Geometry 2
Non-Euclidean Geometries

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Lecture notes for Geometry 2, Summer 2021.

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Week 1: Möbius geometry, projective model

Chapter 10

Möbius geometry

10.1 Elementary model

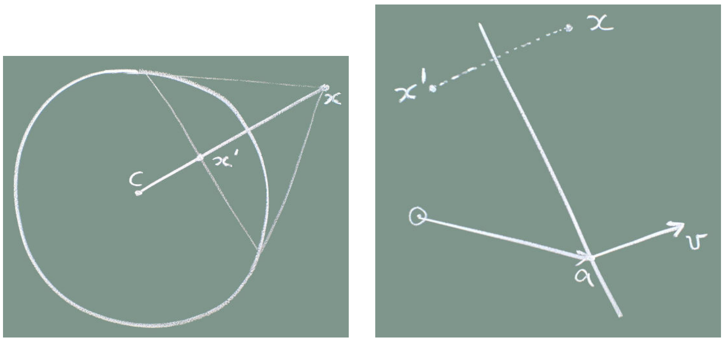


Figure 10.1. *Left:* Inversion in a hypersphere. *Right:* Reflection in a hyperplane.

Consider $\mathbb{R}^n$ with the standard Euclidean scalar product $\langle x, y\rangle = \sum_1^n x_i y_i$.

Definition 10.1.1. *Inversion in a hypersphere* with center c and radius r is the map

$$\mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n\{\infty\}, \quad x \mapsto x' = c + \frac{r^2}{\|x-c\|^2}(x-c), \quad c \leftrightarrow \infty.$$

Note that x' lies on the same ray emanating from c and $\|x-c\| \cdot \|x'-c\| = r^2$. Inversion in a sphere is an involution on $\mathbb{R}^n$, except that the center c has no image and no preimage in $\mathbb{R}^n$. We fix this by adding one extra point, ∞, to $\mathbb{R}^n$ and we declare it to be the image and preimage of c.

Theorem 10.1.2. *Inversions in spheres are conformal and map hyperspheres (or hy-*

perplanes) to hyperspheres (or hyperplanes).

Proof 1. Let us first give an elementary geometric proof. Since the geometry is rotationally symmetric to prove the claim about the hyperspheres it is enough to prove it for circles in a plane. The case of straight lines is limiting and the proof is analogous. Let ℓ be the line through the inversion circle S_0 and a circle S, and A and B be the intersection points of S and ℓ. the cirlce S can be characterized as the set of vertices C of all right angle triangles ΔABC, see Fig. 10.2 (left). Let A', B', C' be the images of A, B, C under the reflection in the sphere S_0 with the center O. We have

$$|OA||OA'| = |OB||OB'| = |OC||OC'|,$$

which implies the similarity of the triangles $\Delta OCB \sim \Delta OB'C'$ and $\Delta OCA \sim \Delta OA'C'$. The corresponding angles are equal $\angle ABC = \angle B'C'O$, $\angle OAC = \angle A'C'O$. Thus we obtain

$$\angle A'C'B' = \angle A'C'O - \angle B'C'O = \angle OAC - \angle ABC = \angle BCA = \frac{\pi}{2}.$$

We see that the point C' lies on the circles with the diameter $A'B'$.

To prove the conformality consider two intersecting lines ℓ_1 and ℓ_2. Their images under inversion i in a hypersphere are two circles $i(\ell_1)$ and $i(\ell_2)$ passing through the center O of the inversion sphere, see Fig. 10.2 (right). For the symmetry reasons the tangent lines ℓ_k to the circles $i(\ell_k)$ at O are parallel to the corresponding lines ℓ_k. Thus the circles $i(\ell_k)$ intersect at the same angle as the lines ℓ_k. □

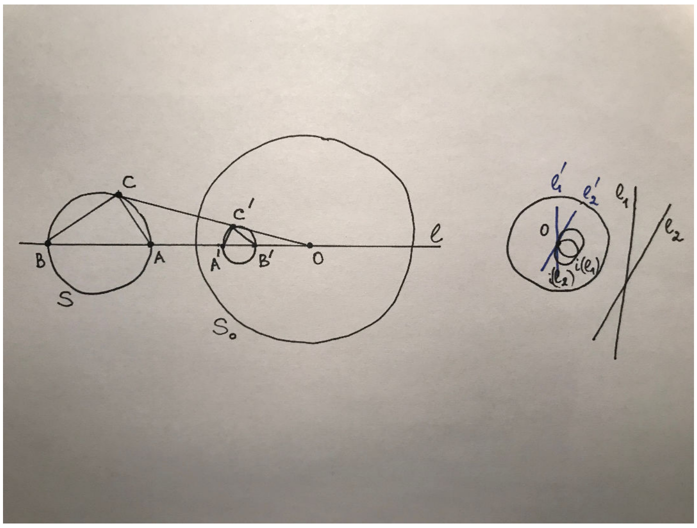


Figure 10.2. *Left:* Inversion in a circle. Circles are mapped to circles. *Right:* To the proof of the conformality.

Proof 2. We give a second algebraic/analytic proof for both statements in the case of inversions in the unit sphere:

We show that the inversion in the unit sphere maps spheres and planes to spheres and planes. One could consider hyperspheres and hyperplanes separately but we will treat both cases simultaneously. Any hypersphere or hyperplane is determined by an equation of the form

$$p\|x\|^2 - 2\langle v, x\rangle + q = 0 \qquad \text{with} \qquad \|v\|^2 - pq > 0.$$

If $p = 0$, the inequality implies $v \neq 0$ so the equation describes a hyperplane. If $p \neq 0$, it describes a hypersphere. Indeed, divide through by p to obtain

$$0 = \|x\|^2 - 2\langle \tfrac{1}{p}v, x\rangle + \tfrac{q}{p} = \|x - \tfrac{1}{p}v\|^2 - \tfrac{1}{p^2}\|v\|^2 + \tfrac{q}{p}.$$

This is a sphere with center $\frac{1}{p}v$ and radius $\sqrt{\frac{1}{p^2}\|v\|^2 - \frac{q}{p}}$. (The assumed inequality ensures that the expression under the square root is positive.) Now for $x' = \frac{1}{\|x\|^2}\,x$ one obtains

$$p\|x'\|^2 - 2\langle v, x'\rangle + q = 0 \qquad \Longleftrightarrow \qquad q\|x\|^2 - 2\langle v, x\rangle + p = 0.$$

So x' is contained in a particular hyperplane or hypersphere if and only if x is contained in some other hyperplane or hypersphere.

We show that inversion in the unit sphere is conformal. Let $t \mapsto \gamma(t)$, $t \mapsto \eta(t)$ be two parameterized curves intersecting in $\gamma(t_0) = \eta(t_0)$. The intersection angle α is determined by

$$\cos\alpha = \frac{\langle \gamma'(t_0), \eta'(t_0)\rangle}{\|\gamma'(t_0)\|\|\eta'(t_0)\|}.$$

Let $\hat\gamma = \frac{1}{\langle\gamma,\gamma\rangle}\,\gamma$, $\hat\eta = \frac{1}{\langle\eta,\eta\rangle}\,\eta$, be the image curves after inversion in the unit sphere. One finds that

$$\hat\gamma' = \frac{1}{\langle\gamma,\gamma\rangle^2}\big(\langle\gamma,\gamma\rangle\gamma' - 2\langle\gamma,\gamma'\rangle\gamma\big),$$

and similarly for $\hat\eta'$. From this one obtains $\langle\hat\gamma',\hat\gamma'\rangle = \frac{1}{\langle\gamma,\gamma\rangle^2}\langle\gamma',\gamma'\rangle$, so $\|\hat\gamma'\| = \frac{1}{\|\gamma\|^2}\|\gamma'\|$, and in the same way $\|\hat\eta'\| = \frac{1}{\|\eta\|^2}\|\eta'\|$. Using $\gamma(t_0) = \eta(t_0) =: p$ one finds that

$$\langle\hat\gamma'(t_0), \hat\eta'(t_0)\rangle = \frac{1}{\|p\|^4}\langle\gamma'(t_0), \eta'(t_0)\rangle$$

and hence

$$\frac{\langle\gamma'(t_0), \eta'(t_0)\rangle}{\|\gamma'(t_0)\|\|\eta'(t_0)\|} = \frac{\langle\hat\gamma'(t_0), \hat\eta'(t_0)\rangle}{\|\hat\gamma'(t_0)\|\|\hat\eta'(t_0)\|}$$

□

Reflection in a hyperplane $\{x : \langle x - a, v\rangle = 0\}$ is the map

$$x \longmapsto x' = x - 2\frac{\langle x - a, v\rangle}{\langle v, v\rangle}v.$$

We also declare that reflections in hyperplanes map ∞ to ∞ and thus are involutions on $\mathbb{R}^n \cup \{\infty\}$. We consider them to be special cases of inversions in a hypersphere when the hypersphere becomes a hyperplane. And same as inversions in hyperspheres, they are conformal and map hyperspheres (or hyperplanes) to hyperspheres (or hyperplanes).

Definition 10.1.3. A *Möbius transformation* of $\mathbb{R}^n \cup \{\infty\}$ is a composition of inversions in hyperspheres and reflections in hyperplanes. The Möbius transformations form a group called the *Möbius group* denoted by $\mathrm{Mob}(n)$.

A Möbius transformation is orientation reversing or preserving depending on whether it is the composition of an odd or even number of reflections. The subgroup of orientation preserving Möbius transformations is called the *special Möbius group* and denoted by $\mathrm{SMob}(n)$ or $\mathrm{Mob}^+(n)$.

The Möbius group contains all similarity transformations:

- A *translation* $x \mapsto x + v$ is the composition of two reflections in parallel hyperplanes.
- An *orthogonal transformation* $x \mapsto Ax$ with $A \in O(n)$ is the composition of at most n reflections in hyperplanes through the origin, see Appendix **??**
- A *scaling transformation* $x \mapsto \lambda x$ with $\lambda > 0$ is the composition of a reflection in the unit sphere followed by a reflection in a sphere with center 0 and radius $\sqrt{\lambda}$.

Corollary 10.1.4. *A Möbius transformation is conformal and maps any hyperplane or hypersphere to a hyperplane or hypersphere.*

From now on we will consider hyperplanes as a special cases of hyperspheres that contain ∞. So hypersphere will mean hypersphere or hyperplane.

Theorem 10.1.5 (Fundamental theorem of Möbius geometry)**.** *Any bijective map $f : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$ which maps hyperspheres to hyperspheres is a Möbius transformation.*

Proof. *(i)* Suppose $f(\infty) = \infty$. Then f maps hyperplanes to hyperplanes. Then it also maps lines to lines, because a line is the intersection of $n - 1$ hyperplanes. By the fundamental theorem of affine geometry (see 3.8.7), the restriction $f|_{\mathbb{R}^n}$ is an affine transformation. Since it also maps spheres to spheres it must be a similarity.
(ii) Suppose $f(\infty) = c \neq \infty$. Let g be the inversion in a sphere with center c. Then

$g \circ f$ also maps hyperspheres to hyperspheres and also ∞ to ∞. By *(iii)* it is a similarity transformation, so $f = g \circ g \circ f$ is a Möbius transformation. □ □

Definition 10.1.6. The map

$$\sigma : S^n \to \mathbb{R}^n, \quad \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} \mapsto \frac{1}{1 - x_{n+1}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \tag{10.1}$$

is called *stereographic projection.*

Proposition 10.1.7. *The stereographic projection σ is the restriction of the inversion*

$$i : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \quad p \mapsto e_{n+1} - 2\frac{p - e_{n+1}}{\|p - e_{n+1}\|^2}.$$

to the unit sphere $\mathbb{S}^n$.

Corollary 10.1.8. *Stereographic projection is conformal and maps hyperspheres of $\mathbb{S}^n$ (hyperplanar sections) to hyperspheres (or hyperplanes) of $\mathbb{R}^n$.*

It is even more natural to consider Möbius geometry in the unit sphere

$$S^n = \{y \in \mathbb{R}^{n+1} | \langle y, y \rangle = 1\}.$$

Indeed, identifying $\mathbb{R}^n \cup \{\infty\}$ with S^n by the stereographic projection, we see that spheres in $\mathbb{R}^n$ are mapped to spheres in S^n, and hyperplanes are mapped to hyperspheres in S^n passing through the north pole $y = e_{n+1}$, which is the center of the stereographic projection. There are several equivalent representations of hypersheres $S \subset S^n$, see Fig. 10.3:

- $S = S^n \cap \hat{S}$ is the intersection of S^n with an orthogonal n-dimensional sphere $\hat{S}$, centered at the point s located outside the unit ball. The hypersphere S is uniquelly represented by this point $s \in \mathbb{R}^{n+1}$ with $\|s\| > 1$.

- $S = S^n \cap \mathcal{P}$ is the intersection of S^n with the hyperplane $\mathcal{P} = \{y \in \mathbb{R}^{n+1} | \langle y, s \rangle = 1\}$, polar to s with respect to S^n,

- S is the contact set of the cone touching the sphere S^n with the tip s.

Möbious transformations of S^n are generated by inversions in hyperspheres $\hat{S}$ orthogonal to S^n:

$$y \mapsto s + \frac{\rho^2}{\|y-s\|^2}(y-s),$$

where ρ is the radius of $\hat{S}$. They preserve S^n, and map hyperspheres to hyperspheres. Their action, similar to the inversions in the spheres centered in $\mathbb{R}^n$, is extended to the whole $\mathbb{R}^{n+1}$.

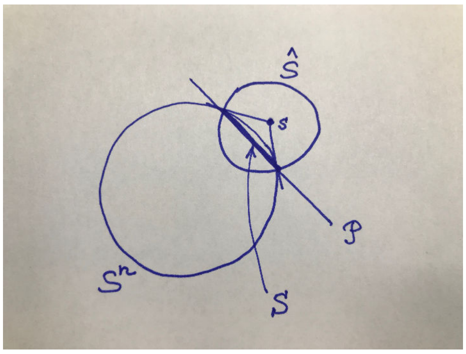


Figure 10.3. Möbius geometry in the sphere S^n. A hypersphere S as the intersection of S^n with an orthogonal sphere $\hat{S}$, with the hyperplane $\mathcal{P}$ polar to the center s of $\hat{S}$, and the tangent cone with the tip at s. It is uniquely represented by a point s outside the unit ball.

10.2 Two-dimensional Möbius geometry

This case is special because we can identify $\mathbb{R}^2$ with $\mathbb{C}$, and $\mathbb{R}^2 \cup \{\infty\}$ with the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, which is the same as $\mathbb{C}\mathrm{P}^1$, the complex projective line (see Example 2.3.2). The orientation preserving and reversing similarity transformations are $z \mapsto az + b$ and $z \mapsto a\bar{z} + b$ $(a \neq 0)$, reflection in the real line is $z \mapsto \bar{z}$, and inversion in the unit circle $|z| = 1$ is the map $z \mapsto \frac{z}{|z|^2} = \frac{1}{\bar{z}}$.

Proposition 10.2.1. *The orientation preserving and reversing Möbius transformations of* $\widehat{\mathbb{C}} = \mathbb{C}\mathrm{P}^1$ *are precisely the maps of the form*

$$z \longmapsto \frac{az+b}{cz+d} \quad \text{and} \quad z \longmapsto \frac{a\bar{z}+b}{c\bar{z}+d} \quad \text{with} \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0.$$

The corresponding group of orientation preserving Möbius transformations is (cf. 3.2.3): ∎

$$\mathrm{SMob}(2) = \mathrm{PGL}(2, \mathbb{C})$$

Proof. First, these transformations form a group: The transformations of the first kind are the projective transformations of $\mathbb{C}\mathrm{P}^1$, and the transformations of the second kind are compositions of these with complex conjugation $z \mapsto \bar{z}$. (Note that first performing a transformation of the first kind and then complex conjugation also leads to a transformation of the second kind.) This groups contains the similarity transformations and inversion in the unit sphere, so it contains the Möbius group. On the other hand, it is not bigger than the Möbius group, because any of these transformations is a composition of reflections and similarity transformations: If $c = 0$, they are just similarity transformations. Otherwise, this follows from

$$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c(cz+d)}$$

and the equation obtained by replacing z by $\bar{z}$. □ □

This has the following immediate consequences:

Corollary 10.2.2.

(i) *The orientation preserving Möbius transformations of the plane preserve the complex cross ratio* $\mathrm{cr}(z_1, z_2, z_3, z_4) = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$. *If f is an orientation reversing Möbius transformation, then* $\mathrm{cr}(f(z_1), f(z_2), f(z_3), f(z_4)) = \overline{\mathrm{cr}(z_1, z_2, z_3, z_4)}$.

(ii) *For any three points z_1, z_2, z_3 and any three points w_1, w_2, w_3, there is a unique orientation preserving Möbius transformation $f \in \mathrm{PGL}(2, \mathbb{C})$ with $f(z_i) = w_i$. There is also a unique orientation reversing one mapping $z_i \mapsto w_i$, namely f followed by an inversion in the circle through w_1, w_2, w_3.*

Proposition 10.2.3. *Four points z_1, z_2, z_3, z_4 lie on a circle if their cross ratio is real. Moreover, they are in that cyclic order on the circle if* $\mathrm{cr}(z_1, z_2, z_3, z_4) < 0$.

Proof. A map $z \mapsto \frac{az+b}{cz+d}$ is a projective transformation of $\mathbb{C}\mathrm{P}^1$ in affine coordinate. It maps circles to circles and preserves cross-ratios. Moreover there exists the one mapping z_1, z_2, z_3 to $0, 1, \infty$. The circle determined by z_1, z_2, z_3 is then mapped to the real line. Let w be the image of z_4. The cross-ratio $\mathrm{cr}(z_1, z_2, z_3, z_4) = \mathrm{cr}(0, 1, \infty, w)$ is real if and only if $w \in \mathbb{R}$, or equivalently, z_1, z_2, z_3, z_4 lie on a circle. The order of the points can be easily controlled for their images $0, 1, \infty, w$ on the real line. □

10.3 The projective model of Möbius geometry

As we have seen at the end of Section 10.1, stereographic projection maps $\mathbb{R}^n \cup \{\infty\}$ to the n-dimensional sphere $S^n \subset \mathbb{R}^{n+1}$. We now embed the sphere into projective space

$$S^n \subset \mathbb{R}^{n+1} \subset \mathrm{P}(\mathbb{R}^{n+1,1}) = \mathbb{R}\mathrm{P}^{n+1}$$

with the Lorentz product

$$\langle v, w\rangle = \textstyle\sum_1^{n+1} v_i w_i - v_{n+2} w_{n+2}.$$

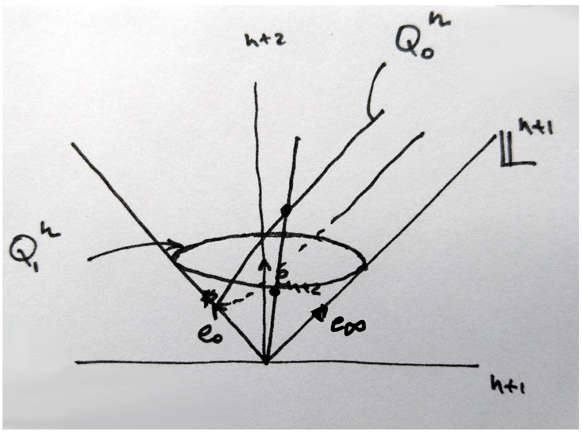
Definition 10.3.1. The set

$$\mathbb{L}^{n+1} := \quad v \in \mathbb{R}^{n+1,1} \mid \langle v, v\rangle = 0\}$$

is called the *light cone*, and

$$\mathcal{Q} := \mathrm{P}(\mathbb{L}^{n+1}) = \quad [v] \in \mathbb{R}\mathrm{P}^{n+1} \mid \langle v, v\rangle = 0\}$$

is called the *Möbius quadric*.



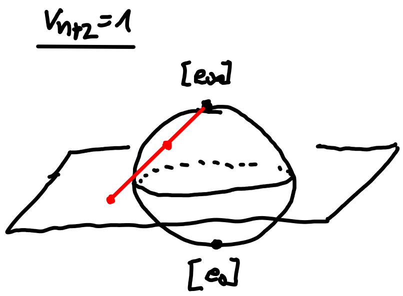


Figure 10.4. Different sections of the light cone lead to different models of Möbius geometry.

