order.

Definition 2.3. Let $x : U \to \mathbb{R}^n$ be a regular surface. Then the plane

$$Tx(u, v) := \{x(u, v) + \alpha x_u(u, v) + \beta x_v(u, v) \mid \alpha, \beta \in \mathbb{R}\}$$

is called the *tangent plane* of x at $(u, v) \in U$.

Similar to curves, we can lift a surfaces $x : U \to \mathbb{R}^n$ to the projective space $\mathbb{RP}^n$ by

$$[\hat{x}] : U \to \mathbb{RP}^n, \qquad \hat{x}(u, v) := \begin{pmatrix} x(u, v) \\ 1 \end{pmatrix}.$$

If x is regular, the partial derivatives

$$\hat{x}_u(u, v) = \begin{pmatrix} x_u(u, v) \\ 0 \end{pmatrix}, \qquad \hat{x}_v(u, v) = \begin{pmatrix} x_v(u, v) \\ 0 \end{pmatrix},$$

describe points at infinity on the lift of the tangent plane

$$\mathrm{P}\{\alpha_1 \hat{x}(u, v) + \alpha_2 \hat{x}_u(u, v) + \alpha_3 x_v(u, v) \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\} = [x(u, v)] \vee [x_u(u, v)] \vee [x_v(u, v)].$$

Generally, we define projective surfaces in the following way.

Definition 2.4. Let $U \subset \mathbb{R}^2$ be an open rectangle and $\hat{x} : U \to \mathbb{R}^{n+1}$ a smooth map. Then

$$[\hat{x}] : U \to \mathbb{RP}^n, \qquad (u, v) \mapsto [\hat{x}(u, v)]$$

is called a *(smooth parametrized) surface (patch)* in $\mathbb{RP}^n$.

The curves

$$u \mapsto [\hat{x}](u, v = \text{const}), \qquad v \mapsto [\hat{x}](u = \text{const}, v)$$

are called *parameter lines* of $[\hat{x}]$.

Consider a surface $[\hat{x}] : U \to \mathbb{R}\mathrm{P}^n$ and a smooth function

$$\lambda : U \to \mathbb{R}\backslash\{0\}.$$

Then $\hat{x}$ and $\tilde{x} := \lambda\hat{x}$ define the same surface in $\mathbb{R}\mathrm{P}^n$,

$$[\hat{x}(u,v)] = [\tilde{x}(u,v)] \qquad \text{for all } (u,v) \in U.$$

Similarly to the considerations for curves, the points represented by the first partial derivatives may change, but the span

$$[\hat{x}(u,v)] \vee [\hat{x}_u(u,v)] \vee [\hat{x}_v(u,v)] = [\tilde{x}(u,v)] \vee [\tilde{x}_u(u,v)] \vee [\tilde{x}_v(u,v)]$$

remains the same. Thus, the following definition of regularity for surfaces in $\mathbb{R}\mathrm{P}^n$ is independent of the choice of representative vectors. Furthermore, in affine coordinates, it coincides with the corresponding definition for surfaces in $\mathbb{R}^n$.

Definition 2.5. A surface $[\hat{x}] : U \to \mathbb{R}\mathrm{P}^n$ is called *regular* if $[\hat{x}(u,v)]$, $[\hat{x}_u(u,v)]$, $[\hat{x}_v(u,v)]$ span a plane, or equivalently, if $\hat{x}(u,v)$, $\hat{x}_u(u,v)$, $\hat{x}_v(u,v)$ are linearly independent.

The same holds for the following definition of the tangent planes for surfaces in $\mathbb{R}\mathrm{P}^n$.

Definition 2.6. Let $[\hat{x}] : U \to \mathbb{R}\mathrm{P}^n$ be a regular surface. Then the plane

$$T[\hat{x}](u,v) := [\hat{x}(u,v)] \vee [\hat{x}_u(u,v)] \vee [\hat{x}_v(u,v)]$$

is called the *tangent plane* of $[\hat{x}]$ at $(u,v) \in U$.

Similar to the considerations for curves, one finds that the introduced notions are also invariant under reparametrization and under projective transformations. We summarize this in the following proposition.

Proposition 2.1. *For a surface $[\hat{x}] : U \to \mathbb{R}\mathrm{P}^n$, regularity and the tangent plane are invariant under*

(i) a change of representative vectors

$$\hat{x}(u,v) \mapsto \lambda(u,v)\hat{x}(u,v)$$

with a smooth non-vanishing function λ.

(ii) reparametrization

$$\hat{x}(u,v) \mapsto \hat{x} \circ \varphi(\tilde{u},\tilde{v})$$

with a smooth bijective map φ.

(iii) projective transformations

$$\hat{x}(u,v) \mapsto F\hat{x}(u,v)$$

with $F \in \mathrm{GL}(n+1,\mathbb{R})$.

Discrete parametrized surfaces

As a discrete version of Definition 2.4, we could consider maps of the form $U \to \mathbb{R}\mathrm{P}^n$ with some rectangle $U \subset \mathbb{Z}$. Again, for simplicity, to avoid treating boundary cases, we mostly use maps defined on all of $\mathbb{Z}^2$.

Definition 2.7. A map

$$x : \mathbb{Z}^2 \to \mathbb{R}\mathrm{P}^n$$

is called a *discrete (parametrized) surface* or *(2-dimensional) discrete net* in $\mathbb{R}\mathrm{P}^n$.

We use subscripts to denote shifts

$$x_{\pm 1}(i,j) = x(i \pm 1, j), \qquad x_{\pm 2}(i,j) = x(i, j \pm 1).$$

The discrete curves

$$i \mapsto x(i, j = \mathrm{const}), \qquad j \mapsto x(i = \mathrm{const}, j)$$

are called *discrete parameter lines.*

A discrete surface $x : \mathbb{Z}^2 \to \mathbb{R}\mathrm{P}^n$ is called *regular* if any three points of each face span a plane.

The plane spanned by three such points

$$x \vee x_{\pm 1} \vee x_{\pm 2}$$

at some $(i,j) \in \mathbb{Z}^2$ may be thought of as a discrete tangent plane, which is assigned to the corresponding corner of the face.

3 Ruled surfaces and developable surfaces

A ruled surface is a surface traced by the movement of a straight line through space.

Definition 3.1. Let $[\hat{a}], [\hat{b}] : I \to \mathbb{R}\mathrm{P}^n$ be two curves such that $[\hat{a}], [\hat{b}], [\dot{\hat{a}}], [\dot{\hat{b}}]$ do not lie on a line. Then the surface

$$[\hat{x}] : I \times \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^n, \qquad (u,v) \mapsto [\hat{a}(u) + v\hat{b}(u)]$$

is a *ruled surfaces* in $\mathbb{R}\mathrm{P}^n$. The lines $[\hat{a}(u)] \vee [\hat{b}(u)]$ on the resulting surface are called *rulings.*

To obtain the entire projective lines as rulings, we use $v \in \mathbb{R}\mathrm{P}^1 = \mathbb{R} \cup \{\infty\}$ in the definition of ruled surfaces. However, in the following, we will only consider $v \in \mathbb{R}$. For completeness, one can check that all the claims in the following also hold for another affine chart of $\mathbb{R}\mathrm{P}^1$.

Example 3.1. A one-sheeted hyperboloid is a doubly ruled surface.

A developable surface is the envelope of a one-parameter family of planes. In $\mathbb{R}\mathrm{P}^3$ a one-parameter family of planes may be described by a regular curve in the dual space

$$[\hat{n}] : I \to (\mathbb{R}\mathrm{P}^3)^*,$$

or a one-parameter family of linear equations

$$T(u) = \left\{ [y] \in \mathbb{R}\mathrm{P}^3 \;\middle|\; \hat{n}(u) \cdot y = 0 \right\}.$$

Then the envelope is the solution of the two equations

$$\begin{aligned} \hat{n} \cdot y &= 0, \\ \dot{\hat{n}} \cdot y &= 0. \end{aligned}$$

For each u these are two independent linear equations, and thus the solution is a line. Thus, in $\mathbb{R}\mathrm{P}^3$, every developable surface is a ruled surface.

Vice versa, for a ruled surface in $\mathbb{R}\mathrm{P}^3$ to be developable its tangent planes must be constant along the rulings, i.e.,

$$[\hat{x}] \vee [\hat{x}_u] \vee [\hat{x}_v] = [\hat{a} + v\hat{b}] \vee [\dot{\hat{a}} + v\dot{\hat{b}}] \vee [\hat{b}]$$

must be independent of v, which is the case, if and and only if

$$\det(\hat{a}, \hat{b}, \dot{\hat{a}}, \dot{\hat{b}}) = 0,$$

or equivalently, if $[\hat{a}], [\hat{b}], [\dot{\hat{a}}], [\dot{\hat{b}}]$ lie in a plane.

This last condition characterizes surfaces which are the envelope of a one-parameter family of planes in any dimension, and thus may be used to define developable ruled surfaces in the following way.

Definition 3.2. Let

$$[\hat{x}] = [\hat{a}] \vee [\hat{b}] : I \times \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^n$$

be a ruled surface. Then $[\hat{x}]$ is *developable* if $[\hat{a}], [\hat{b}], [\dot{\hat{a}}], [\dot{\hat{b}}]$ lie in a plane for every $u \in I$.

Infinitesimally, this condition means that close rulings intersect, and thus, if they do not all go through one point, they envelope a curve in space.

Proposition 3.1. *A ruled surface is developable if and only if it is a cone (over an arbitrary curve) or the trace of tangent lines of a regular curve.*

More specifically, let

$$[\hat{x}] = [\hat{a}] \vee [\hat{b}] : I \times \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^n$$

be a ruled surface. Then there exists a unique curve

$$[\hat{c}] : I \to \mathbb{R}\mathrm{P}^n, \qquad [\hat{c}(u)] \in [\hat{a}(u)] \vee [\hat{b}(u)]$$

such that

$$[\dot{\hat{c}}(u)] \in [\hat{a}(u)] \vee [\hat{b}(u)]$$

for all $u \in I$. In particular, at regular points $u \in I$ of $[\hat{c}]$, it holds

$$[\hat{c}(u)] \vee [\dot{\hat{c}}(u)] = [\hat{a}(u)] \vee [\hat{b}(u)].$$

*The curve $[\hat{c}]$ is called the **line of striction** (or edge of regression) of $[\hat{x}]$.*

Proof. For a cone or the trace of tangent lines of a regular curve, one easily checks that they constitute developable surfaces.

Let $[\hat{x}] = [\hat{a}] \vee [\hat{b}]$ be a developable surface, and

$$\hat{c}(u) = \lambda(u)\hat{a}(u) + \mu(u)\hat{b}(u)$$

with some functions λ, μ. Then

$$\dot{\hat{c}} = \dot{\lambda}\hat{a} + \lambda\dot{\hat{a}} + \dot{\mu}\hat{b} + \mu\dot{\hat{b}},$$

and $[\dot{\hat{c}}] \in [\hat{a}] \vee [\hat{b}]$ if and only if

$$\lambda\dot{\hat{a}} + \mu\dot{\hat{b}} \in \operatorname{span}\{\hat{a}, \hat{b}\}.$$

Such λ, μ exists, since $\hat{a}, \hat{b}, \dot{\hat{a}}, \dot{\hat{b}}$ lie in a 3-dimensional subspace. This choice is unique (up to a common scalar multiple) since they do not lie in a 2-dimensional subspace (regularity for ruled surfaces). □

Discrete ruled surfaces and discrete developable surfaces

A discrete one-parameter family of lines may be thought of as a *discrete ruled surface.* To obtain a map $\mathbb{Z}^2 \to \mathbb{RP}^n$ the lines can be sampled and corresponding points connected.

A discrete one-parameter family of lines such that adjacent lines intersect may be thought of as a *discrete developable surface.* Similar to the smooth case, they are discrete cones or the (edge) tangent lines of a regular discrete curve.

4 Conjugate line parametrizations

We now study special parametrizations, in the sense that the parameter lines satisfy some geometric condition. We start with conjugate line parametrizations, which we first introduce for surfaces in $\mathbb{R}^3$. Conjugate line paramtrizations are geometrically characterized by the following condition: Along each parameter line of the surface, the tangent planes rotate around the tangent line in the other coordinate direction. Put differently: The tangent planes along one parameter line envelope a surface that is ruled by the tangent lines in the other coordinate direction.

The tangent planes of a surface in $\mathbb{R}^3$ can be described in terms of a normal field.

Definition 4.1. Let $x : U \to \mathbb{R}^3$ be a regular surface. Then a smooth map

$$n : U \to \mathbb{R}^3 \backslash \{0\}$$

is called a *normal field* of x if

$$\begin{aligned} n \cdot x_u &= 0, \\ n \cdot x_v &= 0. \end{aligned}$$

Given a normal field n of a surface x, the tangent planes are given by

$$Tx(u,v) = \left\{ y \in \mathbb{R}^3 \;\middle|\; n(u,v) \cdot (y - x(u,v)) = n(u,v) \cdot y + h(u,v) = 0 \right\}$$

with some function $h(u,v) = -n(u,v) \cdot x(u,v)$.

Definition (and Proposition) 4.1. Let $x : U \to \mathbb{R}^3$ be a regular surface, and $n : U \to \mathbb{R}^3$ a normal field of x. Then the following conditions are equivalent:

(i) $n_v \cdot x_u = 0$
(ii) $n_u \cdot x_v = 0$
(iii) $n \cdot x_{uv} = 0$
(iv) $x_{uv} \in \operatorname{span}(x_u, x_v)$
(v) $x_{uv} = \alpha x_u + \beta x_v$ with smooth functions $\alpha, \beta : U \to \mathbb{R}$

If one and thus all of these conditions are satisfied, x is called a *conjugate line parameterization.*

Proof. Taking the v-derivative of $n \cdot x_u = 0$ and the u-derivative of $n \cdot x_v = 0$, we obtain

$$\begin{aligned} n_v \cdot x_u &= -n \cdot x_{uv} \\ n_u \cdot x_v &= -n \cdot x_{vu} \end{aligned}$$

and since $x_{uv} = x_{vu}$ by the symmetry of second derivatives, conditions (i), (ii), and (iii) are equivalent.

Condition (iii) implies (iv) because (x_u, x_v) is a basis for the orthogonal subspace to n. This also means that the equation of condition (v) determines the functions α and β uniquely. In fact, by Cramer's rule,

$$\alpha = \frac{\det(n, x_{uv}\ x_v)}{\det(n, x_u\ x_v)}, \quad \beta = \frac{\det(n, x_u\ x_{uv})}{\det(n, x_u\ x_v)}, \tag{1}$$

which also shows that α and β are smooth because x is. Finally, condition (v) clearly implies (iii) and (iv). □

Conditions (iv) and (v) of Definition 4.1 do not involve the normal field n. We may use them to define conjugate line parametrizations in $\mathbb{R}^n$. However, we cannot use equations (1) to see that α and β are smooth if $n > 3$, because the normal field is not defined. But instead we may use

$$\alpha = \frac{\det\begin{pmatrix} x_{uv} \cdot x_u & x_v \cdot x_u \\ x_{uv} \cdot x_v & x_v \cdot x_v \end{pmatrix}}{\det\begin{pmatrix} x_u \cdot x_u & x_v \cdot x_u \\ x_u \cdot x_v & x_v \cdot x_v \end{pmatrix}}, \quad \beta = \frac{\det\begin{pmatrix} x_u \cdot x_u & x_{uv} \cdot x_u \\ x_u \cdot x_v & x_{uv} \cdot x_v \end{pmatrix}}{\det\begin{pmatrix} x_u \cdot x_u & x_v \cdot x_u \\ x_u \cdot x_v & x_v \cdot x_v \end{pmatrix}}.$$

Definition 4.2. A regular surface $x : U \to \mathbb{R}^n$ is called a *conjugate line parameterization* if it satisfies one and hence both equivalent conditions (iv) and (v) of Definition 4.1.

The definition for conjugate line parametrizations translates as follows to surfaces in $\mathbb{RP}^n$:

Proposition 4.2. *Let $x : U \to \mathbb{R}^n$ be a regular surface. Let*

$$\hat{x} := \lambda \cdot (x, 1) : U \to \mathbb{R}^{n+1}$$

be an arbitrary lift to homogeneous coordinates with a smooth function $\lambda : U \to \mathbb{R}\backslash\{0\}$.

Then x is a conjugate line parametrization if and only if $\hat{x}$ satisfies

$$\hat{x}_{uv} = \alpha \hat{x}_u + \beta \hat{x}_v + \gamma \hat{x} \tag{2}$$

with some smooth functions α, β, γ.

Proof. In the standard lift $\hat{x} = (x, 1)$ the equivalence is obvious. Show that the form of equation (2) is invariant under replacing $\hat{x} \mapsto \lambda\hat{x}$ (Exercise). $\square$

Equation (2) states the linear dependence of four representative vectors, or equivalently that four points lie in a plane. While the four points are not projectively well-defined (the points defined by the derivatives are not invariant under scaling $\hat{x}$) this property is.

Definition 4.3. Let $[\hat{x}] : U \to \mathbb{RP}^n$ be a regular surface. Then $[\hat{x}]$ is called a *conjugate line parametrization* if the four points $[\hat{x}]$, $[\hat{x}_u]$, $[\hat{x}_v]$, $[\hat{x}_{uv}]$ lie in a plane for every $(u, v) \in U$.

We have seen that this property is projectively well-defined. Furthermore, it is a property of the coordinate lines. Thus, it is invariant under reparametrization of the surface along the coordinate lines. Finally, it is also invariant under applying a projective transformation to the surface. We summarize these properties in the following proposition.

Proposition 4.3. *A regular surface $[\hat{x}] : U \to \mathbb{RP}^n$ being a conjugate line parametrization is invariant under*

(i) a change of representative vectors

$$\hat{x}(u, v) \mapsto \lambda(u, v)\hat{x}(u, v)$$

with a smooth non-vanishing function λ.

(ii) reparametrization along the coordinate lines

$$\hat{x}(u, v) \mapsto \hat{x}(\varphi(\tilde{u}), \chi(\tilde{v}))$$

with two smooth bijective functions φ, χ.

(iii) projective transformations

$$\hat{x}(u, v) \mapsto F\hat{x}(u, v)$$

with $F \in \mathrm{GL}(n + 1, \mathbb{R})$.

Example 4.1. Any paramatrization of a developable ruled surface that contains its rulings in one direction is a conjugate line parametrization.

Discrete conjugate nets

For a discrete surface $x : \mathbb{Z}^2 \to \mathbb{R}^n$ consider the following discretization of condition (v) in Proposition 4.1

$$\begin{aligned} & \Delta_1\Delta_2 x = \alpha\Delta_1 x + \beta\Delta_2 x \\ \Leftrightarrow \qquad & x_{12} = x + (\alpha + 1)\Delta_1 x + (\beta + 1)\Delta_2 x, \end{aligned}$$

where Δ_i denotes the *difference operator*:

$$\Delta_i x = x_i - x.$$

This motivates the following definition.

Definition 4.4. A regular discrete net $x : \mathbb{Z}^2 \to \mathbb{RP}^n$ is called a *discrete conjugate net* (or *Q-net*) if the four points of every face x, x_1, x_{12}, x_2 lie in a plane.

Again, this condition is immediately seen to be projectively invariant. Furthermore, in the case of a discrete conjugate net, we have a unique choice for a tangent plane for every face of $\mathbb{Z}^2$. For two adjacent faces the intersection of two such tangent planes is the tangent line of the common edge.

Example 4.2. Discrete developable ruled surfaces as obtained from a discrete one-parameter family of lines such that adjacent lines intersect are discrete conjugate nets.

5 Asymptotic line parametrizations

Asymptotic line parametrizations are geometrically characterized by the following condition: Along each parameter line of the surface patch, the tangent planes rotate around the tangent line of that parameter line. Put differently: The tangent planes along each parameter line envelop a surface that is ruled by the tangent lines of that same parameter line. This leads to a description of asymptotic line parametrizations analogous to conjugate line parametrizations.

Definition (and Proposition) 5.1. Let $x : U \to \mathbb{R}^3$ be a regular surface, and $n : U \to \mathbb{R}^3$ a normal field of x. Then x is a called an *asymptotic line parameterization* if one and hence all of the following equivalent conditions hold:

(i) $n_u \cdot x_u = n_v \cdot x_v = 0$
(ii) $n \cdot x_{uu} = n \cdot x_{vv} = 0$
(iii) $x_{uu}, x_{vv} \in \mathrm{span}(x_u, x_v)$
(iv) $x_{uu} = \alpha x_u + \beta x_v$ for smooth functions $\alpha, \beta : U \to \mathbb{R}$, and
$x_{vv} = \tilde{\alpha} x_u + \tilde{\beta} x_v$ for smooth functions $\tilde{\alpha}, \tilde{\beta} : U \to \mathbb{R}$

Same as for conjugate line parametrizations, conditions (iv) and (v) of Definition 5.1 do not involve the normal field n, and we may use them to define asymptotic line parametrizations in $\mathbb{R}^n$:

Definition 5.1. A regular surface $x : U \to \mathbb{R}^n$ is called an *asymptotic line parameterization* if it satisfies one and hence both equivalent conditions (iv) and (v) of Definition 5.1.

The definition for asymptotic line parametrizations translates as follows to surfaces in $\mathbb{RP}^n$:

Proposition 5.2. *Let $x : U \to \mathbb{R}^n$ be a regular surface. Let*

$$\hat{x} := \lambda \cdot (x, 1) : U \to \mathbb{R}^{n+1}$$

be an arbitrary lift to homogeneous coordinates with a smooth function $\lambda : U \to \mathbb{R}\backslash\{0\}$.

Then x is an asymptotic line parametrization if and only if $\hat{x}$ satisfies

$$\hat{x}_{uu} = \alpha \hat{x}_u + \beta \hat{x}_v + \gamma \hat{x}$$
$$\hat{x}_{vv} = \tilde{\alpha} \hat{x}_u + \tilde{\beta} \hat{x}_v + \tilde{\gamma} \hat{x}$$

with some smooth functions $\alpha, \beta, \gamma, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$.

And thus, it generalizes to the following definition.

Definition 5.2. Let $[\hat{x}] : U \to \mathbb{R}\mathrm{P}^n$ be a regular surface. Then $[\hat{x}]$ is called an *asymptotic line parametrization* if the points $[\hat{x}_{uu}]$ and $[\hat{x}_{vv}]$ both lie in the tangent plane $[\hat{x}] \vee [\hat{x}_u] \vee [\hat{x}_v]$ for every $(u, v) \in U$.

Note that the condition $[\hat{x}_{uu}] \in [\hat{x}] \vee [\hat{x}_u] \vee [\hat{x}_v]$ is equivalent to

$$[\hat{x}] \vee [\hat{x}_u] \vee [\hat{x}_{uu}] = [\hat{x}] \vee [\hat{x}_u] \vee [\hat{x}_v].$$

This means that the osculating plane of the u-parameter line coincides with the tangent plane. And similarly for the v-parameter line.

Proposition 5.3. *Let $[\hat{x}] : U \to \mathbb{R}\mathrm{P}^n$ be a regular surface. Then $[\hat{x}]$ is an asymptotic line parametrization if and only if the two osculating planes for the two parameter lines coincide at every point:*

$$[\hat{x}] \vee [\hat{x}_u] \vee [\hat{x}_{uu}] = [\hat{x}] \vee [\hat{x}_v] \vee [\hat{x}_{vv}].$$

In particular they both coincide with the tangent plane at that point.

The statements from Proposition 4.3 similarly hold for asymptotic line parametrizations.

Discrete asymptotic nets

To obtain a discretization of asymptotic line parametrizations consider the characerization in Proposition 5.3. At every vertex of a discrete surface, we have two osculating planes of the two discrete parameter lines that contain this vertex. These two osculating planes coincide if and only if all five points of the vertex star are coplanar.

Definition 5.3. A regular discrete net $x : \mathbb{Z}^2 \to \mathbb{R}\mathrm{P}^n$ is called a *discrete asymptotic net* (or *A-net*) if the five points of every vertex star $x, x_{\bar{1}}, x_1, x_{\bar{2}}, x_2$ lie in a plane.

Again, this condition is immediately seen to be projectively invariant. Furthermore, in the case of a discrete asymptotic net, we have a unique choice for a tangent plane for every vertex of $\mathbb{Z}^2$. For two adjacent vertices the intersection of two such tangent planes is the tangent line of the common edge.

