Chapter 1

Curves and surfaces in projective geometry

1.1 Curves in projective geometry

First, we introduce the notions of regularity, tangent lines, and osculating plane for curves in the Euclidean space $\mathbb{R}^n$. Then we consider their lift and corresponding definitions in the projective space $\mathbb{RP}^n$, and check in which sense these notions are projectively well-defined and invariant.

We start by recalling the definition of a curve in $\mathbb{R}^n$.

Definition 1.1.1. Let $I \subset \mathbb{R}$ be an interval. Then a smooth map

$$\gamma : I \to \mathbb{R}^n$$

is called a *(smooth parametrized) curve* in $\mathbb{R}^n$.

We usually denote the curve parameter by t and the derivatives with respect to t by

$$\gamma(t), \dot{\gamma}(t), \ddot{\gamma}(t), \ldots$$

We will mostly deal with regular curves, for which the first derivative does not vanish.

Definition 1.1.2. A curve $\gamma : I \to \mathbb{R}^n$ is called *regular* if

$$\dot{\gamma}(t) \neq 0 \qquad \text{for all } t \in I.$$

For a regular curve the tangent line is well-defined at every point of the curve.

Definition 1.1.3. Let $\gamma : I \to \mathbb{R}^n$ be a regular curve. Then the line

$$T(t) := \{\gamma(t) + \alpha\dot{\gamma}(t) \mid \alpha \in \mathbb{R}\}$$

is called the *tangent line* of γ at $t \in I$.

The tangent line is the line that best approximates the curve at some point up to first order. Similarly, the osculating plane is the plane that best approximates the curve at some point up to second order.

Definition 1.1.4. Let $\gamma : I \to \mathbb{R}^n$ be a regular curve. If additionally $\dot{\gamma}(t)$ and $\ddot{\gamma}(t)$ are linearly independent, then the plane

$$\{\gamma(t) + \alpha\dot{\gamma}(t) + \beta\ddot{\gamma}(t) \mid \alpha, \beta \in \mathbb{R}\}$$

is called the *osculating plane* of γ at $t \in I$.

We can lift a curve $\gamma : I \to \mathbb{R}^n$ to the projective space $\mathbb{RP}^n$ by

$$[\hat{\gamma}] : I \to \mathbb{RP}^n, \quad \hat{\gamma}(t) := \begin{pmatrix} \gamma(t) \\ 1 \end{pmatrix}$$

If γ is regular, then

$$\dot{\hat{\gamma}}(t) = \begin{pmatrix} \dot{\gamma}(t) \\ 0 \end{pmatrix}$$

describes a point at infinity on the lift of the tangent line

$$T(t) = \left\{[\alpha_1\hat{\gamma}(t) + \alpha_2\dot{\hat{\gamma}}(t)] \,\middle|\, \alpha_1, \alpha_2 \in \mathbb{R}\right\} = [\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)].$$

What happens if we choose different representative vectors for the lift of the curve? Are the point $[\dot{\hat{\gamma}}(t)]$ and the tangent line well-defined by the curve?

Generally, we define a projective curve in the following way.

Definition 1.1.5. Let $I \subset \mathbb{R}$ be an interval and $\hat{\gamma} : I \to \mathbb{R}^{n+1}$ a smooth map. Then

$$[\hat{\gamma}] : I \to \mathbb{RP}^n, \qquad t \mapsto [\hat{\gamma}(t)]$$

is called a *(smooth parametrized) curve* in $\mathbb{RP}^n$.

Consider a curve $[\hat{\gamma}] : I \to \mathbb{RP}^n$ and a smooth function

$$\lambda : I \to \mathbb{R}\backslash\{0\}.$$

Then $\hat{\gamma}$ and $\tilde{\gamma} := \lambda\hat{\gamma}$ define the same curve in $\mathbb{R}\mathrm{P}^n$,

$$[\hat{\gamma}(t)] = [\tilde{\gamma}(t)] \qquad \text{for all } t \in I.$$

But, the first derivative changes in the following way

$$\dot{\tilde{\gamma}}(t) = \dot{\lambda}(t)\hat{\gamma}(t) + \lambda(t)\dot{\hat{\gamma}}(t).$$

Thus, in general, $[\dot{\tilde{\gamma}}(t)] \neq [\dot{\hat{\gamma}}(t)]$. However, the point $[\dot{\tilde{\gamma}}(t)]$ still lies on the span $[\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)]$, and we have

$$[\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)] = [\tilde{\gamma}(t)] \vee [\dot{\tilde{\gamma}}(t)]$$

In particular, $[\hat{\gamma}(t)] = [\dot{\hat{\gamma}}(t)]$ if and only if $[\tilde{\gamma}(t)] = [\dot{\tilde{\gamma}}(t)]$. Thus, the following definition of regularity is independent of the choice of representative vectors. Furthermore, in affine coordinates, it coincides with the corresponding definition for curves in $\mathbb{R}^n$.

Definition 1.1.6. A curve $[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n$ is called *regular* if

$$[\hat{\gamma}(t)] \neq [\dot{\hat{\gamma}}(t)] \qquad \text{for all } t \in I.$$

For a regular curve the span $[\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)]$ is a line, which again is independent of the choice of representative vectors. Indeed, in this case, by choice of the function λ, the point $[\dot{\tilde{\gamma}}(t)]$ can become any point on this line except $[\hat{\gamma}(t)] = [\tilde{\gamma}(t)]$. In affine coordinates, it coincides with the tangent line as defined for curves in $\mathbb{R}^n$.

Definition 1.1.7. Let $[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n$ be a regular curve. Then the line

$$T(t) := [\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)]$$

is called the *tangent line* of $[\hat{\gamma}]$ at $t \in I$.

Similarly, for higher derivatives, in general, $[\ddot{\tilde{\gamma}}(t)] \neq [\ddot{\hat{\gamma}}(t)]$, but

$$[\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)] \vee [\ddot{\hat{\gamma}}(t)] = [\tilde{\gamma}(t)] \vee [\dot{\tilde{\gamma}}(t)] \vee [\ddot{\tilde{\gamma}}(t)],$$

and thus, this plane is independent of the choice of representative vectors. In affine coordinates, it coincides with the osculating plane as defined for curves in $\mathbb{R}^n$.

Definition 1.1.8. Let $[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n$ be a regular curve. If additionally, $[\ddot{\hat{\gamma}}(t)] \notin [\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)]$ Then the plane

$$[\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)] \vee [\ddot{\hat{\gamma}}(t)]$$

is called the *osculating plane* of $[\hat{\gamma}]$ at $t \in I$.

Thus, we have found that regularity, tangent line, and osculating plane are well-defined for a curve in $\mathbb{R}\mathrm{P}^n$ in the sense that their definition is independent of the choice of representative vectors.

Proposition 1.1.9. *For a curve $[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n$, regularity, tangent line, and osculating plane are independent of the choice of representative vectors*

$$\hat{\gamma}(t) \to \lambda(t)\hat{\gamma}(t)$$

with a smooth function non-vanishing function λ.

Next, we investigate how these properties of a curve depend on the parametrization. Thus, let $I, \tilde{I} \subset \mathbb{R}$ be two intervals, $[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n$ a curve, and

$$\varphi : \tilde{I} \to I$$

a smooth bijective map. Then, $\tilde{\gamma} := \hat{\gamma} \circ \varphi$ defines a reparametrization

$$[\tilde{\gamma}] : \tilde{I} \to \mathbb{R}\mathrm{P}^n, \qquad s \mapsto [\hat{\gamma} \circ \varphi(s)].$$

Its derivative

$$\dot{\tilde{\gamma}}(s) = (\dot{\hat{\gamma}} \circ \varphi)(s)\, \varphi'(s)$$

defines the same point

$$[\dot{\tilde{\gamma}}(s)] = [\dot{\hat{\gamma}} \circ \varphi(s)].$$

Thus, regularity, the tangent line

$$[\tilde{\gamma}(s)] \vee [\dot{\tilde{\gamma}}(s)] = [\hat{\gamma}(s)] \vee [\dot{\hat{\gamma}} \circ \varphi(s)],$$

and similarly the osculating plane are invariant under reparametrization.

Proposition 1.1.10. *For a curve $[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n$, regularity, tangent line, and osculating plane are invariant under reparametrization*

$$\hat{\gamma}(t) \to \hat{\gamma} \circ \varphi(s)$$

with a smooth bijective function φ.

Finally, how do these properties change under projective transformations? Let $[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n$ be a curve, and

$$F \in \mathrm{GL}(n+1, \mathbb{R}),$$

i.e.,

$$f := [F] \in \mathrm{PGL}(n+1, \mathbb{R})$$

is a projective transformation. Then $\tilde{\gamma} := F\hat{\gamma}$ defines the transformed curve

$$[\tilde{\gamma}] : I \to \mathbb{R}\mathrm{P}^n, \qquad t \mapsto f([\hat{\gamma}(t)]) = [F\hat{\gamma}(t)]$$

Its derivative

$$\dot{\tilde{\gamma}}(t) = F\dot{\hat{\gamma}}(t)$$

defines a point, which is transformed by the same projective transformation

$$[\dot{\tilde{\gamma}}(t)] = [F\dot{\hat{\gamma}}(t)] = f([\dot{\hat{\gamma}}(t)])$$

Thus, regularity, the tangent line

$$[\tilde{\gamma}(t)] \vee [\dot{\tilde{\gamma}}(t)] = f\left([\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)]\right),$$

and similarly the osculating plane are invariant under reparametrization.

Proposition 1.1.11. *For a curve* $[\hat{\gamma}] : I \to \mathbb{R}\mathrm{P}^n$*, regularity, tangent line, and osculating plane are invariant under projective transformations*

$$\hat{\gamma}(t) \to F\hat{\gamma}(t)$$

with $F \in \mathrm{GL}(n+1, \mathbb{R})$.

1.2 Surfaces in projective geometry

Definition 1.2.1. Let $U \subset \mathbb{R}^n$ be a open set. Then a smooth map

$$f : U \to \mathbb{R}^n, \qquad (u, v) \mapsto f(u, v)$$

is called a *(smooth parametrized) surface (patch)* in $\mathbb{R}^n$.

The curves

$$u \mapsto f(u, v), \qquad v \mapsto f(u, v)$$

are called *parameter lines* of f.

We usually denote the two parameters by u and v. and the partial derivatives with respect to u and v by

$$f_u := \frac{\partial f}{\partial u}, \quad f_v := \frac{\partial f}{\partial v}.$$

Regularity is defined for surface patches by the linear independence of the first partial derivatives.

Definition 1.2.2. A surface $f : U \to \mathbb{R}^n$ is called *regular* if $f_u(u,v)$ and $f_v(u,v)$ are linearly independent at every point $(u,v) \in U$.

For a regular surface the parameter lines are regular curves, and the tangent plane is well-defined at every point. It is the plane that best approximates the surface patch at some point up to first order.

Definition 1.2.3. Let $f : U \to \mathbb{R}^n$ be a regular surface. Then the plane

$$Tf(u,v) := \{f(u,v) + \alpha f_u(u,v) + \beta f_v(u,v) \mid \alpha, \beta \in \mathbb{R}\}$$

is called the *tangent plane* of f at $(u,v) \in U$.