For $y \in \mathbb{R}^{n+1}$

$$\|y\|_{\mathbb{R}^{n+1}} = 1 \quad \Leftrightarrow \quad \langle (y,1),(y,1)\rangle = 0.$$

Thus, the unit sphere S^n and the quadric $\mathcal{Q}$ can be identified by choosing the following normalization

$$S^n \cong Q_1^n := \quad v \in \mathbb{L}^{n+1} \mid v_{n+2} = 1\},$$

since

$$v \in Q_1^n \quad \Leftrightarrow \quad v = (y,1) \text{ with } y \in S^n \subset \mathbb{R}^{n+1}, \text{ i.e. } \|y\|_{\mathbb{R}^{n+1}} = 1.$$

Let $e_1, \dots, e_n, e_{n+1}, e_{n+2}$ be an orthonormal basis of $\mathbb{R}^{n+1,1}$:

$$\langle e_i, e_j \rangle = \begin{cases} \delta_{ij}, & 1 \leqslant i \leqslant n+1 \\ -\delta_{ij}, & i = n+2 \end{cases}$$

Define

$$e_0 := \tfrac{1}{2}(e_{n+2} - e_{n+1}), \quad e_\infty := \tfrac{1}{2}(e_{n+2} + e_{n+1}).$$

Then

$$\langle e_0, e_0 \rangle = \langle e_\infty, e_\infty \rangle = 0, \quad \langle e_0, e_\infty \rangle = -\tfrac{1}{2},$$

$-\frac{1}{2} = \langle x + v_0 e_0 + v_\infty e_\infty, e_\infty \rangle = -\frac{1}{2} v_0 \iff v_0 = 1$

and we can indentify $\mathbb{R}^n \cup \{\infty\}$ and the qudaric $\mathcal{Q}$ by the following normalization

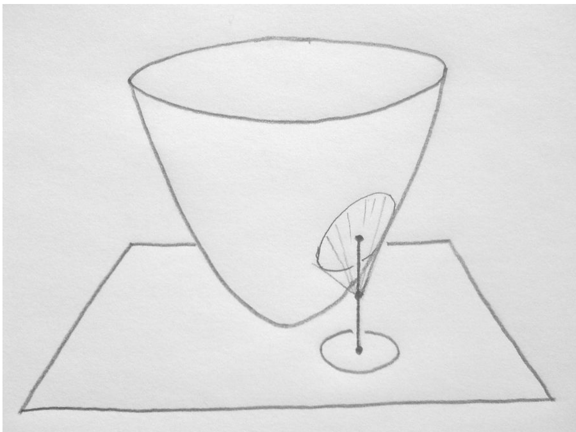
$$\mathbb{R}^n \cong Q_0^n := \left\{ v \in \mathbb{L}^{n+1} \,\middle|\, \langle v, e_\infty \rangle = -\tfrac{1}{2} \right\} = \left\{ v \in \mathbb{L}^{n+1} \,\middle|\, v_{n+2} - v_{n+1} = 1 \right\},$$

i.e., the e_0-th component is normalized to be equal to 1. We find

$0 = \langle x + e_0 + v_\infty e_\infty, x + e_0 + v_\infty e_\infty \rangle = \|x\|^2 - \frac{1}{2} v_\infty - \frac{1}{2} v_\infty \iff v_\infty = \|x\|^2$

$$v \in Q_0^n \quad \Leftrightarrow \quad v = x + e_0 + \|x\|^2 e_\infty \text{ with } x \in \mathbb{R}^n.$$

The point $\infty \in \mathbb{R}^n \cup \{\infty\}$ is indentified with e_∞, which is the only point of $\mathbb{L}^{n+1}$ with the e_0-th component equal to 0. In this way $\mathbb{R}^n$ is modeled as a paraboloid in an $(n+1)$-dimensional affine space (see Figure 10.5): $e_0 + \operatorname{span}\{e_1, \dots, e_n, e_\infty\}$.



$v_0 = 1$

Figure 10.5. Paraboloid model of Möbius geometry.

Theorem 10.3.2. *In the projective model of Möbius geometry the points are identified with the points of the quadric*

$$\mathcal{Q} = \left\{ [v] \in \mathrm{P}(\mathbb{R}^{n+1,1}) \,\middle|\, \langle v, v \rangle = 0 \right\}.$$

The identification $\mathcal{Q} \leftrightarrow S^n$ with the spherical model is given by the coordinate normalization

$$[\hat{y}] \in \mathcal{Q} \quad \leftrightarrow \quad \hat{y} = y + e_{n+2} \in Q_1^n \quad \leftrightarrow \quad y \in S^n$$

The identification $\mathcal{Q} \leftrightarrow \mathbb{R}^n \cup \{\infty\}$ with the Euclidean model is given by the coordinate normalization

$$[\hat{x}] \in \mathcal{Q} \backslash \{[e_\infty]\} \quad \leftrightarrow \quad \hat{x} = x + e_0 + \|x\|^2 e_\infty \in Q_0^n \quad \leftrightarrow \quad x \in \mathbb{R}^n.$$

The corresponding map

$$S^n \to \mathbb{R}^n \cup \{\infty\}, \quad y \mapsto x$$

given by the identification of Q_1^n and Q_0^n along the straight line generators of $\mathbb{L}^{n+1}$

$$\hat{y} \mapsto \hat{x} \quad \textit{with} \quad [\hat{y}] = [\hat{x}]$$

is the stereographic projection.

Proof. The identification is given by the formula

$$[\hat{x} = x + e_0 + \|x\|^2 e_\infty] = [\hat{y} = y + e_{n+1}]$$

with $x \in \mathbb{R}^n$ and $y \in S^n$. We write

$$\hat{y} = y + e_{n+1} = \tilde{y} + y_{n+1}e_{n+1} + e_{n+2} = \tilde{y} + (1 - y_{n+1})e_0 + (1 + y_{n+1})e_\infty$$

with $\tilde{y} \in \mathbb{R}^n$. Then $\hat{x} = \lambda\hat{y}$ implies

$$\lambda = \frac{1}{1 - y_{n+1}}.$$

Thus,

$$x = (x_1, \ldots, x_n) = \frac{y}{1 - y_{n+1}} = \frac{1}{1 - y_{n+1}}(y_1, \ldots, y_n)$$

and

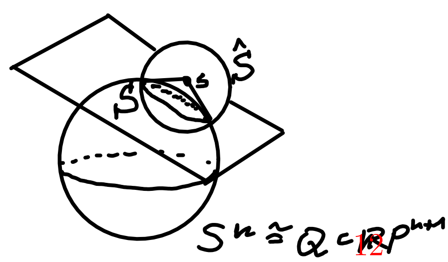
$$\|x\|^2_{\mathbb{R}^n} = \frac{1 + y_{n+1}}{1 - y_{n+1}} \quad \Leftrightarrow \quad \|y\|_{\mathbb{R}^{n+1}} = 1.$$

□

10.4 Spheres in the projective model

Consider the spherical model of Möbius geometry. Let $s \in \mathbb{R}^{n+1}$, $\|s\| > 1$ be the center of the sphere $\hat{S} \perp S^n$. We identified spheres $S = \hat{S} \cap S^n$ in S^n with such points s, and thus with elements

$$[(s, 1)] \in \mathrm{P}(\mathbb{R}^{n+1,1}).$$



These points build the exterior of the quadric $\mathcal{Q}$:

$$\mathcal{Q}_+ := \quad [v] \in \mathrm{P}(\mathbb{R}^{n+1,1}) \,\big|\, \langle v, v \rangle > 0 \big\} \,.$$

Indeed, we have

$$\langle (s,1),(s,1) \rangle = \|s\|^2_{\mathbb{R}^{n+1}} - 1 > 0$$

Thus, the spheres in $S^n \cong \mathbb{R}^n \cup \{\infty\}$ are identified with the exterior $\mathcal{Q}_+$ of the quadric $\mathcal{Q}$.

The identification with the spherical model is given by the normalization

$$Q^n_{1,+} := \quad v \in \mathbb{R}^{n+1,1} \,\big|\, \langle v, v \rangle > 0,\ v_{n+2} = 1 \big\} \,.$$

Thus, $v = \hat{s} = s + e_{n+2}$ where s is the center of the sphere $\hat{S}$ orthogonal to S^n.

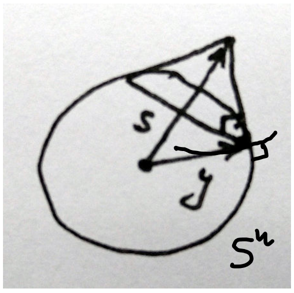


Figure 10.6. Spheres as polar planes of a point outside the Möbius quadric.

Proposition 10.4.1. *Points on the sphere $S \subset S^n$ corresponding to $[\hat{s}] \in \mathcal{Q}_+$ with $\hat{s} \in Q^n_{1,+}$ are given by*

$$\mathcal{Q} \cap [\hat{s}]^\perp$$

i.e., points $[v] \in \mathcal{Q}$ with

$$\langle \hat{s}, v \rangle = 0.$$

Proof. With $\hat{s} = s + e_{n+2}$ and $\hat{y} = y + e_{n+2}$ we find

$$\langle \hat{s}, \hat{y} \rangle = \langle s, y \rangle_{\mathbb{R}^{n+1}} - 1 = 0 \quad \Leftrightarrow \quad \langle s, y \rangle_{\mathbb{R}^{n+1}} = 1$$

□

Remark 10.4.2. The points $[\hat{s}] \in \mathcal{Q}_+$, $\hat{s} \in Q^n_{1,+}$ with the last coordinate equal to 0 correspond to great hyperspheres.

The identification with the Euclidean model is given by the section

$$Q^n_{0,+} := \quad v \in \mathbb{R}^{n+1,1} \,\big|\, \langle v, v\rangle > 0,\ \langle e_\infty, v\rangle = -\tfrac{1}{2}\}$$

This implies the following general form

$$\hat{s} = c + e_0 + (\|c\|^2 - r^2)e_\infty, \quad c \in \mathbb{R}^{n+1}, r > 0$$

Proposition 10.4.3. *Points on the sphere $S \subset \mathbb{R}^n$ corresponding to $[\hat{s}] \in \mathcal{Q}_+$ with $\hat{s} \in Q^n_{0,+}$ are given by*

$$\mathcal{Q} \cap [\hat{s}]^\perp$$

i.e., points $[v] \in \mathcal{Q}$ with

$$\langle \hat{s}, v\rangle = 0.$$

Proof. This follows from Theorem 10.3.2 and Proposition 10.4.1. We verify by computation anyway:

$$\begin{aligned} \langle \hat{x}, \hat{s}\rangle &= \langle x + e_0 + \|x\|^2 e_\infty, c + e_0 + (\|c\|^2 - r^2)e_\infty\rangle \\ &= \langle x, c\rangle_{\mathbb{R}^n} - \tfrac{1}{2}(\|c\|^2 - r^2) - \tfrac{1}{2}\|x\|^2 = 0 \\ \Leftrightarrow \quad & \|x - c\|^2 = r^2. \end{aligned}$$

□

Remark 10.4.4. The points $[\hat{s}] \in \mathcal{Q}_+$, $\hat{s} \in Q^n_{0,+}$ with e_0-component equal to 0, i.e.,

$$\hat{s} = v + 2\langle v, a\rangle_{\mathbb{R}^n} e_\infty,$$

correspond to hyperplanes in $\mathbb{R}^n$. Indeed,

$$\begin{aligned} \langle \hat{s}, \hat{x}\rangle &= \langle v + 2\langle v, a\rangle_{\mathbb{R}^n} e_\infty, x + e_0 + \|x\|^2 e_\infty\rangle \\ &= \langle x, v\rangle_{\mathbb{R}^n} - \langle v, a\rangle_{\mathbb{R}^n} = 0 \\ \Leftrightarrow \quad & \langle x - a, v\rangle_{\mathbb{R}^n} = 0. \end{aligned}$$

We summarize in the following theorem:

Theorem 10.4.5. *In the projective model of Möbius geometry the spheres are identified with the exterior of the Möbius quadric*

$$\mathcal{Q}_+ = \quad [v] \in \mathrm{P}(\mathbb{R}^{n+1,1}) \,\big|\, \langle v, v\rangle > 0\}.$$

The incidence $x \in S$ of a point x lying on a sphere S is given by polarity

$$\langle \hat{s}, \hat{x} \rangle = 0$$

for $[\hat{x}] \in \mathcal{Q}$, $[\hat{s}] \in \mathcal{Q}_+$.

The identification with the spherical model S^n is given by the normalization of the e_{n+2}-component:

$$\begin{aligned} \hat{s} = s + e_{n+2} \quad &\leftrightarrow \quad \text{hyperspheres}, \\ \hat{s} = s + 0 \cdot e_{n+2} \quad &\leftrightarrow \quad \text{great hyperspheres}. \end{aligned}$$

The identification with the Euclidean model $\mathbb{R}^n \cup \{\infty\}$ is given by the normalization of the e_0-component:

$$\begin{aligned} \hat{s} = c + e_0 + (\|c\|^2 - r^2)e_\infty \quad &\leftrightarrow \quad \text{hyperspheres with center } c \in \mathbb{R}^n \text{ and radius } r > 0, \\ \hat{s} = v + \langle v, a \rangle_{\mathbb{R}^n} e_\infty \quad &\leftrightarrow \quad \text{hyperplanes through } a \in \mathbb{R}^n \text{ and normal vector } v \in \mathbb{R}^n. \end{aligned}$$

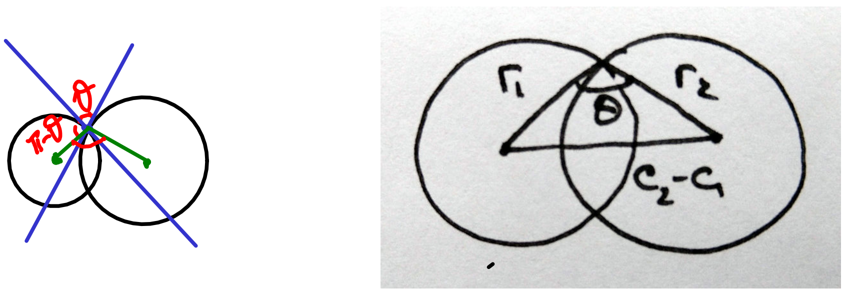


Figure 10.7. Intersection angle of two spheres.

Proposition 10.4.6. *In the spherical model and in the Euclidean model, two intersecting spheres corresponding to the two points $[\hat{s}_1], [\hat{s}_2] \in \mathcal{Q}_+$ intersect at an angle θ (defined up to $\theta \to \pi - \theta$) given by*

$$\cos^2 \theta = \frac{\langle \hat{s}_1, \hat{s}_2 \rangle^2}{\langle \hat{s}_1, \hat{s}_1 \rangle \langle \hat{s}_2, \hat{s}_2 \rangle}.$$

Proof. The formula is well-defined for projective elements. We start with the Euclidean model:

$$\hat{s}_i = c_i + e_0 + (\|c_i\|^2 - r_i^2)e_\infty, \quad i = 1, 2.$$

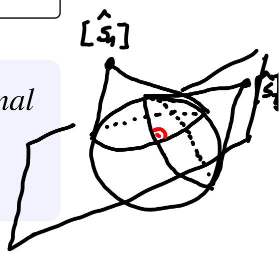
Then

$$\begin{aligned}\langle \hat{s}_1, \hat{s}_2 \rangle &= \langle c_1 + e_0 + (\|c_1\|^2 - r_1^2)e_\infty, c_2 + e_0 + (\|c_2\|^2 - r_2^2)e_\infty \rangle \\ &= \langle c_1, c_2 \rangle_{\mathbb{R}^n} - \tfrac{1}{2}(\|c_1\|^2 - r_1^2 + \|c_2\|^2 - r_2^2) \\ &= -\tfrac{1}{2}(\|c_1 - c_2\|^2 - r_1^2 - r_2^2),\end{aligned}$$

and

$$\langle \hat{s}_i, \hat{s}_i \rangle = \|c_i\|^2 - (\|c_i\|^2 - r^2) = r^2.$$

Now the formula follows from the cosine rule. Similarly, if either or both of $\hat{s}_i$ represent a hyperplane. The spherical and Euclidean model are related by stereographic projection, which is a conformal map. Thus the claim also holds for the spherical model by Corollary 10.1.8. □



Corollary 10.4.7. *Two hyperspheres corresponding to* $[\hat{s}_1], [\hat{s}_2] \in \mathcal{Q}_+$ *are orthogonal if and only if*

$$\langle \hat{s}_1, \hat{s}_2 \rangle = 0.$$

Definition 10.4.8. A family of hyperspheres corresponding to the points on a line in the projective model of Möbius geometry is called a *pencil of hyperspheres*.

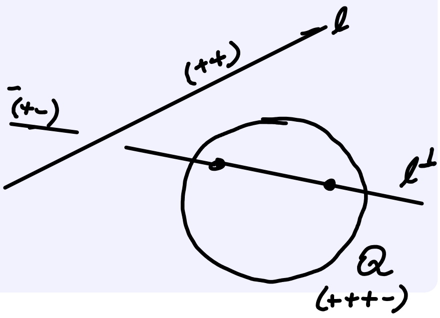
Pencils of spheres are classified by the relative location of the line with respect to the Möbius quadric.

Definition 10.4.9. Let $g \subset \mathrm{P}(\mathbb{R}^{n+1,1})$ be a line in the projective model of Möbius geometry.

(i) g *elliptic* $:\Leftrightarrow \langle \cdot, \cdot \rangle|_g$ has signature $(++)$

(ii) g *hyperbolic* $:\Leftrightarrow \langle \cdot, \cdot \rangle|_g$ has signature $(+-)$

(iii) g *parabolic* $:\Leftrightarrow \langle \cdot, \cdot \rangle|_g$ has signature $(+0)$



Proposition 10.4.10. *Elliptic, parabolic, and hyperbolic pencils are characterized by the following properties: Let* $g \subset \mathrm{P}(\mathbb{R}^{n+1,1})$ *be a line in the projective model of Möbius geometry. Then*

(i) g elliptic $\Leftrightarrow$ *all spheres of* g *intersect*

(ii) g hyperbolic $\Leftrightarrow$ *all spheres of* g *are disjoint*

(iii) g parabolic $\Leftrightarrow$ *all spheres of* g *touch in a common point*

Proof. We prove (i). The other claims are proven analogously.

The signature of g is $(++)$ if and only if g does not intersect $\mathcal{Q}$. In this case, $g^\perp$ intersects $\mathcal{Q}$. But all spheres of the pencil g contain all points of $\mathcal{Q} \cap g^\perp$. □

In the case of planar Möbius geometry ($n = 2$), the polar of a line g is another line $g^\perp$, and thus defines another pencil of circles, containing all circles that are orthogonal to all circles of g (see Figure 10.8).

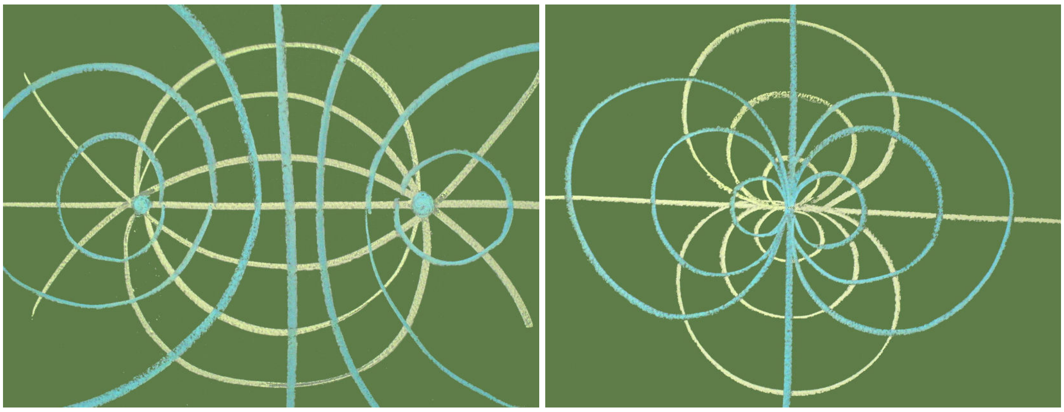


Figure 10.8. *Left:* An orthogonal pair of elliptic and hyperbolic pencils of circles. *Right:* An orthogonal pair of two parabolic pencils of circles.

Corollary 10.4.11.

(i) A hyperbolic pencil of circles consists of all circles that are orthogonal to an elliptic pencil of circles, and vice versa.

(ii) A parabolic pencil of circles consists of all circles that are orthogonal to another parabolic pencil of circles.

10.5 Möbius transformation group

By Theorem 10.1.5, Möbius transformations are characterized by the properties of mapping hyperspheres in $\mathbb{R}^n \cup \{\infty\}$ to hyperspheres in $\mathbb{R}^n \cup \{\infty\}$. In the projective model of Möbius geometry, hyperspheres are represented by sections of the Möbius quadric $\mathcal{Q}$ with hyperplanes of $\mathbb{RP}^{n+1}$. Thus, the projective transformations of $\mathbb{RP}^{n+1}$ that map $\mathcal{Q}$ to itself maps hyperspheres to hyperspheres. Hence $\mathrm{PO}(n+1,1) \subset \mathrm{Mob}(n)$. In this section we use this as an alternative definition for Möbius transformations and then, in Theorem 10.5.3, show that both definitions coincide.

Week 2: Möbius transformations, hyperbolic model, associated points, Miquel's theorem

10.5 Möbius transformation group

By Theorem 10.1.5, Möbius transformations are characterized by the properties of mapping hyperspheres in $\mathbb{R}^n \cup \{\infty\}$ to hyperspheres in $\mathbb{R}^n \cup \{\infty\}$. In the projective model of Möbius geometry, hyperspheres are represented by sections of the Möbius quadric $\mathcal{Q}$ with hyperplanes of $\mathbb{RP}^{n+1}$. Thus, the projective transformations of $\mathbb{RP}^{n+1}$ that map $\mathcal{Q}$ to itself maps hyperspheres to hyperspheres. Hence $\mathrm{PO}(n+1,1) \subset \mathrm{Mob}(n)$. In this section we use this as an alternative definition for Möbius transformations and then, in Theorem 10.5.3, show that both definitions coincide.

Definition 10.5.1. *Möbius transformations* are projective transformations

$$\tau : \mathrm{P}(\mathbb{R}^{n+1,1}) \to \mathrm{P}(\mathbb{R}^{n+1,1})$$

that preserve the Möbius quadric:

$$\tau(\mathcal{Q}) = \mathcal{Q}.$$

By Theorem 7.6.2, a projective transformation τ that maps $\mathcal{Q}$ to itself is a porjective orthogonal transformation, i.e.,

$$\tau \in \mathrm{PO}(n+1,1).$$

Thus, it comes from a linear map

$$T : \mathbb{R}^{n+1,1} \to \mathbb{R}^{n+1,1}$$

which is orthogonal with respect to the Lorentz product $\langle\cdot,\cdot\rangle$, i.e.,

$$T \in \mathrm{O}(n+1,1).$$

In the basis $e_0, e_1, \dots, e_n, e_\infty$ of $\mathbb{R}^{n,1}$ the orthogonality condition for the matrix T reads

$$T^\intercal E T = E$$

with

$$E = (\langle e_i, e_j \rangle)_{i,j=0,1,\dots,n,\infty} = \left(\begin{array}{c|c|c} 0 & 0 & -\frac{1}{2} \\ \hline 0 & I_n & 0 \\ \hline -\frac{1}{2} & 0 & 0 \end{array}\right)$$

Example 10.5.2. We use the Euclidean normalization of Möbius geometry, i.e., points $x \in \mathbb{R}^n$ are represented by

$$\hat{x} = x + e_0 + \|x\|^2 e_\infty = \begin{pmatrix} 1 \\ x \\ \|x\|^2 \end{pmatrix}$$

in the basis $e_0, e_1, \dots, e_n, e_\infty$.