6 Conjugate nets and fundamental transformations

Coming back to conjugate line parametrizations, we may ask the question whether it is possible to construct higher dimensional parametrization / coordinate systems consisting of conjugate lines. That is a map

$$[\hat{x}] : \mathbb{R}^N \supset U \to \mathbb{R}\mathrm{P}^n, \qquad (u_1, \ldots, u_N) \mapsto [\hat{x}](u_1, \ldots, u_N)$$

such that the 2-dimensional restrictions

$$[\hat{x}]\big|_{u+\mathcal{B}_{ij}}, \qquad u \in \mathbb{R}^N, 0 \leqslant i < j \leqslant N$$

where

$$\mathcal{B}_{ij} = \left\{ u \in \mathbb{R}^N \;\middle|\; u_k = 0 \text{ for all } k \neq i, j \right\}$$

are conjugate line parametrizations, i.e.,

$$[\hat{x}], [\partial_i \hat{x}], [\partial_j \hat{x}], [\partial_i \partial_j \hat{x}] \text{ coplanar for all } 0 \leqslant i < j \leqslant N.$$

Thus, in affine coordinates $\hat{x} = (x, 1)$, a conjugate net is governed by a system of (linear partial) differential equations of the form

$$\partial_i \partial_j x = c_{ji} \partial_i x + c_{ij} \partial_j x, \qquad i \neq j \tag{3}$$

with some coefficient functions $c_ij, c_ji : U \supset \mathbb{R}^N \to \mathbb{R}$. The compatibility of these equations, i.e., the requirement $\partial_i(\partial_j \partial_k)x = \partial_j(\partial_i \partial_k)x$, is expressed by the following system of (non-linear partial) differential equations in the coefficients

$$\partial_i c_{jk} = c_{ij} c_{jk} + c_{ji} c_{ik} - c_{jk} c_{ik}, \qquad i \neq j \neq k \neq i. \tag{4}$$

The system 3, 4 is *hyperbolic*, that means that solutions for the following Cauchy problem generically exist and determine a conjugate net uniquely:
(Q_1) the values of x on the coordinate axes $\mathcal{B}_i$,
(Q_2) the values of c_{ij}, c_{ji} on the coordinate planes $\mathcal{B}_{ij}$.

Fundamental transformations

Instead of extending a conjugate line parametrized surface to triply conjugate coordinates, one can define the following transformation.

Definition 6.1. Let $[\hat{x}], [\hat{x}^+] : \mathbb{R}^2 \supset U \to \mathbb{RP}^n$ be two conjugate line parametrizations. Then $[\hat{x}^+]$ is called a *fundamental transform (F-transform)* of $[\hat{x}]$ if $[\hat{x}], [\hat{x}_u], [\hat{x}^+], [\hat{x}^+_u]$ are coplanar and $[\hat{x}], [\hat{x}_v], [\hat{x}^+], [\hat{x}^+_v]$ are coplanar at every point $u \in U$.

Equivalently, the ruled surfaces that are obtained by joining corresponding parameter lines of $[\hat{x}]$ and $[\hat{x}^+]$ are developable.

Thus, while triply conjugate coordinates are built from planar infinitesimal quadrilaterals in all directions, a conjugate line parametrized surfaces and its fundamental transformation have planar infinitesimal quadrilaterals in one pair of directions and planar infinitesimally narrow quadrilaterals in all other pairs of directions.

In affine coordinates, $\hat{x} = (x, 1)$, given a conjugate line parametrization x, its fundamental transformations are governed by the (linear partial) differential equations

$$\begin{aligned} x^+_u &= a_1 x_u + b_1(x^+ - x), \\ x^+_v &= a_2 x_v + b_2(x^+ - x). \end{aligned}$$

Together with the corresponding compatibility conditions for the coefficient functions a_i, b_i, the following data uniquely determines a fundamental transformation:
(F_1) a point $x^+(0, 0)$,
(F_2) the values a_i, b_i on the coordinate axes $\mathcal{B}_i$, i = 1, 2.

The fundamental transformation satisfies the following permutability theorem.

Theorem 6.1. *Let x be a conjugate line parametrized surface and let $x^{(1)}$ and $x^{(1)}$ be two of its fundamental transforms. Then there exists a two-parameter family of conjugate line parametrized surfaces $x^{(12)}$ that are fundamental transforms of both $x^{(1)}$ and $x^{(12)}$. Corresponding points of the four conjugate nets x, $x^{(1)}$, $x^{(2)}$, $x^{(12)}$ are coplanar.*

Small patches of the four surfaces may be embedded into four of the faces of a 4D-cube in such a way that all faces of the cube are either planar quads, planar infinitesimal quads, or planar infinitesimally narrow quads.

Remark 6.1. The definition of fundamental transformations as well as all properties discussed in this section immediately generalize to higher-dimensional conjugate nets $\mathbb{R}^N \supset U \to \mathbb{R}\mathrm{P}^n$.

7 Discrete conjugate nets and multi-dimensional consistency

Definition 7.1. A map
$$x : \mathbb{Z}^N \to \mathbb{R}\mathrm{P}^n$$
is called a *(N-dimensional) discrete net* in $\mathbb{R}\mathrm{P}^n$.

In the following we will always assume that all the points of a net that we are considering are in general position within any given constraints. In particular, this implies the regularity constaint that we previously defined for 2-dimensional discrete nets.

In the discrete case, the two processes of adding additional coordinate directions to a conjugate net and defining fundamental transformations become the same, and are captured by adding planar quadrilaterals to the lattice.

Definition 7.2. A discrete net $x : \mathbb{Z}^N \to \mathbb{R}\mathrm{P}^n$ is called a *discrete conjugate net* (or *Q-net*) if for every elementary quadrilateral its four points x, x_1, x_{12}, x_2 lie in a plane.

The following theorem describes how for every elementary 3-cube of a discrete conjugate net, any seven points determine the eighth.

Theorem 7.1. *Given seven points $x, x_1, x_2, x_3, x_{12}, x_{23}, x_{31} \in \mathbb{R}\mathrm{P}^n$ such that each of the three quadrilaterals (x, x_i, x_{ij}, x_j) is planar, there exists a unique point $x_{123} \in \mathbb{R}\mathrm{P}^n$ such that the three quadrilaterals $(x_k, x_{ik}, x_{123}, x_{jk})$ are planar.*

Proof. The seven initial points lie in the 3-dimensional space Π_{123} spanned by x, x_1, x_2, x_3. Define the three planes
$$\tau_k \Pi_{ij} := x_k \vee x_{ik} \vee x_{jk}.$$
Then, generically, these three planes intersect in Π_{123} in a unique point
$$x_{123} := \tau_1 \Pi_{23} \cap \tau_2 \Pi_{13} \cap \tau_3 \Pi_{12}. \qquad \square$$

Thus, for $N = 3$, a discrete conjugate net can be constructed from 2-dimensional discrete conjugate nets on the coordinate planes. This means that discrete conjugate nets constitute what is called a *3D-system*.

In an elementary 4-cube of a discrete conjugate net, we may use the previous theorem to construct the four points x_{ijk} from the 2-dimensional initial data x, x_i, x_{ij}. Now, the point x_{1234} is determined in three different ways using the same method. Does this always lead to a uniquely defined point?

Definition 7.3. A 3D-system is called 4D-consistent if it can be imposed on all elementary 3-cubes of an elementary 4-cube in $\mathbb{Z}^4$ (without contradiction).

Theorem 7.2. *The 3D-system of discrete conjugate nets is 4D-consistent.*

Proof. See [BS08]. □

The N-dimensional conistency of a 3D-system is defined analogously. Generally, 4D-consistency of a 3D-system already implied consistency in all higher-dimensions.

Theorem 7.3. *A 3D-system that is 4D-consistent is also ND-consistent for all $N > 4$.*

Proof. See [BS08]. □

Discrete conjugate nets may be viewed as a discretization of (smooth) conjugate nets as well as their fundamental transformations:

- The 3D-system of discrete conjugate nets discretizes how a fundamental transform is determined by a conjugate line parametrized surfaces and one point of the transform.
- The 4D-consistency discretizes the permutability theorem of fundamental transformations (Theorem 6.1).

Theorem 7.1 describes the propagation of the 3D-system from initial data in geometric terms. Starting with the difference equations governing discrete conjugate nets (in affine coordinates)

$$\Delta_i \Delta_j x = c_{ji} \Delta_i x + c_{ij} \Delta_j x, \qquad i \neq j \tag{5}$$

we may also obtain a map describing the propagation of the coefficient functions c_{ij}, c_{ji} (defined on the 2-faces of $\mathbb{Z}^N$ by computing the compatibility conditions $\Delta_i(\Delta_j \Delta_k x) = \Delta_j(\Delta_i \Delta_k x)$ similar to the smooth case. This yields

$$\Delta_i c_{jk} = (\tau_k c_{ij}) c_{jk} + (\tau_k c_{ji}) c_{ik} - (\tau_i c_{jk}) c_{ik}, \qquad i \neq j \neq k \neq i, \tag{6}$$

which defines a birational map from three faces of a cube to the three opposite faces. Thus the following Cauchy problem determines a discrete conjugate net uniquely:
(Q_1^Δ) the values of x on the coordinate axes $\mathcal{B}_i$,
(Q_2^Δ) the values of c_{ij}, c_{ji} on all elementary quads of the coordinate planes $\mathcal{B}_{ij}$.

Point equation

While (5) and (6) are very similar to the corresponding partial differential equations, in the discrete case it is more natural to consider the *point equations*. At every face of $\mathbb{Z}^N$ the homogeneous coordinate vectors satisfy

$$A\hat{x} + B\hat{x}_i + C\hat{x}_{ij} + D\hat{x}_j = 0$$

with some coefficients A, B, C, D, which are unique per face up to a common scalar multiple. The propagation of the discrete conjugate net can then be formulated in terms of these coefficients. In affine coordinates the equation remains the same with the additional constraint $A + B + C + D = 0$ and we find

$$c_{ji} = \frac{A + D}{C}, \qquad c_{ij} = \frac{A + B}{C}.$$

Cube flip

A different way of formulating the propagation of the 3D-system of discrete conjugate nets is in terms of cube flips. For this we consider discrete conjugate nets on quad-graphs. A *quad-graph* is a planar graph such that all faces consist of four vertices and four edges.

Definition 7.4. Consider a quad-graph with vertex set V. A map $x : V \to \mathbb{R}\mathrm{P}^n$ is called a *discrete conjugate net* (or *Q-net*) if the image of every quad is contained in a plane.

Now for every vertex with three adjacent faces, we can interpret these faces as three faces of the "back side" of a cube. Thus, we may replace the point at this vertex in a unique way, by replacing the three faces by the three faces of the "front side" of that cube. This move is called a *cube flip* and the result is a new discrete conjugate net.

8 Laplace transforms and Laplace invariants

Laplace transforms

Let $[\hat{x}] : \mathbb{R}^2 \supset U \to \mathbb{R}\mathrm{P}^n$ be a regular conjugate line parametrization, i.e.,

$$\hat{x}_{uv} = \alpha\hat{x}_u + \beta\hat{x}_v + \gamma\hat{x}.$$

Consider the tangent lines in u-direction:

$$\ell^{(1)}(u,v) := [\hat{x}(u,v)] \vee [\hat{x}_u(u,v)].$$

Along a v-coordinate line

$$v \mapsto \ell(u = \mathrm{const}, v)$$

these tangent lines trace a ruled developable surface. By Proposition 3.1, this developable surface has a line of striction (the curve of intersection points of infinitesimally close rulings). Together this one-parameter family of lines of striction form another surfaces, a so called *Laplace transform* of $[\hat{x}]$.

Alternatively, we may view the two-parameter family of lines $\ell^{(1)}$ as a *(torsal) line congruence*. Then the points of the Laplace transform are the *focal points* of $\ell^{(1)}$.

Let us compute the Laplace transform as the limit of the intersection of two close rulings:

$$\begin{aligned}
\ell^{(1)}(u,v) &= [\hat{x}(u,v)] \vee [\hat{x}_u(u,v)],\\
\ell^{(1)}(u,v+\varepsilon) &= [\hat{x}(u,v+\varepsilon)] \vee [\hat{x}_u(u,v+\varepsilon)] \approx [\hat{x} + \varepsilon\hat{x}_v] \vee [\hat{x}_u + \varepsilon\hat{x}_{uv}]\\
&= [\hat{x} + \varepsilon\hat{x}_v] \vee [(1+\varepsilon\alpha)\hat{x}_u + \varepsilon\beta\hat{x}_v + \varepsilon\gamma\hat{x}]
\end{aligned}$$

We see that the rulings are coplanar up to first order, and one easily checks that the point of intersection is given by

$$[(1+\varepsilon\alpha)\hat{x}_u + (\varepsilon\gamma - \beta)\hat{x}] \to [\hat{x}_u - \beta\hat{x}] \quad (\varepsilon \to 0).$$

The same considerations hold for the tangent lines in v-direction, which yields a second Laplace transform.

Definition 8.1. Let $[\hat{x}] : \mathbb{R}^2 \supset U \to \mathbb{RP}^n$ be a regular conjugate line parametrization with

$$\hat{x}_{uv} = \alpha \hat{x}_u + \beta \hat{x}_v + \gamma \hat{x}.$$

Then

$$\mathcal{L}^{(1)}[\hat{x}] := [\hat{x}_u - \beta \hat{x}], \qquad \mathcal{L}^{(2)}[\hat{x}] := [\hat{x}_v - \alpha \hat{x}]$$

are called the *Laplace transforms* of $[\hat{x}]$.

In affine coordinates $\hat{x} = (x, 1)$, we obtain the following formulas for the Laplace transforms:

$$\mathcal{L}^{(1)} x := x - \frac{1}{\beta} x_u, \qquad \mathcal{L}^{(2)} x := x - \frac{1}{\alpha} x_v.$$

The Laplace transforms are itself conjugate line parametrizations.

Proposition 8.1. *Let $[\hat{x}] : \mathbb{R}^2 \supset U \to \mathbb{RP}^n$ be a regular conjugate line parametrization. The Laplace transforms $\mathcal{L}^{(1)}[\hat{x}]$ and $\mathcal{L}^{(2)}[\hat{x}]$ are conjugate line parametrizations.*

Proof. We check this for $\mathcal{L}^{(1)}$:

$$\begin{aligned}
L &:= \hat{x}_u - \beta \hat{x}, \\
L_u &= \hat{x}_{uu} - \beta \hat{x}_u - \beta_u \hat{x}, \\
L_v &= \hat{x}_{uv} - \beta \hat{x}_v - \beta_v \hat{x} = \alpha \hat{x}_u + (\gamma - \beta_v)\hat{x}, \\
L_{uv} &= \alpha \hat{x}_{uu} + (\alpha_u - \beta_v + \gamma)\hat{x}_u + (\gamma_u - \beta_{uv})\hat{x},
\end{aligned}$$

and thus

$$L, L_u, L_v, L_{uv} \in \operatorname{span}\{\hat{x}, \hat{x}_u, \hat{x}_{uu}\}.$$

□

In particular, note that the osculating plane of the u-direction is the tangent plane of the Laplace transform $\mathcal{L}^{(1)}$.

Since $\mathcal{L}^{(1)}$ is conjugate (and if we assume that it is regular again), we may consider further Laplace transforms. It turns out that taking $\mathcal{L}^{(2)}$ of $\mathcal{L}^{(1)}$ brings us back to the orgininal parametrized surface.

Proposition 8.2. *The Laplace transformations of different directions are inverses of each other:*

$$\mathcal{L}^{(1)} \circ \mathcal{L}^{(2)} = \mathcal{L}^{(2)} \circ \mathcal{L}^{(1)} = \operatorname{id}.$$

Proof. Exercise. □

Thus, one may view iterated Laplace transforms as a sequence with a "forward" and "backward" direction:

$$\cdots \leftarrow \mathcal{L}^{(2)}\mathcal{L}^{(2)}x \leftarrow \mathcal{L}^{(2)}x \leftarrow x \to \mathcal{L}^{(1)}x \to \mathcal{L}^{(1)}\mathcal{L}^{(1)}x \to \cdots$$

Discrete Laplace transforms

Let $[\hat{x}] : \mathbb{Z}^2 \to \mathbb{RP}^n$ be a discrete conjugate net, i.e.,

$$A\hat{x} + B\hat{x}_1 + C\hat{x}_{12} + D\hat{x}_2 = 0.$$

Then the lines

$$\ell^{(1)}(i,j) := [\hat{x}(i,j)] \vee [\hat{x}(i+1,j)].$$

form a *discrete line congruence*, whose focal points are given by

$$\ell^{(1)}(i,j) \cap \ell^{(1)}(i,j+1) = ([\hat{x}] \vee [\hat{x}_1]) \cap ([\hat{x}_2] \vee [\hat{x}_{12}]) = [A\hat{x} + B\hat{x}_1] = [C\hat{x}_{12} + D\hat{x}_2].$$

Similarly, for the other direction.

Definition 8.2. Let $[\hat{x}] : \mathbb{Z}^2 \to \mathbb{RP}^n$ be a discrete conjugate net with

$$A\hat{x} + B\hat{x}_1 + C\hat{x}_{12} + D\hat{x}_2 = 0.$$

Then

$$\begin{aligned} \mathcal{L}^{(1)}[\hat{x}] &:= ([\hat{x}] \vee [\hat{x}_1]) \cap ([\hat{x}_2] \vee [\hat{x}_{12}]) = [A\hat{x} + B\hat{x}_1] = [C\hat{x}_{12} + D\hat{x}_2], \\ \mathcal{L}^{(2)}[\hat{x}] &:= ([\hat{x}] \vee [\hat{x}_2]) \cap ([\hat{x}_1] \vee [\hat{x}_{12}]) = [A\hat{x} + D\hat{x}_2] = [B\hat{x}_1 + C\hat{x}_{12}] \end{aligned}$$

are called the *Laplace transforms* of $[\hat{x}]$.

By this definition the Laplace transforms are maps $\mathbb{Z}^2 \to \mathbb{RP}^n$. However, combinatorially it is more natural to assign the points of the Laplace transforms to the faces F of $\mathbb{Z}^2$. E.g., for $f = ((i,j),(i+1,j),(i+1,j+1),(i,j+1))$ one assigns

$$\mathcal{L}^{(1)}[\hat{x}](f) = ([\hat{x}] \vee [\hat{x}_1]) \cap ([\hat{x}_2] \vee [\hat{x}_{12}])$$

and obtains a map $F \to \mathbb{RP}^n$.

We obtain the following two propositions in analogy with the smooth case:

Proposition 8.3. *The Laplace transforms of a discrete conjugate net are discrete conjugate nets.*

Proof. For $\mathcal{L}^{(1)}$ this follows from the fact that $\ell^{(1)}$ and $\tau_1\ell^{(1)}$ are coplanar. $\square$

Similarly, Proposition 8.2 also holds in the discrete case. With the original definition of discrete Laplace transforms one obtains the identity up to a shift in indices. Using the definition in which the Laplace transform is defined on faces, and defining the Laplace transforms of discrete conjugate nets defined on faces accordingly, one obtains the identity (without shifts).

Laplace invariants

The coefficients of the Laplace equation

$$\hat{x}_{uv} = \alpha\hat{x}_u + \beta\hat{x}_v + \gamma\hat{x}.$$

can be used to construct a conjugate net from initial data on the coordinate axes. However, they are not projectively well-defined (gauge invariant).

Since conjugate nets are projectively invariant, it makes sense to find local invariants to describe (the space of) conjugate nets in a projectively invariant way. It turns out that the quantities

$$h := \alpha_u - \alpha\beta - \gamma, \qquad k := \beta_v - \alpha\beta - \gamma$$

are gauge invariant, and called *Laplace invariants* (classically: *point invariants*) of $[\hat{x}]$.

Discrete Laplace invariants

In the discrete case it is more easily seen how to obtain a local projective invariant. Consider the line $\ell^{(1)}$. It contains the points $x := [\hat{x}]$ and $x_1 = [\hat{x}_1]$ as well as the focal points $F^{(1)} := \mathcal{L}^{(1)}[\hat{x}]$ and $F^{(1)}_{-2} := \tau_{-2}\mathcal{L}^{(1)}[\hat{x}]$. Thus, the cross-ratio of these four points is well-defined and projectively invariant.

We assign this cross-ratio to the edge $e = ((i,j),(i+1,j))$:

$$H(e) := \operatorname{cr}(x, F^{(1)}, x_1, F^{(1)}_{-2}).$$

With

$$F^{(1)} = [A\hat{x} + B\hat{x}_1], \qquad F^{(1)}_{-2} = [C_{-2}\hat{x}_1 + D_{-2}\hat{x}]$$

we obtain

$$H(e) = \frac{\det(\hat{x}, A\hat{x} + B\hat{x}_1) \cdot \det(\hat{x}_1, C_{-2}\hat{x}_1 + D_{-2}\hat{x})}{\det(A\hat{x} + B\hat{x}_1, \hat{x}_1) \cdot \det(C_{-2}\hat{x}_1 + D_{-2}\hat{x}, \hat{x})} = \frac{BD_{-2}}{AC_{-2}}.$$

Definition 8.3. For a discrete conjugate net $x = [\hat{x}] : \mathbb{Z}^2 \to \mathbb{RP}^n$ with

$$A\hat{x} + B\hat{x}_1 + C\hat{x}_{12} + D\hat{x}_2 = 0$$

we define the *Laplace invariants* as

$$\begin{aligned} H(e) &:= \operatorname{cr}(x, F^{(1)}, x_1, F^{(1)}_{-2}) = \frac{D_{-2}B}{C_{-2}A} && \text{for horizontal edges} \\ H(e) &:= \operatorname{cr}(x, F^{(2)}_{-1}, x_2, F^{(2)}) = \frac{AC_{-1}}{DB_{-1}} && \text{for vertical edges} \end{aligned}$$

Remark 8.1. Note that the definition is completely symmetric with respect to horizontal and vertical edges. In both cases the Laplace invariant is the alternating ratio of the coefficients in the Laplace equation starting with the "bottom right" coefficient. The asymmetry in the equations is just an artifact of the shift notation.

The following proposition shows how the discrete Laplace invariants approximate the corresponding smooth ones:

Proposition 8.4. *Let $x : \mathbb{R}^2 \supset U \to \mathbb{RP}^n$ a conjugate line parametrization. Then*

$$\begin{aligned} \operatorname{cr}(x(u,v), F^{(1)}(u,v), x(u+\varepsilon,v), F^{(1)}(u,v-\varepsilon)) &= 1 - \varepsilon^2 k(u,v) + o(\varepsilon^2), \\ \operatorname{cr}(x(u,v), F^{(2)}(u-\varepsilon,v), x(u,v+\varepsilon), F^{(2)}(u,v)) &= 1 + \varepsilon^2 h(u,v) + o(\varepsilon^2) \\ &= \frac{1}{1-\varepsilon^2 h(u,v)} + o(\varepsilon^2). \end{aligned}$$

Remark 8.2. Note how different permutations of the cross-rations would lead to a more direct discretization of the smooth Laplace invariants:

$$\operatorname{cr}(x, x_1, F^{(1)}, F^{(1)}_{-2}) \approx \varepsilon^2 k, \qquad \operatorname{cr}(x, F^{(2)}, F^{(2)}_{-1}, x_2) \approx \varepsilon^2 h.$$

However, the representation in terms of alternating ratios of coefficients for the chosen permutation turns out to be very useful.

9 Kœnigs nets

Kœnigs nets are conjugate nets with equal Laplace invariants.

Definition 9.1. Let $x : \mathbb{R}^2 \supset U \to \mathbb{RP}^n$ be a conjugate line parametrization. Then x is called a *Kœnigs net* (classically: *conjugate net with equal point invariants*) if it has equal Laplace invariants $h = k$ at every point $(u, v) \in U$.

Since conjugate nets as well as the Laplace invariant are projectively invariant, so is the notion of Kœnigs nets.

The condition $h = k$ is equivalent to $\alpha_u = \beta_v$, which is the compatibility condition for the system

$$\begin{aligned} f_u &= \beta, \\ f_v &= \alpha. \end{aligned}$$

Thus it is equivalent to the existence of such a function f, which is usually represented as a logarithm $f = \log \nu$. With this observation we obtain the following characterization of Kœnigs nets:

Proposition 9.1. *A parametrization $x : \mathbb{R}^2 \supset U \to \mathbb{R}^n$ is a Kœnigs net if and only if there exists a function $\nu : U \to \mathbb{R}_+$ such that*

$$x_{uv} = (\log \nu)_v x_u + (\log \nu)_u x_v. \tag{7}$$

The function ν is unique up to a global multiplicative constant.

By a special choice of the homogeneous lift $\hat{x} = \frac{1}{\nu}(x, 1)$, equation (7) can be simplified to become the *Moutard equation* (Exercise):

$$\hat{x}_{uv} = a\hat{x}, \qquad a := \nu \left(\tfrac{1}{\nu}\right)_{uv}.$$

Another characterization for Kœnigs nets is given in terms of the existence of the Kœnigs dual.

Proposition 9.2. *A conjugate net $x : \mathbb{R}^2 \supset U \to \mathbb{R}^n$ is a Kœnigs net if and only if there exists a function $\nu : U \to \mathbb{R}_+$ and a net $x^* : U \to \mathbb{R}^n$ such that*

$$\begin{aligned} x^*_u &= \frac{1}{\nu^2} x_u, \\ x^*_v &= -\frac{1}{\nu^2} x_v. \end{aligned} \tag{8}$$

*In this case, x^* called the **Kœnigs dual** of x. It is uniquely determined up to scaling (freedom for ν) and up to translation (from integration of* (8)*). Furthermore, x^* is itself a Kœnigs net with potential $\frac{1}{\nu}$.*

Proof. The compatibility conditions for the system (8) is equivalent to (7). □

A different way to formulate (8) is

$$x^*_u \parallel x_u, \quad x^*_v \parallel x_v, \quad x^*_u + x^*_v \parallel x_u - x_v, \quad x^*_u - x^*_v \parallel x_u + x_v.$$

Note that the notion of the Kœnigs dual is invariant under affine transformations, not projective transformations, while its existence is projectively invariant (since Kœnigs nets are).