Similar to curves, we can lift a surfaces $f : U \to \mathbb{R}^n$ to the projective space $\mathbb{R}\mathrm{P}^n$ by

$$[\hat{f}] : U \to \mathbb{R}\mathrm{P}^n, \qquad \hat{f}(u,v) := \begin{pmatrix} f(u,v) \\ 1 \end{pmatrix}.$$

If f is regular, the partial derivatives

$$\hat{f}_u(u,v) = \begin{pmatrix} f_u(u,v) \\ 0 \end{pmatrix}, \qquad \hat{f}_v(u,v) = \begin{pmatrix} f_v(u,v) \\ 0 \end{pmatrix},$$

describe points at infinity on the lift of the tangent plane

$$\begin{aligned} Tf(u,v) &= \left\{ \alpha_1 \hat{f}(u,v) + \alpha_2 \hat{f}_u(u,v) + \alpha_3 f_v(u,v) \;\middle|\; \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\} \\ &= [f(u,v)] \vee [f_u(u,v)] \vee [f_v(u,v)]. \end{aligned}$$

Generally, we define projective surfaces in the following way.

Definition 1.2.4. Let $U \subset \mathbb{R}^2$ be a open set and $\hat{f} : U \to \mathbb{R}^{n+1}$ a smooth map Then

$$[\hat{f}] : U \to \mathbb{R}\mathrm{P}^n, \qquad (u,v) \mapsto [\hat{f}(u,v)]$$

is called a *(smooth parametrized) surface (patch)* in $\mathbb{R}\mathrm{P}^n$.

The curves

$$u \mapsto f(u,v), \qquad v \mapsto f(u,v)$$

are called *parameter lines* of $[\hat{f}]$.

Consider a surface $[\hat{f}] : U \to \mathbb{R}\mathrm{P}^n$ and a smooth function

$$\lambda : U \to \mathbb{R}\backslash\{0\}.$$

Then $\hat{f}$ and $\tilde{f} := \lambda \hat{f}$ define the same surface in $\mathbb{R}\mathrm{P}^n$,

$$[\hat{f}(u,v)] = [\tilde{f}(u,v)] \qquad \text{for all } (u,v) \in U.$$

Similar to the considerations for curves, the points described by the first partial derivatives may change, but the span

$$[\hat{f}(u,v)] \vee [\hat{f}_u(u,v)] \vee [\hat{f}_v(u,v)] = [\tilde{f}(u,v)] \vee [\tilde{f}_u(u,v)] \vee [\tilde{f}_v(u,v)]$$

remains the same. Thus, the following definition of regularity for surfaces in $\mathbb{R}\mathrm{P}^n$ is independent of the choice of representative vectors. Furthermore, in affine coordinates, it coincides with the corresponding definition for surfaces in $\mathbb{R}^n$.

Definition 1.2.5. A surface $[\hat{f}] : U \to \mathbb{R}\mathrm{P}^n$ is called *regular* if $[\hat{f}(u,v)]$, $[\hat{f}_u(u,v)]$, $[\hat{f}_v(u,v)]$ span a plane, or equivalently, if $\hat{f}(u,v)$, $\hat{f}_u(u,v)$, $\hat{f}_v(u,v)$ are linearly independent.

The same holds for the following definition of the tangent planes for surfaces in $\mathbb{R}\mathrm{P}^n$.

Definition 1.2.6. Let $[\hat{f}] : U \to \mathbb{R}\mathrm{P}^n$ be a regular surface. Then the plane

$$T[\hat{f}](u,v) := [\hat{f}(u,v)] \vee [\hat{f}_u(u,v)] \vee [\hat{f}_v(u,v)]$$

is called the *tangent plane* of $[\hat{f}]$ at $(u,v) \in U$.

Similar to the considerations for curves, one finds that the introduced notions are also invariant under reparametrization and under projective transformations. We summarize in the following proposition.

Proposition 1.2.7. *For a surface $[\hat{f}] : U \to \mathbb{R}\mathrm{P}^n$, regularity, and the tangent plane are invariant under*

(i) a change of representative vectors

$$\hat{f}(u,v) \to \lambda(u,v)\hat{f}(u,v)$$

with a smooth non-vanishing function λ.

(ii) reparametrization

$$\hat{f}(u,v) \to \hat{f} \circ \varphi(\tilde{u},\tilde{v})$$

with a smooth bijective map φ.

(iii) projective transformations

$$\hat{f}(u,v) \to F\hat{f}(u,v)$$

with $F \in \mathrm{GL}(n+1,\mathbb{R})$.

1.3 Ruled surfaces and developable surfaces

A ruled surface is a surface traced out by the movement of a straight line through space.

Definition 1.3.1. Let $[\hat{a}], [\hat{b}] : I \to \mathbb{R}\mathrm{P}^n$ be two curves such that $[\hat{a}], [\hat{b}], [\dot{\hat{a}}], [\dot{\hat{b}}]$ do not lie on a line. Then the surfaces

$$[\hat{f}] : I \times \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^n, \qquad (u,v) \mapsto [a(u) + vb(u)]$$

is a *ruled surfaces* in $\mathbb{R}\mathrm{P}^n$. The lines $[a(u)] \vee [b(u)]$ on the resulting surface are called *rulings*.

Example 1.3.2. A one-sheeted hyperboloid is a doubly ruled surface.

A developable surface is the envelope of a one-parameter family of planes.

In $\mathbb{R}\mathrm{P}^3$ a one-parameter family of planes may be described by a regular curve in the dual space

$$[\hat{n}] : I \to (\mathbb{R}\mathrm{P}^3)^*,$$

or a one-parameter family of linear equations

$$T(u) = \left\{ [x] \in \mathbb{R}\mathrm{P}^3 \,\middle|\, \hat{n}(u) \cdot x = 0 \right\}.$$

Then the envelope is the solution of the two equations

$$\begin{aligned} \hat{n} \cdot x &= 0, \\ \dot{\hat{n}} \cdot x &= 0. \end{aligned}$$

For each u these are two independent linear equations, and thus the solution is a line. Thus, in $\mathbb{R}\mathrm{P}^3$, every developable surface is a ruled surface.

Vice versa, for a ruled surface in $\mathbb{R}\mathrm{P}^3$ to be developable its tangent planes must be constant along the rulings, i.e.,

$$[\hat{f}] \vee [\hat{f}_u] \vee [\hat{f}_v] = [\hat{a} + v\hat{b}] \vee [\dot{\hat{a}} + v\dot{\hat{b}}] \vee [\hat{b}]$$

must be independent of v, which is the case, if and and only if

$$\det(\hat{a}, \hat{b}, \dot{\hat{a}}, \dot{\hat{b}}) = 0,$$

or equivalently, if $[\hat{a}], [\hat{b}], [\dot{\hat{a}}], [\dot{\hat{b}}]$ lie in a plane.

This last condition characterizes surfaces which are the envelope of a one-parameter family of planes in any dimension, and thus may define developable ruled surface in the following way.

Definition 1.3.3. Let

$$[\hat{f}] = [\hat{a}] \vee [\hat{b}] : I \times \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^n$$

be a ruled surface. Then $[\hat{f}]$ is *developable* if $[\hat{a}], [\hat{b}], [\dot{\hat{a}}], [\dot{\hat{b}}]$ lie in a plane for every $u \in I$.

Infinitesimally, this condition means that close rulings intersect, and thus, if they don't all go through one point, they envelope a curve in space.

Proposition 1.3.4. *A ruled surface is developable if it is a cone (over an arbitrary curve) or the trace of tangent lines of a regular curve.*

More specifically, let

$$[\hat{f}] = [\hat{a}] \vee [\hat{b}] : I \times \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^n$$

be a ruled surface, which is not a cone. Then there exists a unique curve

$$[\hat{c}] : I \to \mathbb{R}\mathrm{P}^n, \qquad [\hat{c}(u)] \in [\hat{a}(u)] \vee [\hat{b}(u)]$$

such that

$$[\hat{a}(u)] \vee [\hat{b}(u)] = [\hat{c}(u)] \vee [\dot{\hat{c}}(u)]$$

for all $u \in I$. The curve $[\hat{c}]$ is called the line of striction *(or edge of regression) of $[\hat{f}]$.*

Proof. For a cone or the trace of tangent lines of a regular curve, one easily checks that they constitute developable surfaces.

Let $[\hat{f}] = [\hat{a}] \vee [\hat{b}]$ be a developable surface, and let

$$\hat{c}(u) = \lambda(u)\hat{a}(u) + \mu(u)\hat{b}(u)$$

for some functions λ, μ. Then

$$\dot{\hat{c}}(u) = \dot{\lambda}\hat{a} + \lambda\dot{\hat{a}} + \dot{\mu}\hat{b} + \mu\dot{\hat{b}},$$

and $[\dot{s}] \in [\hat{a}] \vee [\hat{b}]$ if and only if

$$\lambda\dot{\hat{a}} + \mu\dot{\hat{b}} \in \operatorname{span}\{\hat{a}, \hat{b}\}.$$

Such λ, μ exists, since $\hat{a}, \hat{b}, \dot{\hat{a}}, \dot{\hat{b}}$ lie in a 3-dimensional subspace. This choice is unique since they do not lie in a 2-dimensional subspace (regularity for ruled surfaces). □

1.4 Dual representation of surfaces

Instead of describing a surface as a two-parameter family of points, we can equivalently describe it as the envelope of its two-parameter family of tangent planes. In particular, for a surface in $\mathbb{R}^3$, the tangent planes can be described in terms of a normal field.

Definition 1.4.1. Let $f : U \to \mathbb{R}^3$ be a regular surface. Then a smooth map

$$n : U \to \mathbb{R}^3 \backslash \{0\}$$

is called a *normal field* of f if

$$\begin{aligned} n \cdot f_u &= 0, \\ n \cdot f_v &= 0. \end{aligned}$$

The tangent plane of a surface f in $\mathbb{R}^3$ can be described in terms of a normal field

$$Tf(u,v) = \left\{ x \in \mathbb{R}^3 \,\middle|\, n(u,v) \cdot (x - f(u,v)) = n(u,v) \cdot (x + h(u,v) = 0 \right\}$$

and some function $h(u,v) = -n(u,v) \cdot f(u,v)$. Thus, the tangent planes of f are described by the tuple (n, h), which is unique up to a common scalar multiple, and determined by the equations

$$\begin{aligned} n \cdot f_u &= 0, \\ n \cdot f_v &= 0, \\ n \cdot f + h &= 0. \end{aligned} \tag{1.1}$$

Differentiating the last equation with respect to u and v, respectively, we find that (1.1) is equivalent to

$$\begin{aligned} f \cdot n_u + h_u &= 0, \\ f \cdot n_v + h_v &= 0, \\ f \cdot n + h &= 0. \end{aligned} \tag{1.2}$$

Note that if we consider the lifts

$$\begin{aligned} \hat{f} &:= (f, 1), \\ \hat{n} &:= (n, h) \end{aligned}$$

to homogeneous coordinates of $\mathbb{RP}^3$ and $(\mathbb{RP}^3)^*$, respectively, then equations (1.1) and (1.2) become the duality relations for tangent planes of the respective surfaces $[\hat{f}]$ and $[\hat{n}]$.

Definition 1.4.2. Let $[\hat{f}] : U \to \mathbb{RP}^3$ be a regular surface. Then

$$[\hat{n}] := ([\hat{f}] \vee [\hat{f}_u] \vee [\hat{f}_v])^\star : U \to (\mathbb{RP}^3)^*$$

is called the *dual surface* of f.

In homogeneous coordinates the dual surface is determined by the three linearly independent equations

$$\begin{aligned} \hat{n} \cdot \hat{f}_u &= 0, \\ \hat{n} \cdot \hat{f}_v &= 0, \\ \hat{n} \cdot \hat{f} &= 0, \end{aligned} \tag{1.3}$$

and satisfies

$$\begin{aligned} \hat{f} \cdot \hat{n}_u &= 0, \\ \hat{f} \cdot \hat{n}_v &= 0, \\ \hat{f} \cdot \hat{n} &= 0. \end{aligned} \tag{1.4}$$

These equations are completely symmetric in $\hat{f}$ and $\hat{n}$.