(i) **inversion in the unit sphere**

$$x \mapsto \frac{x}{\|x\|^2}.$$

In the Euclidean normalization of Möbius geometry we obtain

$$\begin{pmatrix} 1 \\ x \\ \|x\|^2 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ \frac{x}{\|x\|^2} \\ \frac{1}{\|x\|^2} \end{pmatrix} \sim \begin{pmatrix} \|x\|^2 \\ x \\ 1 \end{pmatrix}$$

Thus, it can be represented by the matrix

$$A = \left(\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & I & 0 \\ \hline 1 & 0 & 0 \end{array}\right),$$

which satisfies

$$A^\intercal E A = E.$$

(ii) **Euclidean motions**

$$x \mapsto Rx + r \quad \text{with } r \in \mathbb{R}^n, R \in \mathrm{O}(n), \text{i.e., } R^\intercal R = I.$$

In the Euclidean normalization of Möbius geometry we obtain

$$\begin{pmatrix} 1 \\ x \\ \|x\|^2 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ Rx+r \\ \|x\|^2+2r^\intercal Rx+\|r\|^2 \end{pmatrix}.$$

Thus, it can be represented by the matrix

$$B = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline r & R & 0 \\ \hline \|r\|^2 & 2r^\intercal R & 1 \end{array}\right),$$

which satisfies

$$B^\intercal E B = E.$$

(iii) **Scaling**

$$x \mapsto \lambda x \quad \text{withe } \lambda \in \mathbb{R}\backslash\{0\}.$$

In the Euclidean normalization of Möbius geometry we obtain

$$\begin{pmatrix} 1 \\ x \\ \|x\|^2 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ \lambda x \\ \lambda^2\|x\|^2 \end{pmatrix} \sim \begin{pmatrix} \frac{1}{\lambda} \\ x \\ \lambda\|x\|^2 \end{pmatrix}$$

Thus, it can be represented by the matrix

$$C = \left(\begin{array}{c|c|c} \frac{1}{\lambda} & 0 & 0 \\ \hline 0 & I & 0 \\ \hline 0 & 0 & \lambda \end{array}\right),$$

which satisfies

$$C^\intercal E C = E.$$

(iv) **Inversion in a sphere**
Let $\hat{s} \in \mathbb{R}^{n+1,1}$ be a space-like vector (representing a sphere). Consider the transformation $\mathbb{R}^{n+1,1} \to \mathbb{R}^{n+1,1}$

$$\hat{x} \mapsto \hat{x} - 2\frac{\langle \hat{x}, \hat{s}\rangle}{\langle \hat{s}, \hat{s}\rangle}\hat{s},$$

which is the reflection in the hyperplane polar to $[\hat{s}]$. This is an orthogonal transformation. We compute its Euclidean representation: With

$$\hat{x} = x + e_0 + \|x\|^2 e_\infty, \quad \hat{s} = c + e_0 + \left(\|c\|^2 - r^2\right) e_\infty$$

we find

$$\langle \hat{s}, \hat{s}\rangle = r^2, \quad \langle \hat{s}, \hat{x}\rangle = -\frac{1}{2}\left(\|c - x\|^2 - r^2\right).$$

Thus,

$$\begin{pmatrix} 1 \\ x \\ \|x\|^2 \end{pmatrix} \mapsto \begin{pmatrix} 1-2\frac{\langle \hat{x},\hat{s}\rangle}{\langle \hat{s},\hat{s}\rangle} \\ x-2\frac{\langle \hat{x},\hat{s}\rangle}{\langle \hat{s},\hat{s}\rangle}c \\ * \end{pmatrix} = \begin{pmatrix} 1+\frac{\|c-x\|^2}{r^2}-1 \\ x+\left(\frac{\|c-x\|^2}{r^2}-1\right)c \\ * \end{pmatrix} \sim \begin{pmatrix} 1 \\ c+\frac{r^2}{\|c-x\|^2}(x-c) \\ * \end{pmatrix},$$

which is inversion in the sphere with center c and radius r.

Theorem 10.5.3. *Definition 10.1.3 and Definition 10.5.1 of Möbius transformations are equivalent:*

$$\mathrm{Mob}(n) = \mathrm{PO}(n+1, 1).$$

Proof. By Theorem 10.1.5, we have seen that $\mathrm{PO}(n+1,1) \subset \mathrm{Mob}(n)$. Vice versa, in Example 10.5.2 (iv), we have seen how inversion in a sphere can be described by a projective orthogonal transformation, and thus, $\mathrm{Mob}(n) \subset \mathrm{PO}(n+1,1)$. □

Thus, we eventually arrive at the following correspondences:

$$\begin{array}{rcl} \underline{\text{elementary model}} & & \underline{\text{projective model}} \\ \mathbb{R}^n \cup \{\infty\} & \longleftrightarrow & S^n \cong \mathcal{Q} \subset \mathbb{RP}^{n+1} \\ \mathrm{Mob}(n) & \longleftrightarrow & \mathrm{PO}(n+1,1) \end{array}$$

$$\text{hypersphere} \subset \mathbb{R}^n \cup \{\infty\} \longleftrightarrow \text{hyperplane} \subset \mathbb{RP}^{n+1} \text{ intersecting } \mathcal{Q} \overset{\text{polarity}}{\longleftrightarrow} \text{point outside } \mathcal{Q}$$

10.6 The hyperbolic model of Möbius geometry

While the spherical model of Möbius geometry can be obtained by normalizing the e_{n+2}-component, a hyperbolic model can be obtained by normalizing the e_{n+1}-component:

$$H^n \cong Q^n_{-1} := \left\{ v \in \mathbb{L}^{n+1} \,\middle|\, v_{n+1} = 1,\ v_{n+2} > 0 \right\}.$$

Indeed, for

$$v \in Q^n_{-1}$$

we find

$$v = z_{1\dots n} + e_{n+1} + z_{n+1}e_{n+2} = z_1 e_1 + \cdots z_n e_n + e_{n+1} + z_{n+1}e_{n+2}$$

with

$$z = (z_{1\dots n}, z_{n+1}) = (z_1, \dots, z_{n+1}) \in \mathbb{R}^{n,1}$$

satisfying

$$\langle z, z \rangle_{\mathrm{h}} = -1, \quad z_{n+1} > 0,$$

where $\langle \cdot, \cdot \rangle_{\mathrm{h}}$ is the Lorentz product on $\mathbb{R}^{n,1}$. Thus,

$$z \in H^n.$$

Note, that due the condition $v_{n+2} > 0$ the set Q^n_{-1} describes only one sheet of a two-sheeted hyperboloid. To recover the whole Möbius quadric $\mathcal{Q}$ the other sheet

$$-Q^n_{-1} \cong -H^n$$

needs to be added, and both sheets glued together at

$$\left\{ v \in \mathbb{L}^{n+1} \,\middle|\, v_{n+1} = 0 \right\} \cong \partial H^n.$$

Now we can add the following to Theorem 10.3.2:

Theorem 10.6.1. *The identification* $\mathcal{Q} \leftrightarrow H^n \cup \partial H^n \cup -H^n$ *with the hyperbolic model is given by the coordinate normalization*

$$
\begin{aligned}
[\hat{z}] \in \mathcal{Q}\backslash[e_{n+1}]^\perp \quad &\leftrightarrow \quad \hat{z} = z_{1\dots n} + e_{n+1} + z_{n+1}e_{n+2} \in Q^n_{-1} \cup -Q^n_{-1} \\
&\leftrightarrow \quad z = (z_{1\dots n}, z_{n+1}) = (z_1, \dots, z_{n+1}) \in H^n \cup -H^n.
\end{aligned}
$$

The corresponding map

$$H^n \to S^n, \quad z \mapsto y$$

given by the identification of Q^n_{-1} *and* Q^n_1 *along the straight line generators of* $\mathbb{L}^{n+1}$

$$\hat{z} \mapsto \hat{y} \quad \textit{with} \quad [\hat{z}] = [\hat{y}]$$

yields the hemisphere model of hyperbolic space, while the corresponding map

$$H^n \to \mathbb{R}^n, \quad z \mapsto x$$

given by the identification of Q^n_{-1} *and* Q^n_0 *along the straight line generators of* $\mathbb{L}^{n+1}$

$$\hat{z} \mapsto \hat{x} \quad \textit{with} \quad [\hat{z}] = [\hat{x}]$$

yields the Poincaré ball model of hyperbolic space.

Same as in the Euclidean and spherical models in the hyperbolic model the hyperbolic hyperspheres of H^n correspond to hyperplanar sections of the Möbius quadric $\mathcal{Q}$ which, by polarity, we identify with the points outside the Möbius quadric $\mathcal{Q}_+$. For these points we employ the normalization

$$Q^n_{-1,+} := \left\{ v \in \mathbb{R}^{n+1,1} \,\middle|\, \langle v, v \rangle > 0, v_{n+1} = 1 \right\}.$$

Theorem 10.6.2. *In the hyperbolic model of Möbius geometry a hyperplanar section*

$$\mathcal{Q} \cap [\hat{s}]^\perp, \quad [\hat{s}] \in \mathcal{Q}_+$$

corresponds to a hyperbolic sphere. The identification with the pole $[\hat{s}]$ *is given by the* *normalization of the e_{n+1}-component:*

$$\hat{s} = \tfrac{1}{\cosh r} c_{1\ldots n} + e_{n+1} + \tfrac{c_{n+1}}{\cosh r} e_{n+2} \quad \leftrightarrow \quad \text{hypersphere with center } c \in H^n \text{ and radius } r > 0,$$

$$\hat{s} = \tfrac{\pm 1}{\sinh r} c_{1\ldots n} + e_{n+1} \pm \tfrac{c_{n+1}}{\sinh r} e_{n+2} \quad \leftrightarrow \quad \text{hypersurface of constant distance to the hyperplane } \langle c, z \rangle_h = 0, \langle c, c \rangle_h = 1$$

$$\hat{s} = c_{1\ldots n} + e_{n+1} \pm c_{n+1} e_{n+2} \quad \leftrightarrow \quad \text{horosphere with center} c, \langle c, c \rangle_h = 0$$

$$\hat{s} = c_{1\ldots n} + 0 \cdot e_{n+1} c_{n+1} e_{n+2} \quad \leftrightarrow \quad \text{hyperplane } \langle c, z \rangle_h = 0$$

Proof. For the first case we find

$$\begin{aligned} & \langle \hat{s}, \hat{z} \rangle = \tfrac{1}{\cosh r} \langle c, z \rangle_{\mathrm{h}} + 1 = 0 \\ \Leftrightarrow \quad & \langle c, z \rangle_{\mathrm{h}} = -\cosh r, \end{aligned}$$

which is the equation for hyperbolic sphere. □

10.7 Relation between Möbius and other geometries

Möbius geometry deals with properties of figures in $S^n \subset \mathbb{R}\mathrm{P}^{n+1}$ that are invariant under the group $PO(n+1,1)$ of projective transformations of $\mathbb{R}\mathrm{P}^{n+1}$ that map $S^n \to S^n$. Thus, n-dimensional Möbius geometry is a subgeometry of $(n+1)$-dimensional projective geometry. The same group, $PO(n+1,1)$, also maps B^{n+1} (the inside of S^n) to itself. This gives the Klein model of $(n+1)$-dimensional hyperbolic geometry. So n-dimensional Möbius geometry can be seen as the geometry of the points in the ideal boundary of $(n+1)$-dimensional hyperbolic space.

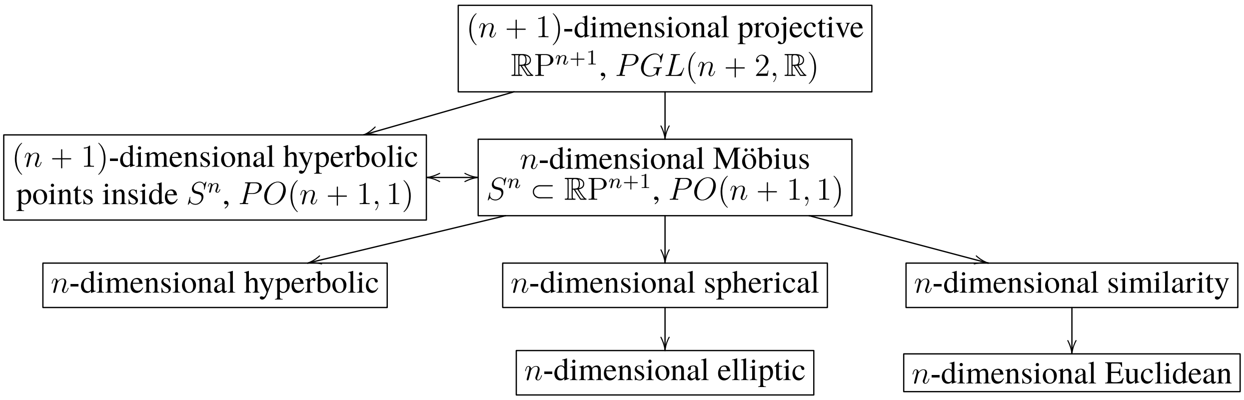
For a point $P = [p] \in \mathbb{R}\mathrm{P}^n$, let G_P be the subgroup of $PO(n+1,1)$ consisting of all projective transformations that map $P \mapsto P$ (in addition to mapping $S^n \to S^n$). These also map the polar plane of P to itself.

If P is outside S^n, then the polar plane intersects B^{n+1}, and the geometry of this intersection with the group G_P is n-dimensional hyperbolic geometry.

If P is the center of S^n, then the polar plane is the plane at infinity, so G_P is the group of affine transformations mapping S^n to itself. This is the group of orthogonal transformations. So the space S^n with the group G_P is n-dimensional spherical geometry. If P is any other point inside S^n, one obtains a Möbius geometrically equivalent model for n-dimensional spherical geometry.

If P is the north pole of S^n, then G_P corresponds (via stereographic projection) to the Möbius transformations of $\mathbb{R}^n \cup \{\infty\}$ that fix ∞. These are the similarity transformations. Thus, S^n with G_P is a model for n-dimensional similarity geometry. If P is any other point in S^n, one obtains a Möbius geometrically equivalent model of similarity geometry.

If $P \in S^n$, the group G_P consists of all projective transformations that come from orthogonal maps $A \in O(n+1,1)$ with $Ap = \lambda p$ for some $\lambda \in \mathbb{R}\backslash\{0\}$. Because p is a lightlike vector λ is not always equal to ± 1. (For example consider the orthogonal transformations that correspond to scalings in $\mathbb{R}^n \cup \{\infty\}$) If instead of G_P, one considers the (projectivized) group of all $A \in O(n+1,1)$ with $Ap = p$, then one obtains a model for n-dimensional Euclidean geometry.



7.5 Associated points

The span of three quadrics $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ in $\mathbb{R}P^3$ (not belonging to a common pencil) is a linear system of quadrics of dimension 2. Its base points are the points common to all conics from the pencil and given by the intersection of any three of them (not belonging to a common pencil). The intersection of three quadrics in $\mathbb{R}P^3$ consists of at most eight points. On the other hand, the family of all quadrics through 7 points in general position already constitutes a linear system of quadrics of dimension 2.

Theorem 7.5.1 (associated points)**.** *Given eight distinct points which are the set of intersections of three quadrics in* $\mathbb{R}P^3$*, all quadrics through any subset of seven of the points must pass through the eighth point. Such sets of points are called* associated points.

Proof. Let $A_1, A_2, \ldots, A_8$ be the set of intersections of three quadrics $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$. Note that no three of the eight points A_k can be collinear, since otherwise the set of intersection of the three quadrics would contain a whole line and not just eight points. For similar reasons no five of the eight points A_k can be coplanar. Indeed, five coplanar points no three of which are collinear determine a unique conic. The intersection of the three quadrics $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ would contain this conic and not just eight points.

Choose any subset of seven points $A_1, A_2, \ldots, A_7$. We show that any quadric $\mathcal{Q}$ through these seven points must belong to the family

$$\mathcal{Q}_1 \vee \mathcal{Q}_2 \vee \mathcal{Q}_3.$$

As a consequence, the eighth intersection point A_8 will automatically lie on $\mathcal{Q}$. Suppose that, on the contrary, $\mathcal{Q}$ is linearly independent of $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$. Consider the family of quadrics

$$\mathcal{P} = \mathcal{Q}_1 \vee \mathcal{Q}_2 \vee \mathcal{Q}_3 \vee \mathcal{Q}.$$

Due to the assumed linear independence, one could find a quadric in this family through any prescribed triple of points in $\mathbb{R}P^3$. We show that this would lead to a contradiction.

First assume that no four points among $A_1, A_2, \ldots, A_7$ are coplanar. Choose three points B, C, D in the plane of A_1, A_2, A_3 so that the six points B, C, D, A_1, A_2, A_3 do not lie on a conic. Find a quadric $\mathcal{Q}'$ in the family $\mathcal{P}$ through B, C, D. This quadric must be reducible, one component being the plane of A_1, A_2, A_3 (indeed, otherwise $\mathcal{Q}'$ would cut this plane in a conic through A_1, A_2, A_3, B, C, D, which contradicts the choice of B, C, D). The other component of $\mathcal{Q}'$ must be a plane containing four points A_4, A_5, A_6, A_7, a contradiction.

The remaining case, when there are four coplanar points among $A_1, A_2, \ldots, A_7$, is dealt with analogously. Let A_1, A_2, A_3 and A_4 be coplanar. Denote the plane through these four points by Π. Take two points B, C in the plane Π so that the six points A_1, A_2, A_3, A_4, B, C do not lie on a conic, and take a point D not coplanar with A_5, A_6, A_7 (which is always possible, because the latter three points are not collinear). Then there exists a quadric $\mathcal{Q}'$ in the family $\mathcal{P}$ through B, C, D. Again, this quadric must be reducible, consisting of two planes, one of them being the plane Π. The other component of $\mathcal{Q}$ must be a plane containing A_5, A_6, A_7, D, a contradiction again (this time to the choice of D). □

Theorem 7.5.2 (Miquel's theorem on quadrics)**.** *Let $\mathcal{Q}$ be a quadric in $\mathbb{R}\mathrm{P}^3$ of rank 3 or 4. Let $x, x_1, x_2, x_3, x_{12}, x_{23}, x_{13}, x_{123} \in \mathcal{Q}$ be eight points of a combinatorial cube (see Figure 7.7), such that five of its faces are coplanar and no two planes coincide. Then its sixth face is coplanar as well.*

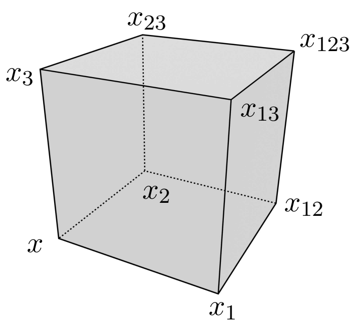


Figure 7.7. Combinatorial cube.

Proof. For $\{i, j, k\} = \{1, 2, 3\}$, $i < j$ define the six planes

$$\Pi^{ij} = x \vee x_i \vee x_j, \quad \Pi^{ij}_k = x_k \vee x_{ik} \vee x_{jk}.$$

The five plane $\Pi^{12}, \Pi^{23}, \Pi^{13}, \Pi^{23}_1, \Pi^{13}_2$ each contain one more of the eight point, and we need to show $x_{123} \in \Pi^{12}_3$.

Consider the two degenerate quadrics

$$\mathcal{Q}_1 = \Pi^{23} \cup \Pi^{23}_1, \quad \mathcal{Q}_2 = \Pi^{13} \cup \Pi^{13}_2.$$

Then, since $\mathcal{Q}$ does not contain any planes, the eight points are exactly the intersection

$$\mathcal{Q} \cap \mathcal{Q}_1 \cap \mathcal{Q}_2.$$

The degenerate quadric

$$\mathcal{Q}_3 = \Pi^{12} \cup \Pi_3^{12}$$

contains seven of the eight points, and therefore, by Theorem 7.5.1, also contains the eighth point x_{123}. This point must be contained in the plane Π_3^{12} since otherwise $x_{12} = x_{123}$. □

10.8 Miquel's theorem, Steiner's theorems and the four coins lemma

We present here a couple of incidence results in Möbius geometry, which turned out to be relevant for modern research.

Theorem 10.8.1 (Miquel)**.** *Given four points x, x_1, x_2, x_{12} on a circle, and four circles passing through each adjacent pair of points, the alternate intersections of these four circles at $x_3, x_{13}, x_{23}, x_{123}$ then lie on a common circle (see Figure 10.9).*

Proof. After stereographic projection to the sphere $S^2 \subset \mathbb{R}^3$ apply Theorem 7.5.2. □

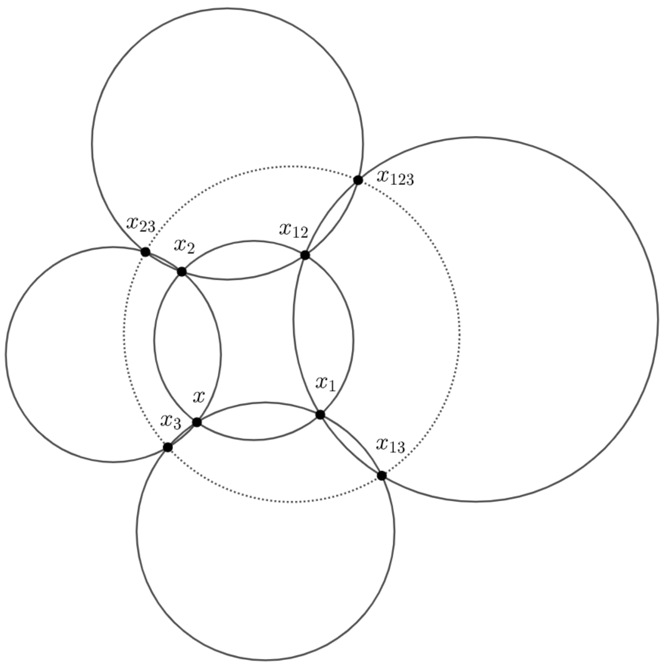


Figure 10.9. Miquel's six circle theorem.

This theorem is crucial for construction of multidimentional circular nets. The latter are maps $f : \mathbb{Z}^n \to \mathbb{R}^N$ where all elementary quadrilaterals are circular. Such nets are discrete analogs of orthogonal coordinate systems, see [BS08, BMS03]. The following formulation of Miquel's theorem is better suited to the construction of circular nets, and we give a more elementary proof.