Discrete Kœnigs nets

Definition 9.2. Let $x : \mathbb{Z}^2 \to \mathbb{RP}^n$ be a discrete conjugate net. Then x is called a *discrete Kœnigs net* if at every vertex $v \in \mathbb{Z}^2$ the Laplace invariants satisfy

$$\prod_{e \sim v} H(e) = 1. \tag{9}$$

Remark 9.1. Note that by Proposition 8.4, the product of the discrete Laplace invariants being equal to 1 may be seen as a discretization of the equality of the smooth Laplace invariants in the two directions: $h = k$.

Using the representation of Laplace invariants in terms of the coefficients of the Laplace equation we find that (9) is equivalent to

$$\frac{B_{12}}{D_{12}} \frac{C_2}{A_2} \frac{D}{B} \frac{A_1}{C_1} = 1. \tag{10}$$

This may be viewed as the closing condition of a discrete multiplicative one-form.

Definition 9.3. Let G be a graph with the set of vertices V and directed edges $\vec{E}$.

(i) A function $q : \vec{E} \to \mathbb{R}\backslash\{0\}$ is called a *(discrete) multiplicative one-form* on G if $q(-e) = \frac{1}{q(e)}$ for every directed edge $e \in \vec{E}$.

(ii) A multiplicative one-form q is called *exact* if for every cycle of directed edges the product of the values of q along this cycle is equal to 1.

(iii) A multiplicative one-form q is called *closed* if for every cycle of directed edges that bounds a face of G the product of the values of q along this cycle is equal to 1.

Remark 9.2. For "trivial topology" such as in $G = \mathbb{Z}^2$, a one-form is exact if and only if it is closed.

Proposition 9.3. *Let $q : \vec{E} \to \mathbb{R}\backslash\{0\}$ be a multiplicative one-form on G. Then q is exact if and only if there exists a function $\nu : V \to \mathbb{R}\backslash\{0\}$ such that for any $e = (v, v') \in \vec{E}$ it holds*

$$q(e) = \frac{\nu(v')}{\nu(v)}.$$

The function ν is called a ***potential*** *for q and it is uniquely determined up to a multiplicative constant.*

Proof. Exercise. □

Now, equation (10) may be viewed as the closing condition for a multiplicative one-form, not on the edges of $\mathbb{Z}^2$, but of the two diagonal grids

$$\begin{aligned}\mathbb{Z}^2_{\text{even}} &= \left\{(m,n) \in \mathbb{Z}^2 \;\middle|\; (m+n) \mod 2 = 0\right\},\\ \mathbb{Z}^2_{\text{odd}} &= \left\{(m,n) \in \mathbb{Z}^2 \;\middle|\; (m+n) \mod 2 = 1\right\}\end{aligned}$$

with edges

$$E(\mathbb{Z}^2_{\text{even/odd}}) = \left\{(v, v') \in \left(\mathbb{Z}^2_{\text{even/odd}}\right)^2 \;\middle|\; \text{there exists } f \in F(\mathbb{Z}^2) \text{ with } v, v' \sim f\right\}.$$

Consider the two one-forms

$$\vec{E}(\mathbb{Z}^2_{\text{even}}) \to \mathbb{R}, \quad e \mapsto \frac{A}{C}$$
$$\vec{E}(\mathbb{Z}^2_{\text{odd}}) \to \mathbb{R}, \quad e \mapsto \frac{B}{D}$$

By condition (10), these two one-forms are closed, and thus there exists a potential on the vertices of $\mathbb{Z}^2_{\text{even}}$ and $\mathbb{Z}^2_{\text{odd}}$, respectively. We summarize this result in the following theorem.

Proposition 9.4. *Let $[\hat{x}] : \mathbb{Z}^2 \to \mathbb{RP}^n$ a discrete conjugate net with*

$$A\hat{x} + B\hat{x}_1 + C\hat{x}_{12} + D\hat{x}_2 = 0.$$

Then $[\hat{x}]$ is a Kœnigs net if and only if there exists a function $\nu : \mathbb{Z}^2 \to \mathbb{R}\backslash\{0\}$ such that for every quad

$$\frac{A}{C} = \frac{\nu}{\nu_{12}}, \qquad \frac{B}{D} = \frac{\nu_1}{\nu_2}. \tag{11}$$

The function ν is unique up to a common scalar multiple on $\mathbb{Z}^2_{\text{even}}$ and $\mathbb{Z}^2_{\text{odd}}$, respectively ("black-white scaling").

Remark 9.3. For the quad (x, x_1, x_{12}, x_2) consider the diagonal intersection

$$M = (x \vee x_{12}) \cap (x_1 \vee x_2).$$

In affine coordinates $\hat{x} = (x, 1)$, the introduced one-forms on the diagonals equal the quotients of directed lengths from M to corners of the quad (Exercise.):

$$\frac{A}{C} = -\frac{\vec{\ell}(M, x_{12})}{\vec{\ell}(M, x)}, \qquad \frac{B}{D} = -\frac{\vec{\ell}(M, x_2)}{\vec{\ell}(M, x_1)}.$$

We can use the potential ν to express all coefficients in the discrete Laplace equation.

Theorem 9.5. *Let $x : \mathbb{Z}^2 \to \mathbb{R}^n$ be a discrete conjugate net Then x is a Kœnigs net if and only if*

$$(\nu_1 + \nu_2)(\nu x + \nu_{12}x_{12}) = (\nu + \nu_{12})(\nu_1 x_1 + \nu_2 x_2). \tag{12}$$

with some non-vanishing function $\nu : V \to \mathbb{R}$.

Proof. In affine coordinates, we have

$$Ax + Bx_1 + Cx_{12} + Dx_2 = 0, \qquad A + B + C + D = 0.$$

The coefficients A, B, C, D are uniquely determined per quad up to a multiplicative scalar. Substituting (11), we obtain

$$\frac{B}{C} = -\frac{\nu_1}{\nu_{12}}\frac{\nu + \nu_{12}}{\nu_1 + \nu_2}, \qquad \frac{D}{C} = -\frac{\nu_2}{\nu_{12}}\frac{\nu + \nu_{12}}{\nu_1 + \nu_2},$$

and thus,

$$\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} \sim \begin{pmatrix} \frac{A}{C} \\ \frac{B}{C} \\ 1 \\ \frac{D}{C} \end{pmatrix} \sim \begin{pmatrix} \frac{\nu}{\nu_{12}} \\ -\frac{\nu_1}{\nu_{12}}\frac{\nu+\nu_{12}}{\nu_1+\nu_2} \\ 1 \\ -\frac{\nu_2}{\nu_{12}}\frac{\nu+\nu_{12}}{\nu_1+\nu_2} \end{pmatrix} \sim \begin{pmatrix} \nu(\nu_1 + \nu_2) \\ -\nu_1(\nu + \nu_{12}) \\ \nu_{12}(\nu_1 + \nu_2) \\ -\nu_2(\nu + \nu_{12}) \end{pmatrix}$$

□

Remark 9.4. The potential ν discretizes the smooth potential $-\nu$ in (7). Equation (12) can be rewritten as a difference equation

$$\Delta_1\Delta_2 x = \alpha\Delta_1 x + \beta\Delta_2 x, \qquad \alpha = \frac{\nu\nu_1 - \nu_2\nu_{12}}{\nu_{12}(\nu_1+\nu_2)}, \qquad \beta = \frac{\nu\nu_2 - \nu_1\nu_{12}}{\nu_{12}(\nu_1+\nu_2)},$$

in which the coefficients α and *beta* can be seen to discretize the logarithmic derivatives of ν.

Similar to the smooth case, one can choose a homogeneous lift $\hat{x} = \nu(x,1)$ to obtain a *discrete Moutard equation* (Exercise):

$$\hat{x} + \hat{x}_{12} = a(\hat{x}_1 + \hat{x}_2), \qquad a := \frac{\nu + \nu_{12}}{\nu + \nu_2}.$$

We now turn towards the characterization of discrete Kœnigs nets in terms of a Kœnigs dual. For this we introduce discrete additive one-forms.

Definition 9.4. Let G be a graph with the set of vertices V and directed edges $\vec{E}$. Let W be a vector space.

(i) A map $p : \vec{E} \to W$ is called a *(discrete additive) one-form* on G if $p(-e) = -p(e)$ for every directed edge $e \in \vec{E}$.

(ii) A one-form p is called *exact* if for every cycle of directed edges the sum of the values of p along this cycle is equal to 0.

(iii) A one-form p is called *closed* if for every cycle of directed edges that bounds a face of G the sum of the values of p along this cycle is equal to 0.

Remark 9.5. For "trivial topology" such as in $G = \mathbb{Z}^2$, a one-form is exact if and only if it is closed.

Proposition 9.6. *Let $p : \vec{E} \to W$ be a one-form on G. Then p is exact if and only if there exists a map $P : V \to W$ such that for any $e = (v, v') \in \vec{E}$ it holds*

$$p(e) = P(v') - P(v).$$

The map P is called a ***potential*** *or* ***integral*** *for p and it is uniquely determined up to an additive constant.*

Equation (12) can be rewritten as

$$\nu\nu_1(x_1 - x) - \nu_1\nu_{12}(x_{12} - x_1) = \nu_2\nu_{12}(x_{12} - x_2) - \nu\nu_2(x_2 - x).$$

This is equivalent to closing condition of a discrete one-form defined on the edges of $\mathbb{Z}^2$. Its integration leads to a discrete Kœnigs dual.

Proposition 9.7. *A discrete conjugate net $x : \mathbb{Z}^2 \to \mathbb{R}^n$ is a Kœnigs net if and only if there exists a non-vanishing function $\nu : U \to \mathbb{R}$ and a net $x^* : \mathbb{Z}^2 \to \mathbb{R}^n$ such that*

$$\begin{aligned} \Delta_1 x^* &= \nu\nu_1\Delta_1 x, \\ \Delta_2 x^* &= -\nu\nu_2\Delta_2 x. \end{aligned} \tag{13}$$

In this case, x^* called the ***Kœnigs dual*** *of x. It is uniquely determined up to scaling (freedom for ν) and up to translation (from integration of* (13)*). Furthermore, x^* is itself a Kœnigs net with potential $\frac{1}{\nu}$ and Kœnigs dual x.*

The conditions (13) for the Kœnigs dual are equivalent to

$$\Delta_1 x^* \parallel \Delta_1 x, \quad \Delta_2 x_v^* \parallel \Delta_2 x, \quad x_{12}^* - x^* \parallel x_1 - x_2, \quad x_1^* - x_2^* \parallel x_{12} - x.$$

For two quads $X = (x, x_1, x_{12}, x_2)$ and $X^* = (x^*, x_1^*, x_{12}^*, x_2^*)$ this means that X and X^* have parallel corresponding sides and parallel not-corresponding diagonals. Given X, such a *dual quad* X^* is uniquely determined up to scaling and translation. Thus, for one quad there always exist a dual quad. For four quads around a vertex, the scaling factors for the dual quads can be chosen in such a way that the dual quads fit together if and only if the original four quads come from a Kœnigs net. Thus, locally the condition for a Kœnigs net involves four quads.

For $n \geqslant 3$, this condition on four quads can be reformulated in the following way.

Proposition 9.8. *Let $x : \mathbb{Z}^2 \to \mathbb{RP}^n$ be a discrete conjugate net such that for every point x its four neighbors x_1, x_2, x_{-1}, x_{-2} are not coplanar. Then x is a Kœnigs net if and only if the intersection points of diagonals build a discrete conjugate net.*

Proof. We proof the claim for $n = 3$. A similar proof holds for $n \geqslant 3$ if the determinants are replaced by exterior products.

Consider the midpoints $[\hat{m}], [\hat{m}_1], [\hat{m}_{12}], [\hat{m}_2]$ of four quads around a vertex x_{12}. The planarity of these four points is characterized by the condition

$$\begin{aligned}
0 &= \det\left(\hat{m}, \hat{m}_1, \hat{m}_{12}, \hat{m}_2\right) \\
&\sim \det\left(B\hat{x}_1 + D\hat{x}_2, A_1\hat{x}_1 + C_1\hat{x}_{112}, B_{12}\hat{x}_{112} + D_{12}\hat{x}_{122}, A_2\hat{x}_2 + C_2\hat{x}_{122}\right) \\
&= \left(BC_1D_{12}A_2 - DC_2B_{12}A_1\right)\det\left(\hat{x}_1, \hat{x}_{112}, \hat{x}_{122}, \hat{x}_2\right).
\end{aligned}$$

The determinant is non-vanishing by the non-planarity assumption on the neighbors of x_{12}. Thus, this condition is equivalent to (10). □

Remark 9.6. While the discretization of Kœnigs nets discussed in this section is due to Bobenko and Suris, the diagonal intersection nets lead to a different discretization of Kœnigs nets due to Doliwa, which satisfies

$$\prod_{e \sim f} H(e) = 1.$$

at each quad f. Together, the Laplace invariants H_x of a Kœnigs net x and the Laplace invariants H_m of the diagonal intersection net m together satisfy

$$H_x(e)H_m(e^*) = 1$$

for every pair of edge e and dual edge e^*. This leads to another discretization of Kœnigs nets defined on $\mathbb{Z}^2 \cup F(\mathbb{Z}^2)$, which generalizes both of the previous notions and is called *Kœnigs binets.*

Consistency of discrete Kœnigs nets

As we have seen, discrete conjugate nets define a 3D-system. Thus, for seven points of an elementary 3-cube the eighth point is already uniquely determined. For any additional constraint on a discrete conjugate net, one may ask the following question: if this condition is satisfied on the initial 2D-data, whether it also hold on all of $\mathbb{Z}^3$ (and then $\mathbb{Z}^N$) after the uniquely determined propagation by the 3D-system of discrete conjugate net. Generally, such a constraint system is called a *consistent reduction*.

For Kœnigs nets on $\mathbb{Z}^N$, we need the condition that the multiplicative one-forms previously defined on the diagonals are closed on all of $\mathbb{Z}^N$. This is equivalent to the condition that for any cycle of three quads around a vertex the product of the Laplace invariants of the three edges incident with that vertex equals 1, or to the conditions that the diagonal intersections of three such quads lie on a line.

For an elementary 3-cube this means that all six diagonal intersections must lie on one line. Such a cube is called a *Kœnigs cube*. It may equivalently be characterized as a cube with planar faces, such that all of its "black" vertices lie in a plane, or equivalently, all its "white" vertices lie in a plane. From this characterization it is easily seen, that Kœnigs nets are a consistent reduction of discrete conjugate nets.

10 Möbius geometry

This section serves as a reminder from Geometry 1.

The elementary model of Möbius geometry

Consider the n-dimensional Euclidean space $\mathbb{R}^n$. The inversion in a hypersphere with center $c \in \mathbb{R}^n$ and radius $r > 0$ can be described in the following way: The point x and its image x' lie on the same ray emanating from c and the distances to c satisfy the relation

$$\| x - c \| \cdot \| x' - c \| \; = \; r^2.$$

This gives rise to an involution on $\mathbb{R}^n$, except that the center c has no image and no preimage. To fix this, we add one extra point to $\mathbb{R}^n$, called ∞, and obtain the *extended Euclidean space*

$$\widehat{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}.$$

Definition 10.1. The *(sphere) inversion* in the hypersphere with center $c \in \mathbb{R}^n$ and radius $r > 0$ is the map defined by

$$\widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}, \qquad \begin{aligned} x &\longmapsto x' = c + \frac{r^2}{\|x - c\|^2}(x - c) \qquad \text{for } x \neq c, \\ c &\longmapsto \infty \\ \infty &\longmapsto c \end{aligned}$$

Sphere inversions preserve angles and map hyperspheres and hyperplanes to hyperspheres and hyperplanes. This statement becomes simpler and more specific at the same time if we consider hyperplanes as special cases of hyperspheres through the point ∞. More precisly, let us adopt the following convention:

Definition 10.2. A *sphere in* $\widehat{\mathbb{R}^n}$ is either a sphere in $\mathbb{R}^n$ or the union of a plane in $\mathbb{R}^n$ with $\{\infty\}$.

Then we can simply say:

Theorem 10.1. *Sphere inversions preserve angles and map hyperspheres in $\widehat{\mathbb{R}^n}$ to hyperspheres in $\widehat{\mathbb{R}^n}$.*

Since circles and, more generally, k-dimensional spheres for $1 \leqslant k < n$ are intersections of $n-k$ hyperspheres, sphere inversions preserve spheres of any dimension:

Corollary 10.2. *Sphere inversions map k-spheres in $\widehat{\mathbb{R}^n}$ to k-spheres in $\widehat{\mathbb{R}^n}$.*

Just as hyperplanes are limiting cases of hyperspheres, reflections in hyperplanes are limiting cases of sphere inversions. The reflection in the hyperplane with equation $\langle x-a, v\rangle = 0$ is the map

$$x \longmapsto x' = x - 2\frac{\langle x-a, v\rangle}{\langle v, v\rangle}\, v,$$

which we extend from $\mathbb{R}^n$ to $\widehat{\mathbb{R}^n}$ by declaring that reflections in hyperplanes map ∞ to ∞.

Definition 10.3. A *Möbius transformation* of $\mathbb{R}^n \cup \{\infty\}$ is a composition of sphere inversions and reflections in hyperplanes. The Möbius transformations form a group called the *Möbius group* and denoted by $\text{Möb}(n)$.

Remark 10.1. A Möbius transformation is orientation reversing or preserving depending on whether it is the composition of an odd or even number of reflections. The subgroup of orientation preserving Möbius transformations is called the *special Möbius group* and denoted by $\text{SMöb}(n)$.

Because reflections preserve angles and map spheres to spheres, Theorem 10.1 extends to Möbius transformations:

Theorem 10.3. *Möbius transformations preserve angles and map spheres in $\widehat{\mathbb{R}^n}$ to spheres in $\widehat{\mathbb{R}^n}$.*

Similarity transformations on $\mathbb{R}^n$ are the transformations of the form $x \mapsto \lambda A x + b$ with $\lambda > 0$, $A \in O(n)$, and $b \in \mathbb{R}^n$. Reflections in hyperplanes are a special case, and like reflections in hyperplanes we extend all similarity transformations from $\mathbb{R}^n$ to $\widehat{\mathbb{R}^n}$ by declaring that ∞ maps to ∞.

Proposition 10.4. *The Möbius group contains all similarity transformations.*

Proof. The group of similarity transformations is generated by translations, orthogonal transformations, and scalings.

- A translation $x \mapsto x + v$ is the composition of two reflections in parallel hyperplanes.
- An orthogonal transformation $x \mapsto Ax$ with $A \in O(n)$ is the composition of at most n reflections in hyperplanes through the origin.
- A scaling transformation $x \mapsto \lambda x$ with $\lambda > 0$ is the composition of a reflection in the unit sphere followed by a reflection in a sphere with center 0 and radius $\sqrt{\lambda}$. □

Conversely, one only needs to add one sphere inversion to the group of similarity transformations to generate the Möbius group:

Proposition 10.5. *Every Möbius transformation is a composition of similarity transformations and inversions in the unit sphere.*

By Theorem 10.3, Möbius transformations map hyperspheres to hypersphers. This property already characterizes all Möbius transformations.

Theorem 10.6 (Fundamental theorem of Möbius geometry)**.** *Any bijective map $f : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ which maps hyperspheres in $\widehat{\mathbb{R}^n}$ to hyperspheres in $\widehat{\mathbb{R}^n}$ is a Möbius transformation.*

The projective model of Möbius geometry

The idea is to transfer Möbius geometry from $\widehat{\mathbb{R}^n}$ to the n-dimensional sphere S^n via stereographic projection

$$\sigma : S^n \to \widehat{\mathbb{R}^n}.$$

Via the standard embedding

$$\mathbb{R}^{n+1} \hookrightarrow \mathbb{RP}^{n+1}, \qquad u \longmapsto \begin{bmatrix} u \\ 1 \end{bmatrix} = [x]$$

we identify the unit sphere $S^n \subset \mathbb{R}^{n+1}$ with a quadric in $\mathbb{RP}^{n+1}$,

$$S^n = \left\{ [x] \in \mathbb{RP}^{n+1} \mid \langle x, x \rangle_{n+1,1} = 0 \right\}, \tag{14}$$

where

$$\langle x, \tilde{x} \rangle_{n+1,1} = x_1 \tilde{x}_1 + \ldots + x_{n+1} \tilde{x}_{n+1} - x_{n+2} \tilde{x}_{n+2}$$

is the Lorentz product. Thus, indeed, in affine coordinates $x_{n+2} = 1$, the quadric S^n is the unit sphere

$$x_1^2 + \ldots + x_{n+1}^2 = 1.$$

Upon identifying the affine part of $\widehat{\mathbb{R}^n}$ with the affine part of the hyperplane

$$E = \left\{ [x] \in \mathbb{RP}^{n+1} \;\middle|\; x_{n+1} = 0 \right\}.$$

the stereographic projection coincides with the central projection of S^n to the hyperplane E from the "north pole" $[e_{n+1} + e_{n+2}]$. A point $[x] \in S^n \backslash [e_{n+1} + e_{n+2}]$ on the quadric is mapped to

$$([x] \vee [e_{n+1} + e_{n+2}]) \cap E = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ 0 \\ x_{n+2} - x_{n+1} \end{bmatrix}.$$

Inversely, a point $[u, 0, 1] \in E$, $u \in \mathbb{R}^n$, is mapped to

$$\left(\begin{bmatrix} u \\ 0 \\ 1 \end{bmatrix} \vee [e_{n+1} + e_{n+2}] \right) \cap S^n = \begin{bmatrix} 2u \\ \|u\|^2 - 1 \\ \|u\|^2 + 1 \end{bmatrix}.$$

Thus, we obtain the following equations for stereographic projection and its inverse:

$$
\begin{array}{llll}
\sigma: & S^n & \longrightarrow & \widehat{\mathbb{R}^n} \\
& \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ x_{n+2} \end{bmatrix} & \longmapsto & \dfrac{1}{x_{n+2}-x_{n+1}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \text{if} \quad x_{n+1} \neq x_{n+2} \\
& [x] & \longmapsto & \infty \qquad\qquad\qquad\qquad\quad \text{if} \quad x_{n+1} = x_{n+2}
\end{array}
$$

$$
\begin{array}{llll}
\sigma^{-1}: & \widehat{\mathbb{R}^n} & \longrightarrow & S^n \\
& u & \longmapsto & \begin{bmatrix} 2u \\ \|u\|^2 - 1 \\ \|u\|^2 + 1 \end{bmatrix} \qquad \text{if} \quad u \in \mathbb{R}^n \\
& \infty & \longmapsto & [e_{n+1} + e_{n+2}]
\end{array}
$$

The stereographic projection σ is the restriction to $S^n \subset \mathbb{R}^{n+1}$ of the inversion in the hypersphere with center e_{n+1} (the north pole of S^n) and radius $\sqrt{2}$ (so that it contains the equatorial sphere $\{x \in S^n \mid x_{n+1}\} = 0)$. Thus, Theorem 10.1 implies, that the stereographic projection preserves angles and maps spheres to spheres.

It is convenient to additionally employ another basis for the projective model, one that contains the center of projection. We change the basis for our homogeneous coordinates from the standard basis $(e_1, \ldots, e_{n+2})$ of $\mathbb{R}^{n+2}$ to the new basis

$$B = (e_1, \ldots, e_n, e_\infty, e_0)$$

with

$$e_\infty = \tfrac{1}{2}\,(e_{n+1} + e_{n+2}), \qquad e_0 = \tfrac{1}{2}\,(-e_{n+1} + e_{n+2}).$$

The points $[e_0]$ and $[e_\infty]$ are the "south pole" and the "north pole" of S^n. We choose the subscripts 0 and ∞ for the new basis vectors because

$$\sigma^{-1}(0) = [e_0] \quad \text{and} \quad \sigma^{-1}(\infty) = [e_\infty].$$

Note that B is not an orthonormal basis of $\mathbb{R}^{n+1,1}$. Rather, we have

$$\langle e_0, e_0\rangle_{n+1,1} = \langle e_\infty, e_\infty\rangle_{n+1,1} = 0, \quad \langle e_0, e_\infty\rangle = -\frac{1}{2}\,.$$

Now the change-of-basis-transformations between coordinate vectors with respect to the standard basis and the new basis B are

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ x_{n+2} \end{pmatrix} = \sum_{k=1}^{n} x_k e_k + x_\infty e_\infty + x_0 e_0 = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \frac{1}{2}(x_\infty - x_0) \\ \frac{1}{2}(x_\infty + x_0) \end{pmatrix},$$

and inversely

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_\infty \\ x_0 \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} + x_{n+2} \\ -x_{n+1} + x_{n+2} \end{pmatrix} = \underbrace{\left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & \begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix} \end{array}\right)}_{F} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ x_{n+2} \end{pmatrix}$$

We may interpret $[x] = [x_1, \ldots, x_n, x_{n+1}, x_{n+2}]$ and $[x_1, \ldots, x_n, x_\infty, x_0]$ as representing the same point in $\mathbb{RP}^{n+1}$ with respect to different bases, or we may interpret $[x_1, \ldots, x_n, x_\infty, x_0]$ as the image of $[x]$ under the projective transformation $f : \mathbb{RP}^{n+1} \to \mathbb{RP}^{n+1}, [x] \mapsto [Fx]$.