Proposition 1.4.3. *If the dual surface of a regular surface $[\hat{f}]$ in $\mathbb{RP}^3$ is itself regular, then the dual surface of a the dual surface is $[\hat{f}]$.*

Remark 1.4.4. The primal surface is regular if it is locally not a curve. The dual surface is regular if the primal surface is locally not developable.

1.5 Conjugate line parametrizations

We now study special parametrizations, in the sense that the parameter lines satisfy some geometric condition. We start with conjugate line parametrizations, which we first introduce for surfaces in $\mathbb{R}^3$. Conjugate line paramtrizations are geometrically characterized by the following condition: Along each parameter line of the surface, the tangent planes rotate around the tangent line in the other coordinate direction. Put differently: The tangent planes along one parameter line envelop a surface that is ruled by the tangent lines in the other coordinate direction.

Definition (and Proposition) 1.5.1. Let $f : U \to \mathbb{R}^3$ be a regular surface, and $n : U \to \mathbb{R}^3$ a normal field of f. Then f is a called a *conjugate line parameterization* if one and hence all of the following equivalent conditions hold:

(i) $n_v \cdot f_u = 0$
(ii) $n_u \cdot f_v = 0$
(iii) $n \cdot f_{uv} = 0$
(iv) $f_{uv} \in \operatorname{span}(f_u, f_v)$
(v) $f_{uv} = \alpha f_u + \beta f_v$ for smooth functions $\alpha, \beta : U \to \mathbb{R}$

Proof. Taking the v-derivative of $n \cdot f_u = 0$ and the u-derivative of $n \cdot f_v = 0$, we obtain

$$n_v \cdot f_u = n \cdot f_{uv}$$
$$n_u \cdot f_v = n \cdot f_{vu}$$

and since $f_{uv} = f_{vu}$ by the symmetry of second derivatives, conditions (i), (ii), and (iii) are equivalent.

Condition (iii) implies (iv) because (f_u, f_v) is a basis for the orthogonal subspace to n. This also means that the equation of condition (v) determines the functions α and β uniquely. In fact, by Cramer's rule,

$$\alpha = \frac{\det(n, f_{uv}\, f_v)}{\det(n, f_u\, f_v)}, \quad \beta = \frac{\det(n, f_u\, f_{uv})}{\det(n, f_u\, f_v)}, \tag{1.5}$$

which also shows that α and β are smooth because f is. Finally, condition (v) clearly implies (iii) and (iv). □

Conditions (iv) and (v) of Definition 1.5.1 do not mention the normal field n. We may use them to define conjugate line parametrizations in $\mathbb{R}^n$:

Definition 1.5.2. A regular surface $f : U \to \mathbb{R}^n$ is called a *conjugate line parameterization* if it satisfies one and hence both equivalent conditions (iv) and (v) of Definition 1.5.1.

Of course we cannot use equations (1.5) to see that α and β are smooth if $n > 3$, because the normal field is not defined. But instead we may use

$$\alpha = \frac{\det\begin{pmatrix} f_{uv}\cdot f_u & f_v \cdot f_u \\ f_{uv}\cdot f_v & f_v\cdot f_v\end{pmatrix}}{\det\begin{pmatrix} f_u\cdot f_u & f_v\cdot f_u \\ f_u\cdot f_v & f_v\cdot f_v\end{pmatrix}}, \quad \beta = \frac{\det\begin{pmatrix} f_u\cdot f_u & f_{uv}\cdot f_u \\ f_u\cdot f_v & f_{uv}\cdot f_v\end{pmatrix}}{\det\begin{pmatrix} f_u\cdot f_u & f_v\cdot f_u \\ f_u\cdot f_v & f_v\cdot f_v\end{pmatrix}}. \tag{1.6}$$

The definition for conjugate line parametrizations translates as follows to surfaces in $\mathbb{RP}^n$:

Proposition 1.5.3. *Let $f : U \to \mathbb{R}^n$ be a regular surface. Let*

$$\hat{f} := \lambda \cdot (f, 1) : U \to \mathbb{R}^{n+1}$$

be an arbitrary lift to homogeneous coordinates with a smooth function $\lambda : U \to \mathbb{R}\backslash\{0\}$.

Then f is a conjugate line parametrization if and only if $\hat{f}$ satisfies

$$\hat{f}_{uv} = \alpha \hat{f}_u + \beta \hat{f}_v + \gamma \hat{f} \tag{1.7}$$

with some smooth functions α, β, γ.

Proof. □

Equation (1.7) states the linear dependence of four representative vectors, or equivalently that four points lie in a plane. While the four points are not projectively well-defined (the points defined by the derivatives are not invariant under scaling $\hat{f}$) this property is.

Definition 1.5.4. Let $[\hat{f}] : U \to \mathbb{RP}^n$ be a regular surface. Then $[\hat{f}]$ is called a *conjugate line parametrization* if the four points $[f]$, $[f_u]$, $[f_v]$, $[f_{uv}]$ lie in a plane for every $(u, v) \in U$.

We have seen that this property is projectively well-defined. Furthermore, it is a property of the coordinate lines. Thus, it is invariant under reparametrization of the surface along the coordinate lines. Finally, it is also invariant under applying a projective transformation to the surface. We summarize these properties in the following proposition.

Proposition 1.5.5. *A regular surface $[\hat{f}] : U \to \mathbb{RP}^n$ being a conjugate line parametrization is invariant under*

(i) a change of representative vectors

$$\hat{f}(u, v) \to \lambda(u, v)\hat{f}(u, v)$$

with a smooth non-vanishing function λ.

(ii) reparametrization along the coordinate lines

$$\hat{f}(u, v) \to \hat{f}(\varphi(\tilde{u}), \chi(\tilde{v}))$$

with two smooth bijective functions φ, χ.

(iii) projective transformations

$$\hat{f}(u,v) \to F\hat{f}(u,v)$$

with $F \in \mathrm{GL}(n+1,\mathbb{R})$.

For surfaces in $\mathbb{R}\mathrm{P}^3$ the property of being a conjugate line parametrization is also invariant under dualization.

Proposition 1.5.6. *A regular surface* $[\hat{f}] : \mathbb{R}^2 \supset U \to \mathbb{R}\mathrm{P}^3$ *is a conjugate line parametrization if and only if its dual surface* $[\hat{n}] : U \to (\mathbb{R}\mathrm{P}^3)^*$ *is a conjugate line parametrization.*

Proof. $[\hat{f}]$ is a conjugate line parametrization if $\hat{f}$ satisfies an equation of the form (1.7), which is equivalent to

$$\hat{f}_{uv} \cdot \hat{n} = 0.$$

From equations (1.3), or equivalently, equations (1.4), we find that this is equivalent to either of the three equations

$$\begin{aligned} \hat{f}_u \cdot \hat{n}_v &= 0, \\ \hat{f}_v \cdot \hat{n}_u &= 0, \\ \hat{f} \cdot \hat{n}_{uv} &= 0, \end{aligned} \tag{1.8}$$

and thus in turn to

$$\hat{n}_{uv} = \tilde{\alpha}\hat{n}_u + \tilde{\beta}\hat{n}_v + \tilde{\gamma}\hat{n},$$

□

Remark 1.5.7. The first two equations of (1.8) state, respectively, that

$$\begin{aligned} [\hat{f}] \vee [\hat{f}_u] &= ([\hat{n}] \vee [\hat{n}_v])^\star, \\ [\hat{f}] \vee [\hat{f}_v] &= ([\hat{n}] \vee [\hat{n}_u])^\star. \end{aligned}$$

which capture the geometric description of conjugate line parametrizations given in the beginning of the section.

1.6 Asymptotic line parametrizations

Asymptotic line parametrizations are geometrically characterized by the following condition: Along each parameter line of the surface patch, the tangent planes rotate around the tangent line of that parameter line. Put differently: The tangent planes along each parameter line envelop a surface that is ruled by the tangent lines of that same parameter line. This leads to a description of asymptotic line parametrizations analogous to conjugate line parametrizations.

Definition (and Proposition) 1.6.1. Let $f : U \to \mathbb{R}^3$ be a regular surface, and $n : U \to \mathbb{R}^3$ a normal field of f. Then f is a called an *asymptotic line parameterization* if one and hence all of the following equivalent conditions hold:

(i) $n_u \cdot f_u = n_v \cdot f_v = 0$

(ii) $n \cdot f_{uu} = n \cdot f_{vv} = 0$

(iii) $f_{uu}, f_{vv} \in \operatorname{span}(f_u, f_v)$

(iv) $f_{uu} = \alpha f_u + \beta f_v$ for smooth functions $\alpha, \beta : U \to \mathbb{R}$, and
$f_{vv} = \tilde{\alpha} f_u + \tilde{\beta} f_v$ for smooth functions $\tilde{\alpha}, \tilde{\beta} : U \to \mathbb{R}$

Same as for conjugate line parametrizations, conditions (iv) and (v) of Definition 1.6.1 do not mention the normal field n, and we may use them to define asymptotic line parametrizations in $\mathbb{R}^n$:

Definition 1.6.2. A regular surface $f : U \to \mathbb{R}^n$ is called an *asymptotic line parameterization* if it satisfies one and hence both equivalent conditions (iv) and (v) of Definition 1.6.1.

The definition for asymptotic line parametrizations translates as follows to surfaces in $\mathbb{RP}^n$:

Proposition 1.6.3. *Let $f : U \to \mathbb{R}^n$ be a regular surface. Let*

$$\hat{f} := \lambda \cdot (f, 1) : U \to \mathbb{R}^{n+1}$$

be an arbitrary lift to homogeneous coordinates with a smooth function $\lambda : U \to \mathbb{R}\backslash\{0\}$.

Then f is an asymptotic line parametrization if and only if $\hat{f}$ satisfies

$$\begin{aligned} \hat{f}_{uu} &= \alpha \hat{f}_u + \beta \hat{f}_v + \gamma \hat{f} \\ \hat{f}_{vv} &= \tilde{\alpha} \hat{f}_u + \tilde{\beta} \hat{f}_v + \tilde{\gamma} \hat{f} \end{aligned} \qquad (1.9)$$

with some smooth functions $\alpha, \beta, \gamma, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$.

And thus, it generalizes to the following definition.

Definition 1.6.4. Let $[\hat{f}] : U \to \mathbb{RP}^n$ be a regular surface. Then $[\hat{f}]$ is called an *asymptotic line parametrization* if the points $[\hat{f}_{uu}]$ and $[\hat{f}_{vv}]$ both lie in the tangent plane $[\hat{f}] \vee [\hat{f}_u] \vee [\hat{f}_v]$ for every $(u, v) \in U$.

Note that the condition $[\hat{f}_{uu}] \in [\hat{f}] \vee [\hat{f}_u] \vee [\hat{f}_v]$ is equivalent to

$$[\hat{f}] \vee [\hat{f}_u] \vee [\hat{f}_{uu}] = [\hat{f}] \vee [\hat{f}_u] \vee [\hat{f}_v].$$

This means that the osculating plane of the u-parameter line coincides with the tangent plane. And similarly for the v-parameter line.

Proposition 1.6.5. *Let $[\hat{f}] : U \to \mathbb{RP}^n$ be a regular surface. Then $[\hat{f}]$ is an asymptotic line parametrization if and only if the two osculating planes for the two parameter lines coincide at every point:*

$$[\hat{f}] \vee [\hat{f}_u] \vee [\hat{f}_{uu}] = [\hat{f}] \vee [\hat{f}_v] \vee [\hat{f}_{vv}].$$

In particular they both coincide with the tangent plane at that point.

The statements from Proposition 1.5.5 and Proposition 1.5.6 similarly hold for asymptotic line parametrizations.