Theorem 10.8.2 (Miquel's theorem for circular nets)**.** *Given seven points f, f_i, and f_{ij} $(1 \leqslant i < j \leqslant 3)$ in $\mathbb{R}^3$, such that each of the three quadruples (f, f_i, f_j, f_{ij}) is inscribed in a circle C_{ij}, define three new circles C'_{jk} as those passing through the triples (f_i, f_{ij}, f_{ik}), respectively. Then these new circles intersect at one point:*

$$f_{123} = C'_{23} \cap C'_{31} \cap C'_{12},$$

see Fig. 10.10.

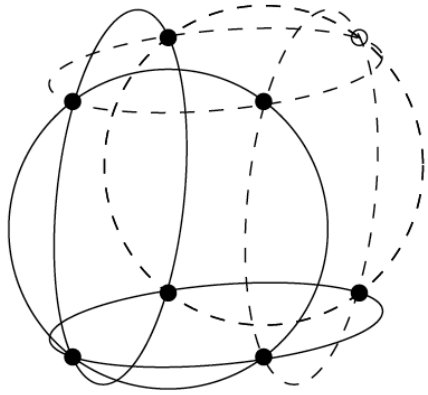


Figure 10.10. Miquel's incidence theorem in three-dimensional space

Proof. Under the conditions of the theorem, the seven points f, f_i, f_{ij} lie on some two-sphere S^2. Indeed, there is a unique sphere S^2 through the four points f, f_i. The circles C_{ij} through the triples (f, f_i, f_j) lie on S^2, and since $f_{ij} \in C_{ij}$, we find that $f_{ij} \in S^2$, as well. Under a stereographic projection of the sphere S^2, the picture becomes planar; see Figure 10.11 (left).

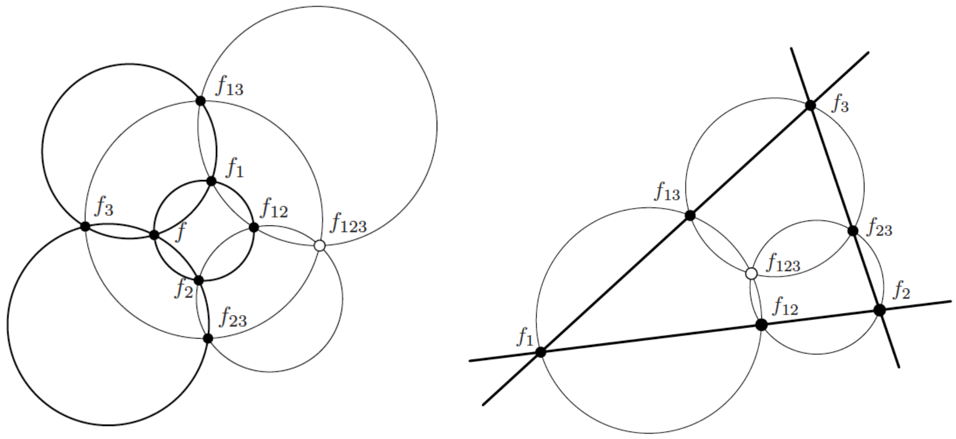


Figure 10.11. Miquel's incidence theorem in a plane: *Left:* general case, *Right:* with one vertex normalized to infinity.

After mapping f to infinity by a Möbius transformation, the circles C_{ij} become the straight lines $(f_i f_j)$; see Figure 10.11 (right). The claim of the theorem is then equivalent to the following claim.

Consider a triangle with the vertices f_1, f_2, f_3, and arbitrary points f_{ij} on each side $(f_i f_j)$. Then the three circles C'_{jk} through (f_i, f_{ij}, f_{ik}) intersect at one point f_{123}.

This result can be proven by elementary geometric methods. Denote the angles of the triangle $\triangle(f_1, f_2, f_3)$ by α_1, α_2, α_3, respectively. The circles C'_{23} through (f_1, f_{12}, f_{13}) and C'_{13} through (f_2, f_{12}, f_{23}) intersect at two points, one of them being f_{12}. Denote the second intersection point by f_{123}. We have to show that this point f_{123} belongs also to the circle C'_{12} through (f_3, f_{13}, f_{23}). For this, note that

$$\measuredangle(f_{12} f_{123} f_{13}) = \pi - \alpha_1, \qquad \measuredangle(f_{12} f_{123} f_{23}) = \pi - \alpha_2,$$

as it follows from the circularity of the quadrilaterals $(f_1, f_{12}, f_{123}, f_{13})$ and $(f_2, f_{12}, f_{123}, f_{23})$. As a consequence, we find:

$$\measuredangle(f_{13} f_{123} f_{23}) = 2\pi - (\pi - \alpha_1) - (\pi - \alpha_2) = \alpha_1 + \alpha_2 = \pi - \alpha_3,$$

and this yields that the quadrilateral $(f_3, f_{13}, f_{123}, f_{23})$ is also circular. □

Other elementary results concern touching spheres and touching circles in three-space.

Lemma 10.8.3. *Whenever four spheres in 3-space touch cyclically their points of contact lie on a circle.*

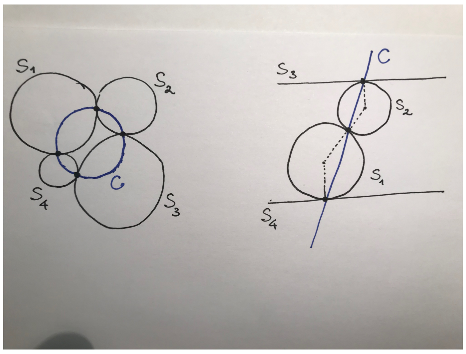


Figure 10.12. To the proof of the touching spheres lemma.

Proof. Mapping one of the touching points to infinity by a Möbius transformation we obtain two parallel planes with two touching spheres between them, see Fig. 10.12. An elementary geometric consideration shows that the points of contact of the spheres with the planes and of the spheres lie on a line. □

We see also that the circle intersects all the spheres at the same angle. A specially interesting is the case when this intersection is orthogonal. The centers of four cyclically touching spheres with an orthogonal circle built a degenerate case of Brianchon's theorem shown in Fig. 6.28.

Two circles touch in three space if they have the same tangent line in the point of contact.

Lemma 10.8.4 (Touching Coins Lemma)**.** *Whenever four circles in 3-space touch cyclically but do not lie on a common sphere, they intersect the sphere which passes through the points of contact orthogonally.*

Proof. Mapping one of the touching points to infinity by a Möbius transformation we obtain two parallel lines C_3, C_4 with two touching circles between them, see Fig. 10.13. The common tangent line ℓ of touching circles is the intersection line of their planes, and therefore is parallel to the lines C_3, C_4. Thus the touching points of circles lie in a plane orthogonal to the lines C_3, C_4, ℓ. □

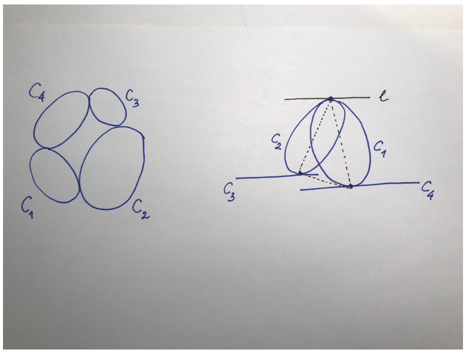


Figure 10.13. To the proof of the touching coins lemma.

Nets of touching circles with the corresponding orthogonal spheres as in Lemma 10.8.4 lie in the core of the concept of S-isothermic surfaces [BHS06, BS08]. The later are defined as nets of touching circles with the combinatorics of the square grid and with the corresponding orthogonal spheres, which also touch. An example of such a surface is a discrete minimal Schwarz P-surface shown in Fig. 10.14.

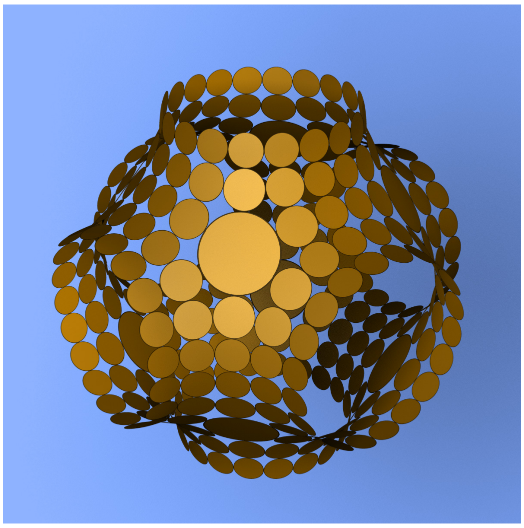


Figure 10.14. A discrete minimal Schwarz P-surface constructed in [BHS06]: four cyclically touching discs intersect the sphere through their points of contact orthogonally. The spheres intersecting a common circle also touch cyclically.

Let K and $\tilde{K}$ be two circles with K inside $\tilde{K}$. Define a sequence of circles $C_1, C_2, \dots$ touching both $\tilde{K}$ and K as follows. Chose a point $P \in \tilde{K}$ and define C_1 as the circle touching $\tilde{K}$ at P and touching K.Construct further circles iteratively so that C_{k+1} touches C_k, see Fig. 10.15.

Theorem 10.8.5 (Steiner's alternative)**.** *If the sequence of circles C_k is periodic, i.e.*

the circle C_n touches the circle C_1 for some point P, then it is periodic for any P.

Proof. Apply a Möbius transformation mapping K and $\tilde{K}$ to two concentric circles. □

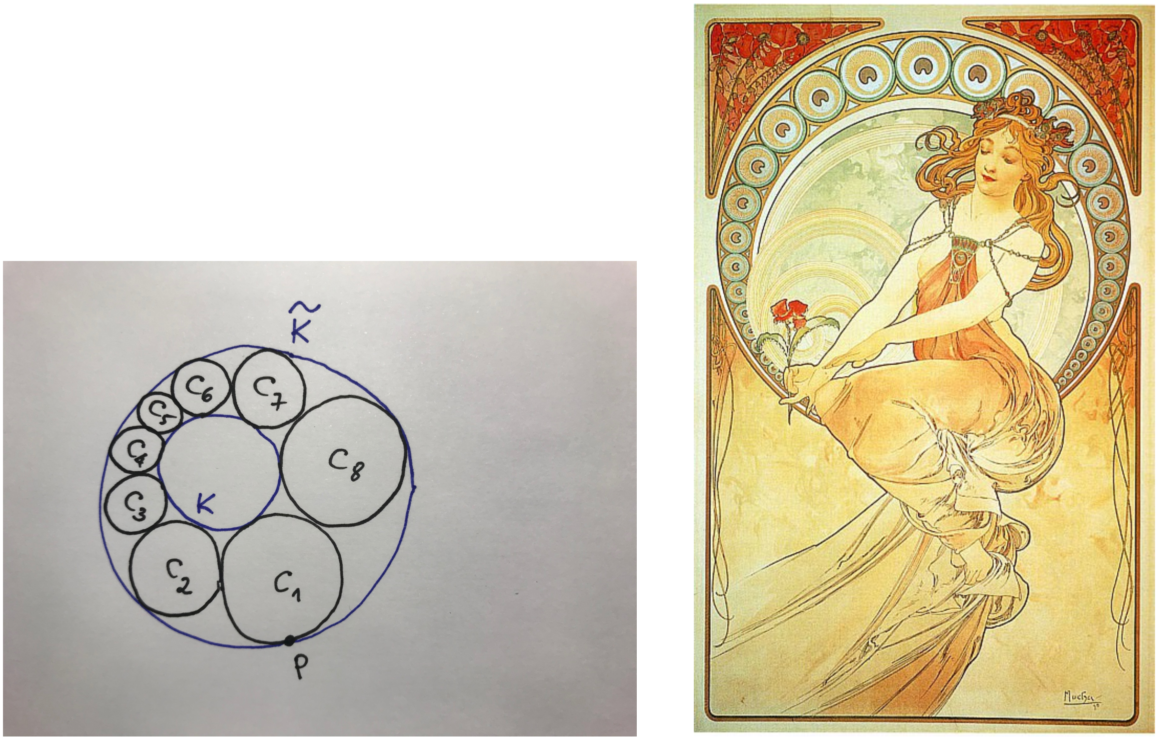
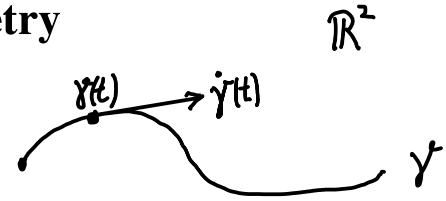


Figure 10.15. Steiner sequence of circles.

Week 3: Curves and surfaces in Möbius geometry

10.9 Curves and surfaces in Möbius geometry



10.9.1 Osculating circle of planar curves

Let

$$\gamma : [a, b] \to \mathbb{R}^2$$

be a smooth planar curve. We further assume that γ is regular, i.e., $\dot\gamma(t) \neq 0$ for all $t \in [a, b]$.

We denote the curve parameter by t and derivatives with respect to t by

$$\gamma(t), \dot\gamma(t), \ddot\gamma(t), \dots$$

For a smooth bijective function

$$\varphi : [\tilde a, \tilde b] \to [a, b]$$

which serves a reparametrization of the curve γ we denote the new curve parameter by s and derivatives with respect to s by

$$\gamma(s) := (\gamma \circ \varphi)(s), \gamma'(s), \gamma''(s), \dots$$

The arc-length of γ is given by

$$s(t) := \int_a^t \|\dot\gamma(t)\|_{\mathbb{R}^2}\, dt = \int_a^t \sqrt{\langle \dot\gamma(t), \dot\gamma(t)\rangle_{\mathbb{R}^2}} dt,$$

which is a strictly monotonically increasing function. By setting $\varphi = s^{-1}$ we find

$$\gamma' = \varphi'\dot\gamma = \frac{\dot\gamma}{\|\dot\gamma\|},$$

which satisfies

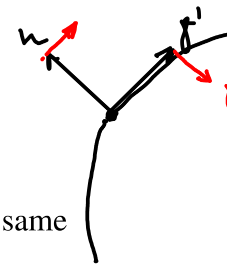
$$\langle \gamma', \gamma' \rangle = 1.$$

Thus, using arc-length as the parameter, the curve is traversed in unit speed. This further implies

$$\langle \gamma'', \gamma' \rangle = 0.$$

The unit normal vector of the curve is given by

$$n(s) := J\gamma'(s) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \gamma_1'(s) \\ \gamma_2'(s) \end{pmatrix} = \begin{pmatrix} -\gamma_2'(s) \\ \gamma_1'(s) \end{pmatrix}$$



In arc-length parametrization the acceleration γ'' and the normal vector n point in the same (or opposite) direction. The (signed) curvature of the curve γ is given by

$$\kappa = \langle \gamma'', n \rangle = -\langle \gamma', n' \rangle,$$

or equivalently,

$$\gamma'' = \kappa n,$$
$$n' = -\kappa\gamma'.$$

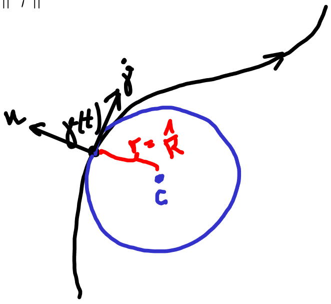
Let us compute the curvature with respect to the general parameter t. The second derivative with respect to arc-length satisfies

$$\gamma'' = \varphi''\dot{\gamma} + (\varphi')^2\ddot{\gamma}.$$

From this we obtain

$$\kappa = \langle \gamma'', n\rangle = \left\langle \varphi''\dot{\gamma} + (\varphi')^2\ddot{\gamma}, n\right\rangle = \frac{\langle \ddot{\gamma}, n\rangle}{\|\dot{\gamma}\|^2} = \frac{\det(\dot{\gamma}, \ddot{\gamma})}{\|\dot{\gamma}\|^3}.$$

The osculating circle of γ at t is the circle with center



$$c(t) := \gamma(t) + \frac{1}{\kappa(t)} n(t)$$

and radius

$$r(t) := \frac{1}{\kappa(t)}.$$

It the unique circle that touches the curve in $\gamma(t)$ with 2nd order contact (equal tangent line and curvature).

Proposition 10.9.1. *Let γ a smooth regular curve in $\mathbb{R}^2$. Let*

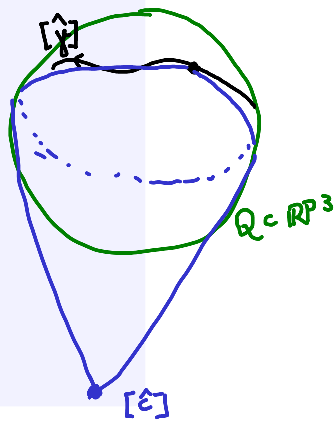
$$\hat{\gamma}(t) := \gamma(t) + e_0 + \|\gamma(t)\|^2 e_\infty$$

be its lift to the Möbius quadric, and

$$\hat{c}(t) := c(t) + e_0 + (\|c(t)\|^2 - r(t)^2) e_\infty$$

be the lift of its osculating circle. Then

$$[\hat{c}] = \mathrm{P}\left(\operatorname{span}\{\hat{\gamma}, \dot{\hat{\gamma}}, \ddot{\hat{\gamma}}\}\right)^{\perp}.$$



Proof. With

$$\hat{c} = \gamma + \tfrac{1}{\kappa} n + e_0 + \left(\|\gamma\|^2 + \tfrac{2}{\kappa}\langle \gamma, n\rangle\right) e_\infty$$

we obtain

$$\langle \hat{\gamma}, \hat{c}\rangle = \left\langle \gamma, \gamma + \tfrac{1}{\kappa} n\right\rangle - \tfrac{1}{2}\|\gamma\|^2 - \tfrac{1}{\kappa}\langle \gamma, n\rangle - \tfrac{1}{2}\|\gamma\|^2 = 0.$$

Now with

$$\dot{\hat{\gamma}} = \dot{\gamma} + 2\langle \gamma, \dot{\gamma}\rangle e_\infty$$

we obtain

$$\left\langle \dot{\hat{\gamma}}, \hat{c}\right\rangle = \left\langle \dot{\gamma}, \gamma + \tfrac{1}{\kappa} n\right\rangle - \langle \gamma, \dot{\gamma}\rangle = 0.$$

Finally, with

$$\ddot{\hat{\gamma}} = \ddot{\gamma} + 2\left(\|\dot{\gamma}\|^2 + \langle \gamma, \ddot{\gamma}\rangle\right) e_\infty$$

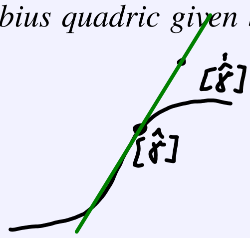
we obtain

$$\left\langle \ddot{\hat{\gamma}}, \hat{c} \right\rangle = \langle \ddot{\gamma}, \gamma + \tfrac{1}{\kappa} n \rangle - \|\dot{\gamma}\|^2 - \langle \gamma, \ddot{\gamma} \rangle = 0.$$

□

One easily checks the following properties of this Möbius geometry representation of the osculating circle:

Proposition 10.9.2. *For a curve* $[\hat{\gamma}] : [a,b] \to \mathcal{Q}$ *on the Möbius quadric given by a smooth representation* $\hat{\gamma} : [a,b] \to \mathbb{L}^{3,1}$ *the point*



$$\mathrm{P}\left(\mathrm{span}\{\hat{\gamma}, \dot{\hat{\gamma}}, \ddot{\hat{\gamma}}\}\right)^{\perp}$$

is invariant under

- *scaling of representative vectors* $\hat{\gamma}(t) \to \lambda(t)\hat{\gamma}(t)$ *with a smooth non-vanishing function* λ,
- *Möbius transformations* $\hat{\gamma}(t) \to A\hat{\gamma}(t)$ *with* $A \in \mathrm{O}(3,1)$,
- *reparametrization* $\hat{\gamma}(t) \to (\hat{\gamma} \circ \varphi)(s)$ *with a smooth bijective function* φ.

Remark 10.9.3. The properties stated in Proposition 10.9.2 are not specific to curves on a quadric or Möbius geometry at all. They hold true for curves in a general projective space if Möbius transformations is replaced by projective transformations.

Corollary 10.9.4. *The osculating circle of a planar curve is Möbius invariant.*

Remark 10.9.5. The circle represented by

$$\mathrm{P}\left(\mathrm{span}\{\hat{\gamma}, \dot{\hat{\gamma}}, \ddot{\hat{\gamma}}\}\right)$$

may also be obtained by considering the circle

$$\mathrm{P}\left(\mathrm{span}\{\hat{\gamma}(t_1), \hat{\gamma}(t), \hat{\gamma}(t_2)\}\right)$$

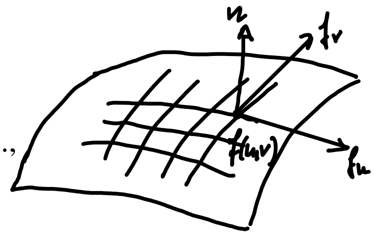
$= P\,\mathrm{sp}\{\hat{\gamma}(t) + \dot{\hat{\gamma}}(t)\varepsilon + \frac{1}{2}\ddot{\hat{\gamma}}(t)\varepsilon^2 + \dots,\ \hat{\gamma}(t),\ \hat{\gamma}(t) + \dot{\hat{\gamma}}(t)\mu + \frac{1}{2}\ddot{\hat{\gamma}}(t)\mu^2\}$

through three close point on the curve and taking the limit $t_1, t_2 \to t$.