In coordinates of the basis B, the Lorentz scalar product is

$$\langle x, \tilde{x} \rangle_{n+1,1} = x_1 \, \tilde{x}_1 \; + \; \ldots \; + \; x_n \, \tilde{x}_n \; - \; \frac{1}{2} \, x_\infty \, \tilde{x}_0 \; - \; \frac{1}{2} \, x_0 \, \tilde{x}_\infty \, .$$

In affine coordinates $x_0 = 1$, the "north pole" $[e_\infty]$ lies at infinity and the quadric S^n becomes a paraboloid

$$x_\infty = x_1^2 + \ldots + x_n^2.$$

Correspondingly the stereographic projection (and its inverse) becomes vertical orthogonal projection from (and onto) this paraboloid (see Fig. **??**).

$$\sigma^{-1}(u) = \begin{bmatrix} 2u \\ \|u\|^2 - 1 \\ \|u\|^2 + 1 \end{bmatrix} = [u + \|u\|^2 \, e_\infty + e_0].$$

Thus, the projective model in homogeneous coordinates with respect to B is sometimes called the *paraboloid model of Möbius geometry*.

Spheres in the projective model

Hyperspheres in S^n are intersections of S^n with hyperplanes in $\mathbb{R}^{n+1}$, or, since we view S^n as a quadric in $\mathbb{RP}^{n+1}$ (see equation (14)) intersections with hyperplanes in $\mathbb{RP}^{n+1}$. The pole-polar relationship provides a bijection between hyperplanes that intersect S^n and points outside S^n.

On the other hand, hyperspheres in S^n correspond to hyperspheres in $\widehat{\mathbb{R}^n}$ via stereographic projection. Thus, we have the following bijections:

$$\text{hyperspheres in } \widehat{\mathbb{R}^n} \quad \xleftrightarrow[\text{projection}]{\text{stereogr.}} \quad \text{hyperspheres in } S^n \quad \xleftrightarrow{\text{polarity}} \quad \text{points outside } S^n.$$

The following proposition provides explicit formulas.

Proposition 10.7. *Let $[y] \in \mathbb{RP}^{n+1}$ be a point outside S^n, i.e.,*

$$\langle y, y \rangle_{n+1,1} > 0.$$

Then the polar plane

$$[y]^\perp = \left\{ \, [x] \, \in \, \mathbb{RP}^{n+1} \, \middle| \, \langle y, x \rangle_{n+1,1} \; = \; 0 \, \right\}$$

intersects S^n in a hypersphere

$$S_{[y]} \; = \; S^n \cap [y]^\perp$$

whose image $\sigma(S_{[y]}) \subset \widehat{\mathbb{R}^n}$ under stereographic projection is a hypersphere in $\widehat{\mathbb{R}^n}$:

- *If $y_{n+1} \neq y_{n+2}$, or equivalently, $y_0 \neq 0$, then $\sigma(S_{[y]})$ is the sphere in $\mathbb{R}^n$ with center c and radius r, where*

$$c = \frac{1}{y_0}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \frac{1}{y_{n+2} - y_{n+1}}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \qquad r = \frac{\sqrt{\langle y, y\rangle_{n+1,1}}}{|y_0|} = \frac{\sqrt{\langle y, y\rangle_{n+1,1}}}{|y_{n+2} - y_{n+1}|}.$$

- *If $y_{n+1} = y_{n+2}$, or equivalently, $y_0 = 0$, then $\sigma(S_{[y]})$ is the union of $\{\infty\}$ and the hyperplane $\{u \in \mathbb{R}^n \mid \langle v, u\rangle = d\}$,*

$$v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \qquad d = \frac{y_\infty}{2} = y_{n+1} = y_{n+2}.$$

Conversely, the inverse projection σ^{-1} maps

- *the sphere in $\mathbb{R}^n$ with center c and radius r to the hypersphere $S_{[y]} \subset S^n$ with*

$$[y] = \left[c + (\|c\|^2 - r^2)e_\infty + e_0\right] = \begin{bmatrix} 2c \\ \|c\|^2 - r^2 - 1 \\ \|c\|^2 - r^2 + 1 \end{bmatrix}.$$

- *the union of $\{\infty\}$ and the hyperplane $\{u \in \mathbb{R}^n \mid \langle v, u\rangle = d\}$ in $\mathbb{R}^n$ to the hypersphere $S_{[y]} \subset S^n$ with*

$$[y] = [v + 2de_\infty] = \begin{bmatrix} v \\ d \\ d \end{bmatrix}.$$

Remark 10.2. Note that the center of the sphere is obtained by the central projection of the point $[y]$ that represents the sphere.

Proof. As an exemplary case, we show that a point $[x] = [u + \|u\|^2 e_\infty + e_0] \in S^n \backslash \{[e_\infty]\}$ in the polar hyperplane of $[y] = \left[c + (\|c\|^2 - r^2)e_\infty + e_0\right]$ is projected to a point $u = \sigma([x]) \neq \infty$ that lies on the sphere in $\mathbb{R}^n$ with center c and radius r.

$$\begin{aligned} 0 = \langle y, x\rangle_{n+1,1} &= \left\langle c + (\|c\|^2 - r^2)e_\infty + e_0, u + \|u\|^2 e_\infty + e_0\right\rangle_{n+1,1} \\ &= \|c - u\|^2 - r^2. \end{aligned}$$

□

Proposition 10.8. *The hyperspheres in S^n corresponding to two points $[y_1]$ and $[y_2]$ outside the sphere S^n intersect if and only if*

$$\langle y_1, y_2\rangle_{n+1,1}^2 \leqslant \langle y_1, y_1\rangle_{n+1,1}\langle y_2, y_2\rangle_{n+1,1}$$

In this case the intersection angle θ is determined up to $\theta \mapsto \pi - \theta$ by the equation

$$\cos^2\theta = \frac{\langle y_1, y_2\rangle_{n+1,1}^2}{\langle y_1, y_1\rangle_{n+1,1}\,\langle y_2, y_2\rangle_{n+1,1}}.$$

Corollary 10.9. *The hyperspheres corresponding to two points $[y_1]$ and $[y_2]$ outside S^n intersect orthogonally if and only if*

$$\langle y_1, y_2\rangle_{n+1,1} = 0.$$

Pencils of spheres

Definition 10.4. In Möbius geometry, a *pencil of spheres* is the family of hyperspheres corresponding to the points of a line in the projective model of Möbius geometry.

There are three different types of sphere pencils, depending on the signature of the corresponding line $\ell \subset \mathbb{RP}^{n+1}$:

`(++)` The line ℓ does not intersect S^n. Then the sphere pencil consists of all hyperspheres that contain a common fixed $(n-2)$-sphere. Indeed, the polar $(n-1)$-plane $\ell^\perp$ has signature `(n-1,1)` and intersects S^n in a $(n-2)$-sphere. The polar planes $Y^\perp$ of points $Y \in \ell$ are precisely the hyperplanes containing $\ell^\perp$. Hence, the pencil of ℓ consists precisely of all spheres containing $\ell^\perp \cap S^n$.

For $n = 2$, these are all circles through two fixed points. For $n = 3$, these are all spheres through a fixed circle.

`(+-)` The line ℓ intersects S^n in two points. Then the polar $(n-1)$-plane $\ell^\perp$ has signature `(n,0)` and does not intersect S^n. By Corollary 10.9, the pencil of ℓ consists precisely of all hyperspheres that intersect all hyperspheres corresponding to points in $\ell^\perp$ orthogonally.

For $n = 2$, these are all circles orthogonal two circles that intersect in two points, and thus all circles from a pencil of type `(++)`. For $n = 3$, these are all spheres orthogonal two three spheres that span a plane of signature `(+++)`.

`(+0)` The line ℓ is a tangent to S^n. Then the polar $(n-1)$-plane $\ell^\perp$ has signature `(n-1,0,1)` and is also tangent to S^n in the same point P. Furthermore, $\ell^\perp$ is the common tangent plane for all spheres in ℓ. Thus, the pencil ℓ consists of hyperspheres that are tangent to each other in a fixed point.

For $n = 2$, these are all circles tangent in a common point, while $\ell^\perp$ corresponds to the orthogonal pencil of the same type. For $n = 2$, these are all spheres tangent in a common point.

Proposition 10.10. *Möbius transformations map sphere pencils to sphere pencils and preserve their type.*

Proof. This follows directly from Theorem 10.3. In the case `(+-)` consider spheres orthogonal to the pencil. □

Möbius transformations in the projective model

In the elementary model, Möbius geometry is the geometry of the Möbius group $\text{Möb}(n)$ generated by sphere inversions acting on the extended n-dimensional space $\widehat{\mathbb{R}^n}$. The Möbius group $\text{Möb}(n)$ is geometrically characterized as the group of all transformations that map spheres in $\widehat{\mathbb{R}^n}$ to spheres in $\widehat{\mathbb{R}^n}$. What is the corresponding group of transformations in the projective model of Möbius geometry?

First, let us consider the subgroup of projective transformations of $\mathbb{RP}^{n+1}$ that map S^n to S^n. This is the projective orthogonal group

$$\mathrm{PO}(n+1,1) \subset \mathrm{PGL}(n+2,\mathbb{R}).$$

Since spheres in S^n are intersections of S^n with planes in $\mathbb{RP}^{n+1}$ and projective transformations map planes to planes, a projective orthogonal transformation $g \in \mathrm{PO}(n+1,1)$ maps spheres in S^n to spheres in S^n. By the Fundamental Theorem of Möbius geometry (Theorem 10.6), the map

$$\sigma \circ g \circ \sigma^{-1} : \widehat{\mathbb{R}^n} \longrightarrow \widehat{\mathbb{R}^n}$$

is a Möbius transformation. Thus we have a group homomorphism

$$\begin{array}{ccc} \mathrm{PO}(n+1,1) & \longrightarrow & \text{Möb}(n), \\ g & \longmapsto & \sigma \circ g \circ \sigma^{-1}. \end{array} \tag{15}$$

It is injective because the identity on $\mathbb{RP}^{n+1}$ is the only projective transformation that fixes the sphere S^n pointwise. However, it is also surjective, and thus:

Theorem 10.11. *The map* (15) *is a group isomorphism.*

With this, we complete the *projective model of Möbius geometry*, which is the space $S^n \subset \mathbb{RP}^{n+1}$ with the action of $\mathrm{PO}(n+1,1)$. Stereographic projection translates between the elementary model and the projective model according to the following dictionary:

elementary model		projective model
$\widehat{\mathbb{R}^n}$	$\longleftrightarrow$	S^n
sphere in $\widehat{\mathbb{R}^n}$	$\longleftrightarrow$	sphere in S^n
Möb(n)	$\longleftrightarrow$	$PO(n+1,1)$

The Möbius group Möb(n) is generated by inversions in spheres and reflections in planes. In the projective model these transformations are given by projective inversions that preserve the quadric.

Proposition 10.12. *Let* $[y] \in \mathbb{RP}^{n+1}$ *be a point outside* S^n*, i.e.,* $\langle y, y\rangle_{n+1,1} > 0$*. Then the map*

$$g : \mathbb{RP}^{n+1} \to \mathbb{RP}^{n+1}, \quad [x] \mapsto \left[x - 2\frac{\langle x, y\rangle_{n+1,1}}{\langle y, y\rangle_{n+1,1}} y \right] \tag{16}$$

is a projective orthogonal transformation, such that $\sigma \circ g \circ \sigma^{-1}$ *is the inversion in the hypersphere (or reflection in the hyperplane) corresponding to the point* $[y]$.

Proof. The map $G : x - 2\frac{\langle x,y\rangle_{n+1,1}}{\langle y,y\rangle_{n+1,1}} y$

- is linear,
- invertible since $G(x) = 0$ implies $x = 0$,
- orthogonal since $\langle G(x), G(x)\rangle_{n+1,1} = \langle x, x\rangle_{n+1,1}$.

Furthermore, $g = [G]$ fixes all points on $[y]^\perp$ and preserves all hyperplanes through $[y]$. □

Remark 10.3. The map (16) for a point $[y] \in \mathbb{RP}^{n+1}$ inside the quadric corresponds to a fixed point free involution, which is still a Möbius transformation, but not a sphere inversion.

11 Curvature line parametrizations

Definition 11.1. Let
$$x : \mathbb{R}^2 \supset U \to \mathbb{R}^n$$
be a regular surface.

(i) x is called *orthogonal* if
$$\langle x_u, x_v \rangle = 0$$

(ii) x is called *curvature line parametrization* if it is orthogonal and conjugate, i.e.,
$$\langle x_u, x_v \rangle = 0, \quad \text{and} \quad x_{uv} = \alpha x_u + \beta x_v.$$

Proposition 11.1. *The property of a parametrization to be orthogonal is Möbius invariant.*

Proof. Möbius transformations are conformal, i.e., preserve angles. □

Conjugate parametrizations, on the other hand, are not Möbius invariant. Are curvature line parametrizations?

Proposition 11.2. *Let $x : \mathbb{R}^2 \supset U \to \mathbb{R}^n$ be a regular surface and $[\hat{x}] : U \to \mathbb{S}^n \subset \mathbb{RP}^{n+1}$ be its Möbius lift*
$$\hat{x} := x + e_0 + \|x\|^2 e_\infty.$$
Then x is a curvature line parametrization if and only if $[\hat{x}]$ is a conjugate line parametrization.

Proof. For the derivatives of the lift we obtain
$$\begin{aligned}
\hat{x}_u &= x_u + 2\langle x, x_u \rangle e_\infty,\\
\hat{x}_v &= x_v + 2\langle x, x_v \rangle e_\infty,\\
\hat{x}_{uv} &= x_{uv} + 2\left(\langle x, x_{uv} \rangle + \langle x_u, x_v \rangle\right) e_\infty.
\end{aligned}$$
Let $\hat{x}$ be a curvature line parametrization. Then
$$\hat{x}_{uv} = x_{uv} + 2\langle x, x_{uv} \rangle e_\infty = \alpha x_u + \beta x_v + 2\left(\alpha \langle x, x_u \rangle + \beta \langle x, x_v \rangle\right) e_\infty = \alpha \hat{x}_u + \beta \hat{x}_v.$$
The reverse direction is shown similarly. □

Corollary 11.3. *Curvature line parametrizations are Möbius invariant.*

Focal surfaces and principal curvature spheres

Let $x : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a regular parametrized surface with unit normal field
$$n(u, v) := \frac{x_u \times x_v}{\|x_u \times x_v\|}$$

Lemma 11.4. *If x is orthogonal and conjugate, then*
$$n_u = -\kappa_1 x_u, \qquad n_v = -\kappa_2 x_v$$
with two smooth functions $\kappa_1, \kappa_2 : U \to \mathbb{R}$.

Proof. Then

$$n_u = \alpha x_u + \beta x_v$$

with two functions α, β. Since x is conjugate and orthogonal, we obtain

$$0 = n_u \cdot x_v = \alpha x_u \cdot x_v + \beta x_v \cdot x_v = \beta x_v \cdot x_v$$

and thus, $\beta = 0$. Similar for n_v. □

The two values, κ_1 and κ_2 are called the *principal curvatures* of the surface x.

Remark 11.1. The principal curvatures can also be characterized as follows. Consider the one parameter family of tangent spheres $S(\kappa)$ with signed curvature κ touching the surface at a point p. κ is positive if the sphere lies at the same side of the tangent plane as the normal vector and negative otherwise. Let M be the set of tangent spheres intersecting any neighborhood U of p in more than one point. The values

$$\kappa_1 \coloneqq \inf_{S \in M} \kappa(S), \quad \kappa_2 \coloneqq \sup_{S \in M} \kappa(S)$$

are called the *principal curvatures* of the surface at p. The spheres $S(\kappa_1)$ and $S(\kappa_2)$ are called *principal curvature spheres* and are in second order contact with the surface along one direction. The contact directions are are the *principal directions* and are orthogonal.

Points with $\kappa_1 = \kappa_2$ are called *umbilic* points. Away from umbilic points the reverse is also true.

Lemma 11.5. *If*

$$n_u = -\kappa_1 x_u, \qquad n_v = -\kappa_2 x_v$$

with two smooth functions $\kappa_1, \kappa_2 : U \to \mathbb{R}$, then at points with $\kappa_1(u,v) \neq \kappa_2(u,v)$, the parametrization x is orthogonal and conjugate.

Proof. From $n_u \cdot x_v = n_v \cdot x_u$ we obtain

$$-\kappa_1 x_u \cdot x_v = n_u \cdot x_v = n_v \cdot x_u = -\kappa_2 x_v \cdot x_u.$$

Thus, if $\kappa_1 \neq \kappa_2$ this implies $x_u \cdot x_v = 0$, and further $n_u \cdot x_v = 0$. □

We can now characterize curvature line parametrizations by their normals.

Definition 11.2. Let $x : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a regular parametrized surface. Then the two-parameter family of lines

$$\ell : U \to \mathrm{Lines}(\mathbb{R}^3), \qquad (u,v) \mapsto \ell(u,v) = \{x(u,v) + \lambda n(u,v) \mid \lambda \in \mathbb{R}\}$$

is called the *normal congruence* of the surface x.

Proposition 11.6. *Let $x : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a regular parametrized surface. Then x is a curvature line parametrization if and only if the ruled surfaces*

$$u \mapsto \ell(u, v = const), \qquad v \mapsto \ell(u = const, v)$$

in the normal congruence are developable.

Proof. Since $n \cdot n = 1$, we have

$$n_u = \alpha x_u + \beta x_v$$

Now

$$0 = \det(n, x_u, n_u) = \beta \det(n, x_u, x_v)$$

if and only if $\beta = 0$. □

Thus, in particular, each of these developable surfaces in the normal congruence has a line of striction. For a developable surfaces in u-direction

$$u \mapsto \ell(u, v = \text{const})$$

it is given by

$$u \mapsto x(u, v) + \frac{1}{\kappa_1(u, v)} n(u, v)$$

Together these lines of striction in u-direction form a surface.

Definition 11.3. Let $x : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a curvature line parametrization. Then the two surfaces given by

$$x^{(1)}(u, v) := x(u, v) + \frac{1}{\kappa_1(u, v)} n(u, v), \qquad x^{(2)}(u, v) := x(u, v) + \frac{1}{\kappa_1(u, v)} n(u, v)$$

are called the *focal surfaces* of the surface x.

Note that the focal surfaces are a generalization of the evolute of a plane curve to surfaces. Correspondingly, they each form the centers of a two-parameter family of sphere, called *curvature spheres.*

Definition 11.4. Let $x : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a curvature line parametrization. Then the two two-parameter families of spheres $S^{(i)}$ with centers x^i and radii $\frac{1}{|\kappa_i|}$ are called *curvature spheres* of the surface x.

Remark 11.2. The (unique) midsphere of the two (touching) curvature spheres $S^{(1)}$ and $S^{(2)}$ at any given point $(u, v) \in U$ is the *mean curvature sphere* of the surface x.

In the Möbius lift the curvature spheres are represented by the points

$$\hat{s}^{(i)} := x^{(i)} + \left(\left\| x^{(i)} \right\|^2 - \frac{1}{\kappa_i^2} \right) e_\infty + e_0 = x + \frac{1}{\kappa_i} n + \left(\|x\|^2 + \frac{2}{\kappa_i} \langle x, n \rangle \right) e_\infty + e_0$$

Proposition 11.7. *Let $x : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a curvature line parametrization, and*

$$\hat{x} = x + \|x\|^2 e_\infty + e_0$$

its Möbius lift. Then the Möbius lift $[\hat{s}^{(1)}]$ of the curvature spheres $S^{(1)}$ is given by

$$[\hat{s}^{(1)}] = \left([\hat{x}] \vee [\hat{x}_u] \vee [x_v] \vee [x_{uu}] \right)^\perp ,$$

and, similarly, the Möbius lift $[\hat{s}^{(2)}]$ of the curvature spheres $S^{(2)}$ is given by

$$[\hat{s}^{(2)}] = \left([\hat{x}] \vee [\hat{x}_u] \vee [x_v] \vee [x_{vv}] \right)^\perp .$$

Proof. One easily checks

$$\left\langle \hat{s}^{(1)}, \hat{x} \right\rangle_{4,1} = \left\langle \hat{s}^{(1)}, \hat{x}_u \right\rangle_{4,1} = \left\langle \hat{s}^{(1)}, \hat{x}_v \right\rangle_{4,1} = 0.$$

And the Lorentz product with the second derivative

$$\hat{x}_{uu} = x_{uu} + 2(\|x_u\|^2 + \langle x, x_{uu} \rangle) e_\infty$$

is given by

$$\left\langle \hat{s}^{(1)}, \hat{x}_{uu} \right\rangle_{4,1} = \frac{1}{\kappa_1} \langle n, x_{uu} \rangle - \|x_u\|^2 = 0,$$

since

$$\langle n, x_{uu} \rangle = -\langle n_u, x_u \rangle = \kappa_1 \|x_u\|^2 .$$

□

Remark 11.3. Note that this shows, that the curvature spheres are tangential to the surface. Furthermore, the intersection with the osculating plane of the corresponding curvature line (which is also a normal plane) is the osculating circle of that curvature line.

Corollary 11.8. *Curvature spheres are Möbius invariant.*

Ribaucour transformations

The fundamental transformations for conjugate line parametrizations can be adapted to curvature line parametrizations.

Definition 11.5. Let $x, x^+ : \mathbb{R}^2 \supset U \to \mathbb{R}^n$ be two curvature line parametrizations. Then x^+ is called a *Ribaucour transform* of x if the corresponding coordinate curves envelope one-parameter families of circles, i.e., if at every point the corresponding tangent lines are interchanged by the reflection in the orthogonal bisector of x and x^+.

Ribaucour transforms satisfy the following permutability theorem, which is a specialized version of Theorem 6.1.

Theorem 11.9. *Let x be a curvature line parametrized surface and let $x^{(1)}$ and $x^{(2)}$ be two of its Ribaucour transforms. Then there exists a one-parameter family of curvature line parametrized surfaces $x^{(12)}$ that are Ribaucour transforms of both $x^{(1)}$ and $x^{(2)}$. Furthermore, corresponding points of x, $x^{(1)}$, $x^{(2)}$, $x^{(12)}$ are concircular.*

12 Orthogonal coordinate systems

Definition 12.1. A (regular) parametrization $x : \mathbb{R}^n \supset U \to \mathbb{R}^n$ is called *orthogonal coordinate system* if at every point

$$\langle \partial_i x, \partial_j x \rangle = 0, \qquad i \neq j. \tag{17}$$

It turns out that all coordinate surfaces of an orthogonal coordinates system are curvature line parametrizations.

Theorem 12.1 (Dupin)**.** *For $n \geqslant 3$ every orthogonal coordinate system $x : \mathbb{R}^n \supset U \to \mathbb{R}^n$ is conjugate.*

Proof. For three distinct $i, j, k = 1, \ldots, n$ differentiating (17) with respect to s_k leads to

$$\langle \partial_i x, \partial_j \partial_k x \rangle + \langle \partial_j x, \partial_k \partial_i x \rangle = 0.$$

By permutation of the indices, these are three equations which sum up to

$$2 \left(\langle \partial_i x, \partial_j \partial_k x \rangle + \langle \partial_j x, \partial_k \partial_i x \rangle + \langle \partial_k x, \partial_i \partial_j x \rangle \right) = 0.$$

Dividing by 2 and subtracting one of the first three equations again leads to

$$\langle \partial_k x, \partial_i \partial_j x \rangle = 0.$$

Thus, for $i, j = 1, \ldots, n$, $j \neq k$,

$$\partial_i \partial_j x \in \operatorname{span} \{ \partial_k x \mid k = 1, \ldots, n, \ k \neq i, j \}^{\perp} = \operatorname{span}\{\partial_i x, \partial_j x\},$$

due to the regularity and orthogonality of x. $\square$

Confocal quadrics

We consider a special family of quadrics in $\mathbb{R}^n$ generalizing confocal conics.

Definition 12.2. Let $a_1 > a_2 > \ldots > a_n > 0$ be given. The one-parameter family of quadrics

$$\mathcal{Q}_\lambda = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} \frac{x_i^2}{a_i + \lambda} = 1 \right\}, \quad \lambda \in \mathbb{R}$$

is called *confocal.*

Remark 12.1. The intersection of a family of confocal quadrics with any coordinate plane constitutes a family of confocal conics (cf. Definition **??**). Conversely, this property characterizes confocal quadrics.

Theorem 12.2. *Precisely n confocal quadrics pass through any point $x = (x_1, \ldots, x_n)$ with $x_1 \cdot \ \ldots \cdot x_n \neq 0$, and the quadrics passing through one point are orthogonal.*

Proof. For a given point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_1 \cdot \ldots \cdot x_n \neq 0$ the equation $\sum_{i=1}^n \frac{x_i^2}{a_i + \lambda} = 1$ is, after clearing the denominators, a polynomial equation of degree n in λ with n real roots $u_1, \ldots, u_n$ lying in the intervals

$$-a_1 < u_1 < -a_2 < u_2 < \ldots - a_n < u_n.$$

This fact follows immediatly from the graph of the function

$$f(\lambda) := \frac{x_1^2}{a_1 + \lambda} + \cdots + \frac{x_n^2}{a_n + \lambda}.$$

It has exactly n different intersection points with the horizontal line $f = 1$. The gradient of f at the point $w = (w_1, \ldots, w_n)$

$$\operatorname{grad}_w f(\lambda) = 2 \left(\frac{w_1}{a_1 + \lambda}, \ldots, \frac{w_n}{a_n + \lambda} \right)$$

is normal (orthogonal) to the quadric at w. Thus, two quadrics $\mathcal{Q}_\lambda, \mathcal{Q}_\mu$ passing through the point w are orthogonal since their gradient vectors are orthogonal:

$$\begin{aligned}\langle \operatorname{grad}_w f(\lambda), \operatorname{grad}_w f(\mu)\rangle &= 4\sum_{i=1}^n \frac{w_i^2}{(a_i+\lambda)(a_i+\mu)} \\ &= \frac{4}{\mu-\lambda}\left(\sum_{i=1}^n \frac{w_i^2}{a_i+\lambda} - \sum_{i=1}^n \frac{w_i^2}{a_i+\mu}\right) = 0\,.\end{aligned}$$

□

The n roots $u_1, \dots, u_n$ of

$$\sum_{i=1}^n \frac{x_i^2}{a_i+\lambda} - 1 = -\frac{\prod_{i=1}^n(\lambda-u_i)}{\prod_{i=1}^n(a_i+\lambda)} \tag{18}$$

correspond to n confocal quadrics $\mathcal{Q}_{u_i}, i = 1, \dots, n$ that intersect at the point x:

$$\sum_{i=1}^n \frac{x_i^2}{a_i+u_k} = 1, k = 1, \dots, n \quad \Leftrightarrow \quad x \in \bigcap_{i=1}^n \mathcal{Q}_{u_i}\,. \tag{19}$$

Each of the quadrics is of a different signature.