Remark 1.6.6. The invariance of asymptotic line parametrizations under dualization can equivalently be stated as

$$[\hat{f}] \vee [\hat{f}_u] = ([\hat{n}] \vee [\hat{n}_u])^\star,$$
$$[\hat{f}] \vee [\hat{f}_v] = ([\hat{n}] \vee [\hat{n}_v])^\star.$$

which capture the geometric description of asymptotic line parametrizations given in the beginning of the section.

1.7 Discrete nets

We study discrete nets as discrete analogues of parametrizations. Discrete nets are maps defined on a subset of $\mathbb{Z}^m$. For simplicity (to avoid special treatment of the boundary) we mostly consider maps defined on the entire $\mathbb{Z}^m$.

Definition 1.7.1. Let $m, n \in \mathbb{N}$. A map

$$f : \mathbb{Z}^m \to \mathbb{RP}^n$$

is called a *(discrete m-dimensional) net* in $\mathbb{RP}^n$.

In particular, 1-dimensional discrete nets may be considered as discrete analogues of parametrized curves, and 2-dimensional discrete nets as discrete analogues of parametrized surfaces.

In the case of a *discrete curve* $\gamma : \mathbb{Z} \to \mathbb{RP}^n$, and $k \in \mathbb{Z}$ we introduce the following notation

$$\gamma_k := \gamma(k) \qquad \text{for } k \in \mathbb{Z}$$

for the point assigned to the vertex k.

Definition 1.7.2. Let $\gamma : \mathbb{Z} \to \mathbb{RP}^n$ be a discrete curve.

- γ is called regular if any two successive points γ_k, γ_{k+1} are distinct.
- The line
$$T_k := \gamma_k \vee \gamma_{k+1}$$
is called the *(edge) tangent line* at the edge $(k, k+1)$.
- The plane
$$\gamma_{k-1} \vee \gamma_k \vee \gamma_{k+1}$$
is called the *(vertex) osculating plane* at k.

Note how the regularity, tangent line, and osculating plane are immediately soon to be projectively invariant.

In the case of a *discrete surface* $f : \mathbb{Z}^2 \to \mathbb{RP}^n$, we use subscripts to denote shifts

$$f_1(i,j) = f(i+1,j), \quad f_{\bar{1}}(i,j) = f(i-1,j),$$
$$f_2(i,j) = f(i,j+1), \quad f_{\bar{2}}(i,j) = f(i,j-1).$$

The discrete curves

$$i \mapsto f(i,j), \qquad j \mapsto f(i,j)$$

may be thought of as *discrete parameter lines*, which come with conditions of discrete regularity, as well as tangent lines and osculating planes.

Definition 1.7.3. A discrete surfaces $f : \mathbb{Z}^2 \to \mathbb{RP}^n$ is called *regular* if any three points of each face span a plane.

The plane spanned by three such points, e.g.,

$$f(i,j) \vee f(i+1,j) \vee f(i,j+1)$$

may be thought of as a discrete tangent plane, which is assigned to the corresponding corner of the face.

1.8 Discrete conjugate nets

For a discrete surface $f : \mathbb{Z}^2 \to \mathbb{R}^n$ consider the following discretization of condition v in Definition 1.5.1

$$\begin{aligned} \Delta_1\Delta_2 f &= \alpha\Delta_1 f + \beta\Delta_2 f \\ \Leftrightarrow \qquad f_{12} &= f + (\alpha+1)\Delta_1 f + (\beta+1)\Delta_2 f. \end{aligned}$$

This motivates the following definition.

Definition 1.8.1. A regular discrete net $f : \mathbb{Z}^2 \to \mathbb{RP}^n$ is called a *discrete conjugate net* (or *Q-net*) if the four points of every face f, f_1, f_{12}, f_2 lie in a plane.

Again, this condition is immediately seen to be projectively invariant. Furthermore, in the case of a discrete conjugate net, we have a unique choice for a tangent plane for every face of $\mathbb{Z}^2$. For two adjacent faces the intersection of two such tangent planes is the tangent line of the common edge.

For a discrete conjugate net in $\mathbb{RP}^3$ this gives rise to a dual net defined on the faces of $\mathbb{Z}^2$ into $(\mathbb{RP}^3)$. It turns out that the dual net is again a discrete conjugate net, since the dual configuration of four planes intersecting in a point is given by four points lying in a plane. More generally, we can obtain a discrete version of Proposition 1.5.6

Proposition 1.8.2. *Let $f : \mathbb{Z}^2 \to \mathbb{RP}^3$ be a regular discrete surface such that its dual discrete surface*

$$n := (f \vee f_1 \vee f_2)^\star : \mathbb{Z}^2 \to (\mathbb{RP}^3)^*$$

is regular. Then, f is a discrete conjugate net if and only if n is a discrete conjugate net.

Proof. Exercise. □

1.9 Discrete asymptotic nets

To obtain a discretization of asymptotic line parametrizations consider the characerization in Proposition 1.6.5. At every vertex of a discrete surface, we have two osculating planes of the two discrete parameter lines that contain this vertex. These two osculating planes coincide if and only if all five points of the vertex star are coplanar.

Definition 1.9.1. A regular discrete net $f : \mathbb{Z}^2 \to \mathbb{RP}^n$ is called a *discrete asymptotic net* (or *A-net*) if the five points of every vertex star $f, f_{\bar{1}}, f_1, f_{\bar{2}}, f_2$ lie in a plane.

Again, this condition is immediately seen to be projectively invariant. Furthermore, in the case of a discrete asymptotic net, we have a unique choice for a tangent plane for every vertex of $\mathbb{Z}^2$. For two adjacent vertices the intersection of two such tangent planes is the tangent line of the common edge.

For a discrete asymptotic net in $\mathbb{R}\mathrm{P}^3$ this gives rise to a dual net defined on the vertices of $\mathbb{Z}^2$ into $(\mathbb{R}\mathrm{P}^3)^*$. It turns out that the dual net is again a discrete asymptotic net.

Chapter 2

Curves and surfaces in Möbius geometry

2.1 Arc-length, curvature, and osculating circles

To introduce the curvature of a curve, we first consider a special parametrization.

Definition 2.1.1. Let $\gamma : \mathbb{R} \supset I \to \mathbb{R}^n$ be a curve

(i) The function
$$v(t) := \|\dot{\gamma}(t)\|$$
is called the *speed* of γ.

(ii) If $v(t) \neq 0$, the vector
$$\tau(t) := \frac{\dot{\gamma}(t)}{v(t)}$$
is called the *unit tangent vector* of γ.

(iii) The function
$$s(t) := \int_{t_1}^{t} v(t)\mathrm{d}t$$
is called the *arc-length* of γ, here $I = [t_1, t_2]$.

(iv) If $v(t) = 1$ for all $t \in I$, then γ is called *arc-length parametrized.*

Note that the derivative of the arc-length is the speed
$$\dot{s}(t) = v(t).$$

For a regular curve γ the arc-length $s(\cdot)$ is strictly monotonically increasing, and thus invertible. We call its inverse function $t(\cdot) = s^{-1}(\cdot)$ and thus write
$$\gamma(s) = (\gamma \circ t)(s).$$

For the derivative w.r.t. arc-length we write

$$\gamma' = \frac{\mathrm{d}}{\mathrm{d}s}\gamma = \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\gamma = \frac{1}{v}\dot{\gamma}.$$

In particular, the parametrization of γ w.r.t. arc-length has unit speed

$$\|\gamma'\| = 1.$$

Thus, the unit tangent vector is equivalently given by the derivative w.r.t. arc-length

$$\tau = \gamma'.$$

Furthermore, the second derivative w.r.t. arc-length defines a unique normal vector in the osculating plane

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\|\gamma'\|^2 = \frac{\mathrm{d}}{\mathrm{d}s}\langle\gamma', \gamma'\rangle = 2\langle\gamma'', \gamma'\rangle.$$

Definition 2.1.2. Let $\gamma : \mathbb{R} \supset I \to \mathbb{R}^n$ be a regular curve

(i) Any vector $n(t)$ orthogonal to $\dot{\gamma}(t)$, i.e.,

$$\langle n(t), \dot{\gamma}(t)\rangle = 0,$$

is called a *normal vector* of γ at $t \in I$

(ii) The hyperplane

$$\gamma(t) + \{n(t) \mid \langle n(t), \dot{\gamma}(t)\rangle = 0\}$$

is called the *normal plane*.

(iii) The normal vector

$$n(s) = \frac{\tau'(s)}{\|\tau'(s)\|} = \frac{\gamma''(s)}{\|\gamma''(s)\|}$$

is called the *principal unit normal vector*.

(iv) Let $n(s)$ be the principal unit normal vector. Then the line

$$N(s) = \{\gamma(s) + \alpha n(s) \mid \alpha \in \mathbb{R}\}.$$

is called the *principal normal line*.

The principal normal line is the intersection of the normal plane and the osculating plane.

Definition 2.1.3. Let γ be a regular curve. Then

$$\kappa(s) = \|\tau'(s)\| = \|\gamma''(s)\|$$

is called the *curvature* of γ at s.

By definition of the principal unit normal vector, the curvature can also be expressed as

$$\kappa(s) = \langle \gamma''(s), n(s) \rangle$$

This leads to the following description in terms of an arbitrary parametrization.

Proposition 2.1.4. *Let γ be a regular curve, and n the principal unit normal vector. Then the curvature of γ is given by*

$$\kappa(t) = \frac{\langle \ddot{\gamma}(t), n(t) \rangle}{\|\dot{\gamma}(t)^2\|}.$$

Proof. With $\frac{\mathrm{d}}{\mathrm{d}s} = \frac{1}{v}\frac{\mathrm{d}}{\mathrm{d}t}$ we find

$$\gamma'' = \frac{1}{v}\left(\frac{\dot{\gamma}}{v}\right)^{\bullet} = \frac{v\ddot{\gamma} - \dot{v}\dot{\gamma}}{v^3}.$$

Using $\langle \dot{\gamma}, n \rangle = 0$ leads to the result. □

It is also useful to have a formula for the curvature in terms of an arbitrary parametrization that does not depend on the principal unit normal vector.

Proposition 2.1.5. *Let γ be a regular curve. Then its curvature is given by*

$$\kappa(t) = \frac{\sqrt{\|\dot{\gamma}\|^2 \|\ddot{\gamma}\|^2 - \langle \dot{\gamma}, \ddot{\gamma} \rangle^2}}{\|\dot{\gamma}\|^3}.$$

Proof.

$$\kappa = \|\gamma''\| = \left\| \frac{1}{v}\left(\frac{\dot{\gamma}}{v}\right)^{\bullet} \right\| = \frac{\|v\ddot{\gamma} - \dot{v}\dot{\gamma}\|}{v^3} = \frac{\|v^2\ddot{\gamma} - \langle \dot{\gamma}, \ddot{\gamma} \rangle \dot{\gamma}\|}{v^4} = \frac{\sqrt{v^2 \|\ddot{\gamma}\|^2 - \langle \dot{\gamma}, \ddot{\gamma} \rangle^2}}{v^3}$$

where we used

$$\dot{v} = \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle} = \frac{\langle \dot{\gamma}, \ddot{\gamma} \rangle}{\sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle}} = \frac{\langle \dot{\gamma}, \ddot{\gamma} \rangle}{v}$$

□

Example 2.1.6. Consider a parametrized circle of radius $r > 0$

$$\gamma(t) = r \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}, \qquad t \in [0, 2\pi].$$

Then

$$\dot{\gamma}(t) = r \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}, \quad \ddot{\gamma}(t) = -r \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix},$$

and

$$\|\dot{\gamma}(t)\| = \|\ddot{\gamma}(t)\| = r, \quad \langle \dot{\gamma}(t), \ddot{\gamma}(t) \rangle = 0.$$

Thus, the curvature of γ is given by

$$\kappa(t) = \frac{1}{r}.$$

We can now assign to every point of a curve a circle which lies in the osculating plane and has the same curvature as the curve

Definition 2.1.7. Let $\gamma : I \to \mathbb{R}^n$ be a regular curve, and let n be the principal unit normal vector of γ. If $\kappa(t) \neq 0$, then the *osculating circle* at $t \in I$ is the circle in the osculating plane of γ at t with center

$$c(t) = \gamma(t) + \frac{1}{\kappa(t)} n(t)$$

and radius

$$r(t) = \frac{1}{\kappa(t)}.$$

If $\kappa(t) = 0$, then we consider the tangent line at $t \in I$ to be the osculating circle.