10.9.2 Curvature line parametrized surfaces

Let

$$f : \mathbb{R}^2 \supset U \to \mathbb{R}^3$$



a smooth parametrized surface patch. We further assume that f is regular, i.e.,

$$f_u := \frac{\partial f}{\partial u}, \quad f_v := \frac{\partial f}{\partial v}$$

are linearly independent at every point $(u, v) \in U$. Thus, we can define the unit normal field of f by

$$n(u, v) := \frac{f_u \times f_v}{\|f_u \times f_v\|}$$

The first and second fundamental forms of f are given by

$\langle f_u, n\rangle = 0 \Rightarrow \underset{\|}{\overset{0}{}}\frac{\partial}{\partial v}\langle f_u, n\rangle = \langle f_{uv}, n\rangle + \langle f_u, n_v\rangle$

$$\mathrm{I} = \begin{pmatrix} \langle f_u, f_u\rangle & \langle f_u, f_v\rangle \\ \langle f_u, f_v\rangle & \langle f_v, f_v\rangle \end{pmatrix}, \quad \mathrm{II} = \begin{pmatrix} \langle f_{uu}, n\rangle & \langle f_{uv}, n\rangle \\ \langle f_{uv}, n\rangle & \langle f_{vv}, n\rangle \end{pmatrix} = -\begin{pmatrix} \langle f_u, n_u\rangle & \langle f_u, n_v\rangle \\ \langle f_v, n_u\rangle & \langle f_v, n_v\rangle \end{pmatrix}.$$

A parametrization is called *orthogonal* if the first fundamental form is diagonal, i.e.,

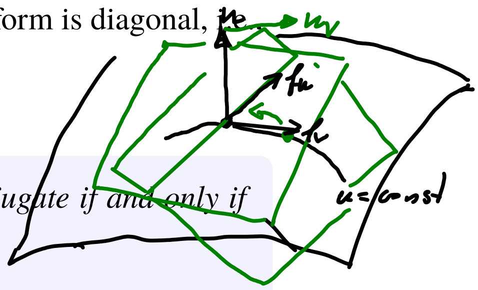
$$\langle f_u, f_v\rangle = 0$$

Proposition 10.9.6. *The property of a parametrization to be orthogonal is Möbius invariant.*

Proof. Möbius transformations are conformal, i.e., preserve angles. □

A parametrization is called *conjugate* if the second fundamental form is diagonal,

$$-\langle f_{uv}, n\rangle = \langle f_u, n_v\rangle = \langle f_v, n_u\rangle = 0.$$



Proposition 10.9.7. *A parametrization $f : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ is conjugate if and only if there exist two functions $\alpha, \beta : U \to \mathbb{R}$ such that*

$$f_{uv} = \alpha f_u + \beta f_v \tag{10.2}$$

Proof. Generally,

$$f_{uv} = \alpha f_u + \beta f_v + \gamma n.$$

with some functions α, β, γ. Thus,

$$\langle f_{uv}, n\rangle = 0 \quad \Leftrightarrow \quad \gamma = 0.$$

□

Note that the condition (10.2) does not depend on the existence of a normal field anymore. It may be used to define conjugate parametrizations in any dimension: Thus, a parametrization $f : \mathbb{R}^2 \supset U \to \mathbb{R}^n$, $n \geqslant 3$ is called *conjugate* if it satisfies an equation of the from (10.2).

Proposition 10.9.8. *Let $f : \mathbb{R}^2 \supset U \to \mathbb{R}^n$ be a conjugate parametrization. Then an arbitrary lift to homogeneous coordinates*

$$\hat{f} := \lambda \cdot (f, 1) : U \to \mathbb{R}^{n+1}$$

with a smooth non-vanishing function $\lambda : U \to \mathbb{R}$, satisfies

$$\hat{f}_{uv} = \alpha \hat{f}_u + \beta \hat{f}_v + \gamma \hat{f} \tag{10.3}$$

with some functions α, β, γ.

Equation (10.3) states the linear dependence of four representative vectors, or equivalently that four points lie in a plane. While the four points are not projectively well-defined (the points defined by the derivatives are not invariant under scaling $\hat{f}$) this property is:

Proposition 10.9.9. *Equation* (10.3) *is invariant under*

- *scaling of representative vectors $\hat{f}(u, v) \to \lambda(u, v)\hat{f}(u, v)$ with a smooth non-vanishing function λ,*
- *projective transformations $\hat{f}(u, v) \to F\hat{f}(u, v)$ with $F \in \mathrm{GL}(n + 1, \mathbb{R})$,*
- *reparametrization along the coordinate lines $\hat{f}(u, v) \to \hat{f}(u(\tilde{u}), v(\tilde{v}))$.*

Corollary 10.9.10. *The property of a parametrization to be conjugate is projectively invariant.*

Returning to parametrized surfaces in $\mathbb{R}^3$: A parametrization

$$f : \mathbb{R}^2 \supset U \to \mathbb{R}^3$$

is called *curvature line parametrization* if the first and second fundamental form are diagonal, or equivalently if it is orthogonal and conjugate, i.e.,

$$\langle f_u, f_v \rangle = 0, \quad \text{and} \quad f_{uv} = \alpha f_u + \beta f_v.$$

Proposition 10.9.11. *Let $f : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a parametrized surface and*

$$\hat{f} := f + e_0 + \|f\|^2 e_\infty$$

its lift to the Möbius quadric. Then f is a curvature line parametrization if and only if $[\hat{f}]$ is a conjugate parametrization.

Proof. For the derivatives of the lift we obtain

$$\begin{aligned} \hat{f}_u &= f_u + 2\langle f, f_u \rangle e_\infty, \\ \hat{f}_v &= f_v + 2\langle f, f_v \rangle e_\infty, \\ \hat{f}_{uv} &= f_{uv} + 2\left(\langle f, f_{uv} \rangle + \langle f_u, f_v \rangle\right) e_\infty. \end{aligned}$$

Let $\hat{f}$ be a curvature line parametrization. Then

$$\hat{f}_{uv} = f_{uv} + 2\langle f, f_{uv} \rangle e_\infty = \alpha f_u + \beta f_v + 2\left(\alpha \langle f, f_u \rangle + \beta \langle f, f_v \rangle\right) e_\infty = \alpha \hat{f}_u + \beta \hat{f}_v.$$

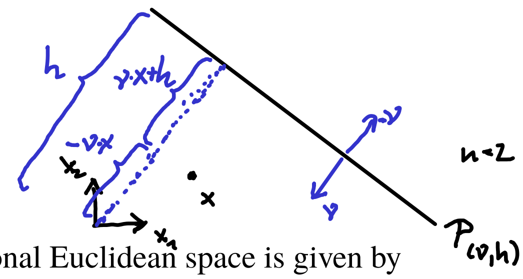
The reverse direction is shown similarly. □

Week 4: Laguerre geometry

Chapter 11

Laguerre geometry

Classically, Laguerre geometry is the geometry of oriented lines and oriented circles in the Euclidean plane, and their oriented contact. More generally, it is the geometry of oriented hyperplanes and oriented spheres in Euclidean space.



11.1 Models of Laguerre geometry

11.1.1 The Blaschke cylinder

Oriented hyperplanes A hyperplane in the n-dimensional Euclidean space is given by

$$P_{(\nu,h)} := \{x \in \mathbb{R}^n \mid \nu \cdot x + h = 0\} \subset \mathbb{R}^n$$

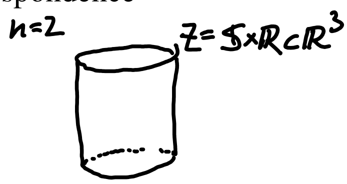
with $\nu \in \mathbb{S}^{n-1}$ and $h \in \mathbb{R}$. The vector ν serves as the unit normal vector of the hyperplane and, by that, induces an orientation on the hyperplane. It distinguishes the two regions that Euclidean space is separated into by $P_{(\nu,h)}$ and points into the region

$$P^+_{(\nu,h)} := \{x \in \mathbb{R}^n \mid \nu \cdot x + h > 0\} \subset \mathbb{R}^n.$$

The left-hand-side $\nu \cdot x + d$ of the equation expresses the signed distance of the point $x \in \mathbb{R}^n$ to the hyperplane $P_{(\nu,h)}$. It is positive if the point lies in $P^+_{(\nu,h)}$. Finally, h is the signed distance of the origin to $P_{(\nu,h)}$. The two tuples (ν, h) and $(-\nu, -h)$ determine the same hyperplane $P_{(\nu,h)}$, but with opposite orientation.

Definition 11.1.1. The *oriented hyperplane* in the n-dimensional Euclidean space with unit normal vector $\nu \in \mathbb{S}^{n-1}$ to which the origin has signed distance $h \in \mathbb{R}$ is denoted by $\vec{P}_{(\nu,h)}$.

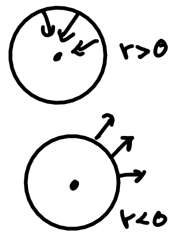
Oriented hyperplanes $\vec{P}_{(\nu,h)}$ in the Euclidean space $\mathbb{R}^n$ are in one-to-one correspondence with points (ν, h) on the *Blaschke cylinder*



$$\mathcal{Z} = \{(\nu, h) \in \mathbb{R}^n \times \mathbb{R} \mid |\nu| = 1\} = \mathbb{S}^{n-1} \times \mathbb{R} \subset \mathbb{R}^{n+1}.$$

Oriented hyperspheres A hypersphere in the n-dimensional Euclidean space is given by

$$S_{(c,r)} := \left\{x \in \mathbb{R}^n \;\middle|\; |x-c|^2 = r^2\right\} \subset \mathbb{R}^n$$



with some center $c \in \mathbb{R}^n$ and signed radius $r \in \mathbb{R}$. The sign of radius induces an orientation on the hypersphere by assigning normal vectors that point towards the center if $r > 0$ and away from it if $r < 0$. The two tuples (c, r) and $(c, -r)$ describe the same hypersphere, but with opposite orientation, where the special case of $r = 0$ describes a point, also called a *null-sphere*, and is non-oriented.

Definition 11.1.2. The *oriented hypersphere* in the n-dimensional Euclidean space with center $c \in \mathbb{R}^n$ and signed radius $r \in \mathbb{R}$ is denoted by $\vec{S}_{(\nu,h)}$. (c,r)

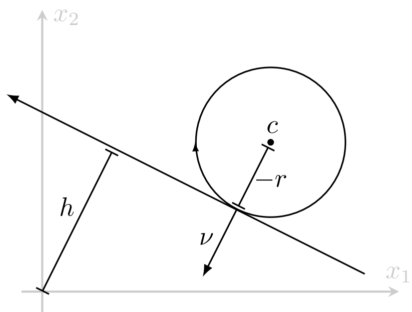


Figure 11.1. An oriented circle and an oriented line in oriented contact in the Euclidean plane.

Oriented contact An oriented hyperplane and an oriented hypersphere are said to be in *oriented contact* if the hyperplane is tangent to the circle and their normal vectors coincide at the point (see Figure 11.1).

Proposition 11.1.3. *An oriented hyperplane $\vec{P}_{(\nu,h)}$, $(\nu, h) \in \mathcal{Z}$, and an oriented hypersphere $\vec{S}_{(c,r)}$, $(c, r) \in \mathbb{R}^{n+1}$, are in oriented contact if and only if*

$$c \cdot \nu + h = r. \tag{11.1}$$

signed dist of c to $\vec{P}_{(\nu,h)}$

Equation (11.1) is linear in (ν, h) and thus describes a plane.

Proposition 11.1.4.

(i) The oriented hyperplanes $\vec{P}_{(\nu,h)}$ in oriented contact with an oriented hypersphere $\vec{S}_{(c,r)}$ correspond to the points of the hyperplanar section of the Blaschke cylinder

$$\{(\nu, h) \in \mathcal{Z} \mid c \cdot \nu + h = r.\}$$

(ii) Vice versa, a hyperplanar section of the Blaschke cylinder with a hyperplane non-parallel to the axis corresponds to all oriented hyperplanes in oriented contact with a fixed oriented hypersphere.

(iii) A hyperplanar section of the Blaschke cylinder with a hyperplane non-parallel to the axis corresponds to all oriented hyperplanes through a point, i.e., describes a null-sphere, if and only if the plane contains the origin.

Proof.

(i) Follows from Proposition 11.1.3.

(ii) A hyperplane non-parallel to the axis is given by

$$\{(\nu, h) \in \mathcal{Z} \mid \alpha \cdot \nu + \beta h = \gamma.\}$$

with $\alpha \in \mathbb{R}^n$, $\beta, \gamma \in \mathbb{R}$, and $\beta \neq 0$. Dividing by β yields an equation of the form (11.1).

(iii) The hyperplane contains the origin if and only if $r = 0$.

□

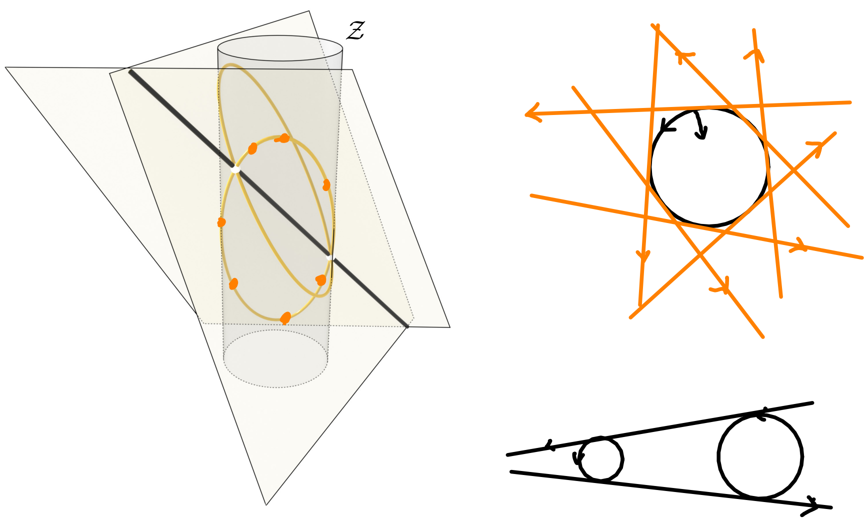


Figure 11.2. The Blaschke cylinder model of 2-dimensional Euclidean Laguerre geometry. An oriented line in the Euclidean plane is represented by a point on the Blaschke cylinder $\mathcal{Z}$. All oriented lines in oriented contact to an oriented circle correspond to the points of a planar section of $\mathcal{Z}$ (non-parallel to the axis).

Thus, generally speaking, in the *Blaschke cylinder model* of Laguerre geometry, oriented hyperplanes correspond to points on the Blaschke cylinder and oriented hyperspheres correspond to hyperplanes (non-parallel to the axis of the Blaschke cylinder).

Corollary 11.1.5. *A section of the Blaschke cylinder with a k-dimensional plane non-parallel to the axis corresponds to all oriented hyperplanes in oriented contact with k fixed oriented hyperspheres.*

In particular, sections with codimension 2 planes describe oriented right circular cones.

What about the planar sections parallel to the axis of the Blaschke cylinder? These contain straight line generators of the Blaschke cylinder.

Proposition 11.1.6.

(i) A generator of the Blaschke cylinder corresponds to a one-parameter family of parallel oriented hyperplanes, where parallel means "with coinciding normal vectors".

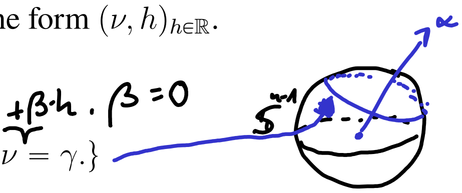
(ii) A hyperplanar section of the Blaschke cylinder with a hyperplane parallel to the axis corresponds to all oriented hyperplanes parallel to an oriented right circular cone.

Proof.

(i) Generators of the Blaschke cylinder are of the form $(\nu, h)_{h\in\mathbb{R}}$.

(ii) A hyperplane parallel to the axis is given by

$$\{(\nu, h) \in \mathcal{Z} \mid \alpha \cdot \nu = \gamma.\}$$



with $\alpha \in \mathbb{R}^n$, $\gamma \in \mathbb{R}$. The solution is given by all ν from a hyperplanar section of $\mathbb{S}^{n-1}$ and arbitrary $h \in \mathbb{R}$.

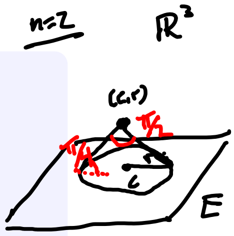
□

11.1.2 The cyclographic model

In the Blaschke cylinder model oriented hyperplanes are the primary objects and described as points, while oriented hyperspheres are described as hyperplanes in the same space. Taking oriented hyperspheres as the primary objects gives rise to the *cyclographic model*.

An oriented hypersphere with center $c \in \mathbb{R}^n$ and signed radius $r \in \mathbb{R}$ corresponds to a tuple $(c, r) \in \mathbb{R}^{n+1}$. We embed the Euclidean space into the same $\mathbb{R}^{n+1}$ by identifying it with the $r = 0$ hyperplane, which is the hyperplane of null-spheres and called the *base plane*.

$$\mathbf{E} := \{(c, r) \in \mathbb{R}^n \times \mathbb{R} \mid r = 0\} \cong \mathbb{R}^n.$$



Definition 11.1.7. For a point $(c, r) \in \mathbb{R}^{n+1}$ the cone

$$\begin{aligned}\mathcal{C}_{(c,r)} &:= \left\{(\hat{x}, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; |\hat{x} - c|^2 - (x_{n+1} - r)^2 = 0\right\}. \\ &= \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=1}^{n}(x_i - c_i)^2 - (x_{n+1} - r)^2 = 0\right\}.\end{aligned}$$

is called its *isotropic cone*.

Isotropic cones are right circular cones with an axis orthogonal to the base plane. Their opening angle is $\frac{\pi}{2}$ and they intersect the base plane in a constant angle of $\frac{\pi}{4}$. The intersection of an isotropic cone $\mathcal{C}_{(c,r)}$ with the base plane yields the hypersphere represented by the point (c, r):

$$S_{(c,r)} = \mathcal{C}_{(c,r)} \cap \mathbf{E},$$

while the orientation of $\vec{S}_{(c,r)}$ has to be inferred from the sign of r. The map

$$(c, r) \mapsto \vec{S}_{(c,r)}$$

is sometimes referred to as the *cyclographic projection*.

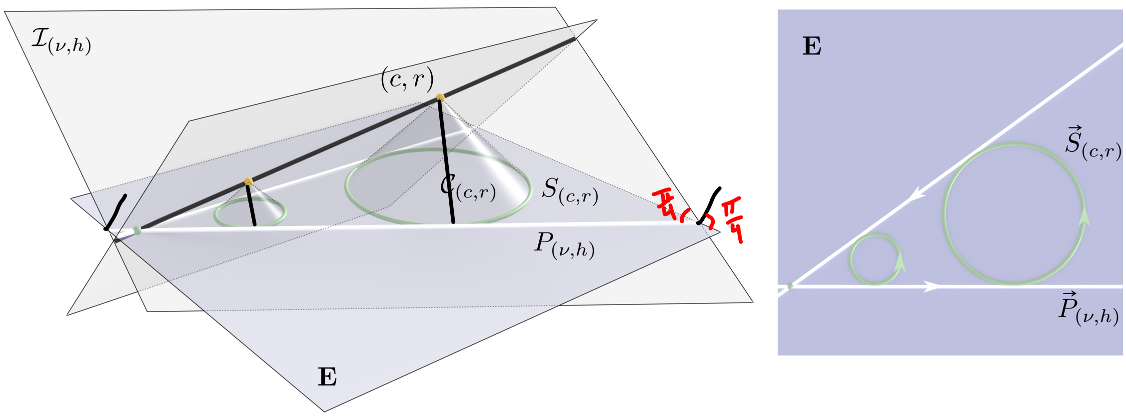


Figure 11.3. The cyclographic model of 2-dimensional Euclidean Laguerre geometry. The Euclidean plane is embedded as the base plane $\mathbf{E}$. An oriented circle in the Euclidean plane is represented by a point $(c, r) \in \mathbb{R}^3$. All oriented circles in oriented contact to an oriented line $\vec{P}_{(\nu,h)}$ correspond to points in an isotropic plane $\mathcal{I}_{(\nu,h)}$.

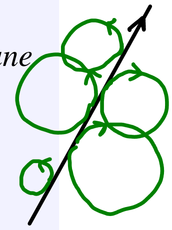
The tangent hyperplanes of an isotropic cone intersect the base plane at an angle of $\frac{\pi}{4}$. Such hyperplanes are called *isotropic hyperplanes*. Reviewing equation (11.1) we find that it is linear in (c, r), and describes an isotropic hyperplane.

Proposition 11.1.8.

(i) The oriented hyperspheres $\vec{S}_{(c,r)}$ in oriented contact with an oriented hyperplane $\vec{P}_{(\nu,h)}$ correspond to the points on the isotropic plane

$$\mathcal{I}_{(\nu,h)} := \{(c, r) \in \mathbb{R}^n \times \mathbb{R} \mid \nu \cdot c - r = -h\}.$$

(ii) Vice versa, an isotropic hyperplane corresponds to all oriented hyperspheres in oriented contact with a fixed oriented hyperplane.



Proof.

(i) Follows from Proposition 11.1.3.

(ii) An isotropic hyperplane is of the form

$$\{(c, r) \in \mathbb{R}^n \times \mathbb{R} \mid \alpha \cdot c + \beta r = \gamma\}$$

with $\alpha \in \mathbb{R}^n$, $\beta, \gamma, \in \mathbb{R}$ where $|\alpha| = \beta^2 \neq 0$.

□

The intersection of an isotropic hyperplane $\mathcal{I}_{(\nu,h)}$ with the base plane yields the hyperplane represented by the point (ν, d):

$$P_{(\nu,h)} = \mathcal{I}_{(\nu,h)} \cap \mathbf{E},$$

while the orientation of $\vec{P}_{(\nu,h)}$ has to be inferred from the direction of ν.

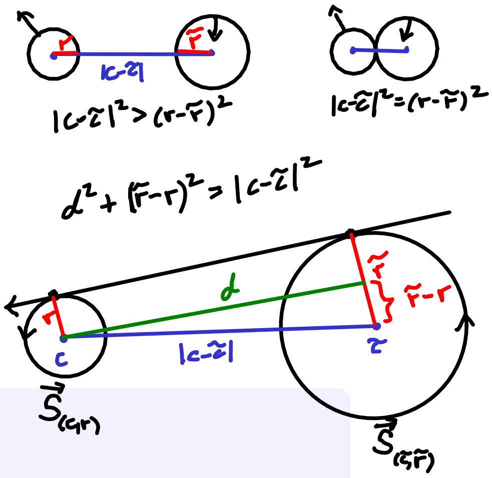
Summarizing, in the cyclographic model of Laguerre geometry, oriented hyperspheres correspond to points in $\mathbb{R}^{n+1}$, or equivalently isotropic cones, and oriented hyperplanes correspond to isotropic hyperplanes.