Exercise 12.1. Consider the case $n = 2$.

Let $-a < u_1 < -b < u_2$. Then

$$\begin{aligned}\frac{x^2}{a+u_1} + \frac{y^2}{b+u_1} &= 1, \\ \frac{x^2}{a+u_2} + \frac{y^2}{b+u_2} &= 1\end{aligned} \tag{20}$$

is an inhomogeneous linear system of two equations in the variables (x^2, y^2). Its solution is given by

$$\begin{aligned}x^2 &= \frac{(a+u_1)(a+u_2)}{a-b}, \\ y^2 &= \frac{(b+u_1)(b+u_2)}{b-a}.\end{aligned}$$

How to obtain the solution to the linear system (20)?

(i) The linear system (20) may be written as

$$A\begin{pmatrix} x^2 \\ y^2\end{pmatrix} = \begin{pmatrix} 1 \\ 1\end{pmatrix}, \qquad \text{with } A := \begin{pmatrix} \frac{1}{a+u_1} & \frac{1}{b+u_1} \\ \frac{1}{a+u_2} & \frac{1}{b+u_2}\end{pmatrix}$$

Then we compute

$$\det A = \tfrac{1}{(a+u_1)(b+u_2)} - \tfrac{1}{(a+u_2)(b+u_1)} = \tfrac{(u_1-u_2)(a-b)}{(a+u_1)(a+u_2)(b+u_1)(b+u_2)},$$

$$A^{-1} = \tfrac{1}{\det A}\begin{pmatrix} \frac{1}{b+u_2} & \frac{-1}{b+u_1} \\ \frac{-1}{a+u_2} & \frac{1}{a+u_1}\end{pmatrix} = \tfrac{1}{(u_1-u_2)(a-b)}\begin{pmatrix} (a+u_1)(a+u_2)(b+u_1) & -(a+u_1)(a+u_2)(b+u_2) \\ -(a+u_1)(b+u_1)(b+u_2) & (a+u_2)(b+u_1)(b+u_2)\end{pmatrix}$$

and thus

$$\begin{pmatrix} x^2 \\ y^2\end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 1\end{pmatrix} = \tfrac{1}{(u_1-u_2)(a-b)}\begin{pmatrix} (a+u_1)(a+u_2)((b+u_1)-(b+u_2)) \\ (b+u_1)(b+u_2)((a+u_1)-(a+u_1))\end{pmatrix} = \tfrac{1}{a-b}\begin{pmatrix} (a+u_1)(a+u_2) \\ -(b+u_1)(b+u_2)\end{pmatrix}.$$

(ii) Alternatively, define

$$g(\lambda) := \frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} - 1.$$

Then $g(u_1) = g(u_2) = 0$ and $g(\lambda)(a+\lambda)(b+\lambda)$ is a polynomial in λ of degree 2 of which the highest order coefficient is -1. Thus,

$$g(\lambda)(a+\lambda)(b+\lambda) = -(\lambda - u_1)(\lambda - u_2),$$

or equivalently,

$$g(\lambda) = \frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} - 1 = -\frac{(\lambda - u_1)(\lambda - u_2)}{(a+\lambda)(b+\lambda)}.$$

Evaluating the residues of g at $-a$ and $-b$, we obtain

$$x^2 = \mathrm{res}_{\lambda=-a} g(\lambda) = \frac{(a+u_1)(a+u_2)}{a-b},$$
$$y^2 = \mathrm{res}_{\lambda=-b} g(\lambda) = \frac{(b+u_1)(b+u_2)}{b-a}.$$

Back to the general case, calculating the residue of (18) at $\lambda = -a_j$ we obtain the formula for x_j through $(u_1, \ldots, u_n)$:

$$x_j^2 = \frac{\prod_{i=1}^n (a_j + u_i)}{\prod_{i \neq j} (a_j - a_i)}, \quad j = 1, \ldots, n. \tag{21}$$

Thus for each point x of $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > 0, \ldots, x_n > 0\}$ there is exactly one solution $(u_1, \ldots, u_n) \in \mathcal{U}$ of (21), where

$$\mathcal{U} = \{(u_1, \ldots, u_n) \in \mathbb{R}^n \mid -a_1 < u_1 < -a_2 < u_2 < \ldots - a_n < u_n\}.$$

On the other hand, for each $(u_1, \ldots, u_n) \in \mathcal{U}$ there are exactly 2^n solutions $(x_1, \ldots, x_n) \in \mathbb{R}^n$, which are mirror symmetric with respect to the coordinate hyperplanes. Thus, we are dealing with a parametrization of the first hyperoctant of $\mathbb{R}^n$,

$$x : \mathcal{U} \ni (u_1, \ldots, u_n) \mapsto (x_1, \ldots, x_n) \in \mathbb{R}^n_+,$$

given by

$$x_k = \frac{\prod_{i=1}^{k-1} \sqrt{-(u_i + a_k)} \prod_{i=k}^n \sqrt{u_i + a_k}}{\prod_{i=1}^{k-1} \sqrt{a_i - a_k} \prod_{i=k+1}^n \sqrt{a_k - a_i}}, \quad k = 1, \ldots, n,$$

such that the coordinate hyperplanes $u_i = \mathrm{const}$ are mapped to (parts of) the respective quadrics given by (19). The coordinates $(u_1, \ldots, u_n)$ are called *confocal coordinates* (or *elliptic coordinates*). For various applications, it is often useful to re-parametrize the coordinate lines according to $u_i = u_i(s_i)$, $i = 1, \ldots, n$.

It turns out that the confocal coordinate systems are already characterized by orthogonality and factorizability of the coordinate functions.

Theorem 12.3 ([BSST18])**.** *If a coordinate system $x : \mathbb{R}^n \supset U \to \mathbb{R}^n$ satisfies two conditions:*

i) $x(s)$ ***factorizes***, *in the sense that*

$$\begin{cases} x_1(s) = f_1^1(s_1)f_2^1(s_2)\cdots f_n^1(s_n), \\ x_2(s) = f_1^2(s_1)f_2^2(s_2)\cdots f_n^2(s_n), \\ \dots \\ x_n(s) = f_1^n(s_1)f_2^n(s_2)\cdots f_n^n(s_n), \end{cases}$$

for some functions f_i^k *with all* $f_i^k(s_i) \neq 0$ *and* $\left(f_i^k\right)'(s_i) \neq 0$*;*

ii) $x(s)$ *is* ***orthogonal***, *that is,*

$$\left\langle \frac{\partial x}{\partial s_i}, \frac{\partial x}{\partial s_j} \right\rangle = 0 \quad for \quad i \neq j,$$

then all coordinate hypersurfaces are confocal quadrics.

13 Circular nets

In the smooth case we have seen that curvature line parametrizations are conjugate line parametrizations in the Möbius quadric. Consider a discrete conjugate net $x : \mathbb{Z}^2 \to \mathbb{S}^n \subset \mathbb{RP}^{n+1}$ in the Möbius quadric. Then the stereographic projection of the four points of every face lie on a circle.

Definition 13.1. A regular discrete net $x : \mathbb{Z}^2 \to \mathbb{R}^n$ is called a *circular net* if the four points of every face x, x_1, x_{12}, x_2 lie on a circle.

The definition immediately implies that circular nets are Möbius invariant.

Proposition 13.1. *Let* $x : \mathbb{Z}^2 \to \mathbb{R}^n$ *be a regular discrete net, and* $[\hat{x}] : \mathbb{Z}^2 \to \mathbb{S}^n \subset \mathbb{RP}^{n+1}$ *be its Möbius lift*

$$\hat{x} := x + e_0 + \|x\|^2 e_\infty.$$

Then x *is a circular net if and only if* $[\hat{x}]$ *is a discrete conjugate net.*

Thus, circular nets maybe viewed as a discretization of curvature line parametrizations.

Remark 13.1. In $\mathbb{R}^3$ the axes of the circles can be interpreted as discrete normals (per face). Adjacent discrete normal lines intersect, and in this sense they form discrete developable surfaces. Furthermore, two adjacent circles are contained in a sphere centered at the intersection point of the corresponding normals. This sphere can be interpreted as a discrete curvature sphere.

Consistency of circular nets

The multi-dimensional consistency of circular nets follows from the classical Miquel's theorem.

Theorem 13.2 (Miquel)**.** *Let* $x, x_1, x_2, x_3, x_{12}, x_{23}, x_{13}, x_{123} \in \hat{\mathbb{R}}^2$ *be eight points in the plane associated to the vertices of a combinatorial cube.*

If for five of its faces the corresponding four points lie on a circle, then the four points corresponding to the sixth face also lie on a circle.

Proof. Geometry 1. □

A different way of stating this theorem is the following.

Theorem 13.3. *Let $x, x_1, x_2, x_3, x_{12}, x_{23}, x_{13} \in \mathbb{R}^n$.*

If the three quadruples x, x_i, x_{ij}, x_j are concircular, then the three circles through x_k, x_{ik}, x_{jk} intersect in a point.

Proof. Consider the sphere through x, x_1, x_2, x_3. Then all seven points lie on this sphere and may be mapped to $\hat{\mathbb{R}}^2$ by stereographic projection. Denote the circle through x_k, x_{ik}, x_{jk} by $\tau_k C_{ij}$. Consider the second intersection point x_{123} of the two circles $\tau_1 C_{23}$ and $\tau_2 C_{13}$. Then by Theorem 13.2, the four points $x_3, x_{13}, x_{23}, x_{123}$ lie on a circle. This circle coincides with $\tau_3 C_{12}$. □

Theorem 13.4. *Circular nets are a consistent reduction of the 3D-system of discrete conjugate nets.*

Proof. Consider an elementary 3-cube of a discrete conjugate net $x, x_1, x_2, x_3, x_{12}, x_{23}, x_{13}, x_{123}$ such that the three quads x, x_i, x_{ij}, x_j are concircular. Then by Theorem 13.3, the three circles through x_k, x_{ik}, x_{jk} intersect in a point. Since the circles are planar this point coincides with the point x_{123} uniquely determined by the 3D-system of discrete conjugate nets. □

14 Principal binets

Let us consider a different discretization of curvature line parametrizations based on binets [AT26b]. Traditionally, (discrete) nets are maps defined on the vertices of $\mathbb{Z}^2$. Binets are pairs of maps on the vertices *and* the faces of $\mathbb{Z}^2$.

Consider $\mathbb{Z}^2$ as an infinite planar graph with

$$\begin{aligned} \text{vertices } V &:= V(\mathbb{Z}^2) = \mathbb{Z}^2, \\ \text{edges } E &:= E(\mathbb{Z}^2) \simeq \mathbb{Z}^2 \cup \mathbb{Z}^2, \\ \text{faces } F &:= F(\mathbb{Z}^2) =: (\mathbb{Z}^2)^* \simeq \mathbb{Z}^2. \end{aligned}$$

We say two vertices $v, v' \in V$ (resp. two faces) are *adjacent* if they share an edge, i.e., $(v, v') \in E$. We say that a vertex $v \in V$ and a face $f \in F$ are *incident* if v is a boundary vertex of f, and write

$$v \sim f.$$

We additionally denote the vertices of the double graph by

$$D := V \cup F.$$

For $d, d' \in D$ the incidence $d \sim d'$ implies $d' \in V, d \in F$ or $d \in V, d' \in F$. We also consider the incident pairs of adjacent vertices and faces, and denote them by

$$C := \{(v, f, v', f') \mid v, v' \in V, \quad f, f' \in F, \quad v, v' \sim f, f'\},$$

which we call the *crosses* of D.

The faces and edges of $\mathbb{Z}^2$ can be identified with the vertices and edges of the dual graph $(\mathbb{Z}^2)^*$, respectively. In this way D may be viewed as the join of $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^*$, and C as pairs of an edge of $\mathbb{Z}^2$ and its corresponding dual edge of $(\mathbb{Z}^2)^*$.

The identification of the faces F with the vertices of the dual graph yields a definition for nets as maps on F analogous to the definition for nets as maps on V. We define binets as maps on D that come from pairs of nets on V and F.

Definition 14.1.

(i) A *binet* is a map $b : D \to \mathbb{RP}^n$, such that the restrictions to V and F are nets.

(ii) A binet b such that the restrictions to V and F are conjugate nets is a *conjugate binet.*

(iii) A binet $b : D \to \mathbb{R}^n$ is an *orthogonal binet* if

$$\langle b(v) - b(v'), b(f) - b(f') \rangle = 0, \qquad \text{for all crosses } (v, f, v', f') \in C.$$

(iv) A *principal binet* is a binet $b : D \to \mathbb{R}^n$ that is both conjugate and orthogonal.

Remark 14.1.

(i) Note that different to circular nets, for binets the orthogonality condition can be considered separate from the conjugacy condition.

(ii) For a conjugate binet, we can associate to every $d \in D$ the plane

$$\square b(d) = b(d_1) \vee b(d_2) \vee b(d_3) \vee b(d_4)$$

where $d_1, d_2, d_3, d_4 \sim d$ are the four vertices/faces incident with d.

(iii) For a conjugate binet b in $\mathbb{R}^3$, we define its *normal bicongruence*

$$N : D \to \mathrm{Lines}(\mathbb{R}^3)$$

where $N(d)$ is the unique line through $b(d)$ orthogonal to $\square b(d)$.

(iv) For a principal binet b in $\mathbb{R}^3$ two neighboring normal lines $N(v), N(v')$ intersect since they are both orthogonal to the line $b(f) \vee b(f')$, where $(v, f, v', f') \in C$.

This leads to a natural definition of focal points, and once we have a Möbius geometric representation of principal binets, to curvature spheres.

Neither orthogonal binets nor principal binets are a priori Möbius invariant. Both are only invariant under similarity transformations. This can be changed by the following additional construction.

Möbius lift of orthogonal binets

In the following, binets that consist of two nets which are related by polarity with respect to a quadric will play an important role.

Recall that a quadric $\mathcal{Q} \subset \mathbb{RP}^n$ is given by a symmetric bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{Q}}$ on $\mathbb{R}^{n+1}$

$$\mathcal{Q} = \left\{ [x] \in \mathbb{RP}^n \;\middle|\; \langle x, x \rangle_{\mathcal{Q}} = 0 \right\}.$$

For a projective subspace $K \subset \mathbb{RP}^n$ its polar subspace is given by

$$K^{\perp} = \{ X \in \mathbb{RP}^n \mid X \perp Y \text{ for all } Y \in K \},$$

where we denote the polarity relation of two points $X = [x], Y = [y] \in \mathbb{RP}^n$ by

$$X \perp Y \quad \Leftrightarrow \quad \langle x, y \rangle_{\mathcal{Q}} = 0.$$

Definition 14.2 (Polar binets)**.** Let $\mathcal{Q} \subset \mathbb{RP}^n$ be a quadric. A *polar binet* is a binet $b : D \to \mathbb{RP}^n$, such that

$$b(d) \perp b(d') \qquad \text{for all incident } d, d' \in D.$$

Equivalently polar binets can be characterized by the condition that the two lines of an edge and its corresponding dual edge are related by polarity, i.e.,

$$(b(v) \vee b(v')) \perp (b(f) \vee b(f')) \qquad \text{for all crosses } (v, f, v', f') \in C.$$

We use the projective model of Möbius geometry, where we denote the Möbius quadric (n-dimensional sphere) by $\mathcal{M} \subset \mathbb{RP}^{n+1}$ and the "north pole" by $B = [e_\infty]$. and identify points in $\mathbb{RP}^{n+1}$ with spheres in $\mathbb{R}^n$ by means of the map $\xi_{\mathcal{S}} : \mathbb{RP}^{n+1} \to \mathcal{S}$. where $\mathcal{S}$ denotes the set of *generalized Euclidean spheres*, that is the set of Euclidean spheres and planes

$$\mathcal{S} := \text{Spheres}(\mathbb{R}^n).$$

Then we obtain a map

$$\xi_{\mathcal{S}} : \mathcal{M}^+ \to \mathcal{S}, \quad X \mapsto \pi_{\mathcal{E}}(X^\perp \cap \mathcal{M}).$$

In this representation of spheres (and planes) the orthogonality of spheres is given by polarity with respect to the Möbius quadric. That is, two points $X, X' \in \mathcal{M}^+$ are polar with respect to $\mathcal{M}$ if and only if the two corresponding spheres $\xi_{\mathcal{S}}(X)$ and $\xi_{\mathcal{S}}(X')$ are orthogonal.

This immediately leads to a representation of a polar binet in $\mathbb{RP}^{n+1}$ in terms of orthogonal spheres, which we call its *orthogonal sphere representation*.

Lemma 14.1. *Let $b_{\mathcal{M}} : D \to \mathbb{RP}^{n+1} \backslash B^\perp$ be a polar binet with respect to the Möbius quadric $\mathcal{M} \subset \mathbb{RP}^{n+1}$. Let*

$$b_{\mathcal{S}} := \xi_{\mathcal{S}} \circ b_{\mathcal{M}}$$

be its orthogonal sphere representation, and

$$b := \pi_{\mathcal{E}} \circ b_{\mathcal{M}}$$

the projection of $b_{\mathcal{M}}$ to Euclidean space $\mathbb{R}^n$. Then

(i) the two spheres $b_{\mathcal{S}}(d)$ and $b_{\mathcal{S}}(d')$ intersect orthogonally for all incident $d, d' \in D$,

(ii) $b(d)$ is the center of $b_{\mathcal{S}}(d)$ for all $d \in D$.

The first key aspect that we want to emphasize is the close relation of polar binets in $\mathbb{RP}^{n+1}$ and orthogonal binets in $\mathbb{R}^n$. More specifically, the stereographic projection of a polar binet is an orthogonal binet. Conversely, an orthogonal binet can always be lifted to a polar binet. We formalize both statements in the following.

Lemma 14.2. *Let $b_{\mathcal{M}} : D \to \mathbb{RP}^{n+1} \backslash B^\perp$ be a polar binet with respect to the Möbius quadric $\mathcal{M} \subset \mathbb{RP}^{n+1}$. Then its projection*

$$b := \pi_{\mathcal{E}} \circ b_{\mathcal{M}},$$

to Euclidean space $\mathbb{R}^n$ is an orthogonal binet.

Proof. Consider a cross $(v, f, v', f') \in C$. By Lemma 14.1 (i), the two spheres $b_{\mathcal{S}}(v)$ and $b_{\mathcal{S}}(v')$ are orthogonal to both spheres $b_{\mathcal{S}}(f)$ and $b_{\mathcal{S}}(f')$. Therefore the line spanned by the two sphere centers $b(f) \vee b(f')$ is in the radical plane of $b_{\mathcal{S}}(v)$ and $b_{\mathcal{S}}(v')$ (Exercise). This radical plane (and thus the line $b(f) \vee b(f')$) is orthogonal to the line $b(v) \vee b(v')$. $\square$

We are now in a position to define the *Möbius lift* of orthogonal binets.

Definition 14.3 (Möbius lift)**.** Let $b : D \to \mathcal{E}$ be a binet. Then a binet $b_{\mathcal{M}} : D \to \mathbb{RP}^{n+1} \backslash B^{\perp}$ is called a *Möbius lift* of b if

(i) $b_{\mathcal{M}}$ is a polar binet with respect to $\mathcal{M}$,

(ii) b is the projection of $b_{\mathcal{M}}$, that is, $\pi_{\mathcal{E}}(b_{\mathcal{M}}) = b$.

Lemma 14.2 implies that the condition that b is an orthogonal binet is necessary for the existence of a Möbius lift $b_{\mathcal{M}}$. The following theorem shows that it is also sufficient, and thus guarantees the existence of a Möbius lift for an orthogonal binet.

Theorem 14.3. *Let $b : D \to \mathbb{R}^n$ be a binet. Then a Möbius lift $b_{\mathcal{M}} : D \to \mathbb{RP}^{n+1} \backslash B^{\perp}$ of b exists if and only if b is an orthogonal binet.*

Proof.
($\Rightarrow$) Follows from Lemma 14.2.
($\Leftarrow$) Assume b is an orthogonal binet, and let us construct a corresponding Möbius lift $b_{\mathcal{M}}$. Because b needs to be the stereographic projection of $b_{\mathcal{M}}$, for all $d \in D$ the point $b_{\mathcal{M}}(d)$ must lie on the line through $b(d)$ and B, that is

$$b_{\mathcal{M}}(d) \in \ell(d) := b(d) \vee B.$$

Moreover, the point $b_{\mathcal{M}}(d)$ determines the point $b_{\mathcal{M}}(d')$ as the intersection of the line $\ell(d)$ with the polar hyperplane $b_{\mathcal{M}}(d)^{\perp}$:

$$b_{\mathcal{M}}(d') = \ell(d) \cap b_{\mathcal{M}}(d)^{\perp}.$$

Therefore, if for one $d \in D$ the lift $b_{\mathcal{M}}(d)$ is chosen arbitrarily on the line $\ell(d)$, then the whole image of $b_{\mathcal{M}}$ is determined.

It remains to show that this construction is well-defined around crosses. Two spheres in $\mathbb{R}^n$ with centers x, x^* and radii r, r^*, respectively, are orthogonal if and only if (Exercise)

$$\langle x, x^* \rangle = \rho + \rho^*,$$

where

$$\rho = \frac{1}{2}\left(|x|^2 - r^2\right), \qquad \rho^* = \frac{1}{2}\left(|x^*|^2 - (r^*)^2\right).$$

Thus, given a binet b, a Möbius lift exists if and only if there is a function $\rho : D \to \mathbb{R}$ with

$$\langle b(d), b(d') \rangle = \rho(d) + \rho(d') \quad \text{for all incident } d, d' \in D.$$

This is the case if and only if the following condition around each cross is satisfied

$$\langle b(v), b(f) \rangle - \langle b(f), b(v') \rangle + \langle b(v'), b(f') \rangle - \langle b(f'), b(v) \rangle = 0.$$

But this equation is equivalent to $\langle b(v) - b(v'), b(f) - b(f')\rangle = 0$, i.e., the condition that b is an orthogonal binet. The Möbius lift of b is then given by

$$b_{\mathcal{M}}(d) = [b(d) + e_0 + 2\rho(d)e_\infty],$$

where the function ρ is related to the centers c and radii r of the spheres of the orthogonal sphere representation by

$$2\rho = |c|^2 - r^2.$$

□

Remark 14.2. From the proof of Theorem 14.3 follows that each orthogonal binet has a 1-parameter family of Möbius lifts. One point $b_{\mathcal{M}}(d)$ can be chosen arbitrarily on the line $b(d) \vee B$ (or correspondingly, the radius of one sphere $b_{\mathcal{S}}(d)$ with center $b(d)$ can be chosen arbitrarily) and then the remaining points of the lift are uniquely determined.

Note that – using the function ρ – a different choice for the Möbius lift is represented by the simple transformation

$$\begin{aligned} \rho(v) &\mapsto \rho(v) + \varepsilon, \quad v \in V, \\ \rho(f) &\mapsto \rho(f) - \varepsilon, \quad f \in F, \end{aligned}$$

with $\varepsilon \in \mathbb{R}$.

Möbius lift of principal binets

Since a principal binet is a special case of an orthogonal binet, let us discuss properties of the Möbius lift of a principal binet. We begin with the specialization of Lemma 14.2.

Lemma 14.4. *Let $b_{\mathcal{M}} : D \to \mathbb{RP}^{n+1} \backslash B^\perp$ be a conjugate and polar binet with respect to the Möbius quadric $\mathcal{M} \subset \mathbb{RP}^{n+1}$. Then its projection*

$$b := \pi_{\mathcal{E}} \circ b_{\mathcal{M}},$$

to Euclidean space $\mathbb{R}^n$ is a principal binet.

Proof. Exercise.

Indeed, conjugate polar binets are (generically) in bijection with principal binets, as captured by the following specialization of Theorem 14.3.

Theorem 14.5. *Let $b : D \to \mathbb{R}^n$ be a orthogonal binet. Then its Möbius lift $b_{\mathcal{M}} : D \to \mathbb{RP}^{n+1} \backslash B^\perp$ is a conjugate binet if and only if b is a principal binet.*

Proof. Exercise. □

The fact that $b_{\mathcal{M}}$ is a conjugate binet implies an additional Möbius geometric structure of principal binets.