The osculating circle touches the curve in the corresponding point, and has the same curvature. Even more, if the curve and its osculating circle are parametrized by arc-length such that the first derivative $\gamma'(s)$ of both curves coincide, then their second derivative $\gamma''(s)$ also coincide.

It can also be shown that it is the best approximating circle in the following sense. Consider the circle through three points of the curve $\gamma(t)$, $\gamma(t - \epsilon)$, and $\gamma(t + \epsilon)$. Then in the limit $\epsilon \to 0$, this circle converges to the osculating circle.

2.2 Osculating circles in Möbius geometry

We first consider the case of a regular plane curve

$$\gamma : I \to \mathbb{R}^2.$$

By inverse stereographic projection, we can map it to the sphere (Möbius lift)

$$[\hat{\gamma}] : I \to S^2 \subset \mathbb{R}\mathrm{P}^3, \qquad \hat{\gamma}(t) := \gamma(t) + \|\gamma(t)\|^2 e_\infty + e_0.$$

The osculating circle of γ at $t \in I$ is the circle with center and radius

$$c(t) := \gamma(t) + \frac{1}{\kappa(t)} n(t), \qquad r(t) := \frac{1}{\kappa(t)}, \tag{2.1}$$

where n is the principal unit normal vector field of γ and

$$\kappa(t) = \frac{\langle \ddot{\gamma}(t), n(t) \rangle}{\|\dot{\gamma}(t)\|^2}$$

is the curvature of γ. Its inverse stereographic projection (Möbius lift) to the sphere is given by

$$[\hat{c}(t)]^\perp \cap S^2, \qquad \hat{c}(t) := c(t) + (\|c(t)\|^2 - r(t)^2) e_\infty + e_0$$

Proposition 2.2.1. *Let $\gamma : I \to \mathbb{R}^2$ be a regular plane curve, and $[\hat{\gamma}] : I \to S^2 \subset \mathbb{R}\mathrm{P}^3$ be its Möbius lift. Then the Möbius lift of the osculating circle of γ is the intersection of the osculating plane of $[\hat{\gamma}]$ with S^2:*

$$[\hat{c}]^\perp \cap S^2 = \Big([\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)] \vee [\ddot{\hat{\gamma}}(t)] \Big) \cap S^2$$

Proof. With

$$\hat{c} = \gamma + \tfrac{1}{\kappa} n + e_0 + \left(\|\gamma\|^2 + \tfrac{2}{\kappa} \langle \gamma, n \rangle \right) e_\infty$$

we obtain

$$\langle \hat{\gamma}, \hat{c} \rangle_{3,1} = \left\langle \gamma, \gamma + \tfrac{1}{\kappa} n \right\rangle - \tfrac{1}{2} \|\gamma\|^2 - \tfrac{1}{\kappa} \langle \gamma, n \rangle - \tfrac{1}{2} \|\gamma\|^2 = 0.$$

Now with

$$\dot{\hat{\gamma}} = \dot{\gamma} + 2 \langle \gamma, \dot{\gamma} \rangle e_\infty$$

we obtain

$$\left\langle \dot{\hat{\gamma}}, \hat{c} \right\rangle_{3,1} = \left\langle \dot{\gamma}, \gamma + \tfrac{1}{\kappa} n \right\rangle - \langle \gamma, \dot{\gamma} \rangle = 0.$$

Finally, with

$$\ddot{\hat{\gamma}} = \ddot{\gamma} + 2 \left(\|\dot{\gamma}\|^2 + \langle \gamma, \ddot{\gamma} \rangle \right) e_\infty$$

we obtain
$$\left\langle \ddot{\hat{\gamma}}, \hat{c} \right\rangle_{3,1} = \langle \ddot{\gamma}, \gamma + \tfrac{1}{\kappa} n \rangle - \|\dot{\gamma}\|^2 - \langle \gamma, \ddot{\gamma} \rangle = 0.$$

□

Thus, the Möbius lift of the osculating circle of a curve is the intersection of osculating plane with the Möbius lift of the curve. In particular, this implies that the osculating plane of the Möbius lift of the curve always intersects the Möbius quadric in a circle, i.e., is of signature $(++-)$.

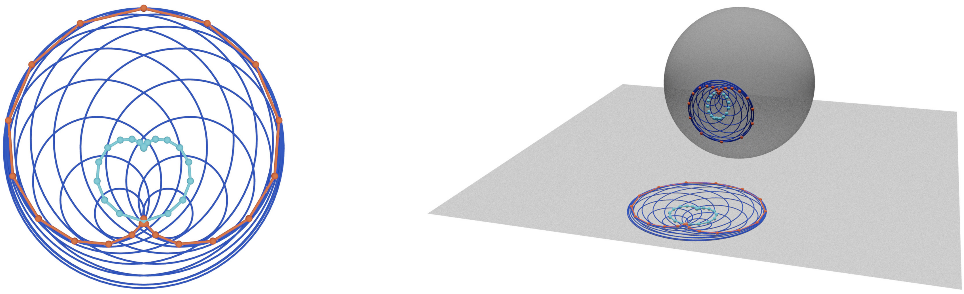

Figure 2.1. Osculating circles of a cardoid and the lift to Möbius geometry.

The same holds true for curves in arbitrary dimension $n \geqslant 2$:

Proposition 2.2.2. *Let $\gamma : I \to \mathbb{R}^n$ be a regular curve, and $[\hat{\gamma}] : I \to S^n \subset \mathbb{RP}^n$ be its Möbius lift*
$$\hat{\gamma}(t) := \gamma(t) + \|\gamma(t)\|^2 e_\infty + e_0.$$
Then the stereographic projection of the intersection of the osculating plane of $[\hat{\gamma}]$ with the Möbius quadric
$$\sigma\left(\left([\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)] \vee [\ddot{\hat{\gamma}}(t)]\right) \cap S^n\right)$$
is the osculating circle of γ.

Proof. The osculating circle of γ is the intersection of osculating plane of γ and the sphere with center and radius given by (2.1). The central projections of three points spanning the osculating plane of $[\hat{\gamma}]$ are
$$\sigma([\hat{\gamma}(t)]) = [\gamma(t), 1], \qquad \sigma([\dot{\hat{\gamma}}(t)]) = [\dot{\gamma}(t), 0], \qquad \sigma([\ddot{\hat{\gamma}}(t)]) = [\ddot{\gamma}(t), 0],$$
which span the osculating plane of γ. Thus the image of the stereographic projection of
$$\left([\hat{\gamma}(t)] \vee [\dot{\hat{\gamma}}(t)] \vee [\ddot{\hat{\gamma}}(t)]\right) \cap S^n$$

lies in the osculating plane of γ.

The remaining part of the proof is analogous to Proposition 2.2.1. □

Since in the projective model of Möbius geometry, Möbius transformations are the projective transformations that preserve the Möbius quadric, this means that the osculating circles of a curve are mapped to the osculating circles of the image curve under a Möbius transformation.

Corollary 2.2.3. *The osculating circles of a regular curve are Möbius invariant.*

Proof. By Proposition 1.1.11, osculating planes are mapped to osculating planes under projective transformations. □

Example 2.2.4. Recall that the evolute of a plane curve consists of the centers of the osculating circles. As an exercise, let we use the Möbius lift to determine the evolute of a parabola

$$\gamma(t) := \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

Its Möbius lift is given by

$$\hat{\gamma}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix} + (t^2 + t^4)e_\infty + e_0,$$

and its first two derivatives by

$$\dot{\hat{\gamma}}(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix} + 2(t + 2t^3)e_\infty, \qquad \ddot{\hat{\gamma}}(t) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 2(1 + 6t^2)e_\infty.$$

We determine the polar point

$$\hat{c}(t) = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} + c_\infty(t)e_\infty + e_0,$$

From

$$0 = \left\langle \hat{c}, \ddot{\hat{\gamma}} \right\rangle = 2c_2 - 1 - 6t^2$$

we obtain

$$c_2(t) = \frac{1}{2}(1 + 6t^2).$$

and from

$$0 = \left\langle \hat{c}, \dot{\hat{\gamma}} \right\rangle = c_1 + t + 6t^3 - t - 2t^3$$

we obtain

$$c_1(t) = -4t^3$$

Thus, the evolute of γ is given by

$$e(t) = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} -4t^3 \\ \frac{1}{2}(1+6t^2), \end{pmatrix}$$

which coincides with the solution from Example **??**. Note, that we don't have to compute c_∞, if we are only interested in the evolute of γ, and not the osculating circles.

2.3 Discrete curves in Möbius geometry

For a discrete curve we can easily introduce a Möbius invariant osculating circle as the circle through three consecutive vertices.

Definition 2.3.1. Let $\gamma : I \to \mathbb{R}^2$ be a regular discrete curve. Then the circle C_k through three successive points $\gamma_{k-1}, \gamma_k, \gamma_{k+1}$ is called the *(vertex) osculating circle* at $k \in I$.

Let r_k be the radius of C_k. Then the discrete *(vertex) curvature* at $k \in I$ is given by

$$\kappa_k = \frac{1}{r_k}$$

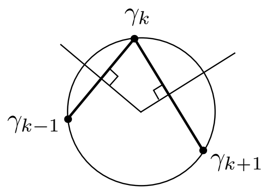

Figure 2.2. Vertex osculating circle.

The osculating circle defined in this way lies in the osculating plane of the corresponding vertex. In this plane, one can introduce the perpendicular bisectors as normal lines on each edge. Then the two normal lines intersect in the center of the osculating circle.

We introduce the *turning angle* at a vertex $k \in I$ by

$$\varphi_k \coloneqq \sphericalangle(\Delta\gamma_k, \Delta\gamma_{k-1}) \in [-\pi, \pi].$$

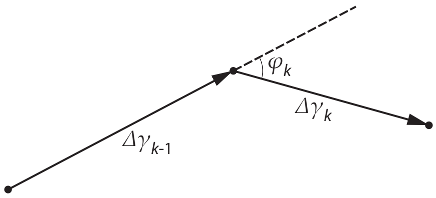

Figure 2.3. Turning angle at a vertex of a discrete curve.

With this the discrete curvature can be expressed in the following way.

Proposition 2.3.2. *Let $\gamma : \mathbb{Z} \to \mathbb{R}^n$ be a regular discrete curve. Then its vertex curvature is given by*

$$\kappa_k = \frac{2 \sin \varphi_k}{\|\gamma_{k+1} - \gamma_{k-1}\|}.$$

Proof. The radius r_k of the osculating circle is given by $\|\gamma_{k+1} - \gamma_{k-1}\| = 2r_k \sin \varphi_k$. □

Remark 2.3.3.

- The vertex osculating circle inherits an orientation from the order of the three points on it. This can be used to also associate a sign to the discrete curvature, which corresponds to the sign in the formula above.
- The vertex osculating circle can also be used to define *vertex tangent lines* as the line tangent to C_k in the point γ_k.

2.4 Curvature line parametrizations

Definition 2.4.1. Let

$$f : \mathbb{R}^2 \supset U \to \mathbb{R}^n$$

be a regular surface.