Tangential distance The appearance of isotropic cones and planes in the cyclographic model makes it natural to introduce a Lorentz product

$$\langle x, y \rangle_{n,1} = \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1},$$

and in particular the corresponding (squared) *Minkowski distance*

$$\|x - y\|_{n,1}^2 = \langle x - y, x - y \rangle_{n,1} = \sum_{i=1}^{n} (x_i - y_i)^2 - (x_{n+1} - y_{n+1})^2$$



for $x, y \in \mathbb{R}^{n+1}$.

Then an isotropic cone with apex $x \in \mathbb{R}^{n+1}$ is given by

$$\left\{ y \in \mathbb{R}^{n+1} \,\middle|\, \|x - y\|_{n,1}^2 = 0 \right\}$$

and an isotropic plane by

$$\left\{ y \in \mathbb{R}^{n+1} \,\middle|\, \langle n, y \rangle_{n,1} = \gamma \right\}$$

with some $n \in \mathbb{R}^{n+1}$, $\|n\|_{n,1} = 0$ and $\gamma \in \mathbb{R}$.

Proposition 11.1.9. *For $x, \tilde{x} \in \mathbb{R}^{n+1}$*

$$\begin{aligned}
\|x - \tilde{x}\|_{n,1}^2 > 0 &\Leftrightarrow \vec{S}_x, \vec{S}_{\tilde{x}} \text{ have multiple hyperplanes in common oriented contact} \\
\|x - \tilde{x}\|_{n,1}^2 = 0 &\Leftrightarrow \vec{S}_x, \vec{S}_{\tilde{x}} \text{ are in oriented contact} \\
\|x - \tilde{x}\|_{n,1}^2 < 0 &\Leftrightarrow \vec{S}_x, \vec{S}_{\tilde{x}} \text{ have no hyperplanes in common oriented contact}
\end{aligned}$$

If $\|x - \tilde{x}\|_{n,1}^2 > 0$, then the Minkowski distance $\|x - \tilde{x}\|_{n,1}$ is equal to the Euclidean distance between the two touching points of any common oriented tangent hyperplane of $\vec{S}_x$ and $\vec{S}_{\tilde{x}}$.

Proof. With $x = (c, r)$ and $\tilde{x} = (\tilde{x}, \tilde{r})$ we obtain

$$\|x - \tilde{x}\|_{n,1}^2 = |c - \tilde{c}|^2 - (r - \tilde{r})^2.$$

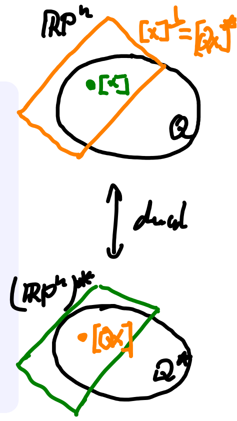
□

In the context of Laguerre geometry the (squared) Minkowski distance in the cyclographic model is also called the (squared) *tangential distance*.

Remark 11.1.10. If one of the two spheres is null-sphere, i.e., describes a point, then the squared Minkowski distance becomes the power of that point with respect to the hypersphere.

Week 5: Dual quadrics, projective models of Laguerre geometry, Miquel's theorem

7.4 Dual quadrics



Theorem 7.4.1. *Let $\mathcal{Q}$ be a non-degenerate quadric in $\mathbb{RP}^n$. Then the set of tangent hyperplanes to $\mathcal{Q}$ forms a non-degenerate quadric in the dual space $(\mathbb{RP}^n)_*$. This quadric $\mathcal{Q}^\star$ is called the* dual quadric *of $\mathcal{Q}$ and has the same signature as $\mathcal{Q}$.*

Furthermore, let Q be a symmetric matrix representing $\mathcal{Q}$ in some basis of $\mathbb{R}^{n+1}$. Then

$$Q^\star := Q^{-1}$$

is a symmetric matrix representing the dual quadric $\mathcal{Q}^\star$ in the dual basis of $(\mathbb{R}^{n+1})^*$.

Proof. The set of tangent hyperplanes of $\mathcal{Q}$ is given by

$$\left\{X^\perp \subset \mathbb{RP}^n \;\middle|\; X \in \mathcal{Q}\right\}.$$

Thus, the dual quadric is given by

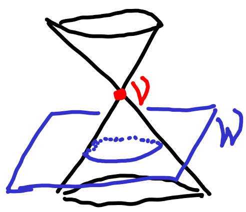
$$\begin{aligned}
\mathcal{Q}^\star &= \left\{(X^\perp)^\star \in (\mathbb{RP}^n)^\star \;\middle|\; X \in \mathcal{Q}\right\} \\
&= \{[Qx] \in (\mathbb{RP}^n)^\star \mid [x] \in \mathbb{RP}^n,\ x^\intercal Q x = 0\} \\
&= \left\{[y] \in (\mathbb{RP}^n)^\star \;\middle|\; y^\intercal Q^{-1} y = 0\right\},
\end{aligned}$$

$= (Qx)^T Q^{-1} Qx = x^T Qx$

which indeed is a quadric represented by the symmetric matrix Q^{-1}. The signs of the eigenvalues are the same for Q and Q^{-1}. Thus, $\mathcal{Q}$ and $\mathcal{Q}^\star$ have the same signature. □

Now consider a (possibly degenerate) quadric $\mathcal{Q} \subset \mathbb{RP}^n$ of signature (r, s, t) represented by the symmetric bilinear form q. Let

$$V := \ker \mathrm{q} = \{v \in \mathbb{R}^{n+1} \mid \forall x \in \mathbb{R}^{n+1}: q(v,x) = 0\}$$



and let W be any complementary linear subspace of V, i.e.,

$$\mathbb{R}^{n+1} = V \oplus W.$$

Thus, $\mathcal{V} := \mathrm{P}(V)$ is the set of singular points of $\mathcal{Q}$

$$\mathbb{RP}^n = \mathcal{V} \vee \mathcal{W}, \qquad \mathcal{V} \cap \mathcal{W} = \varnothing,$$

where $\mathcal{W} := \mathrm{P}(W)$.

For a point $X \in \mathcal{Q} \backslash \mathcal{V}$ the *tangent hyperplane* of $\mathcal{Q}$ at X is given by its polar hyperplane $X^\perp$. By duality each tangent hyperplane corresponds to a point in the dual space $(\mathbb{RP}^n)^*$.

Definition 7.4.2. For a quadric $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^n$ its *dual quadric* is given by

$$\mathcal{Q}^\star := \left\{ (X^\perp)^\star \in (\mathbb{R}\mathrm{P}^n)^* \;\middle|\; X \in \mathcal{Q} \backslash \mathcal{V} \right\}.$$

where $\mathcal{V}$ is the set of singular points of $\mathcal{Q}$.

The bilinear form q induces a linear map

$$\mathrm{Q} : \mathbb{R}^{n+1} \to (\mathbb{R}^{n+1})^*, \quad x \mapsto \mathrm{q}(x, \cdot) : y \mapsto \mathrm{q}(x, y).$$

Note that $V = \ker \mathrm{Q}$. For a point $[x] \in \mathbb{R}\mathrm{P}^n$ the image $[\mathrm{Q}\, x]$ represents the dual of the polar hyperplane of $[x]$:

$$[x] = [\mathrm{Q}\, x].$$

With this map the quadric $\mathcal{Q}$ can be written as

$$\mathcal{Q} = \{ [x] \mid (\mathrm{Q}\, x)(x) = 0 \}.$$

The dual of the tangent hyperplane at the point $[x] \in \mathbb{R}\mathrm{P}^n$ is given by $[\mathrm{Q}\, x] \in (\mathbb{R}\mathrm{P}^n)^*$. With the decomposition

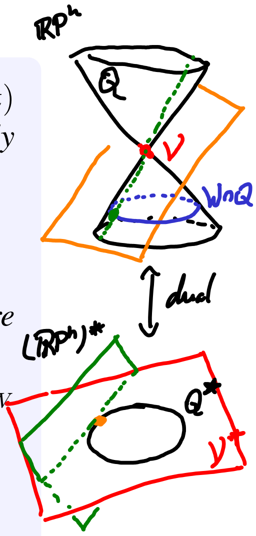
$$x = v + w$$

where $v \in V$ and $w \in W$ this yields

$$[\mathrm{Q}\, x] = [\mathrm{Q}\, w],$$

and thus the dual quadric of $\mathcal{Q}$ can be written as

$$\mathcal{Q}^\star = \{ [\mathrm{Q}\, x] \mid [x] \in \mathcal{Q} \backslash \mathcal{V} \} = \{ [\mathrm{Q}\, w] \mid [w] \in \mathcal{Q} \cap \mathcal{W} \}.$$



Theorem 7.4.3. *Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^n$ be a (possibly degenerate) quadric of signature (r, s, t) with singular points $\mathcal{V} = \mathrm{P}(V)$. Then its dual quadric $\mathcal{Q}^\star \subset (\mathbb{R}\mathrm{P}^n)^*$ is entirely contained in the projective subspace $\mathcal{V}^\star$ of dimension $n - t - 1$:*

$$\mathcal{Q}^\star \subset \mathcal{V}^\star.$$

In $\mathcal{V}^\star \subset (\mathbb{R}\mathrm{P}^n)^*$ the dual quadric $\mathcal{Q}^\star$ constitutes a non-degenerate quadric of signature (r, s).

Furthermore, let $\mathcal{W} = \mathrm{P}(W) \subset \mathbb{R}\mathrm{P}^n$ be a complementary subspace of $\mathcal{V}$, and Q_W be a symmetric matrix representing $\mathcal{Q}$ in some basis of W. Then

$$Q^\star := Q_W^{-1}$$

is a symmetric matrix representing the dual quadric $\mathcal{Q}^\star$ in the dual basis of $V^\star$.

Proof. The restriction of the map $\mathrm{Q} : \mathbb{R}^{n+1} \to (\mathbb{R}^{n+1})^*$ to W is a bijective map to $V^\star$

$$\mathrm{Q}_W := \mathrm{Q}\big|_W : W \to V^\star.$$

In particular, this implies $\mathcal{Q}^\star \subset \mathcal{V}$.

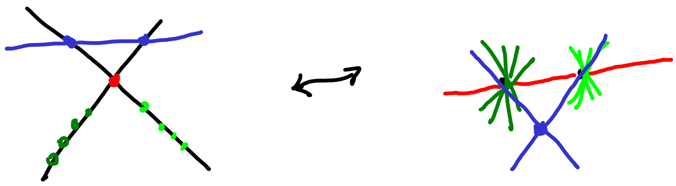
Furthermore, for $[w] \in \mathcal{W}$ and $\alpha = \mathrm{Q}\,w$ we find

$$\alpha(\mathrm{Q}_W^{-1}\,\alpha) = (\mathrm{Q}\,w)(\mathrm{Q}_W^{-1}\,\mathrm{Q}\,w) = (\mathrm{Q}\,w)(w),$$

and thus

$$\mathcal{Q}^\star = \left\{[\alpha] \in \mathcal{V}^\star \;\middle|\; \alpha(\mathrm{Q}_W^{-1}\,\alpha) = 0\right\}.$$

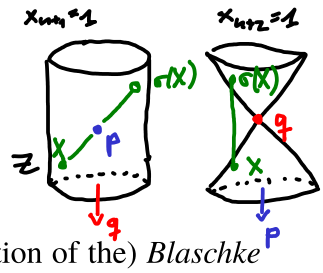
A basis representation of Q_W and the consideration of the signs of its eigenvalues yields the remaining claims. □



11.1.3 The projective models and their duality

In the Blaschke cylinder model oriented hyperplanes correspond to (special) points and oriented hyperspheres correspond to hyperplanes, while in the cyclographic the roles of points and hyperplanes are reversed. Yet in both models the oriented contact is given by the incidence of a point lying on a hyperplane. Embedding both models into projective space will reveal that they are related by duality.

The Blaschke cylinder Let $\langle \cdot, \cdot \rangle$ be the standard degenerate symmetric bilinear form of signature $(n, 1, 1)$, i.e.,



$$\langle x, y \rangle = x_1 y_1 + \ldots + x_n y_n - x_{n+1} y_{n+1}$$

for $x, y \in \mathbb{R}^{n+2}$, and

$\langle x,x \rangle \overset{x_{n+1}=1}{=} x_1^2 + \ldots + x_n^2 - 1 = 0$

$$\mathcal{Z} := \{ [x] \in \mathbb{RP}^{n+1} \mid \langle x, x \rangle = 0 \} \subset \mathbb{RP}^{n+1}$$

the corresponding quadric in $\mathbb{RP}^{n+1}$, which we call the (projectivization of the) *Blaschke cylinder*. Projectively $\mathcal{Z}$ is a cone with apex

$$q = [e_{n+2}] = [0, \ldots, 0, 1].$$

The Blaschke cylinder as described in Section 11.1.1 is recovered by introducing affine coordinates $x_{n+1} = 1$.

Thus, an oriented hyperplane $\vec{P}_{(\nu,h)}$ in the n-dimensional Euclidean space with unit normal vector $\nu \in \mathbb{S}^{n-1}$ and signed distance $h \in \mathbb{R}$ corresponds to the point

$$[\nu, 1, h] \in \mathcal{Z} \subset \mathbb{RP}^{n+1}.$$

The only point not captured in the affine picture is the apex q of the Blaschke cylinder. It can be interpreted as the (non-oriented) hyperplane at infinity.

Orientation reversion $\vec{P}_{(\nu,h)} \mapsto \vec{P}_{(-\nu,-h)}$ is given by the projective involution

$$\sigma : \mathbb{RP}^{n+2} \to \mathbb{RP}^{n+2}, \quad [x_1, \ldots, x_n, x_{n+1}, x_{n+2}] \mapsto [x_1, \ldots, x_n, -x_{n+1}, x_{n+2}].$$

It preserves $\mathcal{Z}$ and fixes the point

$$p = [e_{n+1}] = [0, \ldots, 0, 1, 0]$$

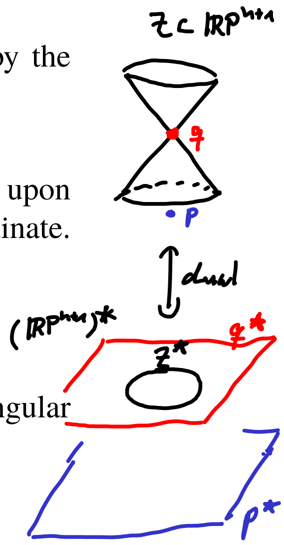
and all points on its polar hyperplane.

An oriented hypersphere $\vec{S}_{(c,r)}$ in the n-dimensional Euclidean space with center $c \in \mathbb{R}^n$ and signed radius $r \in \mathbb{R}$ corresponds to the intersection of the Blaschke cylinder with the hyperplane

$$[c, -r, 1]^\star \subset \mathbb{RP}^{n+1}.$$

corresponding plane equation
$c \cdot \nu - r \cdot 1 + 1 \cdot h = 0$

It is a null-sphere if and only if it contains the point p. The hyperplanes of $\mathbb{RP}^{n+1}$ that, in the affine picture ($x_{n+1} = 1$), appear as hyperplanes parallel to the axis of the Blaschke cylinder, are exactly the hyperplanes that contain the point q.



The cyclographic model Dually, an oriented hypersphere $\vec{S}_{(c,r)}$ is represented by the point

$$[c, -r, 1] \in (\mathbb{R}\mathrm{P}^{n+1})^*.$$

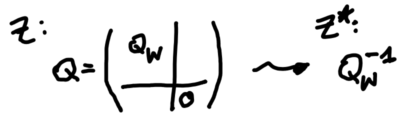
Thus, we may identify the points of the cycligraphic model with the points of $(\mathbb{R}\mathrm{P}^{n+1})^*$ upon introducing affine coordinates $x_{n+2} = 1$ and reversing the sign of the $(n+1)$-coordinate. The base plane of the cyclographic model is then embedded as

$$\mathbf{E} = p^\star = \left\{ [x] \in (\mathbb{R}\mathrm{P}^{n+1})^* \;\middle|\; x_{n+1} = 0 \right\}.$$

The Blaschke cylinder $\mathcal{Z}$ is a degenerate quadric of signature $(n, 1, 1)$. Its singular points consist exactly of the apex q. By Theorem 7.4.3, its dual quadric is given by

$$\mathcal{Z}^\star = \left\{ [x] \in (\mathbb{R}\mathrm{P}^{n+1})^* \;\middle|\; x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = 0,\ x_{n+2} = 0 \right\}.$$

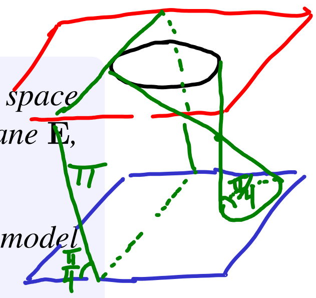
It is contained in the hyperplane



$$q^\star = \left\{ [x] \in (\mathbb{R}\mathrm{P}^{n+1})^* \;\middle|\; x_{n+2} = 0 \right\},$$

which, in affine coordinates $x_{n+2} = 1$, is the hyperplane at infinity. In this hyperplane, $\mathcal{Z}^\star$ constitutes a quadric of signature $(n, 1)$.

The dual of a point on the Blaschke cylinder is a hyperplane in $(\mathbb{R}\mathrm{P}^{n+1})^*$ that touches the dual quadric $\mathcal{Z}^\star$. Now the following proposition establishes the correspondence of the dual of the Blaschke cylinder model and the cyclographic model.



Proposition 11.1.11. *Upon introducing affine coordinates $x_{n+2} = 1$ on the dual space $(\mathbb{R}\mathrm{P}^{n+1})^*$, which contains the dual of the Blaschke cylinder $\mathcal{Z}^\star$ and the base plane $\mathbf{E}$, the following correspondence holds:*

(i) A hyperplane in $(\mathbb{R}\mathrm{P}^{n+1})^*$ is an isotropic hyperplane of the cyclographic model if and only if it touches $\mathcal{Z}^\star$.

(ii) A cone in $(\mathbb{R}\mathrm{P}^{n+1})^*$ is an isotropic cone of the cyclographic model if and only if it contains $\mathcal{Z}^\star$.

Proof.
(i) A plane $\Pi = [N]^\star \subset (\mathbb{R}\mathrm{P}^{n+1})^*$ touches $\mathcal{Z}^\star$ if and only if $[N] \in \mathcal{Z}$, i.e.

$$\langle N, N \rangle = N_1^2 + \cdots N_n^2 - N_{n+1}^2 = 0$$

On the other hand, in affine coordinates $x_{n+2} = 1$, the angle between the normal vector $\tilde{N} = (N_1, \ldots, N_{n+1})$ of the plane Π and the normal vector

$\tilde{P} = (0, \ldots, 0, 1)$ of the base plane $\mathbf{E}$ is equal to $\gamma = \frac{\pi}{4}$ if and only if

$$\begin{aligned} & \frac{1}{2}(N_1^1 + \cdots + N_{n+1}^2) = \left|\tilde{N}\right|^2 \cos^2\gamma = (\tilde{N} \cdot \tilde{P})^2 = N_{n+1}^2 \\ \Leftrightarrow \quad & \langle N, N \rangle = 0. \end{aligned}$$

(ii) A cone containing $\mathcal{Z}^\star$ consists of all lines through its apex and points on $\mathcal{Z}^\star$. For a line with point $[v_1, \ldots, v_{n+1}, 0]$ at infinity the vector $(v_1, \ldots, v_{n+1})$ gives the direction of the line in affine coordinates $x_{n+2} = 1$. Thus, a line contains a point of $\mathcal{Z}^\star$ if and only if it direction vector satisfies

$$v_1^2 + \cdots v_n^2 - v_{n+1}^2 = 0.$$

Yet, by a similar argument as above, this is equivalent to the line intersecting the base plane in an angle of $\frac{\pi}{4}$.

□

Thus, the dual of the Blaschke cylinder model yields the cyclographic model.

Note that orientation reversion acts on the dual space as

$$\sigma^* : (\mathbb{R}\mathrm{P}^{n+1})^* \to (\mathbb{R}\mathrm{P}^{n+1})^*, \quad [x_1, \ldots, x_n, x_{n+1}, x_{n+2}] \mapsto [x_1, \ldots, x_n, -x_{n+1}, x_{n+2}].$$

In particular, it preserves the base plane

$$\mathbf{E} = p^\star = \left\{ [x] \in (\mathbb{R}\mathrm{P}^{n+1})^* \;\middle|\; x_{n+1} = 0 \right\},$$

which we identified with the base plane.

The tangential distance How do we recover the tangential distance in the projective version of the cyclographic model?

We first note that in the hyperplane $q^\star$, the quadric $\mathcal{Z}^\star$ is described by the Lorentz product $\langle \cdot, \cdot \rangle_{n,1}$, which we used to describe the tangential distance. For two oriented hyperspheres $\vec{S}_{(c,r)}, \vec{S}_{(\tilde{c},\tilde{r})}$ the two corresponding points $[c, -r, 1], [\tilde{c}, -\tilde{r}, 1] \subset \mathbb{R}\mathrm{P}^{n+1}$ span a line, which intersects the hyperplane at infinity in a point with representative vector

$$(c, -r, 1) - (\tilde{c}, -\tilde{r}, 1) = (c - \tilde{c}, -r + \tilde{r}, 0)$$

For this point at infinity, we can use the Lortentz product to obtain the tangential distance of $\vec{S}_{(c,r)}$ and $\vec{S}_{(\tilde{c},\tilde{r})}$:

$$\|(c - \tilde{c}, -r + \tilde{r}, 0)\|_{n,1}.$$

But the result depends on the representative vectors chosen for the two oriented hyperspheres.