Definition 14.4 (principal binet circles)**.** Let $b : D \to \mathbb{R}^n$ be a principle binet, and $b_{\mathcal{M}} : D \to \mathbb{RP}^{n+1} \backslash B^\perp$ its Möbius lift. For any vertex or face $d \in D$, we denote by

$$K(d) := \pi_{\mathcal{E}}(\square b_{\mathcal{M}}(d) \cap \mathcal{M}), \tag{22}$$

the (possibly imaginary) circle given by the planar section of the Möbius quadric with the plane $\square b_{\mathcal{M}}(d)$, projected to Euclidean space.

Proposition 14.6. *Let $b : D \to \mathbb{R}^3$ be a principal binet, and $N : D \to \mathrm{Lines}(\mathbb{R}^3)$ be its normal bicongruence. For every vertex or face $d \in D$*

(i) the circle $K(d)$ is orthogonal to all four spheres $b_\mathcal{S}(d')$ with d' incident to d,

(ii) $K(d)$ is contained in the plane $\square b(d)$ and the sphere $b_\mathcal{S}(d)$, and thus,

$$K(d) = \square b(d) \cap b_\mathcal{S}(d),$$

(iii) and the axis of $K(d)$ is the normal line $N(d)$. □

Proof.

(i) $b_\mathcal{M}(d') \in \square b_\mathcal{M}(d)$ implies that $K(d)$ is the circle contained in every sphere that is orthogonal to all four incident spheres $b_\mathcal{S}(d')$. This is equivalent to $K(d)$ being orthogonal to those four incident spheres.

(ii) $K(d) \subset b_\mathcal{S}(d)$ follows from $b_\mathcal{M}(d) \perp \square b_\mathcal{M}(d)$, while $K(d) \subset \square b(d)$ follows from $\pi_\mathcal{E}(\square b_\mathcal{M}(d)) = \square b(d)$.

(iii) From (ii), it follows that the axis of $K(d)$ is the unique line that contains the center $b(d)$ of $b_\mathcal{S}(d)$ and is orthogonal to $\square b(d)$. □

We briefly introduce the notion of principal curvature spheres for principal binets.

Definition 14.5 (Principal curvature spheres)**.** Let b be a principal binet in $\mathbb{R}^3$ with Möbius lift $b_\mathcal{M}$. Let $(v, f, v', f') \in C$ be a cross of D, and let

$$H(v, v') := \square b_\mathcal{M}(v) \vee \square b_\mathcal{M}(v'), \qquad H(f, f') := \square b_\mathcal{M}(f) \vee \square b_\mathcal{M}(f'),$$

be the two 3-dimensional subspaces in $\mathbb{RP}^4$ spanned by the two pairs of adjacent faces of the cross. Then we call the spheres

$$\mathfrak{S}(v, v') := \xi_\mathcal{S}(H(v, v')^\perp), \qquad \mathfrak{S}(f, f') := \xi_\mathcal{S}(H(f, f')^\perp),$$

the *principal curvature spheres* of the edges (v, v') and (f, f') respectively.

The definition of the principal curvature spheres is invariant under Möbius transformations. It exhibits the following geometric properties.

Proposition 14.7. *Let b be a principal binet in $\mathbb{R}^3$ with Möbius lift $b_\mathcal{M}$ and orthogonal sphere representation $b_\mathcal{S}$. Let $(v, f, v', f') \in C$ be a cross of D. Then the following properties hold for the principal curvature sphere $\mathfrak{S}(v, v')$:*

(i) $\mathfrak{S}(v, v')$ is orthogonal to the six spheres $b_\mathcal{S}(\tilde{f})$, where $\tilde{f} \in F$ is any face incident to v or v',

(ii) $\mathfrak{S}(v, v')$ contains the two circles $K(v)$ and $K(v')$, as defined in (22)*,*

(iii) the intersection of the lines of the normal congruence $N(v) \cap N(v')$ is the center of $\mathfrak{S}(v, v')$.

Analogous properties hold for the principal curvature sphere $\mathfrak{S}(f, f')$.

Proof. Exercise. □

Property (iii) of Proposition 14.7 is the discrete analog of the centers of the principal curvature spheres being the focal points of the normal congruence of a surface, while properties (i) and (ii) describe the tangentiality to the surface.

Consistency of principal binets

We generalize binets from $\mathbb{Z}^2$ to $\mathbb{Z}^N$, defining them as maps on the vertices and (2-dimensional) faces of $\mathbb{Z}^N$ [AT26a].

Let V_N, E_N, F_N denote the vertices, edges, and (2-dimensional) faces of $\mathbb{Z}^N$, respectively. Moreover, let $D_N = V_N \cup F_N$ denote the union of vertices and faces of $\mathbb{Z}^N$. We say that two vertices $v, v' \in V_N$ (resp. two faces $f, f' \in F_N$) are *adjacent* if they share an edge, i.e., $(v, v') \in E_N$. A vertex $v \in V_N$ and a face $f \in F_N$ are called *incident* if v is a boundary vertex of f, and we write $v \sim f$.

Definition 14.6. We define three types of conjugate nets as follows.

(i) A *conjugate vertex-net* is a map $g : V_N \to \mathbb{R}\mathrm{P}^n$ such that the span

$$\bigvee_{v\sim f} g(v) \quad \text{is a plane for every } f \in F_N.$$

(ii) A *conjugate face-net* is a map $h : F_N \to \mathbb{R}\mathrm{P}^n$ such that the span

$$\bigvee_{f\sim v} h(f) \quad \text{is a plane for every } v \in V_N.$$

(iii) A *conjugate binet* is a map $b : D_N \to \mathbb{R}\mathrm{P}^n$ such that the restriction to V_N is a conjugate vertex-net and the restriction to F_N is a conjugate face-net.

Under the assumption of non-degeneracy, we introduce the following notation for the planes occurring in conjugate nets.

Definition 14.7.

(i) Let $g : V_N \to \mathbb{R}\mathrm{P}^n$ be a conjugate vertex-net. Then for every face $f \in F_N$ we denote the plane spanned by the images of the incident vertices by

$$\Box g(f) := \bigvee_{v\sim f} g(v).$$

(ii) Let $h : F_N \to \mathbb{R}\mathrm{P}^n$ be a conjugate face-net. Then for every vertex $v \in V_N$ we denote the plane spanned by the images of the incident faces by

$$\Box h(v) := \bigvee_{f\sim v} h(f).$$

Remark 14.3. In comparison with the smooth theory of conjugate systems, we may view a conjugate face-net, and hence a conjugate binet, as a collection of points of a conjugate system together with some of its focal points.

Instead of the points of a conjugate face-net, one may regard the planes defined on vertices as the primary objects, i.e., as a map

$$\Box h : F_N \to \mathrm{Planes}(\mathbb{R}\mathrm{P}^n).$$

In this way one easily sees that conjugate face-nets constitute a 3D-system. For an elementary 3-cube, the plane corresponding to the eighth vertex is determined by completing the octahedron formed by the seven other planes. This 3D-system is multi-dimensionally consistent (see [AT26a]).

Proposition 14.8. *Conjugate face-nets are multi-dimensionally consistent 3D-systems.*

Definition 14.8. Let $\mathcal{Q}$ be a quadric in $\mathbb{RP}^n$. A *polar conjugate binet* $b : D_N \to \mathbb{RP}^n$ is a conjugate binet such that for every pair of incident $v \in V_N$ and $f \in F_N$ the points $b(v)$ and $b(f)$ are related by polarity with respect to $\mathcal{Q}$.

For $N = 2$ the definition of conjugate polar binets coincides with that given in [AT26b].

Theorem 14.9. *Polar conjugate binets are a consistent reduction of conjugate binets.*

Proof. Let $\mathcal{Q}$ be a quadric in $\mathbb{RP}^n$ and let $b\colon D_N \to \mathbb{RP}^n$ be a conjugate binet. A polar conjugate binet is characterized by the polarity constraint

$$b(v) \in b(f)^\perp \quad \text{for all incident } v \in V_N,\ f \in F_N,$$

or, equivalently, by

$$\Box b(d) \subset b(d)^\perp \quad \text{for all } d \in D_N,$$

where $(\cdot)^\perp$ denotes the polar subspace with respect to $\mathcal{Q}$. In the following, we use this latter characterization and propagate the binet together with its associated planes.

Consider the eight vertices

$$v,\ v_1,\ v_2,\ v_3,\ v_{12},\ v_{23},\ v_{13},\ v_{123}$$

of a 3-cube in $\mathbb{Z}^N$ and its six faces

$$f^{12},\ f^{23},\ f^{13},\ f_3^{12},\ f_1^{23},\ f_2^{13}.$$

Assume that the polarity constraint holds at all vertices except v_{123} and on the three faces f^{12}, f^{23}, f^{13}. To complete the cube, it suffices to verify that

$$\begin{aligned} &\Box\, b(f_3^{12}) \subset b(f_3^{12})^\perp, \quad \Box b(f_1^{23}) \subset b(f_1^{23})^\perp, \quad \Box b(f_2^{13}) \subset b(f_2^{13})^\perp,\\ &\Box\, b(v_{123}) \subset b(v_{123})^\perp \end{aligned} \tag{23}$$

holds. We first consider the face f_3^{12}. By definition of a conjugate binet,

$$\begin{aligned} \Box b(f_3^{12}) &= b(v_{23}) \vee b(v_3) \vee b(v_{13}),\\ b(f_3^{12}) &= \Box b(v_{23}) \cap \Box b(v_3) \cap \Box b(v_{13}). \end{aligned}$$

Since the polarity constraint holds on the initial data, we have

$$b(v_{23}) \in \Box b(v_{23})^\perp, \quad b(v_3) \in \Box b(v_3)^\perp, \quad b(v_{13}) \in \Box b(v_{13})^\perp,$$

and therefore

$$\begin{aligned} \Box b(f_3^{12}) &\subset \Box b(v_{23})^\perp \vee \Box b(v_3)^\perp \vee \Box b(v_{13})^\perp\\ &= \big(\Box b(v_{23}) \cap \Box b(v_3) \cap \Box b(v_{13})\big)^\perp\\ &= b(f_3^{12})^\perp. \end{aligned}$$

The same argument applies to the faces f_1^{23} and f_2^{13}. Finally, for the vertex v_{123} we have

$$\begin{aligned} \Box b(v_{123}) &= b(f_3^{12}) \vee b(f_2^{13}) \vee b(f_1^{23}),\\ b(v_{123}) &= \Box b(f_3^{12}) \cap \Box b(f_2^{13}) \cap \Box b(f_1^{23}), \end{aligned}$$

and, using the polarity of the three faces just established,

$$\begin{aligned} \Box b(v_{123}) &\subset \Box b(f_3^{12})^\perp \vee \Box b(f_2^{13})^\perp \vee \Box b(f_1^{23})^\perp\\ &= \big(\Box b(f_3^{12}) \cap \Box b(f_2^{13}) \cap \Box b(f_1^{23})\big)^\perp\\ &= b(v_{123})^\perp. \end{aligned}$$

This verifies all conditions in (23) and completes the proof. $\Box$

Definition 14.9. Let $b : D_N \to \mathbb{R}^n$ be a principal binet. A binet $b_{\mathcal{M}} : D_N \to \mathbb{RP}^{n+1}$ is called a *Möbius lift* of b if

(i) $b_{\mathcal{M}}$ is a polar conjugate binet with respect to $\mathcal{M}$,

(ii) b is the projection of $b_{\mathcal{M}}$, that is $\pi \circ b_{\mathcal{M}} = b$.

The properties for the Möbius lift in the case $N = 2$ carry over to the general case. Let us recollect two relevant properties of Möbius lifts.

Lemma 14.10.
(i) Every principal binet has a one-parameter family of Möbius lifts.

(ii) The projection π of any polar conjugate binet (with respect to $\mathcal{M}$) is a principal binet.

Theorem 14.11. *Principal binets are a consistent reduction of conjugate binets.*

Proof. Let $b : D_3 \to \mathbb{R}^n$ be a conjugate binet. Assume that on the union of the 2-dimensional coordinate planes, b satisfies the constraints defining a principal binet. Note that this implies additional conditions at the corner where the coordinate planes meet. By Lemma 14.10 (i), we may choose a Möbius lift $b_{\mathcal{M}}$ on the initial data. The Möbius lift is a polar conjugate binet in $\mathbb{RP}^{n+1}$. Therefore, by Theorem 14.9, we can extend the Möbius lift from the initial data to a polar conjugate binet $b_{\mathcal{M}}$ on all of D_3. Since both b and $b_{\mathcal{M}}$ are conjugate binets and are uniquely determined by their initial data, we have $\pi \circ b_{\mathcal{M}} = b$ on D_3. Thus, by Lemma 14.10 (ii), b is a principal binet on all of D_3. $\square$

Discrete orthogonal coordinate systems

In the following we consider a combinatorially different extension of binets to $\mathbb{Z}^N$. Previously we considered binets with points on the vertices and on the 2-faces of $\mathbb{Z}^N$, which we will now call *fc-binets* (face-centered binets). In this section we consider binets with points on the vertices and the N-cubes of $\mathbb{Z}^N$, which we will call *bc-binets* (body-centered binets). Thus, we consider them to be defined on pairs of dual lattices. For the square lattice $\mathbb{Z}^N$ we call $(\mathbb{Z}^N)^* \cong \left(\mathbb{Z} + \frac{1}{2}\right)^N$ its *dual square lattice*, and say that any two edges

$$[\boldsymbol{n}, \boldsymbol{n} + \boldsymbol{e}_i] \subset \mathbb{Z}^N, \qquad [\boldsymbol{n} + \tfrac{1}{2}\boldsymbol{\sigma}, \boldsymbol{n} + \tfrac{1}{2}\boldsymbol{\sigma} + \boldsymbol{e}_j] \subset \left(\mathbb{Z} + \tfrac{1}{2}\right)^N$$

are *dual edges*, where $\boldsymbol{n} \in \mathbb{Z}^N$ and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_M) \in \{\pm 1\}^N$ with $\sigma_i = 1$ and $\sigma_j = -1$. Furthermore, for a point $\boldsymbol{n} \in \mathbb{Z}^N$, we call the 2^N points $\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma} \in \left(\mathbb{Z} + \frac{1}{2}\right)^N$, $\boldsymbol{\sigma} \in \{\pm 1\}^N$, its *adjacent points from the dual lattice*.

Definition 14.10.
(i) A map

$$\boldsymbol{x} : \mathbb{Z}^N \cup \left(\mathbb{Z} + \tfrac{1}{2}\right)^N \to \mathbb{R}^n$$

is called a *bc-binet*.

(ii) A pair of dual discrete nets $\boldsymbol{x} : \mathbb{Z}^N \cup \left(\mathbb{Z} + \frac{1}{2}\right)^N \to \mathbb{R}^n$ is called *orthogonal* if every pair of dual edges is orthogonal in $\mathbb{R}^n$, i.e.,

$$\langle \Delta_i \boldsymbol{x}(\boldsymbol{n}), \Delta_j \boldsymbol{x}(\boldsymbol{n}^*) \rangle = 0, \tag{24}$$

for all distinct $i, j = 1, \ldots, N$ and $\boldsymbol{n} \in \mathbb{Z}^N$, $\boldsymbol{n}^* = \boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma} \in \left(\mathbb{Z} + \frac{1}{2}\right)^N$, where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_M) \in \{\pm 1\}^N$ with $\sigma_i = 1$ and $\sigma_j = -1$.

In the case of $N = n$, we obtain a discrete version of Dupin's theorem.

Theorem 14.12 ("discrete Dupin's theorem")**.** *Let $N \geqslant 3$ and $\boldsymbol{x} : \mathbb{Z}^N \cup \left(\mathbb{Z} + \frac{1}{2}\right)^N \to \mathbb{R}^N$ be an orthogonal bc-binet, which we also call a **discrete orthogonal coordinate system (dOCS)**. Then its two discrete subnets $\boldsymbol{x}\big|_{\mathbb{Z}^N}$ and $\boldsymbol{x}\big|_{\left(\mathbb{Z}+\frac{1}{2}\right)^N}$ are discrete conjugate nets.*

Proof. The edge vectors of an elementary quadrilateral

$$(\boldsymbol{x}(\boldsymbol{n}), \boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_i), \boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_i + \boldsymbol{e}_j), \boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_j))$$

lie in the orthogonal complement of the $N-2$ linearly independent vectors

$$\bar{\Delta}_k \boldsymbol{x}(\boldsymbol{n} + \tfrac{1}{2}\boldsymbol{\sigma}), \qquad \boldsymbol{\sigma} = (1, \dots, 1), \qquad k \neq i, j,$$

which is of dimension 2. $\square$

For a dOCS the Moebius lift, Moebius invariance, and circles per face are defined analogous to principal binets. We will consider dOCS in the the case $N = 3$, and in particular the relation to principal binets, in more detail in the following. For this we denote the vertices of $\mathbb{Z}^3$ again by V_3, while we denote the 3-cubes of $\mathbb{Z}^3$ (i.e. the dual lattice) by $\cube_3$.

Definition 14.11. Let $x : V_3 \cup \cube_3 \to \mathbb{R}^3$ be a dOCS. A map $x_{\mathcal{M}} : V_3 \cup \cube_3 \to \mathbb{RP}^4$ is called a *Möbius lift* of x if

(i) for every incident $v \in V_3$ and $c \in \cube_3$, the points $x(v)$ and $x(c)$ are related by polarity with respect to $\mathcal{M}$,

(ii) x is the projection of $x_{\mathcal{M}}$, that is, $\pi \circ x_{\mathcal{M}} = x$.

The polarity constraint immediately implies that the restriction of a Möbius lift $x_{\mathcal{M}}$ of a dOCS to V_3 (and to $\cube_3 \cong V_3$, respectively) is again a conjugate vertex-net. The existence and completeness results for Möbius lifts of principal binets extend directly to Möbius lifts of dOCS.

Lemma 14.13.

(i) Every dOCS has a one-parameter family of Möbius lifts.

(ii) The projection π of any map $V_3 \cup \cube_3 \to \mathbb{RP}^4$ that satisfies the polarity constraint (i) from Definition 14.11 is a dOCS.

Moreover, the polarity constraint (i) from Definition 14.11 implies that the restriction of $x_{\mathcal{M}}$ to V_3 (or, respectively, to $\cube_3$) uniquely determines the entire lift via

$$x_{\mathcal{M}}(c) = \left(\bigvee_{v \sim c} x(v) \right)^{\perp}.$$

Hence, on the level of Möbius lifts, there is an immediate correspondence between principal binets and dOCS: starting from a conjugate vertex-net on V_3, one may extend either to faces F_3 or to cubes $\cube_3$. The extension to $\cube_3$ is unique, whereas the extension to F_3 depends on initial data. We now make this correspondence explicit and describe its effect on the projections.

Let $b_{\mathcal{M}} \colon V_3 \cup F_3 \to \mathbb{RP}^4$ be a Möbius lift of a principal binet $b \colon V_3 \cup F_3 \to \mathbb{R}^3$. Since the restriction of $b_{\mathcal{M}}$ to V_3 is a conjugate vertex-net, the span of the vertices of every elementary 3-cube is three-dimensional. Consequently, its polar subspace is a single point. Geometrically, this point can be interpreted as the center of a sphere associated with each 3-cube. We therefore define a map $x_{\mathcal{M}} \colon V_3 \cup \cube_3 \to \mathbb{RP}^4$ by

$$\begin{aligned} x_{\mathcal{M}}(v) &:= b_{\mathcal{M}}(v) && \text{for all } v \in V_3, \\ x_{\mathcal{M}}(c) &:= \Big(\bigvee_{v \sim c} b_{\mathcal{M}}(v)\Big)^{\perp} && \text{for all } c \in \cube_3. \end{aligned}$$

By construction, $x_{\mathcal{M}}$ satisfies the polarity constraint (i) from Definition 14.11 and therefore defines, by Lemma 14.13 (ii), a Möbius lift of a dOCS.

We now describe how this dOCS arises directly from the principal binet b. The point $x_{\mathcal{M}}(c)$ is the intersection point of all lines $x_{\mathcal{M}}(c) \vee x_{\mathcal{M}}(c')$ with $c' \in \cube_3$ adjacent to c. Each such line is polar to the plane $\square b_{\mathcal{M}}(f)$, where $f \in F_3$ is the face incident to c and c'. Under projection, this line becomes the normal line of the principal binet (or, equivalently, the axis of the circle $C(f)$),c

$$\pi\big(x_{\mathcal{M}}(c) \vee x_{\mathcal{M}}(c')\big) = N(f),$$

where $N(f)$ denotes the line through $b(f)$ orthogonal to $\square b(f)$. Consequently, the dOCS $x \colon V_3 \cup \cube_3 \to \mathbb{R}^3$ associated with b is given by

$$\begin{aligned} x(v) &:= b(v) && \text{for all } v \in V_3, \\ x(c) &:= \bigcap_{f \sim c} N(f) && \text{for all } c \in \cube_3. \end{aligned} \tag{25}$$

In particular, the resulting points of x depend only on the principal binet b and not on the choice of Möbius lift.

Conversely, let $x_{\mathcal{M}} : V_3 \cup \cube_3 \to \mathbb{RP}^4$ be the Möbius lift of a dOCS $x : V_3 \cup \cube_3 \to \mathbb{R}^3$. First, we set

$$b_{\mathcal{M}}(v) := x_{\mathcal{M}}(v) \qquad \text{for all } v \in V_3.$$

Furthermore, by the polarity condition on the Möbius lift, we have

$$b_{\mathcal{M}}(f) \in \big(\square b_{\mathcal{M}}(f)\big)^{\perp} = x_{\mathcal{M}}(c) \vee x_{\mathcal{M}}(c') \qquad \text{for all } f \in F_3,$$

where $c, c' \in \cube_3$ are the two cubes incident to f. These points may be chosen arbitrarily on one-dimensional initial data, for instance along coordinate axes in the coordinate planes. All remaining points in the coordinate planes are then uniquely determined on these lines by the conjugacy condition, as described in [AT26b, Section 20]. Finally, by Theorem 14.9, polar conjugate binets form a consistent reduction of conjugate binets, and thus all remaining points in F_3 are uniquely determined by the 3D-system of conjugate face-nets.

We now describe how this principal binet arises directly from the dOCS x. The construction carries over directly to the projection. First, we set

$$b(v) := x(v) \qquad \text{for all } v \in V_3. \tag{26}$$

We then construct the remaining points

$$b(f) \in x(c) \vee x(c') \qquad \text{for all } f \in F_3, \tag{27}$$

from one-dimensional initial data, where $c, c' \in \mathcal{C}_3$ are the two cubes incident to f. The construction proceeds first to two-dimensional data by the conjugacy condition, as described in [AT26b, Section 20], and then to all of F_3 using the 3D-system of conjugate face-nets. This yields a principal binet by Theorem 14.11.

Note that although the constructions for the Möbius lift and for the projection look identical, for the Möbius lift the restriction to V_3 uniquely determines all remaining points, while for the projection the restriction to F_3 is still used.

We summarize this correspondence in the following theorem.

Theorem 14.14. *There is a natural correspondence between principal binets $b : V_3 \cup F_3 \to \mathbb{R}^3$ and dOCS $x : V_3 \cup \mathcal{C}_3 \to \mathbb{R}^3$ on $\mathbb{Z}^3$:*

(i) Every principal binet defines a unique dOCS by (25)*.*

(ii) Every dOCS defines a principal binet by (26) *and by* (27) *on one-dimensional initial data, and propagation as described above (via the 3D-system of conjugate face-nets).*

Discrete confocal quadrics

To obtain a discretization of confocal quadrics we apply the characterizing properties from Theorem 12.3 for (smooth) confocal coordinates to Definition 14.10 (ii) of orthogonal bc-binets [BSST18].

Thus, consider applying the factorizability condition

$$x_k(\boldsymbol{n}) = f_1^k(n_1) f_2^k(n_2) \cdots f_N^k(n_N), \qquad k = 1, \dots, N,$$

to an orthogonal bc-binet $\boldsymbol{x} : \mathbb{Z}^N \cup \left(\mathbb{Z} + \frac{1}{2}\right)^N \to \mathbb{R}^N$ defined on the dual pair of square lattices $\mathbb{Z}^N$ and $\left(\mathbb{Z} + \frac{1}{2}\right)^N$. Then the functions f_i^k must each be defined on $\frac{1}{2}\mathbb{Z}$, and thus, the net $\boldsymbol{x}$ can be extended to all of $\left(\frac{1}{2}\mathbb{Z}\right)^N$. The two dual lattices $\mathbb{Z}^M$ and $\left(\mathbb{Z} + \frac{1}{2}\right)^M$ are just one pair of dual sublattices of $\left(\frac{1}{2}\mathbb{Z}\right)^M$. More generally we call two lattices

$$\mathbb{Z}^M + \tfrac{1}{2}\boldsymbol{\delta}, \qquad \mathbb{Z}^M + \tfrac{1}{2}\bar{\boldsymbol{\delta}},$$

a *pair of dual sublattices* of $\left(\frac{1}{2}\mathbb{Z}\right)^M$, where

$$\boldsymbol{\delta} = (\delta_1, \dots, \delta_M) \in \{0,1\}^M, \qquad \bar{\boldsymbol{\delta}} = (1 - \delta_1, \dots, 1 - \delta_M) \in \{0,1\}^M.$$

The stepsize $\frac{1}{2}$ square lattice $\left(\frac{1}{2}\mathbb{Z}\right)^M$ has 2^{M-1} such pairs of dual sublattices.

Definition 14.12.