(i) f is called *orthogonal* if

$$\langle f_u, f_v \rangle = 0$$

(ii) f is called *curvature line parametrization* if it is orthogonal and conjugate, i.e.,

$$\langle f_u, f_v \rangle = 0, \quad \text{and} \quad f_{uv} = \alpha f_u + \beta f_v.$$

Proposition 2.4.2. *The property of a parametrization to be orthogonal is Möbius invariant.*

Proof. Möbius transformations are conformal, i.e., preserve angles. □

Conjugate parametrizations, on the other hand, are not Möbius invariant. Are curvature line parametrizations?

Proposition 2.4.3. *Let $f : \mathbb{R}^2 \supset U \to \mathbb{R}^n$ be a regular surface and $[\hat{f}] : U \to S^n \subset \mathbb{RP}^{n+1}$ be its Möbius lift*

$$\hat{f} := f + e_0 + \|f\|^2 e_\infty.$$

Then f is a curvature line parametrization if and only if $[\hat{f}]$ is a conjugate line parametrization.

Proof. For the derivatives of the lift we obtain

$$\begin{aligned} \hat{f}_u &= f_u + 2\langle f, f_u\rangle e_\infty, \\ \hat{f}_v &= f_v + 2\langle f, f_v\rangle e_\infty, \\ \hat{f}_{uv} &= f_{uv} + 2\left(\langle f, f_{uv}\rangle + \langle f_u, f_v\rangle\right) e_\infty. \end{aligned}$$

Let $\hat{f}$ be a curvature line parametrization. Then

$$\hat{f}_{uv} = f_{uv} + 2\langle f, f_{uv}\rangle e_\infty = \alpha f_u + \beta f_v + 2\left(\alpha\langle f, f_u\rangle + \beta\langle f, f_v\rangle\right) e_\infty = \alpha \hat{f}_u + \beta \hat{f}_v.$$

The reverse direction is shown similarly. □

Corollary 2.4.4. *Curvature line parametrizations are Möbius invariant.*

2.5 Circular nets

In the smooth case we have seen that curvature line parametrizations are conjugate line parametrizations in the Möbius quadric. Consider a discrete conjugate net $f : \mathbb{Z}^2 \to S^n \subset \mathbb{RP}^{n+1}$ in the Möbius quadric. Then the stereographic projection of the four points of every face lie on a circle.

Definition 2.5.1. A regular discrete net $f : \mathbb{Z}^2 \to \mathbb{R}^n$ is called a *circular net* if the four points of every face f, f_1, f_{12}, f_2 lie on a circle.

The definition immediately implies that circular nets are Möbius invariant.

Remark 2.5.2. The axes of the circles can be interpreted as discrete normals (per face). Adjacent discrete normal lines intersect, and in this sense they form discrete developable surfaces.

Proposition 2.5.3. *Let $f : \mathbb{Z}^2 \to \mathbb{R}^n$ be a regular discrete net, and $[\hat{f}] : \mathbb{Z}^2 \to S^n \subset \mathbb{R}\mathrm{P}^{n+1}$ be its Möbius lift*

$$\hat{f} := f + e_0 + \|f\|^2 e_\infty.$$

Then f is a circular net if and only if $[\hat{f}]$ is a discrete conjugate net.

Thus, circular nets maybe viewed as a discretization of curvature line parametrizations.

Chapter 3

Curves and surfaces in Laguerre geometry

3.1 Planar curves in Laguerre geometry

Let

$$\gamma : [a, b] \to \mathbb{R}^2$$

be a smooth regular curve in the Euclidean plane. Its unit tangent and normal vector are given by

$$T(t) := \frac{\dot{\gamma}}{\|\gamma\|}, \qquad N(t) := JT(t), \quad \text{where } J := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right).$$

Then the tangent line at the point $\gamma(t)$ is given by

$$P_{(N(t),h(t))} = \left\{ x \in \mathbb{R}^2 \;\middle|\; N(t) \cdot x + h(t) = 0 \right\}, \quad h(t) := -N(t) \cdot \gamma(t).$$

The oriented tangent lines $\vec{P}_{(N(t),h(t))}$ yield a curve on the Blaschke cylinder. We have seen this in the example of circles which correspond to curves on the Blaschke cylinder given by planar sections. On the other hand, the curve γ can be uniquely reconstructed from its tangent lines as the envelope.

Proposition 3.1.1. *Let γ be a smooth regular curve in $\mathbb{R}^2$. Then*

$$\hat{\gamma}(t) := (N(t), 1, h(t)), \quad h(t) := -N(t) \cdot \gamma(t)$$

defines a curve on the Blaschke cylinder. The corresponding oriented lines are the oriented tangent lines of γ, i.e.,

$$\begin{aligned} N \cdot \gamma + h &= 0, \\ N \cdot \dot{\gamma} &= 0. \end{aligned}$$

Furthermore, the curve γ is the envelope of those lines, i.e.,

$$\begin{aligned} N \cdot \gamma + h &= 0, \\ \dot{N} \cdot \gamma + \dot{h} &= 0. \end{aligned} \tag{3.1}$$

Vice versa, given a smooth regular curve $t \mapsto (N(n), 1, h(t))$ on the Blaschke cylinder not tangent to a generator, equations (3.1) *determine a unique curve as the envelope of the corresponding oriented lines in the plane.*

3.2 Osculating circle of planar curves

The osculating circle of the planar curve γ at the point $\gamma(t)$ is the circle $\vec{S}_{(c(t),r(t))}$ with center

$$c(t) := \gamma(t) + \frac{1}{\kappa(t)} N(t)$$

and radius

$$r(t) := \frac{1}{\kappa(t)}$$

where $\kappa(t)$ is the curvature at $\gamma(t)$.

Proposition 3.2.1. *Let γ a smooth regular curve in $\mathbb{R}^2$. Let*

$$\hat{\gamma}(t) = (N(t), 1, h(t))$$

be its lift to the Blaschke cylinder, and let

$$\hat{c}(t) := (c(t), -r(t), 1)$$

be the lift of its osculating circle to the cyclographic model. Then

$$[\hat{c}(t)]^\star = \mathrm{P}\left(\operatorname{span}\{\hat{\gamma}, \dot{\hat{\gamma}}, \ddot{\hat{\gamma}}\}\right).$$

Proof. Show that

$$\hat{c}^\intercal \hat{\gamma} = \hat{c}^\intercal \dot{\hat{\gamma}} = \hat{c}^\intercal \ddot{\hat{\gamma}} = 0$$

where one uses $\ddot{N} \cdot N = -\dot{N} \cdot \dot{N}$ and $\dot{N} = -\kappa \dot{\gamma}$. □

To apply a Laguerre transformation to a curve it is applied to its oriented tangent lines. Then the image curve is reconstructed as the envelope of the image tangent lines.

Corollary 3.2.2. *The osculating circle of a planar curve is Laguerre invariant.*

3.3 Conics and hypercycles

We will now study which curves on the Blaschke cylinder correspond to conics (more precisely ellipses and hyperbolas).

By means of a rotation and a translation (which constitute special Laguerre transformations) an ellipse or a hyperbola may be brought into the form

$$C = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a} + \frac{y^2}{b} = 1 \right\} \tag{3.2}$$

with some $a, b \neq 0$. The case $a > 0, b > 0$ corresponds to an ellipse and the case $ab < 0$ to a hyperbola.

Proposition 3.3.1. *The curve on the Blaschke cylinder $\mathcal{Z}$ corresponding to the tangent lines (with both orientations) of the conic C is given by the intersection of $\mathcal{Z}$ with the cone*

$$\mathcal{C} = \left\{ [x_1, x_2, x_3, x_4] \in \mathbb{RP}^3 \;\middle|\; a x_1^2 + b x_2^2 - x_4^2 = 0 \right\}. \tag{3.3}$$

Proof. The tangent line to C at a point $(x_0, y_0) \in C$ is given by

$$\left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \frac{x x_0}{a} + \frac{y y_0}{b} = 1 \right\},$$

and its two lifts to the Blaschke cylinder by

$$\left[\frac{x_0}{a}, \frac{y_0}{b}, \pm\sqrt{\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}}, -1 \right] = \left[\frac{x_0 h}{a}, \frac{y_0 h}{b}, \pm 1, h \right] \in \mathcal{Z}$$

where

$$h := \frac{1}{\sqrt{\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}}}.$$

□

In particular, we found that the curve on the Blaschke cylinder corresponding to an ellipse or hyperbola is given by the intersection with a quadric.

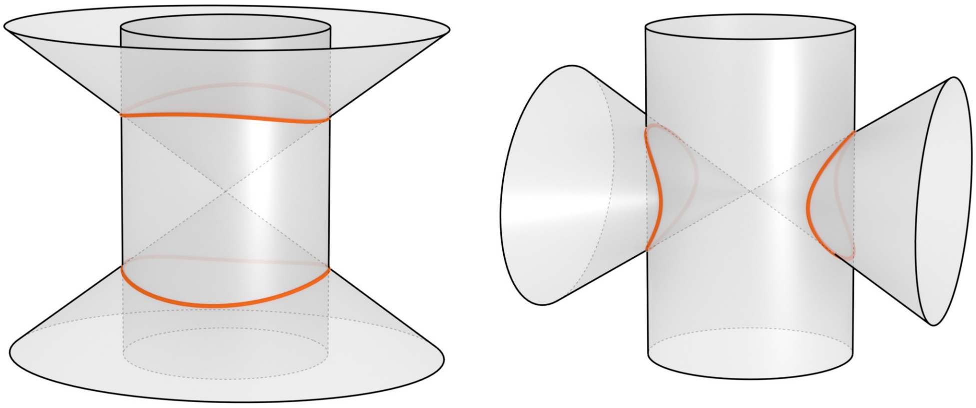

Figure 3.1. Hypercycle base curves corresponding to an ellipse and hyperbola respectively.

Definition 3.3.2. The intersection curve of the Blaschke cylinder $\mathcal{Z}$ with another quadric $\mathcal{Q}$ is called a *hypercycle base curve*. The envelope of the corresponding lines in the plane is called a *hypercycle*.

Corollary 3.3.3. *Conics (considered with both orientations) are hypercycles.*

The hypercycle base curve is the base curve of the pencil of quadrics spanned by $\mathcal{Z}$ and $\mathcal{Q}$. The intersection of any quadric from this pencil with the Blaschke cylinder yields the same curve $\mathcal{Z} \cap \mathcal{Q}$.

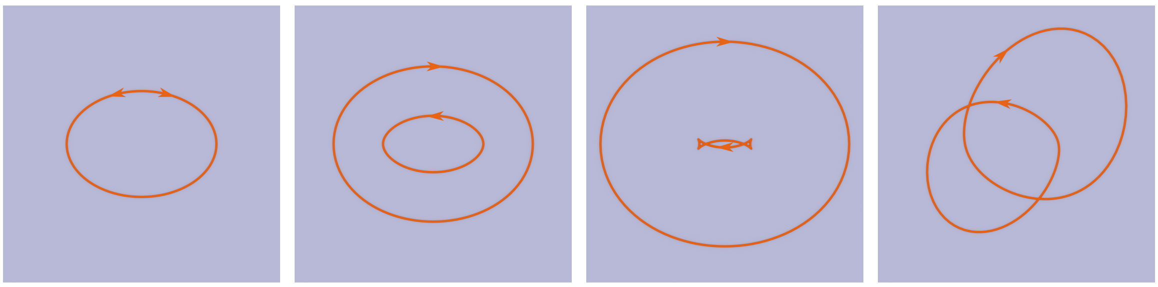

Figure 3.2. A conic under Laguerre transformations.

3.4 Discrete curves in Laguerre geometry

Definition 3.4.1. Let $\gamma : I \to \mathbb{R}^2$ be a regular discrete curve. Then the circle C_k that touches three consecutive edge tangent lines T_{k-1}, T_k, T_{k+1} is called the *edge osculating circle* at $(k, k+1) \in I$.