In the cyclographic model, the hyperplane at infinity is given by

$$q^\star = \left\{[x] \in (\mathbb{R}\mathrm{P}^{n+1})^* \;\middle|\; \alpha(x) = x_{n+2} = 0\right\}$$

where $\alpha : (\mathbb{R}^{n+2})^* \to \mathbb{R}$ is a corresponding linear functional on $(\mathbb{R}^{n+2})^*$. The quadric $\mathcal{Z}^\star \subset q^\star$ is given by this functional and the Lorentz product $\langle \cdot, \cdot \rangle_{n,1}$:

$$\mathcal{Z}^\star = \left\{[x] \in (\mathbb{R}\mathrm{P}^{n+1})^* \;\middle|\; \alpha(x) = 0,\ \langle x, x \rangle_{n,1} = 0\right\}.$$

For two points $[x], [y] \in (\mathbb{R}\mathrm{P}^{n+1})^* \backslash q^\star$ we recovered the tangential distance as

$$\left\| \frac{x}{\alpha(x)} - \frac{y}{\alpha(y)} \right\|_{n,1}.$$

It is invariant under rescaling $x \to \lambda x$ with $\lambda \neq 0$, and also invariant under rescaling $y \to \lambda y$, but not invariant under rescaling $\alpha \to \lambda\alpha$. Yet for three points $[x], [y], [z] \in (\mathbb{R}\mathrm{P}^{n+1})^* \backslash q^\star$ the quotient of tangential distances

$$\frac{\left\| \frac{x}{\alpha(x)} - \frac{y}{\alpha(y)} \right\|_{n,1}}{\left\| \frac{x}{\alpha(x)} - \frac{z}{\alpha(z)} \right\|_{n,1}}.$$

is invariant under rescaling $\alpha \to \lambda\alpha$. Thus, it is a well-defined quantity in $(\mathbb{R}\mathrm{P}^{n+1})^*$, which is entirely determined by $\mathcal{Z}^\star$.

11.2 Miquel's theorem in Laguerre geometry

Theorem 7.6.2 yields a Laguerre geometric version of Miquel's theorem (Theorem 10.8.2).

Theorem 11.2.1 (Miquel's theorem in Laguerre geometry).
Let $\ell_1, \ell_2, \ell_3, \ell_4, m_1, m_2, m_3, m_4$ be eight oriented lines in Euclidean plane. If the five quadrilaterals $(\ell_1, \ell_2, m_1, m_2)$, $(\ell_1, \ell_2, m_3, m_4)$, $(\ell_3, \ell_4, m_1, m_2)$, $(\ell_3, \ell_4, m_3, m_4)$, $(\ell_2, \ell_3, m_2, m_3)$ are circumscribed (each quadruple of lines touches a common oriented circle), then so is the quadrilateral $(\ell_1, \ell_4, m_1, m_4)$ (cf. Figure 11.11).

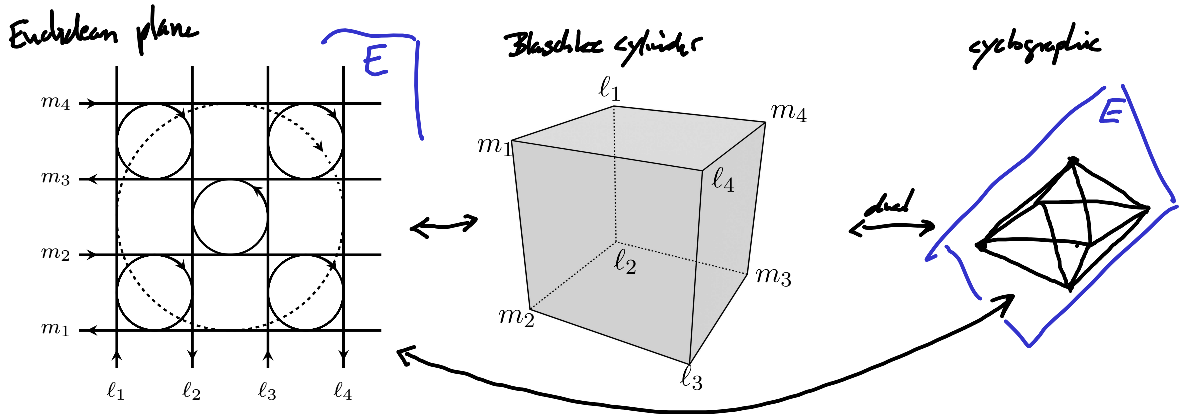
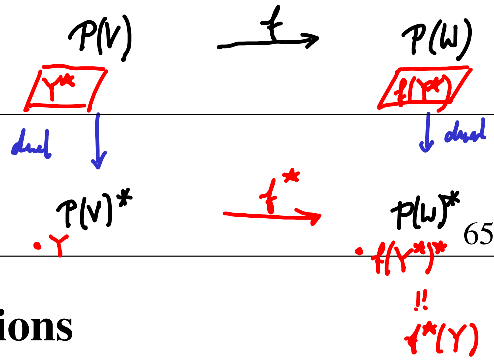


Figure 11.4. Combinatorial pictures on Miquel's theorem in Laguerre geometry. *Left:* The eight oriented lines and six incircles in the plane. *Right:* The eight corresponding points on the Blaschke cylinder and how to associate them with the vertices of a cube.

Proof. The eight oriented lines correspond to eight points on the Blaschke cylinder. Associate them with the vertices of a combinatorial cube (see Figure 11.11). Coplanarity of the bottom and side faces corresponds to the assumed circumscribility. By Theorem 10.8.2 the top face is planar as well. □

Remark 11.2.2. Under duality the cube in the Blaschke cylinder with planar faces becomes an octahedron in the cyclographic model with isotropic face planes.

Week 6: Dual projective transformations, Laguerre transformations



4.2 Dual projective transformations

A projective transformation maps hyperplanes to hyperplanes. Thus it induces a corresponding map on the dual spaces.

Definition 4.2.1. Let $f : \mathrm{P}(V) \to \mathrm{P}(W)$ be a projective transformation. Then the map

$$f^\star : \mathrm{P}(V)^* \to \mathrm{P}(W)^*, \quad Y \mapsto f(Y^\star)^\star.$$

is called the *dual transformation* of f.

Theorem 4.2.2. *The dual transformation is a projective transformation. In particular, it satisfies*

$$f^\star(K^\star) = f(K)^\star$$

for every projective subspace $K \subset \mathrm{P}(V)$.

Proof. The dual map is invertible with inverse $Z \mapsto f^{-1}(Z^\star)^\star$, and maps k-planes to k-planes as shown in the following. Let $X_1, \ldots, X_{k+1} \in \mathrm{P}(V)$ such that $K = X_1 \vee \ldots \vee X_{k+1}$. Then $K^\star = X_1^\star \cap \ldots \cap X_{k+1}^\star$ and

$$\begin{aligned} f(K^\star)^\star &= \big(f(X_1^\star) \cap \ldots \cap f(X_{k+1}^\star)\big)^\star \\ &= f(X_1^\star)^\star \vee \ldots \vee f(X_{k+1}^\star)^\star \\ &= f^\star(X_1) \vee \ldots \vee f^\star(X_{k+1}) \\ &= f^\star(X_1 \vee \ldots \vee X_{k+1}) \\ &= f^\star(K) \end{aligned}$$

$(K_1 \cap K_2)^\star = K_1^\star \vee K_2^\star$ for arbitrary subspaces K_1, K_2

□

Proposition 4.2.3. *If the projective transformation $f : \mathrm{P}(V) \to \mathrm{P}(W)$ is represented by a matrix F with respect to some chosen bases of V and W, then the dual projective transformation $f^\star$ is represented by the matrix*

$$F^\star = F^{-\intercal}.$$

with respect to the corresponding dual bases of V^* and W^*.

Proof. Let $[y] \in (\mathbb{R}\mathrm{P}^n)^*$ with $y \in \mathbb{R}^{n+1}$ the representative vector in the chosen dual basis. Then

$$\begin{aligned} f^\star([y]) &= f([y]^\star)^\star \\ &= f\left(\{[x] \in \mathbb{R}\mathrm{P}^n \mid y^\intercal x = 0\}\right)^\star \\ &= \{[Fx] \in \mathbb{R}\mathrm{P}^n \mid y^\intercal x = 0\}^\star \\ &= \left\{[\tilde{x}] \mid y^\intercal F^{-1}\tilde{x} = 0\right\}^\star \\ &= [F^{-\intercal} y]. \end{aligned}$$

□

Remark 4.2.4. In a basis-free way, this proposition may be formulated as: The dual map is represented by the inverse of the adjoint map.

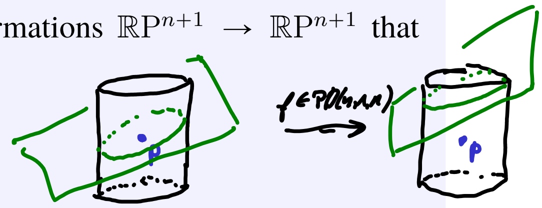
11.3 Laguerre transformations

The transformation group of Laguerre geometry of the n-dimensional Euclidean space consists of all transformations that map oriented hyperplanes to oriented hyperplanes, oriented hyperspheres to oriented hyperspheres, while preserving their oriented contact. Thus, in the Blaschke cylinder model, Laguerre transformations are given by transformations of $\mathbb{RP}^{n+1}$ that preserve the Blaschke cylinder $\mathcal{Z}$ and map hyperplanes to hyperplanes.

Definition 11.3.1. The group of projective transformations $\mathbb{RP}^{n+1} \to \mathbb{RP}^{n+1}$ that preserve the Blaschke cylinder $\mathcal{Z}$

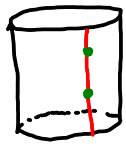
$$\mathrm{PO}(n,1,1)$$

is also called the group of *Laguerre transformations*.



In Laguerre geometry points of the n-dimensional Euclidean space are described as special hyperspheres, namely null-spheres. Thus, points are not generally mapped to points by a Laguerre transformations, but to hyperspheres.

Proposition 11.3.2. *Laguerre transformations map parallel oriented hyperplanes to parallel oriented hyperplanes.*



Proof. Parallel oriented hyperplanes are described by points on the Blaschke cylinder contained in the same generator. As projective transformations that preserve the Blaschke cylinder Laguerre transformations map straight line generators to straight line generators. □

Theorem 11.3.3.

(i) Every Laguerre transformation $f \in \mathrm{PO}(n,1,1)$ in the Blaschke cylinder model is of the form

$$f = \left[\begin{array}{c|c} A & 0 \\ \hline c^\intercal & d \end{array}\right]$$

with some $A \in \mathrm{O}(n,1)$, $c \in \mathbb{R}^{n+1}$, and $d \neq 0$.

(ii) Dually, every Laguerre transformation $f^\star \in \mathrm{PO}(n,1,1)^\star$ in the cyclographic model is of the form

$$f^\star = \left[\begin{array}{c|c} \tilde{A} & \tilde{b} \\ \hline 0 & \tilde{d} \end{array}\right]$$

with some $\tilde{A} \in \mathrm{O}(n,1)$, $\tilde{b} \in \mathbb{R}^{n+1}$, and $\tilde{d} \neq 0$.

Proof.

(i) Let $f = [F]$, $F \in \mathrm{GL}(n+2,\mathbb{R})$, $A \in \mathrm{GL}(n+1,\mathbb{R})$, $b, c \in \mathbb{R}^{n+1}$, $d \in \mathbb{R}$ with

$$F = \left(\begin{array}{c|c} A & b \\ \hline c^\intercal & d \end{array}\right).$$

Furthermore, let

$$Z := \left(\begin{array}{c|c} \tilde{Z} & 0 \\ \hline 0 & 0 \end{array}\right), \qquad \tilde{Z} := \mathrm{diag}(1,\dots,1,-1) \in \mathbb{R}^{(n+1)\times(n+1)}$$

denote the Gram matrix of the Blaschke cylinder. Then,

$$F^\intercal Z F = \left(\begin{array}{c|c} A^\intercal & c \\ \hline b^\intercal & d \end{array}\right)\left(\begin{array}{c|c} \tilde{Z} & 0 \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} A & b \\ \hline c^\intercal & d \end{array}\right) = \left(\begin{array}{c|c} A^\intercal \tilde{Z} A & A^\intercal \tilde{Z} b \\ \hline b^\intercal \tilde{Z} A & b^\intercal \tilde{Z} b \end{array}\right) = \left(\begin{array}{c|c} \tilde{Z} & 0 \\ \hline 0 & 0 \end{array}\right)$$

$\begin{cases} A^\intercal \tilde{Z} A = \tilde{Z} \\ A^\intercal \tilde{Z} b = 0 \Leftrightarrow b = 0 \end{cases}$

Thus, $F^\intercal Z F = Z$ is equivalent to $A \in \mathrm{O}(n,1)$ and $b = 0$. To ensure $F \in \mathrm{GL}(n+2,\mathbb{R})$ this further implies $d \neq 0$.

(ii) We find that

$$F^{-\intercal} = \left(\begin{array}{c|c} A^{-\intercal} & -\frac{1}{d}A^{-\intercal}c \\ \hline 0 & \frac{1}{d} \end{array}\right) =: \left(\begin{array}{c|c} \tilde{A} & \tilde{b} \\ \hline 0 & \tilde{d} \end{array}\right)$$

where $A \in \mathrm{O}(n,1) \Leftrightarrow \tilde{A} \in \mathrm{O}(n,1)$, $c \in \mathbb{R}^{n+1} \Leftrightarrow \tilde{b} \in \mathbb{R}^{n+1}$, $d \neq 0 \Leftrightarrow \tilde{d} \neq 0$.

$A^\intercal \tilde{Z} A = \tilde{Z} \Leftrightarrow A^{-1} \tilde{Z} A^{-\intercal} = \tilde{Z} \Leftrightarrow (A^{-\intercal})^\intercal \tilde{Z} A^{-\intercal} = \tilde{Z}$

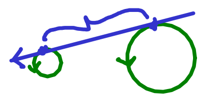
□

We found that, in the cyclographic model, Laguerre transformations are special affine transformations. In affine coordinates $x_{n+2} = 1$ a Laguerre transformation $f^\star \in \mathrm{PO}(n, 1, 1)$ takes the form

$$g : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \quad x = (x_1, \dots, x_{n+1}) \mapsto \lambda A x + b$$

with some $A \in \mathrm{O}(n, 1)$, $b \in \mathbb{R}^{n+1}$, and $\lambda \neq 0$.

Thus, it preserves the ratios of tangential distances



$$\| g(x) - g(\tilde{x}) \|^2_{n,1} = \lambda^2 \| x - \tilde{x} \|^2_{n,1} ,$$

similar to similarity transformations preserving the ratios of Euclidean distances.

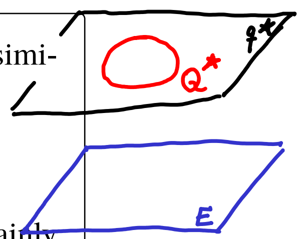
Corollary 11.3.4. *Laguerre transformations in the cyclographic model are exactly the affine transformations preserving ratios of the tangential distance.*

To better understand the group of Laguerre transformations we first establish that it contains the group of similarity transformations.

Proposition 11.3.5. *A Laguerre transformation $f \in \mathrm{PO}(n, 1, 1)$ is a similarity transformation if and only if it fixes the point*

$$p = [e_{n+1}] = [0, \cdots, 0, 1, 0].$$

Proof. Dually, this means that a Laguerre transformation $f^\star \in \mathrm{PO}(n, 1, 1)^\star$ is a similarity transformation if and only if it preserves the base plane



$$\mathbf{E} = p^\star = \left\{ [x] \in (\mathbb{RP}^{n+1})^* \;\middle|\; x_{n+1} = 0 \right\} ,$$

Thus, if and only if it maps points of the Euclidean space to points, which is certainly a necessary requirement for a similarity transformation.

In affine coordinates $x_{n+2} = 1$, the condition $g(\mathbf{E}) = \mathbf{E}$ on the transformation

$$g : x = (x_1, \dots, x_{n+1}) \mapsto \lambda A x + b$$

reads

$$\lambda a_{n+1,1} x_1 + \lambda a_{n+1,n} x_n + b_{n+1} = 0$$

for all $x_1, \dots, x_n \in \mathbb{R}$, which yields

$$b_{n+1} = a_{n+1,1} = \dots = a_{n+1,n} = 0.$$

Since $A \in \mathrm{O}(n,1)$ this implies

$$a_{1,n+1} = \cdots = a_{n,n+1} = 0,$$

and further

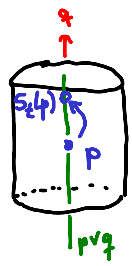
$$a_{n+1,n+1} = 1.$$

Thus, we obtain

$$A = \left(\begin{array}{c|c} R & 0 \\ \hline 0 & 1 \end{array}\right), \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \\ 0 \end{pmatrix} \in \mathbb{R}^{n+1}, \qquad \lambda \neq 0$$

with $R \in \mathrm{O}(n)$, which describes a similarity transformation on $\mathbf{E}$. □

As examples of Laguerre transformations which are not similarity transformations we introduce the following two families of transformations:



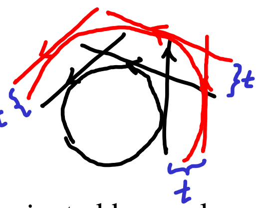
Laguerre offset Consider the family of Laguerre transformations

$$S_t = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & \begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix} \end{array}\right], \qquad t \in \mathbb{R}$$

with $I = \mathrm{diag}(1, \cdots, 1) \in \mathbb{R}^{n \times n}$. Note that S_t preserves the line $p \vee q = \mathrm{span}\{e_{n+1}, e_{n+2}\}$, and maps p to any point on this line except q.

It acts on an oriented hyperplane $\vec{P}_{(\nu,h)}$ by

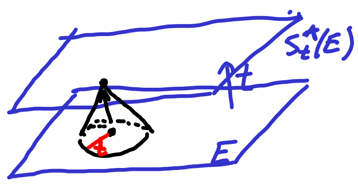
$$S_t \begin{bmatrix} \nu \\ 1 \\ h \end{bmatrix} = \begin{bmatrix} \nu \\ 1 \\ h+t \end{bmatrix}$$



and thus maps every oriented hyperplane to a parallel oriented hyperplane at distance t.

Dually, in the cyclographic model, this family is described by

$$S_t^\star = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & \begin{smallmatrix} 1 & -t \\ 0 & 1 \end{smallmatrix} \end{array}\right], \qquad t \in \mathbb{R}.$$

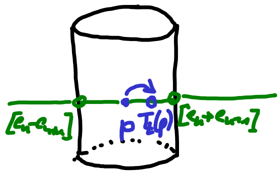


It acts on an oriented hypersphere $\vec{S}_{(c,r)}$ by

$$S_t^\star \begin{bmatrix} c \\ -r \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ -r-t \\ 1 \end{bmatrix}$$

and thus maps every oriented hypersphere with radius r to a concentric oriented hypersphere with radius $r + t$.

Laguerre boost Consider the family of Laguerre transformations



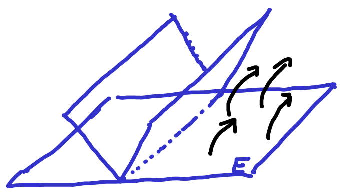
$$T_t = \left[\begin{array}{c|ccc} I & & 0 & \\ \hline 0 & \cosh t & \sinh t & 0 \\ & \sinh t & \cosh t & 0 \\ & 0 & 0 & 1 \end{array}\right], \qquad t \in \mathbb{R},$$

with $I = \operatorname{diag}(1, \cdots, 1) \in \mathbb{R}^{(n-1)\times(n-1)}$. It preserves the line $\operatorname{span}\{e_n, e_{n+1}\}$, which intersects the Blaschke cylinder in the two points

$$[e_n \pm e_{n+1}] = [0, \cdots, 0, 1, \pm 1, 0],$$

and it maps p to any point on this line inside the Blaschke cylinder.

Dually, in the cyclographic model, this family is described by



$$T_t^\star = \left[\begin{array}{c|ccc} I & & 0 & \\ \hline 0 & \cosh t & -\sinh t & 0 \\ & -\sinh t & \cosh t & 0 \\ & 0 & 0 & 1 \end{array}\right], \qquad t \in \mathbb{R},$$

It maps the base plane $\mathbf{E} = p^\star$ to any space-like hyperplane in the pencil of hyperplanes spanned together with $[e_n \pm e_{n+1}]^\star$.

It turns out, that up to similarity transformations a Laguerre transformation is a either a Laguerre offset or a Laguerre boost.

Theorem 11.3.6. *Let $f \in \mathrm{PO}(n,1,1)$ be a Laguerre transformation. Then there exist two similarity transformations $\Phi, \Psi \in \mathrm{PO}(n,1,1)$ such that either*

$$f = \Phi \circ S_t = S_t \circ \Psi$$

for some $t \in \mathbb{R}$, or

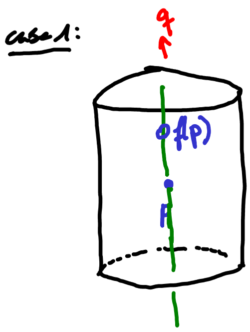
$$f = \Phi \circ T_t \circ \Psi$$

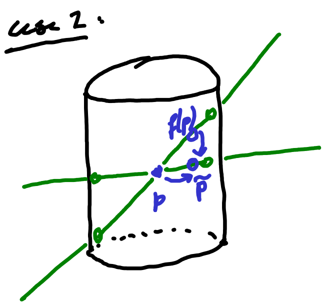
for some $t \in \mathbb{R}$.

Proof. Consider the line $L = p \vee f(p)$.

- If L contains the point q, let $t \in \mathbb{R}$ such that $S_t(p) = f(p)$. Then $\Phi = S_t^{-1} \circ f$ fixes the point p and thus is a similarity transformation.
- If L does not contain the point q, it is a line of signature $(+-)$ and intersects the Blaschke cylinder $\mathcal{Z}$ in two points. Let Ψ be a similarity transformation that maps $r := [e_n + e_{n+1}]$ to one of the intersection points $L \cap \mathcal{Z}$. Then it maps the line $\tilde{L} = p \vee r$ to the line L, and thus $\tilde{p} = \Psi^{-1} \circ f(p) \in \tilde{L}$. Let $t \in \mathbb{R}$ such that $T_t(p) = \tilde{p}$. Then $\Phi = T_t^{-1} \circ \Psi^{-1} \circ f$ fixes the point p and thus is a similarity transformation.