(i) A map

$$\boldsymbol{x} : \left(\tfrac{1}{2}\mathbb{Z}\right)^M \to \mathbb{R}^N$$

is called a *stepsize $\frac{1}{2}$ discrete net.*

(ii) A stepsize $\frac{1}{2}$ discrete net is called *regular* if all of its 2^M (stepsize 1) discrete subnets are regular.

(iii) A stepsize $\frac{1}{2}$ discrete net is called *orthogonal* if all of its 2^{M-1} pairs of dual discrete subnets (bc-binets) are orthogonal.

Remark 14.4. For a general stepsize $\frac{1}{2}$ discrete net, the discrete orthogonality constraint (24) only correlates the two nets from each bc-binet. The 2^{M-1} different bc-binets are not mutually correlated by this condition unless an additional constraint, like the factorizability, is introduced.

Theorem 14.15. *Let*

$$\boldsymbol{x} : \left(\tfrac{1}{2}\mathbb{Z}\right)^N \supset \mathcal{U} = I_1 \times \ldots \times I_N \to \mathbb{R}^N, \quad \boldsymbol{x}(s_1, \ldots, s_N) = \begin{pmatrix} x_1(\boldsymbol{n}) \\ \vdots \\ x_N(\boldsymbol{n}) \end{pmatrix}$$

be a discrete coordinate system that satisfies the following two conditions:

(i) $\boldsymbol{x}$ *factorizes:*

$$x_k(\boldsymbol{n}) = \prod_{i=1}^{N} f_i^k(n_i), \qquad k = 1, \ldots, N$$

with some smooth functions $f_i^k : I_i \to \mathbb{R}, i = 1, \ldots, N$ *that do not vanish:*

$$f_i^k(n_i) \neq 0, \quad \textit{for all} \quad n_i \in I_i, \quad i = 1, \ldots, N$$

and whose differences do not constantly vanish:

$$\Delta(f_i^k) \neq 0, \quad \textit{for all} \quad i = 1, \ldots, N$$

(ii) $\boldsymbol{x}$ *is orthogonal, in the sense of Definition 14.12.*

Then there exist $a_1, \ldots, a_N \in \mathbb{R}$ *and sequences* $u_i : \left(\frac{1}{2}\mathbb{Z} + \frac{1}{4}\right) \to \mathbb{R}$, *such that*

$$\sum_{k=1}^{N} \frac{x_k(\boldsymbol{n}) x_k(\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma})}{a_k + u_i} = 1, \quad u_i = u_i(n_i + \tfrac{1}{4}\sigma_i), \qquad i = 1, \ldots, N$$

for any $\boldsymbol{n} \in \mathcal{U}$ *and* $\boldsymbol{\sigma} \in \{\pm 1\}^N$ *(apart from degenerate cases which arise as limits in which two or more of the values* a_k *coincide).*

Proof. See e.g. [BSST18]. □

15 Isothermic nets

A parametrization $x : \mathbb{R}^2 \supset U \to \mathbb{R}^n$ is called *conformal* (angle preserving) if

$$\langle x_u, x_v \rangle = 0, \qquad \|x_u\| = \|x_v\|.$$

A surface is called *isothermic* if it possesses a conformal curvature line parametrization, i.e., if it possess a parametrization that satisfies

$$x_{uv} \in \operatorname{span}\{x_u, x_v\}, \qquad \langle x_u, x_v \rangle = 0, \qquad \|x_u\| = \|x_v\|,$$

where $s := \|x_u\| = \|x_v\|$ is called them *conformal metric coefficient* of x. Locally and away from umbilic points the curvature line parametrization of a surface is uniquely determined up to reparametrization along the coordinate lines

$$x(u, v) \mapsto x(\varphi(\tilde{u}), \chi(\tilde{v})).$$

The parametrization is conformal in the new parameters $\tilde{u}, \tilde{v}$ if

$$0 = \langle x_{\tilde{u}}, x_{\tilde{v}} \rangle = \varphi' \chi' \langle x_u, x_v \rangle \;\Leftrightarrow\; \langle x_u, x_v \rangle = 0,$$
$$\|x_{\tilde{u}}\| = \|x_{\tilde{v}}\| \;\Leftrightarrow\; |\varphi'| \, \|x_u\| = |\chi'| \, \|x_{\tilde{v}}\| \;\Leftrightarrow\; \frac{\|x_u\|}{\|x_v\|} = \frac{|\chi'|}{|\varphi'|}.$$

Definition 15.1. Let $x : \mathbb{R}^2 \supset U \to \mathbb{R}^n$ be a curvature line parametrization, i.e.,

$$x_{uv} \in \operatorname{span}\{x_u, x_v\}, \qquad \langle x_u, x_v \rangle = 0.$$

Then x is called an *isothermic net* if x is conformal up to a reparametrization along the parameter lines, i.e.,

$$\frac{\|x_u\|^2}{\|x_v\|^2} = \frac{\alpha(u)}{\beta(v)}$$

at every point $(u, v) \in U$.

The two functions α, β are unique up to a common global scalar. The conformal metric coefficient is determined (again up to that global scalar) by

$$s^2 = \frac{\|x_u\|^2}{\alpha} = \frac{\|x_v\|^2}{\beta}.$$

Proposition 15.1. *A parametrization $x : \mathbb{R}^2 \supset U \to \mathbb{R}^n$ is an isothermic net if x is an orthogonal Kœnigs net.*

Proof. Let x be a curvature line parametrization. We show that $\frac{\|x_u\|}{\|x_v\|}$ factorizes if and only if x has equal Laplace invariants.

Since x is a conjugate line parametrization, we have

$$x_{uv} = \lambda x_u + \mu x_v$$

with some functions $\lambda, \mu : U \to \mathbb{R}$. Taking the scalar product with x_u and using orthogonality, we obtain

$$\langle x_{uv}, x_u \rangle = \lambda \, \|x_u\|^2 \;\Leftrightarrow\; \lambda = (\log \|x_u\|)_v,$$

and similarly

$$\mu = (\log \|x_v\|)_u.$$

Thus, x has equal Laplace invariants ($\lambda_u = \mu_v$) if and only if

$$(\log \|x_u\|)_{uv} = (\log \|x_u\|)_{uv} \;\Leftrightarrow\; \left(\log \tfrac{\|x_u\|}{\|x_v\|}\right)_{uv} = 0,$$

which is equivalent to $\frac{\|x_u\|}{\|x_v\|}$ factorizing. $\square$

Recall that a conjugate line parametrization is a Kœnigs net if and only if there exists a function ν such that

$$x_{uv} = (\log \nu)_v x_u + (\log \nu)_u x_v.$$

The function ν is unique up to a global multiplicative constant. In the case of isothermic nets the function ν is given by the conformal metric coefficient s.

Proposition 15.2. *A parametrization $x : \mathbb{R}^2 \supset U \to \mathbb{R}^n$ is an isothermic net if and only if*

$$x_{uv} = (\log s)_v x_u + (\log s)_u x_v, \qquad \langle x_u, x_v \rangle = 0$$

with some function $s : U \to \mathbb{R}_+$, which is the conformal coefficient of x.

Furthermore, Kœnigs nets are characterized by the existence of the Kœnigs dual, which was given by

$$x_u^* = \frac{1}{\nu^2} x_u, \qquad x_v^* = -\frac{1}{\nu^2} x_v.$$

Proposition 15.3. *Let $x : \mathbb{R}^2 \supset U \to \mathbb{R}^n$ be a curvature line parametrization. Then x is an isothermic net if and only there exists functions $\alpha(u), \beta(v)$ and a net $x^* : U \to \mathbb{R}^n$ such that*

$$x_u^* = \alpha \frac{x_u}{\|x_u\|^2}, \qquad x_v^* = -\beta \frac{x_v}{\|x_v\|^2}.$$

In this case, x^* is called the ***Christoffel dual*** *of x. It is uniquely determined up to scaling and translation. The conformal metric coefficient is given by $s^2 = \frac{\|x_u\|^2}{\alpha} = \frac{\|x_v\|^2}{\beta}$ such that*

$$x_u^* = \frac{x_u}{s^2}, \qquad x_v^* = -\frac{x_v}{s^2}.$$

and x^* is itself an isothermic net with conformal metric coefficient $\frac{1}{s}$.

Conformal parametrizations as well as curvature line parametrizations are invariant under Möbius transformations, thus, so are isothermic nets. Therefore, it makes sense to derive a projective characterization of isothermic lifts in the projective model of Möbius geometry.

Proposition 15.4. *Let $x : \mathbb{R}^2 \supset U \to \mathbb{R}^n$ be a parametrized surface and $[\hat{x}] : U \to \mathcal{M} \subset \mathbb{RP}^{n+1}$ its Möbius lift*

$$\hat{x} := x + \|x\|^2 e_\infty + e_0.$$

Then x is an isothermic net if and only if $[\hat{x}]$ is a Kœnigs net.

Proof. Recall that for a curvature line parametrization x with

$$x_{uv} = \lambda x_u + \mu x_v$$

the Möbius $\hat{x}$ is a conjugate line parametrization with the same coefficients

$$\hat{x}_{uv} = \lambda \hat{x}_u + \mu \hat{x}_v.$$

Thus, $[\hat{x}]$ has the same Laplace invariants as x, from which the claim follows. □

Darboux transformations

The Ribaucour transformations for curvature line parametrizations can be further specialized for isothermic nets.

Definition 15.2. Let $x, x^+ : \mathbb{R}^2 \supset U \to \mathbb{R}^n$ be two isothermic nets. Then x^+ is called a *Darboux transform* of x if x^+ is a Ribaucour transform of x and both are conformal up to the same reparametrization, i.e.,

$$\frac{\|x_u^+\|^2}{\|x_v^+\|^2} = \frac{\|x_u\|^2}{\|x_v\|^2} = \frac{\alpha(u)}{\beta(v)}$$

at every point and with the same functions α, β.

Again Darboux transformations satisfy corresponding permutability theorems.

16 Discrete isothermic nets

We have seen that in the smooth case, isothermic nets are orthogonal Kœnigs nets, or equivalently, stereographic projections of Kœnigs nets in the Möbius quadric. This leads to the following definition.

Definition 16.1. A *discrete isothermic net* is a circular Kœnigs net.

We obtain the following Möbius geometric description, which immediately implies the Möbius invariance of discrete isothermic nets.

Proposition 16.1. *Let $x : \mathbb{Z}^2 \to \mathbb{R}^n$ be a discrete net and $[\hat{x}] : \mathbb{Z}^2 \to \mathcal{M} \subset \mathbb{R}\mathrm{P}^{n+1}$ its lift to the Möbius quadric*

$$\hat{x} := x + \|x\|^2 e_\infty + e_0.$$

Then x is a discrete isothermic net if and only if $[\hat{x}]$ is a discrete Kœnigs net.

Proof. Exercise. □

Remark 16.1. While we have also studied other discretizations of orthogonal nets and also mentioned other discretizations of Kœnigs nets, we will restrict our studies of discrete isothermic nets here to the circular Kœnigs type.

We derive a characterization of discrete isothermic nets in terms of cross-ratios, for which the proof is based on the following generalized version of Menelaus theorem.

Theorem 16.2 (Generalized Menelaus' theorem)**.** *Let $A_1, \ldots, A_{n+1}$ be $n+1$ points in general position in $\mathbb{R}\mathrm{P}^n$. Let $P_{i,i+1}$ be some points on the lines A_iA_{i+1} different from A_i, A_{i+1} (indices are taken modulo $n+1$). The $n+1$ points $P_{i,i+1}$ lie in an $(n-1)$-dimensional projective subspace if and only if the following relation for the quotients of the directed lengths holds:*

$$\prod_{i=1}^{n+1} \frac{\ell(A_i, P_{i,i+1})}{\ell(P_{i,i+1}, A_{i+1})} = (-1)^{n+1}.$$

Proof. Geometry 1. □

For a planar quadrilateral (x, x_1, x_{12}, x_2), we can identify its plane with the complex plane $\mathbb{C}$ and define its cross-ratio

$$q = \mathrm{cr}(x, x_1, x_{12}, x_2) = \frac{(x - x_1)(x_{12} - x_2)}{(x_1 - x_{12})(x_2 - x)}.$$

This value is invariant under Möbius transformations of the plane, and also under Möbius transformations of the ambient space (though in this case the planar quadrilateral might become non-planar). For a circular net, this cross-ratio is a real number, and under Möbius transformations of the ambient space the quadrilateral stays circular and thus planar. In this case, up to sign the value of q coincides with

$$|q| = \frac{\|x - x_1\| \, \|x_{12} - x_2\|}{\|x_1 - x_{12}\| \, \|x_2 - x\|},$$

which discretizes $\frac{\|x_u\|^2}{\|x_v\|^2}$. Furthermore, a circular quadrilateral is embedded if $q < 0$ and non-embedded if $q > 0$. Thus, the following theorem on discrete isothermic nets can be viewed as a discrete version of the characterization from the definition of isothermic nets in the smooth case (Definition 15.1).

Theorem 16.3. *Let $x : \mathbb{Z}^2 \to \mathbb{R}^n$ be a circular net. Then x is a discrete isothermic net if and only if the cross-ratios of its elementary quadrilaterals satisfy*

$$\operatorname{cr}(x, x_1, x_{12}, x_2) = -\frac{\alpha(n_1)}{\beta(n_2)}$$

where α, β constitute a real-valued labelling of the edges of $\mathbb{Z}^2$, i.e., α is a real-valued function that only depends on n_1 and β

Proof. Such a labelling of the edge exists if and only if the function $q := \operatorname{cr}(x, x_1, x_{12}, x_2)$ satisfies

$$qq_{-1-2} = q_{-1}q_{-2}. \tag{28}$$

Consider the four quadrilaterals incident with x. Apply a Möbius transformation that maps x to ∞. Then the cross-ratios become quotients of directed lengths

$$\operatorname{cr}(x, x_1, x_{12}, x_2) = -\frac{\ell(x_{12}, x_2)}{\ell(x_1, x_{12})}.$$

Thus, (28) is equivalent to

$$\frac{\ell(x_2, x_{12})}{\ell(x_{12}, x_1)} \frac{\ell(x_1, x_{1,-2})}{\ell(x_{1,-1}, x_{-2})} \frac{\ell(x_{-2}, x_{-1,-2})}{\ell(x_{-1,-2}, x_{-1})} \frac{\ell(x_{-1}, x_{-1,2})}{\ell(x_{-1,2}, x_2)} = 1, \tag{29}$$

which is equivalent to the multi-ratio condition in Theorem 16.2 for the case $n = 3$.

Assume the four points $x_{\pm 1}, x_{\pm 2}$ span a 3-dimensional subspace (which is generically the case). Then, (29) is equivalent to the four points $x_{\pm 1, \pm 2}$ lying in a plane. Thus, before the Möbius transformation the five points $x, x_{\pm 1, \pm 2}$ lie on a sphere, and in the Möbius lift they lie in a 3-dimensional subspace. Therfore, in the Möbius lift the diagonal intersection points of the four quadrilaterals lie in the intersection of two 3-dimensional subspaces in a 4-dimensional subspace, which is a plane. This is equivalent to the Möbius lift being a Kœnigs net. □

Conformal metric coefficient for discrete isothermic nets

Recall that a discrete conjugate net $[\hat{x}] : \mathbb{Z}^2 \to \mathbb{RP}^n$ with

$$A\hat{x} + B\hat{x}_1 + C\hat{x}_{12} + D\hat{x}_2 = 0.$$

is a Kœnigs net if the two discrete multiplicative 1-forms given by

$$\vec{E}(\mathbb{Z}^2_{\text{even}}) \to \mathbb{R}, \quad e \mapsto \frac{A}{C} = -\frac{\vec{\ell}(M, x_{12})}{\vec{\ell}(M, x)}$$

$$\vec{E}(\mathbb{Z}^2_{\text{odd}}) \to \mathbb{R}, \quad e \mapsto \frac{B}{D} = -\frac{\vec{\ell}(M, x_2)}{\vec{\ell}(M, x_1)}$$

are closed, where M is the intersection of the diagonals of the quad. Equivalently, there exists a function $\nu : \mathbb{Z}^2 \to \mathbb{R}\backslash\{0\}$ such that

$$\frac{\vec{\ell}(M, x_{12})}{\vec{\ell}(M, x)} = -\frac{\nu}{\nu_{12}}, \qquad \frac{\vec{\ell}(M, x_2)}{\vec{\ell}(M, x_1)} = -\frac{\nu_1}{\nu_2}.$$

For a discrete isothermic net, and in comparison with the smooth theory, we might consider the function $s := \frac{1}{\nu}$ a discretization of the conformal metric coefficient. It has further analogous properties:

Proposition 16.4. *For a discrete isothermic net $x : \mathbb{Z}^2 \to \mathbb{R}^n$ the relations*

$$\frac{\vec{\ell}(M, x_{12})}{\vec{\ell}(M, x)} = -\frac{s_{12}}{s}, \qquad \frac{\vec{\ell}(M, x_2)}{\vec{\ell}(M, x_1)} = -\frac{s_2}{s_1}.$$

define a function $s : \mathbb{Z}^2 \to \mathbb{R}\backslash\{0\}$ uniquely, up to a black-white scaling ($s \mapsto \lambda s$ at black vertices, $s \mapsto \mu s$ at white vertices).

There exists a labelling α, β of horizontal and vertical edges such that

$$\|x_1 - x\|^2 = \alpha s s_1, \qquad \|x_2 - x\|^2 = \beta s s_2.$$

A back-white scaling of the function s results in the rescaling $\alpha \mapsto \frac{\alpha}{\lambda\mu}$, $\beta \mapsto \frac{\beta}{\lambda\mu}$.

Furthermore, with this edge labelling, the cross-ratios satisfy

$$\mathrm{cr}(x, x_1, x_{12}, x_2) = -\frac{\alpha}{\beta}.$$

Proof. The existence of the function s follows from our previous considerations on Kœnigs nets. Define α and β by

$$\alpha = \frac{\|x_1 - x\|^2}{s s_1}, \qquad \beta = \frac{\|x_2 - x\|^2}{s s_2}.$$

To see that they define an edge labelling (do only depend on one direction each), we show

$$\frac{\|x_1 - x\|^2}{s s_1} = \frac{\|x_{12} - x_2\|^2}{s_2 s_{12}}, \qquad \frac{\|x_2 - x\|^2}{s s_2} = \frac{\|x_{12} - x_1\|^2}{s_1 s_{12}}.$$

For a circular quadrilateral (x, x_1, x_{12}, x_2), one has two pairs of similar triangles:

$$\Delta(x, x_1, M) \sim \Delta(x_2, x_{12}, M), \qquad \Delta(x, x_2, M) \sim \Delta(x_1, x_{12}, M).$$

Hence,

$$\frac{\|M - x_{12}\|}{\|M - x_1\|} = \frac{\|M - x_2\|}{\|M - x\|} = \frac{\|x_{12} - x_2\|}{\|x_1 - x\|}, \qquad \frac{\|M - x_{12}\|}{\|M - x_2\|} = \frac{\|M - x_1\|}{\|M - x\|} = \frac{\|x_{12} - x_1\|}{\|x_2 - x\|},$$

and thus,

$$\frac{\|M - x_{12}\|}{\|M - x\|} \cdot \frac{\|M - x_2\|}{\|M - x_1\|} = \frac{\|x_{12} - x_2\|^2}{\|x_1 - x\|^2}, \qquad \frac{\|M - x_{12}\|}{\|M - x\|} \cdot \frac{\|M - x_1\|}{\|M - x_2\|} = \frac{\|x_{12} - x_1\|^2}{\|x_2 - x\|^2}.$$

We can replace the quotients of lengths by the quotients of directed length to obtain

$$\frac{\vec{\ell}(M, x_{12})}{\vec{\ell}(M, x)} \cdot \frac{\vec{\ell}(M, x_2)}{\vec{\ell}(M, x_1)} = \frac{\|x_{12} - x_2\|^2}{\|x_1 - x\|^2}, \qquad \frac{\vec{\ell}(M, x_{12})}{\vec{\ell}(M, x)} \cdot \frac{\vec{\ell}(M, x_1)}{\vec{\ell}(M, x_2)} = \frac{\|x_{12} - x_1\|^2}{\|x_2 - x\|^2}.$$

Indeed, the fractions on the left-hand side of both equations are either both negative (for an embedded quad) or both positive (for a non-embedded quad). Substituting the defining relations for s, we obtain

$$\frac{s_{12}}{s} \cdot \frac{s_2}{s_1} = \frac{\|x_{12} - x_2\|^2}{\|x_1 - x\|^2}, \qquad \frac{s_{12}}{s} \cdot \frac{s_1}{s_2} = \frac{\|x_{12} - x_1\|^2}{\|x_2 - x\|^2},$$

which implies that α and β are edge labellings.

Now the cross-ratio of e circular quad is given by ($\varepsilon = -1$ for an embedded quad, $\varepsilon = 1$ for a non-embedded quad)

$$\begin{aligned}
\mathrm{cr}(x, x_1, x_{12}, x_2) &= \varepsilon \frac{\|x_1 - x\| \, \|x_{12} - x_2\|}{\|x_2 - x\| \, \|x_{12} - x_1\|} \\
&= \varepsilon \frac{\|x_1 - x\|^2}{\|x_2 - x\|^2} \cdot \frac{\|x_{12} - x_2\|}{\|x_1 - x\|} \cdot \frac{\|x_2 - x\|}{\|x_{12} - x_1\|} \\
&= \varepsilon \frac{\alpha s s_1}{\beta s s_2} \cdot \frac{\|M - x_2\|}{\|M - x\|} \cdot \frac{\|M - x\|}{\|M - x_1\|} \\
&= \frac{\alpha s_1}{\beta s_2} \cdot \frac{\vec{\ell}(M, x_2)}{\vec{\ell}(M, x_1)} = \frac{\alpha s_1}{\beta s_2} \cdot \left(-\frac{s_2}{s_1} \right) = -\frac{\alpha}{\beta}.
\end{aligned}$$

□

Similar to the smooth case, the Kœnigs dual (Christoffel dual) for a discrete isothermic net can now be expressed by

$$\Delta_1 x^* = \alpha \frac{\Delta_1 x}{\|\Delta_1 x\|^2} = \frac{1}{s s_1} \Delta_1 x, \qquad \Delta_2 x^* = -\beta \frac{\Delta_2 x}{\|\Delta_2 x\|^2} = -\frac{1}{s s_2} \Delta_2 x.$$

Consistency of discrete isothermic nets

We have already studies higher-dimensional analogues of circular nets and Kœnigs nets. Thus, we only need to bring both definitions together to obtain a higher-dimensional analogue for *discrete isothermic nets* on $\mathbb{Z}^N$.

They can analogously be characterized by the cross-ratios on the faces of $\mathbb{Z}^N$ such that

$$\mathrm{cr}(x, x_i, x_{ij}, x_j) \cdot \mathrm{cr}(x, x_j, x_{jk}, x_k) \cdot \mathrm{cr}(x, x_k, x_{ki}, x_i) = 1,$$

or equivalently, by the existence of an edge labelling $\alpha_1, \ldots, \alpha_N$ such that

$$\mathrm{cr}(x, x_i, x_{ij}, x_j) = \frac{\alpha_i}{\alpha_j}. \tag{30}$$

Note how the factorization does not involve a minus sign. While this the sign does not matter in the 2-dimensional case, for $N \geqslant 3$ the consistency can only be achieved with a plus sign. In particular this means, that in a 3-cube of a discrete isothermic net not all of the (circular) faces can be embedded.

Theorem 16.5. *Discrete isothermic nets are a 3D-consistent (and thus multi-dimensionally consistent) 2D-system.*

Proof. Discrete isothermic nets constitute a 2D-system, since they can be constructed from 1-dimensional initial data such as an edge labelling $\alpha_1, \ldots, \alpha_N$ together with points on the coordinate axes of $\mathbb{Z}^N$. From this all points on the 2-dimensional coordinate planes can be generated via the cross-ratio equation (30).

The cube flip is then determined by the 3D-system of conjugate nets of which circular nets as well as Kœnigs nets are a consistent reduction. □

17 Normal congruences, offsets, and curvature

Line congruences and torsal directions

A *(smooth) line congruence* $\mathcal{L}$ is a smooth 2-parameter family of lines described locally by lines $\ell(u,v)$ which connect corresponding points $x(u,v)$ and $y(u,v)$ of two parametrized surfaces. $n(u,v) \coloneqq y(u,v) - x(u,v)$ indicates the direction of the line. We will only consider line congruences in $\mathbb{R}^3$.

A *volume parametrization* of $\mathcal{L}$ is given by

$$p(u,v,\lambda) \coloneqq x(u,v) + \lambda n(u,v) = (1-\lambda)x(u,v) + \lambda y(u,v)$$

A curve $u(t)$, $v(t)$ in the domain describes a ruled surface $\mathcal{R} \subset \mathcal{L}$ by

$$(t,\lambda) \mapsto p(u(t),v(t),\lambda).$$

$\mathcal{R}$ is *developable* if and only if x_t, y_t, n are coplanar, i.e., $\det(x_t,y_t,n) = 0$, or equivalently, $\det(n_t,x_t,n) = 0$. The corresponding equation for u_t and v_t is

$$\det(n_u,x_u,n)u_t^2 + (\det(n_u,x_v,n) + \det(n_v,x_u,n))u_t v_t + \det(n_v,x_v,n)v_t^2 = 0.$$

This quadratic equation has up to two solutions $\frac{u_t}{v_t}$ which are called the *torsal directions* of $\mathcal{L}$.