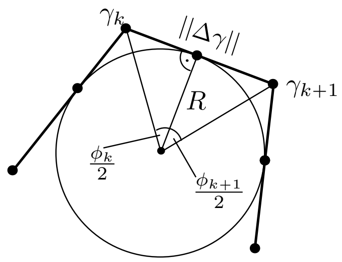

Figure 3.3. Edge osculating circle.

- For three (non-concurrent) lines in $\mathbb{R}^3$ there are four circles touching them. By endowing the tangent lines with the orientation coming from the order of the points of the curve on them, this choice can be made unique.

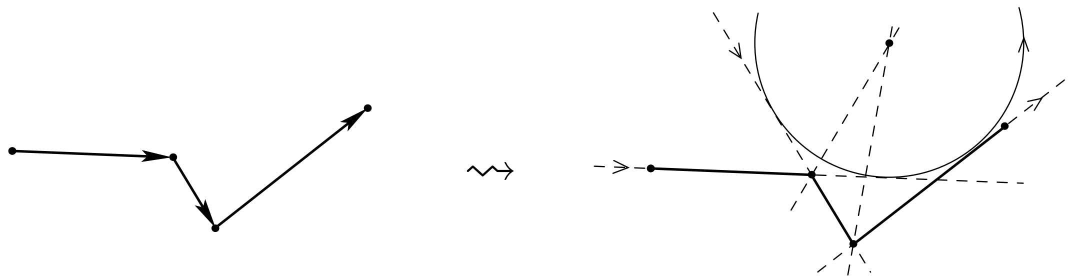

Figure 3.4. Edge osculating circle from oriented tangent lines.

- Note that the (correctly chosen) angle bisectors of successive edge tangent lines contain the center of the edge osculating circle. Thus, the edge osculating circle can be used to define *edge normal lines*.
- The (oriented) edge osculating circle can be used to define a (signed) discrete curvature at the edge $(k, k+1)$. The radius is given by $\|\Delta\gamma_k\| = R_k(\tan\frac{\varphi_k}{2} + \tan\frac{\varphi_{k+1}}{2})$. This leads to the curvature
$$\kappa_k = \frac{\tan\frac{\varphi_k}{2} + \tan\frac{\varphi_{k+1}}{2}}{\|\Delta\gamma_k\|}.$$

3.5 Surfaces in Laguerre geometry

Let
$$f : \mathbb{R}^2 \supset U \to \mathbb{R}^3$$

be a smooth regular parametrized surface patch. Let

$$n : U \to \mathbb{R}^3$$

be an arbitrary smooth normal field of f such that at every point $(u, v) \in U$

$$n = \lambda(f_u \times f_v)$$

with some positive scalar $\lambda > 0$, and let

$$\sigma := \|n\| > 0$$

denote the norm of n. Furthermore, let h be such that

$$n \cdot f + h = 0.$$

Then the lift of f to the Blaschke cylinder is given by

$$\hat{f} := (n, \sigma, h).$$

Recall that f is a curvature line parametrization if and only if f is orthogonal and conjugate. In Section 1.4 we have established that f is conjugate if and only if its dual surface $[n, h]$ is conjugate. Thus, to describe curvature line parametrizations in Laguerre geometry we should determine how to express the orthogonality in the homogeneous coordinates (n, σ, h).

Lemma 3.5.1. *For a parametrized surface f the lift to the Blaschke cylinder (n, σ, h) satisfies*

$$\begin{aligned} \sigma^2 &= n \cdot n, \\ \sigma\sigma_u &= n \cdot n_u, \\ \sigma\sigma_v &= n \cdot n_v, \\ \sigma\sigma_{uv} + \sigma_u\sigma_v &= n \cdot n_{uv} + n_u \cdot n_v. \end{aligned} \tag{3.4}$$

Lemma 3.5.2. *Let f be a conjugate line parametrized surface. Then f is orthogonal if and only if its lift to the Blaschke cylinder (n, σ, h) satisfies*

$$\sigma\sigma_{uv} = n \cdot n_{uv}.$$

Proof. Since f is conjugate, we have

$$f_u \cdot n_v = 0,$$

and thus f_u is proportional to $n_v \times n$. Similarly, f_v is proportional to $n \times n_u$, the orthogonality condition

$$f_u \cdot f_v = 0$$

is equivalent to

$$\begin{aligned}
&(n_v \times n) \cdot (n \times n_u) = 0 \\
\Leftrightarrow\ &(n \cdot n_u)(n \cdot n_v) = (n \cdot n)(n_u \cdot n_v) \\
\Leftrightarrow\ &\sigma_u \sigma_v = n_u \cdot n_v \\
\Leftrightarrow\ &\sigma \sigma_{uv} = n \cdot n_{uv},
\end{aligned}$$

where we used Lemma 3.5.1. □

Theorem 3.5.3. *Let $f : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a parametrized surface and*

$$\hat{f} := (n, \sigma, h)$$

a lift to the Blaschke cylinder. Then f is a curvature line parametrization if and only if $[\hat{f}]$ is a conjugate parametrization.

Proof. f is a conjugate line parametrization if and only if $[n, h]$ is a conjugate line parametrization, i.e., if

$$\begin{aligned}
n_{uv} &= \alpha n_u + \beta n_v + \gamma n, \\
h_{uv} &= \alpha h_u + \beta h_v + \gamma h \\
&.
\end{aligned}$$

with some functions $\alpha, \beta, \gamma : U \to \mathbb{R}$.

Now if f is orthogonal, then by Lemma 3.5.1 and Lemma 3.5.2

$$\sigma \sigma_{uv} = n_{uv} \cdot n = \alpha n_u \cdot n + \beta n_v \cdot n + \gamma n \cdot n = \alpha \sigma \sigma_u + \beta \sigma \sigma_v + \gamma \sigma^2$$

and thus

$$\sigma_{uv} = \alpha \sigma_u + \beta \sigma_v + \gamma \sigma.$$

Vice versa, if σ satisfies the previous equation, the argument may be reversed. □

Corollary 3.5.4. *Curvature line parametrizations are Laguerre invariant.*

3.6 Conical nets

In the smooth case we have seen that curvature line parametrizations can be represented by conjugate line parametrizations in the Blaschke cylinder. Consider a discrete conjugate net $\mathbb{Z}^2 \to \mathcal{Z} \subset \mathbb{RP}^4$ in the Blaschke cylinder Then for every face the four oriented planes corresponding to its four vertices are in oriented contact with an oriented cone.

Definition 3.6.1. A regular discrete conjugate net $f : \mathbb{Z}^2 \to \mathbb{R}^n$ is called a *conical net* if the planes on the faces can be oriented such that for each vertex the four incident oriented planes are in oriented contact with an oriented cone.

Thus, we obtain an another discretization of curvature line parametrizations. The definition immediately implies that conical nets are Laguerre invariant.

Remark 3.6.2. The axes of the cones can be interpreted as discrete normals (per face). Adjacent discrete normal lines intersect, and in this sense they form discrete developable surfaces.

There is a well-known *reflection construction* to obtain a circular net from a conical net and vice-versa

(i) A conical net is obtained from a circular net by reflecting an initial plane through the point of a vertex about the planes that are spanned by adjacent circle-axes. The composition of the four reflections incident to a vertex is the identity, and thus this construction yields a well-defined plane per face. The four planes corresponding to four faces incident to a vertex intersect in a common point on the circle-axis. These points constitute a conical net.

(ii) A circular net is obtained from a conical net by reflecting an initial point in the plane of a vertex about the planes that are spanned by adjacent cone-axes. The composition of the four reflections incident to a vertex is the identity, and thus this construction yields a well-defined point per vertex. These points constitute a circular net.

The two constructions are symmetric in the following sense: A conical net h can be obtained from a circular net g by construction (i) if and only if g can be obtained from h by construction (ii). Indeed, this holds since the net of the reflection planes coincide.

Chapter 4

Curves and surfaces in Lie geometry

4.1 Planar curves in Lie geometry

Let

$$\gamma : [a, b] \to \mathbb{R}^2$$

be a regular planar curve. Its unit tangent and normal vector are given by

$$\tau(t) := \frac{\dot{\gamma}}{\|\gamma\|}, \qquad n(t) := J\tau(t), \quad \text{where } J := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right).$$

We can lift the points of the curve as well as the oriented tangent lines to the Lie quadric.

$$\begin{aligned} s_p(t) &:= \gamma(t) + e_0 + \|\gamma(t)\|^2 e_\infty, \\ s_q(t) &:= n(t) - 2h(t)e_\infty + e_5. \end{aligned}$$

Neither a point nor an oriented line are Lie invariant objects. But if the point lies on the line, together they span a contact element, which corresponds to an isotropic line in the Lie quadric. Thus, we can lift the curve γ to a one-parameter family of lines (a ruled surface) in the Lie quadric:

$$\ell(t) := [s_p(t)] \vee [s_q(t)]$$

The condition for the oriented lines to be the tangent lines of the curve becomes

$$\langle \dot{s}_p, s_q \rangle = \dot{\gamma} \cdot n = 0, \tag{4.1}$$

or equivalently

$$\langle s_p, \dot{s}_q \rangle = \gamma \cdot \dot{n} + \dot{h} = 0.$$

Lemma 4.1.1. *Let*

$$\ell(t) := [s_1(t)] \vee [s_2(t)] \subset \mathcal{L} \subset \mathbb{R}\mathrm{P}^4$$

be a smooth regular one-parameter family of lines in the Lie quadric. Then the following are equivalent:

(i) $\langle \dot{s}_1, s_2 \rangle = 0$.

(ii) $\langle s_1, \dot{s}_2 \rangle = 0$.

(iii) $[\dot{s}_1], [\dot{s}_2] \in ([s_1] \vee [s_2])^\perp$

(iv) ℓ *is a developable surface, i.e.,* $[s_1], [s_2], [\dot{s}_1], [\dot{s}_2]$ *lie in a plane.*

Proof. (i) and (ii) are equivalent since

$$\langle s_1, s_2 \rangle = 0$$

implies

$$\langle \dot{s}_1, s_2 \rangle + \langle s_1, \dot{s}_2 \rangle = 0.$$

Thus, both are equivalent to (iii). $([s_1] \vee [s_2])^\perp$ is a plane, which also contains $[s_1], [s_2]$. Vice versa, if $[s_1], [s_2], [\dot{s}_1], [\dot{s}_2]$ lie in a plane, then

$$\alpha s_1 + \beta s_2 + \gamma \dot{s}_1 + \delta \dot{s}_2 = 0$$

where neither γ nor δ can be zero. Taking the scalar product with s_2 yields (i). □

Proposition 4.1.2. *Let*

$$\ell(t) := [s_1(t)] \vee [s_2(t)] \subset \mathcal{L} \subset \mathbb{R}\mathrm{P}^4$$

be a developable surface in the Lie quadric. Then its sections with the point complex and plane (line) complex

$$[s_p(t)] := \ell(t) \cap p^\perp,$$
$$[s_q(t)] := \ell(t) \cap q^\perp$$

define a planar curve in the Euclidean plane together with its oriented tangent lines.

Proof. By Lemma 4.1.1, ℓ is developable if and only if

$$\langle \dot{s}_1, s_2 \rangle = 0$$

Furthermore, the equivalence in this lemma, implies that this condition is invariant under a change of choice of points spanning the lines ℓ, which is easily checked independently. Indeed, for

$$\tilde{s}_1 := \lambda_1 s_1 + \lambda_2 s_2,$$
$$\tilde{s}_2 := \mu_1 s_1 + \mu_2 s_2$$

with smooth $\lambda_1, \lambda_2, \mu_1, \mu_2$, we find

$$\left\langle \dot{\tilde{s}}_1, \tilde{s}_2 \right\rangle = \left\langle \dot{\lambda}_1 s_1 + \lambda_1 \dot{s}_1 + \dot{\lambda}_2 s_2 + \lambda_2 \dot{s}_2, \mu_1 s_1 + \mu_2 s_2 \right\rangle = 0.$$

Thus, in particular

$$\langle \dot{s}_p, s_q \rangle = 0$$

which by (4.1) is equivalent to the claimed tangency condition. □

Each contact element along the curve contains the osculating circle of the curve. We show that the corresponding points on the isotropic lines in the Lie quadric constitute the line of striction of the developable surface.