Normal congruences and principal directions

A line congruence formed by the lines orthogonal to a surface (in $\mathbb{R}^3$) is called a *normal congruence.*

First, we investigate whether this yields an additional restriction on a line congruence. A general line congruence

$$p(u,v,\lambda) = x(u,v) + \lambda n(u,v)$$

might be a normal congruence of a yet unknown surface $x^*(u,v)$ with normal vectors $n(u,v)$. In order to find it we have to solve

$$x^*(u,v) = x(u,v) + \lambda(u,v)n(u,v)$$

for $\lambda(u,v)$. We restrict $n(u,v)$ to $\|n\| = 1$. Then the orthogonality conditions $\langle n, x_u^* \rangle = \langle n, x_v^* \rangle = 0$ are equivalent to

$$\lambda_u = -\langle x_u, n \rangle, \quad \lambda_v = -\langle x_v, n \rangle$$

This equation has a solution if and only if $\lambda_{uv} = \lambda_{vu}$, i.e.

$$\langle x_u, n_v \rangle = \langle x_v, n_u \rangle.$$

Now, consider a normal congruence with $n(u,v)$ having unit length. We have seen before that curvature line parametrizations may be characterized in the following way:

$$\begin{aligned} &(u,v) \mapsto x(u,v) \text{ curvature line parametrization} \\ &\qquad \Leftrightarrow (u,\lambda) \mapsto x(u, v = \text{const}, \lambda) \text{ and } (v,\lambda) \mapsto p(u = \text{const}, v, \lambda) \\ &\qquad\quad \text{are developable surfaces} \\ &\qquad \Leftrightarrow n_u \parallel x_u,\ n_v \parallel x_v \end{aligned}$$

Thus a normal line congruence always has torsal directions. These directions are orthogonal and are called *principal directions.*

Offset surfaces

Definition 17.1. Let $x : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a parametrized surfaces with unit normal field $n : U \to \mathbb{S}^2$. Then a surface $(u, v) \mapsto x^\rho(u, v) = x(u, v) + \rho n(u, v)$ of constant distance $\rho \in \mathbb{R}$ in normal direction is called an *offset surface* of x.

Any offset surface x^ρ defines the same normal congruence $p^\rho(u, v, \lambda) = p(u, v, \lambda + \rho)$. Note that the tangent vectors of x^ρ

$$(x^\rho)_u = x_u + \rho n_u, \qquad (x^\rho)_v = x_v + \rho n_v$$

do not change direction, i.e, $(x^\rho)_u \parallel x_u$, $(x^\rho)_v \parallel x_v$ if and only if x is a curvature line parametrization.

Definition 17.2. Let $x, x^+ : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be two parametrized surfaces. Then x^+ is called *parallel* to x if corresponding tangent vectors are parallel, i.e,

$$x^+_u \parallel x_u, \qquad x^+_v \parallel x_v.$$

If x and x^+ are conjugate line parametrizations, then it is sufficient for the tangent planes to be parallel.

Proposition 17.1. *Let $x, x^+ : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be two conjugate line parametrizations. Then x and x^+ are parallel if and only if corresponding tangent planes are parallel. In this case x^+ is called a* ***Combescure transform*** *of x.*

Proof. Exercise. □

From our previous considerations, we can now see that for a curvature line parametrization, we have parallel offsets as well as a parallel parametrization on the sphere. The parallel parametrization on the sphere is its normal field, also called the *Gauß map*.

Proposition 17.2. *Let $x : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a parametrized surface. Then (away from umbilical points) the following three statements are equivalent:*

(i) x is a curvature line parametrization.

(ii) x has a parallel offset (and thus all offsets are parallel).

(iii) x has a parallel net on the sphere, i.e, its Gauß map is parallel.

Proof. Exercise. □

Note that the parallel directions in the offset of a curvature line parametrization immediately implies that the extension $(u, v, \rho) \mapsto x^\rho(u, v)$ is an orthogonal coordinate system.

Corollary 17.3. *Every curvature line parametrized surfaces can be extended to an orthogonal coordinate system.*

Remark 17.1. This implies that curvature lines are mapped to curvature lines by a conformal transformation of $\mathbb{R}^3$, and in particular, umbilic points are mapped to umbilic points. Indeed, a curvature line parametrization can be extended to an orthogonal coordinates system, which is mapped to an orthogonal coordinate system by a conformal transformation. By Dupin's theorem the image is again a curvature line parametrization.

Together with the fact that every totally umbilical surface is (a piece of) a sphere or a plane, this implies that conformal transformations of $\mathbb{R}^3$ map spheres to spheres and thus by the fundamental theorem of Möbius geometry are Möbius transformations:

Every conformal map $f : \mathbb{R}^3 \supset U \to \mathbb{R}^3$ is (the restriction of) a Möbius transformation.

This is *Liouville's theorem.*

Surface area under offset

Let $x : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a curvature line parametrization and $n : U \to \mathbb{S}^2$ its Gauß map. Let $\kappa_1, \kappa_2 : U \to \mathbb{R}$ be the principal curvatures, i.e.,

$$n_u = -\kappa_1 x_u, \qquad n_v = -\kappa_2 x_v$$

The area element of x is given by

$$\mathrm{d}A = x_u \times x_v \mathrm{d}u\mathrm{d}v.$$

We investigate the change of the area element under the offset x^ρ:

$$\begin{aligned}(x^\rho)_u \times (x^\rho)_v &= (x_u + \rho n_u) \times (x_v + \rho n_v) \\ &= x_u \times x_v + (x_u \times n_v + x_v \times n_u)\rho + n_u \times n_v \rho^2 \\ &= \left(1 - (\kappa_1 + \kappa_2)\rho + \kappa_1\kappa_2\rho^2\right) x_u \times x_v.\end{aligned}$$

Thus, the area changes by a quadratic polynomials for which the coefficients are determined by the principal curvatures.

The *Gaussian curvature* and *mean curvature* are defined as

$$K \coloneqq \kappa_1\kappa_2, \quad H \coloneqq \frac{1}{2}\left(\kappa_1 + \kappa_2\right).$$

With this we obtain:

$$\mathrm{d}A^\rho = \left(1 - 2H\rho + K\rho^2\right) \mathrm{d}A,$$

which is the differential version of *Steiner's formula.*

Remark 17.2.

(i) The resulting formula does not depend on the paramatrization (such as the curvature line parametrization, we used to derive it).

(ii) The area element of the Gauß map is given by $\mathrm{d}A(n) = n_u \times n_v \mathrm{d}u\mathrm{d}v$, and thus the Gauß curvature is the quotient of the area of Gauß map and the surface:

$$K = \frac{\mathrm{d}A(n)}{\mathrm{d}A(x)} = \lim_{\varepsilon \to 0} \frac{A\left(n\left(U_\varepsilon\right)\right)}{A\left(x(U_\varepsilon)\right)},$$

for some small ε-neighborhood of the point (u, v). Similarly, the mean curvature may be understood as a quotient of a mixed area of the Gauß map and the surface and the area of the surface:

$$H = \frac{\mathrm{d}A(x, n)}{\mathrm{d}A(x)}.$$

(iii) The area becomes critical if $H = 0$. Such surfaces are called *minimal surfaces.*

Minimal and CMC surfaces

A *minimal surface* is a surface with $H = 0$ at every point. Minimal surfaces are isothermic surfaces, and their Christoffel dual is the Gauß map.

Theorem 17.4. *A minimal surface without umbilic points $x : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ is the Christoffel dual of its Gauß map $n : U \to \mathbb{S}^2$:*

$$x = n^*.$$

Remark 17.3. Thus, minimal surfaces can be constructed from isothermic parametrizations of (pieces of) the sphere, or by stereographic projection, from holomorphic maps in the plane.

Minimal surfaces are a special case of *constant mean curvature surfaces (CMC surfaces)* which are surfaces with $H = \text{const}$. This class of surfaces is still isothermic and satisfies the following remarkable properties with respect to its parallel offsets.

Theorem 17.5. *Let $x : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a CMC surfaces with $H \neq 0$ and without umbilic points. Let $n : U \to \mathbb{S}^2$ be its Gauß map.*

(i) Every parallel offset $x^\rho = x + \rho n$ is linear Weingarten, i.e., its mean and Gaußian curvature functions H^ρ, K^ρ satisfy a linear relation $\alpha H^\rho + \beta K^\rho = 1$ with constant coefficients α, β.

(ii) The parallel offset

$$x^{\frac{1}{H}} = f + \frac{1}{H} n$$

is the Christoffel dual of x and has constant mean curvature H.

(iii) The mid-surface

$$x^{\frac{1}{2H}} = f + \frac{1}{2H} n$$

has constant positive Gaußian curvature $K = 4H^2$.

18 Discrete normal congruences, offsets, and curvature

Discrete torsal line congruences

We directly consider discretizations of line congruences, which are already parametrized in torsal directions. A special case of polyhedral surfaces are Q-nets.

Definition 18.1. A *(torsal) line congruence* is a map

$$\ell : \mathbb{Z}^2 \to \text{Lines}(\mathbb{R}^3)$$

from $\mathbb{Z}^2$ to the space of lines such that each two adjacent lines are coplanar.

Remark 18.1.

(i) Discrete line congruences constitute a 3D-system that is 4D-consistent.

(ii) The intersections of the lines of a discrete line congruence form its *focal nets*, which are discrete conjugate nets.

Parallel offsets

Definition 18.2. Let $x, x^+ : \mathbb{Z}^2 \to \mathbb{R}^3$ be two parametrized surfaces. Then x^+ is called *parallel* to x if corresponding edge lines are parallel, i.e,

$$\Delta_1 x^+ \parallel \Delta_1 x, \qquad \Delta_2 x^+ \parallel \Delta_2 x.$$

If x and x^+ are discrete conjugate nets, then it is sufficient for the tangent planes to be parallel.

Proposition 18.1. *Let $x, x^+ : \mathbb{Z}^2 \to \mathbb{R}^3$ be two conjugate line parametrizations. Then x and x^+ are parallel if and only if corresponding face planes are parallel. In this case x^+ is called a* ***(discrete) Combescure transform*** *of x.*

The goal is to consider a pair (x, ℓ) of a discrete conjugate net $x : \mathbb{Z}^2 \to \mathbb{R}^3$ and a torsal line congruence ℓ such that the lines of ℓ pass through the corresponding vertices of x

$$x(n) \in \ell(n) \qquad \text{for } n \in \mathbb{Z}^2,$$

and interpret this pair as a discrete curvature line parametrization together with its normal congruence. For such a pair, we can always define a family of parallel offsets.

Proposition 18.2. *Let (x, ℓ) be a pair of a discrete conjugate net and a torsal line congruence such that $x(n) \in \ell(n)$ for $n \in \mathbb{Z}^2$. Then there exists a one-parameter family of discrete conjugate nets x^+ parallel to x such that $x^+(n) \in \ell(n)$ for $n \in \mathbb{Z}^2$.*

Proof. For any point point $x^+ \in \ell$, the plane through x^+ parallel to $x \vee x_1 \vee x_{12} \vee x_2$ uniquely determines the three points x_1^+, x_{12}^+, x_2^+ such that corresponding edges are parallel. Choosing one point $x^+(0,0) \in \ell(0,0)$ arbitrarily, this can be used to obtain the parallel nets. ☐

Discrete nets parallel to a given net build a vector space. In particular, given two parallel nets x and x^+ , the formula $x^\rho = \rho x^+ + (1 - \rho)x$ gives an interpolating family of parallel nets. The difference $n = x^+ - x$ is also parallel to both x^+ and x may be interpreted as a discrete Gauß map, while the family

$$x^\rho = x + \rho n, \quad t \in \mathbb{R}$$

may be interpreted as the (parallel) offset nets of x.

However,

- it is unclear how the offset is of "constant distance";
- the discrete "normal congruence" only discretizes a parametrized line congruence in torsal directions, but satisfies no additional constraints such as a normal congruence in the smooth case;
- the discrete "Gauß map" n is conjugate (since parallel to x), but satisfies no further relation with respect to a sphere.

These issues can be fixed for certain classes of discrete conjugate nets, such as *circular nets*:

Proposition 18.3. *Let $x : \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete conjugate net. Then the following three statements are equivalent:*

(i) x is a circular net.

(ii) x has a parallel net of constant vertex distance (which is not a translation).

(iii) x has a parallel net on the sphere.

Proof.

(i) ⇒ (iii) Consider the following construction for one quad of a parallel net on the sphere. For a point $n \in \mathbb{S}^2$ on the sphere, the points $n_1, n_2 \in \mathbb{S}^2$ are determined by parallelity

$$n_1 - n \parallel x_1 - x, \qquad n_2 - n \parallel x_2 - x.$$

The fourth point $n_{12} \in \mathbb{R}^3$ is already uniquely determined by parallelity. Since the quad n, n_1, n_{12}, n_2 is parallel to the circular quad x, x_1, x_{12}, x_2, it is itself circular. Thus, the point n_{12} lies on the circle through the three points $n, n_1, n_2 \in \mathbb{S}^2$ and therefore on the sphere. This construction can be used for the propagation from one arbitrary starting point $n(0,0) \in \mathbb{S}^2$

(i) ⇒ (iii) A discrete conjugate net on the sphere is circular, and thus so is x.

(ii) ⇒ (iii) The difference between x and a net of constant vertex distance x^+ yields a parallel map on the sphere.

(iii) ⇒ (ii) The sum of x and a parallel map n on the sphere yields a parallel net of constant vertex distance.

□

Remark 18.2. Note that there is a one-parameter family of Gauß maps defined in this way. It can also be obtained from one initial choice by reflection in the orthogonal bisectors of the edges.

A similar statement holds for *conical nets*, which are discrete conjugate nets such that for every vertex the four planes are tangent to a common cone of revolution.

Proposition 18.4. *Let $x : \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete conjugate net. Then the following three statements are equivalent:*

(i) x is a conical net.

(ii) x has a parallel net of constant face distance

(iii) x has a parallel net for which all faces are tangent to the unit sphere.

Proof. Exercise. □

Surface area under offset

In the following we investigate the surface area of a discrete net under parallel offset. We start with a theory of planar polygons with parallel edges. Let

$$v_1, \dots, v_k \in \mathbb{RP}^1 = \mathbb{S}^1/\{\pm I\}$$

be a sequence of tangent directions of a k-gon $P = (p_1, \dots, p_k)$ in a plane; $p_{i+1} - p_i \parallel v_i$. Denote by $\mathcal{P}(v)$, $v = (v_1, \dots, v_k)$, the space of k-gons with edges parallel to $v_1, \dots, v_k$.

The polygons are not necessarily convex nor embedded, and may have degenerate edges. $\mathcal{P}(v)$ is a k-dimensional vector space. Factoring out translations (for example, normalizing $p_1 = 0$), we obtain a $(k-2)$-dimensional vector space $\tilde{\mathcal{P}}(v)$.

Let $A(P)$ be the oriented area of the polygon P. The oriented area of a k-gon with vertices $p_1, \dots, p_k$, $p_{k+1} = p_1$ is equal to

$$A(P) = \frac{1}{2}\sum_{i=1}^{k}[p_i, p_{i+1}],$$

where we write $[\cdot,\cdot] = \det(\cdot,\cdot)$ The oriented area A is a quadratic form on the vector space $\tilde{\mathcal{P}}(v)$. Its corresponding symmetric bilinear form $A(\cdot,\cdot)$ is of central importance for the following theory.

Definition 18.3 (Mixed area)**.** Let P and Q be two k-gons with parallel corresponding (possibly degenerate) edges. Their mixed area is given by the bilinear symmetric form

$$A(P,Q) = \frac{1}{2}(A(P+Q) - A(P) - A(Q)).$$

The area of a linear combination of two polygons is given by the quadratic polynomial

$$A(P + tQ) = A(P) + 2tA(P,Q) + t^2 A(Q). \tag{31}$$

Now let us restrict our considerations to quadrilaterals. For a quadrilateral $P = (p_1, p_2, p_3, p_4)$ with oriented edges $a = p_2 - p_1$, $b = p_3 - p_2$, $c = p_4 - p_3$, $d = p_1 - p_4$, we have

$$A(P) = \frac{1}{2}([a,b] + [c,d]).$$

We call the space $\tilde{\mathcal{P}}(v)$, $v = (v_1, v_2, v_3, v_4)$ (and all the quadrilaterals in this space) non-degenerate if every two of its consecutive tangent directions are different, $v_{i+1} \neq v_i$. The signature of the area form $A : \tilde{\mathcal{P}}(v) \to \mathbb{R}$ depends on the quadruple $v = (v_1, v_2, v_3, v_4) \in (\mathbb{RP}^1)^4$ and can be also characterized in terms of the quadrilaterals in $\tilde{\mathcal{P}}(v)$. It turns out that

- A is indefinite if and only if all vertices of P are extremal points of their convex hull ("convex embedded or non-embedded quadrilateral").
- A is definite if and only if one of the vertices of P lies in the interior of the convex hull of the other three vertices ("non-convex quadrilateral").

Now consider a discrete conjugate net $x : \mathbb{Z}^2 \to \mathbb{R}^3$ with a line congruence ℓ such that $x(n) \in \ell(n)$ for all $n \in \mathbb{Z}^2$. Let $n : V(D) \to \mathbb{R}^3$ be the corresponding generalized Gauss map, which is determined up to a constant factor and is fixed as soon as the length of the normal at one vertex is prescribed (difference of x and one parallel offset in the line congruence).

Theorem 18.5. *The area of the parallel surface $x^\rho := x + \rho n$ satisfies*

$$A(x^\rho(f)) = (1 - 2\rho H_f + \rho^2 K_f) A(x(f)),$$

for every face $f \in F(\mathbb{Z}^2)$, where

$$H_f = -\frac{A(x(f), n(f))}{A(x(f))}, \quad K_f = \frac{A(n(f))}{A(x(f))}. \tag{32}$$

and $x(f)$ and $n(f)$ are the corresponding faces of the surface x and its generalized Gauss map n.

Proof. Since the corresponding faces and edges of discrete surfaces x and n are parallel, the claim follows from formula (31). □

Having in mind Steiner's formula, we come to the following natural definition of the curvatures in the discrete case.

Definition 18.4. Let (x, n) be two parallel discrete conjugate nets, where we consider n as the generalized Gauss map of x. The functions H_f and K_f on the faces given by 32 are the *mean* and the *Gaussian curvature* of the pair (x, n).

Since for a given polyhedral surface with a line congruence the map n is defined up to a common factor, the curvatures at the faces are also defined up to multiplication by a constant.

The *principal curvatures* κ_1, κ_2 at the faces are naturally defined using the formulas $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ and $K = \kappa_1\kappa_2$ as the zeros of the quadratic polynomial

$$A(x^\rho) = (1 - 2\rho H + \rho^2 K)A(x) = (1 - \rho\kappa_1)(1 - \rho\kappa_2)A(x).$$

If the area form A is indefinite then κ_1, κ_2 are real-valued. In particular, for a circular net, real principal curvatures exist for any Gauss map.

Discrete minimal and CMC surfaces

We investigate the minimal surface condition $H = 0$ in the discrete case of a discrete conjugate net x with parallel Gauss-map n.

Proposition 18.6. *Two quadrilaterals $P = (p_1, p_2, p_3, p_4)$ and $Q = (q_1, q_2, q_3, q_4)$ with parallel corresponding edges have vanishing mixed area $A(P, Q) = 0$ if and only if their non-corresponding diagonals are parallel, i.e. they are Kœnigs dual*

$$(p_1p_3) \parallel (q_2q_4) \quad \text{and} \quad (p_2p_4) \parallel (q_1q_3).$$

Proof. Denote the edges of the quadrilaterals P and Q as in fig:dual-quadrilaterals. Formula 31 implies that the area of the quadrilateral $P + tQ$ is given by

$$A(P + tQ) = \frac{1}{2}([a + ta^*, b + tb^*] + [c + tc^*, d + td^*]).$$

Identifying the linear terms in t and using the identity $a + b + c + d = 0$, we get

$$\begin{aligned} 4A(P, Q) &= [a, b^*] + [a^*, b] + [c, d^*] + [c^*, d] \\ &= [a + b, b^*] + [a^*, a + b] + [c + d, d^*] + [c^*, c + d] \\ &= [a + b, b^* - a^* - d^* + c^*]. \end{aligned}$$

Vanishing of the last expression is equivalent to the parallelism of the non-corresponding diagonals, $(a + b) \parallel (b^* + c^*)$. □

Remark 18.3. If two non-corresponding diagonals of two quadrilaterals with parallel edges are parallel then the other non-corresponding diagonals are also parallel.

We have already obtained the following statements in the context of Kœnigs nets, however the characterization of dual quadrilaterals in terms of the mixed area leads to a simpler proof.

Proposition 18.7. *For any planar quadrilateral P, a dual quadrilateral P^* exists and is unique up to scaling and translation.*

Proof. Let $\tilde{\mathcal{P}}$ be the non-degenerate vector space of quadrilaterals with edges parallel, factorized by translations such that $P \in \tilde{\mathcal{P}}$. Then $\dim \tilde{\mathcal{P}} = 2$ and $A(\cdot, \cdot)$ defines a non-degenerate bilinear form on $\tilde{\mathcal{P}}$. So $\dim\{P\}^{\perp} = 1$, i.e. there is $P^* \in \tilde{\mathcal{P}}$ with $A(P, P^*) = 0$, unique up to scaling. $\square$

For a pair (x, n) of a discrete conjugate net $x : \mathbb{Z}^2 \to \mathbb{R}^3$ and parallel Gauß map $n : \mathbb{Z}^2 \to \mathbb{R}^3$ we had defined its mean and Gaussian curvature by

$$K = \frac{A(n)}{A(x)} \qquad H = -\frac{A(x,n)}{A(x,x)}$$

as functions on the faces.
We use this to define minimal and constant mean curvature surfaces by

$$\begin{aligned} (x,n) \text{ minimal } &:\Leftrightarrow \ H = 0 \\ (x,n) \text{ CMC } &:\Leftrightarrow \ H = const. \end{aligned}$$

Theorem 18.8.

$$\begin{aligned} (x,n) \ \textit{minimal} \ &\Leftrightarrow \ x \ \textit{Kœnigs and } n = x^* \\ (x,n) \ \textit{CMC with } H = const \neq 0 \ &\Leftrightarrow \ x \ \textit{Koenigs and } n = (x^* - x)H \end{aligned}$$

Proof. The statement about minimal surfaces follows immediately from our definitions. Let $H = H_0 \neq 0$. Then

$$\begin{aligned} H = H_0 &\Leftrightarrow A(x,n) + H_0 A(x,x) = 0 \\ &\Leftrightarrow A(x, n + H_0 x) = 0 \\ &\Leftrightarrow A(x, x + \frac{1}{H_0} n) = 0 \\ &\Leftrightarrow x^* = x + \frac{1}{H_0} n \\ &\Leftrightarrow n = H_0(x^* - x). \end{aligned}$$

$\square$

Corollary 18.9. *Let (x, n) be a discrete conjugate net with parallel Gauß map and constant mean curvature $H = const \neq 0$. Then $x^* = x + \frac{1}{H}n$ and the parallel surface $x + \frac{1}{2H}n$ has constant Gaußian curvature $K = 4H^2$.*

Proof. Let $H = H_0 \neq 0$. It only remains to show that the mid-surface $x + \frac{1}{2H_0}n$ satisfies $K = 4H_0^2$.

$$\begin{aligned} A\left(x + \frac{1}{2H_0}n\right) &= A\left(x + \frac{1}{2H_0}n, x\right) + A\left(x + \frac{1}{2H_0}n, \frac{1}{2H_0}n\right) \\ &= A\left(x + \frac{1}{2H_0}n, x\right) + A\left(x, \frac{1}{2H_0}n\right) + \frac{1}{4H_0^2}A(n) \\ &= A\left(x + \frac{1}{H_0}n, x\right) + \frac{1}{4H_0^2}A(n) = \frac{1}{4H_0^2}A(n) \end{aligned}$$

So

$$K = \frac{A(n)}{A\left(x + \frac{1}{2H_0} n\right)} = 4H_0^2.$$

□

Note that any discrete Koenigs net x can be extended to a discrete minimal or discrete CMC surface by an appropriate choice of the Gauss map n. Indeed,

(x, n) is minimal for $n = x^*$;
(x, n) has constant mean curvature for $n = x^* - x$.

Thus, n defined in such generality can lead us too far away from the smooth theory. It is natural to look for additional requirements which bring it closer to the Gauss map of a surface. In the smooth case, n is a map to the unit sphere. The following three discrete versions of this fact are natural to consider:

(1) n is a polyhedral surface with all vertices on the unit sphere $\mathbb{S}^2$. This implies that all faces of n are circular. This condition holds also for any parallel surface. In particular, x is also a circular net.

(2) n is a polyhedral surface with all faces touching the unit sphere $\mathbb{S}^2$. This implies that for any vertex p there is a cone of revolution with the tip p touching all faces of n incident to p. This property holds true for any parallel surface. In particular, x is also a conical net.

(3) n is a polyhedral surface with all edges touching the unit sphere $\mathbb{S}^2$. Polyhedra with this property are called *Koebe polyhedra*. For any vertex p, all edges incident to p lie on a cone of revolution with the tip p. Also this property holds true for any parallel surface, in particular for x. Such nets are called nets of *Koebe type*.

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