Theorem 4.1.3. *Let γ a regular curve in $\mathbb{R}^2$. Let*

$$\ell(t) := [s_p(t)] \vee [s_q(t)]$$

with

$$s_p(t) := \gamma(t) + e_0 + \|\gamma(t)\|^2 e_\infty,$$
$$s_q(t) := n(t) - 2h(t)e_\infty + e_5.$$

be its lift to the Lie quadric $\mathcal{L} \subset \mathbb{R}\mathrm{P}^4$, and let

$$s(t) := c(t) + e_0 + (\|c(t)\|^2 - r(t)^2)e_\infty + r(t)e_5$$

be the lift of its osculating circles. Then $[s(t)]$ is the edge of regression of the developable surface $\ell(t)$, i.e.

$$[s] \vee [\dot{s}] = \ell.$$

Proof. We first check that

$$s = s_p + r s_q$$

and thus $[s] \in \ell$.

As a linear combination of s_p and s_q the curve s satsifies

$$\langle \dot{s}, s_p \rangle = \langle \dot{s}, s_q \rangle = 0,$$

and thus $[\dot{s}] \in \ell^\perp$. We check that furthermore, $[\dot{s}] \in \mathcal{L}$, and thus $[\dot{s}] \in \ell$. Indeed, with

$$\dot{s} = \dot{c} + 2(\dot{c} \cdot c - \dot{r}r)e_\infty + \dot{r}e_5$$

we find

$$\langle \dot{s}, \dot{s} \rangle = \|\dot{c}\|^2 - (\dot{r})^2 = \|\dot{\gamma} + \dot{r}n + r\dot{n}\|^2 - (\dot{r})^2 = 0,$$

where we used $\|n\|^2 = 1$ and $\dot{\gamma} = -r\dot{n}$. □

Corollary 4.1.4. *The osculating circles of a planar curve are Lie invariant.*

4.2 Surfaces in Lie geometry

Let

$$f : \mathbb{R}^2 \supset U \to \mathbb{R}^3$$

be a smooth regular parametrized surface patch. Let

$$n : U \to S^2$$

be the unit normal field of f such that at every point $(u, v) \in U$

$$n = \frac{f_u \times f_v}{\|f_u \times f_v\|}.$$

Furthermore, let h be such that

$$n \cdot f + h = 0.$$

At each point of the surface this point together with the oriented tangent plane defines a contact element. The lift of f to the Lie quadric is given by the two-parameter family of isotropic lines representing these contact elements:

$$\ell(u, v) := [s_p(u, v)] \vee [s_q(u, v)]$$

where

$$s_p(u, v) := f(u, v) + e_0 + \|f(u, v)\|^2 e_\infty,$$
$$s_q(u, v) := n(u, v) - 2h(u, v)e_\infty + e_6.$$

The conditions for oriented planes to be tangent planes of the surface becomes

$$\langle \partial_u s_p, s_q \rangle = f_u \cdot n = 0,$$
$$\langle \partial_v s_p, s_q \rangle = f_v \cdot n = 0,$$

or equivalently,

$$\langle s_p, \partial_u s_q \rangle = \langle s_p, \partial_v s_q \rangle = 0.$$

The following lemma may be proven in a similar way to Lemma 4.1.1.

Lemma 4.2.1. *Let*

$$\ell(t) := [s_1(t)] \vee [s_2(t)] \subset \mathcal{L} \subset \mathbb{R}\mathrm{P}^5$$

be a smooth regular two-parameter family of lines in the Lie quadric. Then the following are equivalent:

(i) $\langle \partial_u s_1, s_2 \rangle = \langle \partial_v s_1, s_2 \rangle = 0$,

(ii) $\langle s_1, \partial_u s_2 \rangle = \langle s_1, \partial_v s_2 \rangle = 0$,

(iii) $[\partial_u s_1], [\partial_u s_2], [\partial_v s_1], [\partial_v s_2] \in ([s_1] \vee [s_2])^\perp$.

Note that here $([s_1] \vee [s_2])^\perp$ is a 3-dimensional subspace. Similar to Lemma 4.1.1, the conditions in Lemma 4.2.1 mean that the tangent planes of the surfaces $[s_1]$ and $[s_2]$ are contained in the 3-dimensional polar subspace of the line $[s_1] \vee [s_2]$. We find that surfaces in Lie geometry are characterized by this condition. The proof is analogous to that of Proposition 4.1.2.

Proposition 4.2.2. *Let*

$$\ell(u,v) := [s_1(u,v)] \vee [s_2(u,v)] \subset \mathcal{L} \subset \mathbb{R}\mathrm{P}^5$$

be a line congruence in the Lie quadric. Then its sections with the point complex and plane complex

$$[s_p(u,v)] := \ell(u,v) \cap p^\perp,$$
$$[s_q(u,v)] := \ell(u,v) \cap q^\perp$$

define a smooth regular surface in Euclidean space $\mathbb{R}^3$ *together with its oriented tangent planes.*

Definition 4.2.3. Let

$$\ell(u,v) := [s_1(u,v)] \vee [s_2(u,v)] \subset \mathbb{R}\mathrm{P}^n$$

be a smooth regular two-parameter family of lines in a projective space $\mathbb{R}\mathrm{P}^n$. Then ℓ is called a *(torsal) line congruence* if the two ruled surfaces given by $u \mapsto \ell(u,v)$ and $v \mapsto \ell(u,v)$ are developable, i.e.,

$$[s_1], [s_2], [\partial_u s_1], [\partial_u s_2] \text{ span a plane, and}$$
$$[s_1], [s_2], [\partial_v s_1], [\partial_v s_2] \text{ span a plane.}$$

Lemma 4.2.4. *Let*

$$\ell(u,v) := [s_1(u,v)] \vee [s_2(u,v)] \subset \mathbb{R}\mathrm{P}^n$$

be a (torsal) line congruence. Then

$$[s_1], [s_2], [\partial_u s_1], [\partial_u s_2], [\partial_v s_1], [\partial_v s_2], [\partial_u \partial_v s_1], [\partial_u \partial_v s_2]$$

lie in a 3-dimensional subspace.

Proof. By the condition for a (torsal) line congruence, the points

$$[s_1], [s_2], [\partial_u s_1], [\partial_u s_2], [\partial_v s_1], [\partial_v s_2]$$

lie in a 3-dimensional subspace Π. Thus, we need to show $[\partial_u \partial_v s_1], [\partial_u \partial_v s_2] \in \Pi$.

Again, since ℓ is a (torsal) line congruence, there exist $\alpha, \beta, \gamma, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ such that

$$\begin{aligned}\partial_u s_2 &= \alpha s_1 + \beta s_2 + \gamma \partial_u s_1,\\ \partial_v s_2 &= \tilde{\alpha} s_1 + \tilde{\beta} s_2 + \tilde{\gamma} \partial_v s_1.\end{aligned}$$

Cross-differentiation leads to

$$\begin{aligned}\partial_u \partial_v s_2 &= \partial_v \alpha s_1 + \partial_v \beta s_2 + \alpha \partial_v s_1 + \beta \partial_v s_2 + \partial_v \gamma \partial_u s_1 + \gamma \partial_u \partial_v s_1,\\ \partial_u \partial_v s_2 &= \partial_u \tilde{\alpha} s_1 + \partial_u \tilde{\beta} s_2 + \tilde{\alpha} \partial_u s_1 + \tilde{\beta} \partial_u s_2 + \partial_u \tilde{\gamma} \partial_v s_1 + \tilde{\gamma} \partial_u \partial_v s_1,\end{aligned}$$

which shows that $[\partial_u \partial_v s_1] \in \Pi$. Similarly, $[\partial_u \partial_v s_2] \in \Pi$. □

With this we can show the following characterization for the Lie lift of curvature line parametrizations.

Theorem 4.2.5. *Let* $f : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ *be a parametrized surface and*

$$\ell(u, v) := [s_p(u, v)] \vee [s_q(u, v)]$$

be its lift to the Lie quadric $\mathcal{L} \subset \mathbb{R}\mathrm{P}^5$*, where*

$$\begin{aligned}s_p(u, v) &:= f(u, v) + e_0 + \|f(u, v)\|^2 e_\infty,\\ s_q(u, v) &:= n(u, v) - 2h(u, v)e_\infty + e_6.\end{aligned}$$

If f *is a curvature line parametrization then* ℓ *is a (torsal) line congruence.*

Vice versa, let

$$\ell(u, v) := [s_1(u, v)] \vee [s_2(u, v)] \subset \mathcal{L} \subset \mathbb{R}\mathrm{P}^5$$

be a (torsal) line congruence in the Lie quadric. Then its section with the point complex

$$[s_p(u, v)] := \ell(u, v) \cap p^\perp$$

is a curvature line parametrization.

Proof. Let f be a parametrized surface. Then

$$\begin{aligned} \partial_u s_p &= f_u + 2(f_u \cdot f)e_\infty, \\ \partial_u s_q &= n_u - 2h_u e_\infty = n_u + 2(f \cdot n_u)e_\infty. \end{aligned}$$

Thus,

$$0 \cdot s_p + 0 \cdot s_q + \kappa_1 \partial_u s_p - \partial_u s_q = 0,$$

where we used $n_u = \kappa_1 f_u$ for some κ_1, since f is a curvature line parametrization. Similarly, for the v direction.

Now let ℓ be a (torsal) line congruence. We first check that conditions (i) of Lemma 4.2.1 are satisfied, so that ℓ actually defines a surface. Indeed, since ℓ is a (torsal) line congruence there exist α, β, γ such that

$$\partial_u s_2 = \alpha s_1 + \beta s_2 + \gamma \partial_u s_1.$$

Thus,

$$\langle s_1, \partial_u s_2 \rangle = 0.$$

Similarly,

$$\langle s_1, \partial_v s_2 \rangle = 0.$$

By Lemma 4.2.4 the points

$$[s_1], [s_2], [\partial_u s_1], [\partial_u s_2], [\partial_v s_1], [\partial_v s_2], [\partial_u \partial_v s_1], [\partial_u \partial_v s_2]$$

lie the 3-dimensional subspace, which here is given by

$$\Pi := ([s_1] \vee [s_2])^\perp.$$

Thus, the four points

$$[s], [\partial_u s], [\partial_v s], [\partial_u \partial_v s] \in \Pi$$

lie in Π for any linear combination $s = \lambda_1 s_1 + \lambda_2 s_2$ such as s_p. On the other hand $[s_p]$ lies in the hyperplane $p^\perp$. The intersection $\Pi \cap p^\perp$ is 2-dimensional. Thus, the four points

$$[s_p], [\partial_u s_p], [\partial_v s_p], [\partial_u \partial_v s_p] \in \Pi \cap p^\perp$$

lie in a plane, i.e., the parametrization $[s_p]$ is conjugate. But a conjugate parametrization in the Möbius quadric represents a curvature line parametrization in $\mathbb{R}^3$. □

Corollary 4.2.6. *Curvature line parametrizations are Lie invariant.*

4.3 Discrete contact element nets

Considering discrete line congruences in the Lie quadric leads to *discrete contact element nets*. Their intersections with the point complex and the plane complex yield circular nets and conical nets, respectively, which are related by the reflection construction.