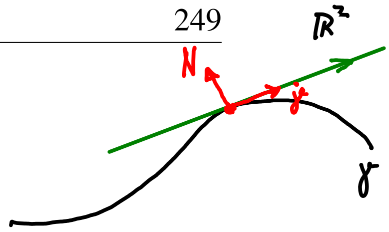
□





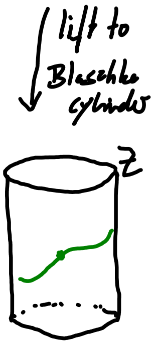
Week 7: Curves in Laguerre geometry, Gingham incircular nets

11.4 Curves in Laguerre geometry



Let

$$\gamma : [a, b] \to \mathbb{R}^2$$



be a smooth regular curve in the Euclidean plane. Its unit tangent and normal vector are given by

$$T(t) := \frac{\dot\gamma}{\|\dot\gamma\|}, \qquad N(t) := JT(t), \quad \text{where } J := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right).$$

Then the tangent line at the point $\gamma(t)$ is given by

$N(t)\cdot(x-\gamma(t))=0$

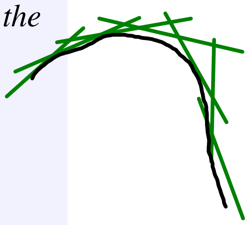
$$P_{(N(t),h(t))} = \left\{ x \in \mathbb{R}^2 \;\middle|\; N(t) \cdot x + h(t) = 0 \right\}, \quad h(t) := -N(t) \cdot \gamma(t).$$

The oriented tangent lines $\vec{P}_{(N(t),h(t))}$ yield a curve on the Blaschke cylinder. We have seen this in the example of circles which correspond to curves on the Blaschke cylinder given by planar sections. On the other hand, the curve γ can be uniquely reconstructed from its tangent lines as the envelope.

Proposition 11.4.1. *Let γ be a smooth regular curve in $\mathbb{R}^2$. Then*

$$\hat\gamma(t) := (N(t), 1, h(t)), \quad h(t) := -N(t) \cdot \gamma(t)$$

defines a curve on the Blaschke cylinder. The corresponding oriented lines are the oriented tangent lines of γ, i.e.,



$$\begin{aligned} N \cdot \gamma + h &= 0, \\ N \cdot \dot\gamma &= 0. \end{aligned}$$

Furthermore, the curve γ is the envelope of those lines, i.e.,

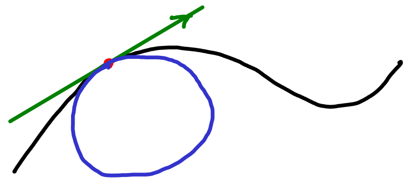
$$\begin{aligned} N \cdot \gamma + h &= 0, \\ \dot N \cdot \gamma + \dot h &= 0. \end{aligned} \tag{11.2}$$

Vice versa, given a smooth regular curve $t \mapsto (N(n), 1, h(t))$ on the Blaschke cylinder not tangent to a generator, equations (11.2) determine a unique curve as the envelope of the corresponding oriented lines in the plane.

11.4.1 Osculating circle of planar curves

The osculating circle of the planar curve γ at the point $\gamma(t)$ is the circle $\vec{S}_{(c(t),r(t))}$ with center

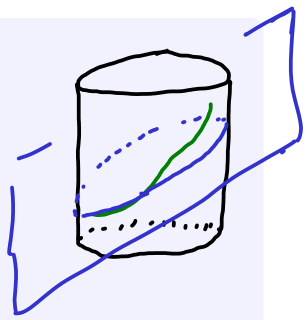
$$c(t) := \gamma(t) + \frac{1}{\kappa(t)} N(t)$$



and radius

$$r(t) := \frac{1}{\kappa(t)}$$

where $\kappa(t)$ is the curvature at $\gamma(t)$.



Proposition 11.4.2. *Let γ a smooth regular curve in $\mathbb{R}^2$. Let*

$$\hat{\gamma}(t) = (N(t), 1, h(t))$$

be its lift to the Blaschke cylinder, and let

$$\hat{c}(t) := (c(t), -r(t), 1)$$

be the lift of its osculating circle to the cyclographic model. Then

$$[\hat{c}(t)]^\star = \mathrm{P}\left(\operatorname{span}\{\hat{\gamma}, \dot{\hat{\gamma}}, \ddot{\hat{\gamma}}\}\right).$$

Proof. Show that

$$\hat{c}^\intercal \hat{\gamma} = \hat{c}^\intercal \dot{\hat{\gamma}} = \hat{c}^\intercal \ddot{\hat{\gamma}} = 0$$

where one uses $\ddot{N} \cdot N = -\dot{N} \cdot \dot{N}$ and $\dot{N} = -\kappa \dot{\gamma}$. □

To apply a Laguerre transformation to a curve it is applied to its oriented tangent lines. Then the image curve is reconstructed as the envelope of the image tangent lines.

Corollary 11.4.3. *The osculating circle of a planar curve is Laguerre invariant.*

11.4.2 Conics and hypercycles

We will now study which curves on the Blaschke cylinder correspond to conics (more precisely ellipses and hyperbolas).

By means of a rotation and a translation (which constitute special Laguerre transformations) an ellipse or a hyperbola may be brought into the form

$$C = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a} + \frac{y^2}{b} = 1 \right\} \tag{11.3}$$

with some $a, b \neq 0$. The case $a > 0, b > 0$ corresponds to an ellipse and the case $ab < 0$ to a hyperbola.

Proposition 11.4.4. *The curve on the Blaschke cylinder $\mathcal{Z}$ corresponding to the tangent lines (with both orientations) of the conic C is given by the intersection of $\mathcal{Z}$ with the cone*

$$\mathcal{C} = \left\{ [x_1, x_2, x_3, x_4] \in \mathbb{RP}^3 \;\middle|\; ax_1^2 + bx_2^2 - x_4^2 = 0 \right\}. \tag{11.4}$$

Proof. The tangent line to C at a point $(x_0, y_0) \in C$ is given by

$$\left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \frac{xx_0}{a} + \frac{yy_0}{b} = 1 \right\},$$

and its two lifts to the Blaschke cylinder by

$$\left[\frac{x_0}{a}, \frac{y_0}{b}, \pm\sqrt{\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}}, -1 \right] = \left[\frac{x_0 h}{a}, \frac{y_0 h}{b}, \pm 1, h \right] \in \mathcal{Z}$$

where

$$h := \frac{1}{\sqrt{\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}}}.$$

□

In particular, we found that the curve on the Blaschke cylinder corresponding to an ellipse or hyperbola is given by the intersection with a quadric.

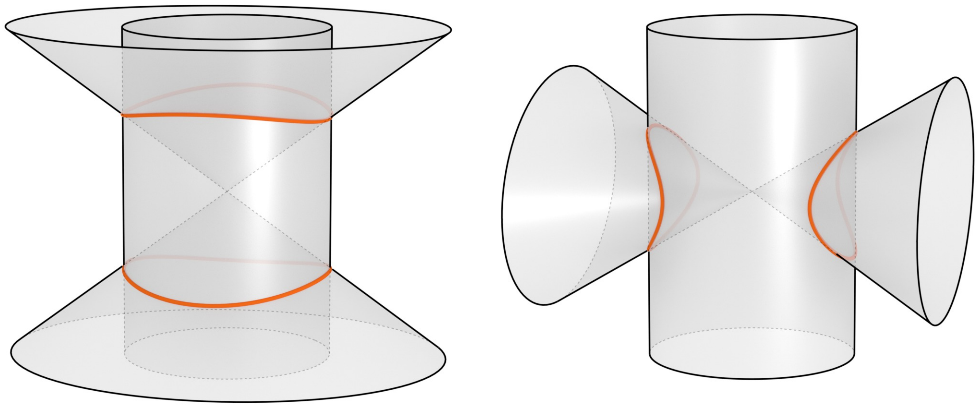


Figure 11.5. Hypercycle base curves corresponding to an ellipse and hyperbola respectively.

Definition 11.4.5. The intersection curve of the Blaschke cylinder $\mathcal{Z}$ with another quadric $\mathcal{Q}$ is called a *hypercycle base curve*. The envelope of the corresponding lines in the plane is called a *hypercycle*.

Corollary 11.4.6. *Conics (considered with both orientations) are hypercycles.*

The hypercycle base curve is the base curve of the pencil of quadrics spanned by $\mathcal{Z}$ and $\mathcal{Q}$. The intersection of any quadric from this pencil with the Blaschke cylinder yields the same curve $\mathcal{Z} \cap \mathcal{Q}$.

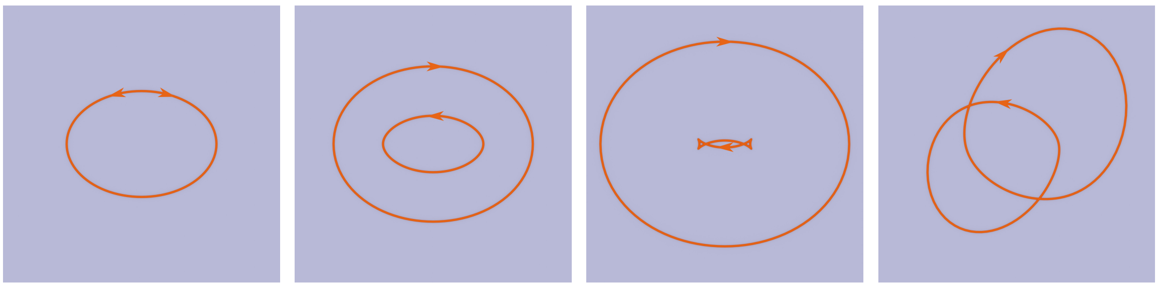
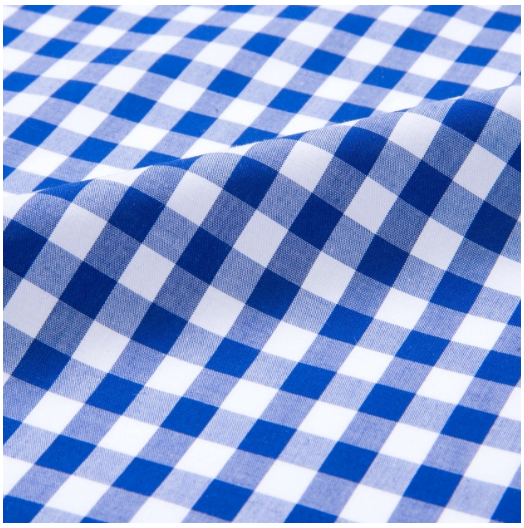


Figure 11.6. A conic under Laguerre transformations.

11.5 Gingham incircular nets

Definition 11.5.1. Two families $(\ell_i)_{i\in\mathbb{Z}}$, $(m_j)_{j\in\mathbb{Z}}$ of oriented lines in the Euclidean plane are called a *gingham incircular net* if for every even $i, j \in \mathbb{Z}$ the four lines $\ell_i, \ell_{i+1}, m_j, m_{j+1}$ touch a common oriented circle.



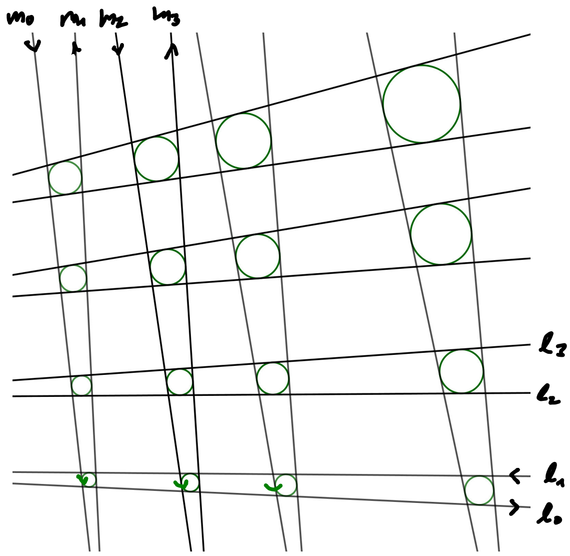


Figure 11.7. *Left:* Gingham fabric. *Right:* A piece of a gingham incircular net.

Theorem 11.5.2. *All lines of a generic gingham incircular net are in oriented contact with a common hypercycle.*

Moreover, let $(\ell_i)_{i\in\mathbb{Z}}$, $(m_j)_{j\in\mathbb{Z}}$ be the points on the Blaschke cylinder $\mathcal{Z} \subset \mathbb{RP}^3$ that correspond to the oriented lines of the gingham incircular net. Consider the lines

$$L_i := \ell_i \vee \ell_{i+1}, \qquad M_j := m_j \vee m_{j+1}.$$

Then, all lines L_{2k}, M_{2l} lie on a common hyperboloid $\mathcal{H} \subset \mathbb{RP}^3$, which intersects the Blaschke cylinder in the hypercycle base curve. The lines L_{2k} are contained in one of the two families of rulings of $\mathcal{H}$ while the lines M_{2l} are contained in the other family of rulings of $\mathcal{H}$.

Proof. The existence of the incircles in a gingham incircular nets is equivalent to every line L_{2k} intersecting every line M_{2l}, and vice versa. Thus, all lines L_{2k}, M_{2l} generically lie on a common hyperboloid $\mathcal{H}$.

Furthermore, this implies that all points $(\ell_i)_{i\in\mathbb{Z}}$, $(m_j)_{j\in\mathbb{Z}}$ lie in the intersection of the Blaschke cylinder and $\mathcal{H}$. Thus, the corresponding oriented lines touch a common hypercycle. □

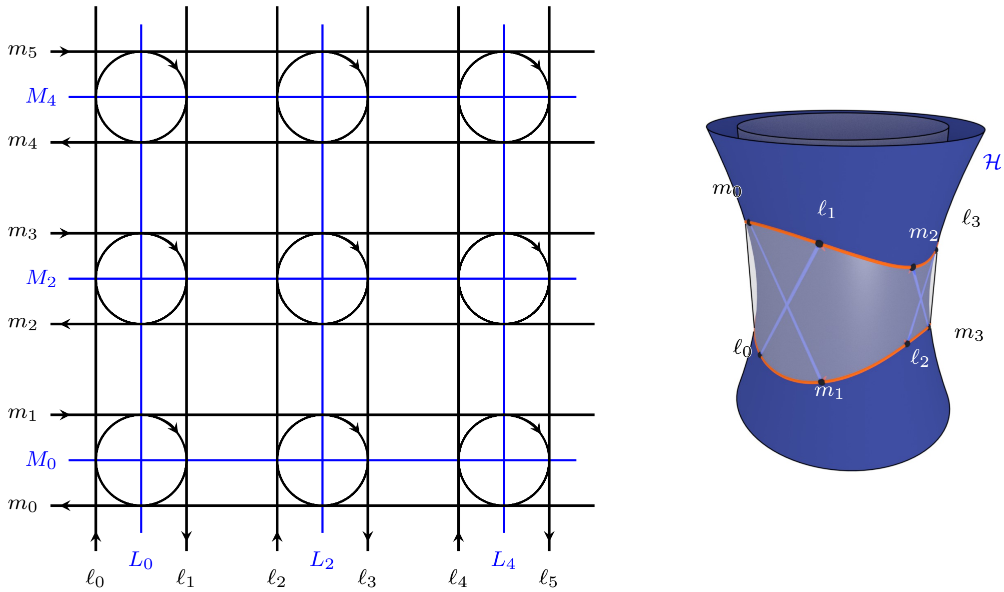


Figure 11.8. Gingham incircular net on the Blaschke cylinder. The lines L_{2k}, M_{2k} are rulings of the hyperboloid $\mathcal{H}$.

This result yields the following construction for gingham incircular nets:

- Choose a hyperboloid $\mathcal{H} \subset \mathbb{R}\mathrm{P}^3$ that intersects the Blaschke cylinder $\mathcal{Z}$. This corresponds to choosing a hypercycle to be the envelope of the gingham incircular net.
- Distinguish the two families of rulings of $\mathcal{H}$ as the L-family and the M-family.
- Choose two arbitrary points ℓ_0 and m_0 on the hypercycle base curve $\mathcal{H} \cap \mathcal{Z}$.
- Let L_0 be the ruling in the L-family that contains the point ℓ_0 and define ℓ_1 as the second intersection point of L_0 with $\mathcal{Z}$. Similarly, let M_0 be the ruling in the M-family that contains the point m_0 and define m_1 as the second intersection point of M_0 with $\mathcal{Z}$.
- Choose two arbitrary points ℓ_2 and m_2 on the hypercycle base curve $\mathcal{H} \cap \mathcal{Z}$, and continue in the same manner.

Remark 11.5.3. A special case of gingham incircular nets is given by *checkerboard incircular nets*. Here all lines lines $\ell_i, \ell_{i+1}, m_j, m_{j+1}$ with $i, j \in \mathbb{Z}$ where $i + j$ is even touch a common oriented circle.

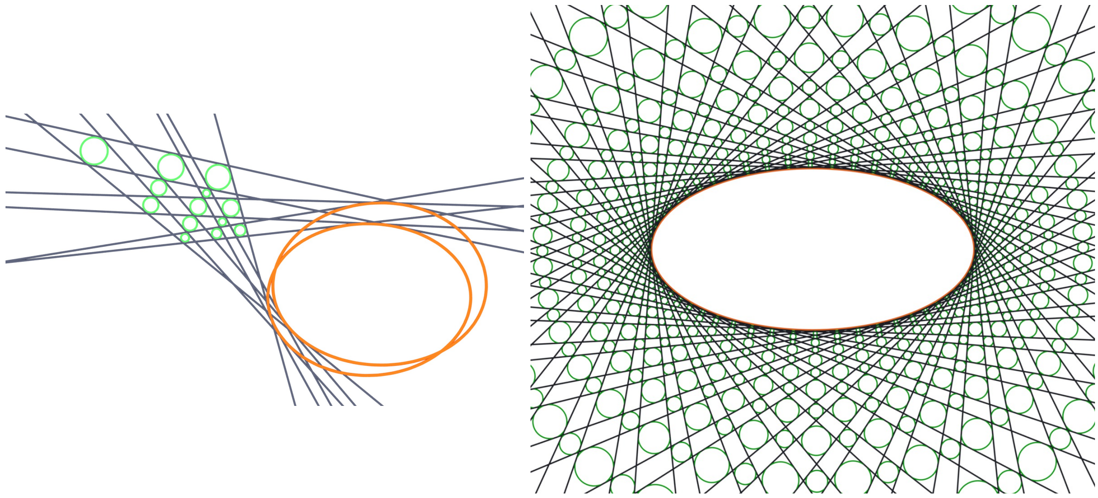
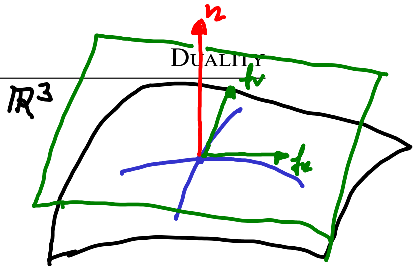


Figure 11.9. *Left:* Piece of a checkerboard incircular net touching a hypercycle. *Right:* Periodic checkerboard incircular net touching an ellipse.

Week 8: Dual representation of surfaces, surfaces in Laguerre geometry, Lie geometry



4.3 Dual representation of surfaces

Let

$$f : \mathbb{R}^2 \supset U \to \mathbb{R}^3$$

be a smooth regular parametrized surface patch,

Instead of describing the surface f as a two-parameter family of points, we can equivalently describe it as the envelope of its two-parameter family of tangent planes.

Let

$$n : U \to \mathbb{R}^3$$

be an arbitrary smooth normal field of f, i.e.,

$$\begin{aligned} n \cdot f_u &= 0, \\ n \cdot f_v &= 0 \end{aligned}$$

The tangent plane of f at the point $(u, v) \in U$ is given by

$$\left\{ x \in \mathbb{R}^3 \,\middle|\, n(u,v) \cdot x + h(u,v) = 0 \right\}$$

with some function $h : U \to \mathbb{R}$. Thus, the tangent planes of f described by the tuple (n, h) (uniquely up to a common scalar multiple) is determined by the set of equations

$$\begin{aligned} n \cdot f_u &= 0, \\ n \cdot f_v &= 0, \\ n \cdot f + h &= 0. \end{aligned} \tag{4.1}$$

$\to$ $n_u \cdot f + n \cdot f_u + h_u = 0$
$n_v \cdot f + n \cdot f_v + h_v = 0$

Differentiating the last equation with respect to u and v, respectively, we find that (4.1) is equivalent to

$$\begin{aligned} f \cdot n_u + h_u &= 0, \\ f \cdot n_v + h_v &= 0, \\ f \cdot n + h &= 0. \end{aligned} \tag{4.2}$$

Note that if we consider the lifts

$$\begin{aligned} \hat{f} &:= (f, 1), \\ \hat{n} &:= (n, h) \end{aligned}$$

to homogeneous coordinates of $\mathbb{RP}^3$ and $(\mathbb{RP}^3)^*$, respectively, then equations (4.1) and (4.2) become the duality relations for tangent planes of the respective surfaces $[\hat{f}]$ and $[\hat{n}]$.

Definition 4.3.1. Let $[\hat{f}] : \mathbb{R}^2 \supset U \to \mathbb{RP}^3$ be a smooth regular parametrized surface in $\mathbb{RP}^3$. Then

$$[\hat{n}] := ([\hat{f}] \vee [\hat{f}_u] \vee [\hat{f}_v])^\star : U \to (\mathbb{RP}^3)^*$$

is called the *dual surface* of f.

In homogeneous coordinates the dual surface is determined by the three linearly independent equations

$$\begin{aligned} \hat{n} \cdot \hat{f}_u &= 0, \\ \hat{n} \cdot \hat{f}_v &= 0, \\ \hat{n} \cdot \hat{f} &= 0, \end{aligned} \tag{4.3}$$

and satisfies

$$\begin{aligned} \hat{f} \cdot \hat{n}_u &= 0, \\ \hat{f} \cdot \hat{n}_v &= 0, \\ \hat{f} \cdot \hat{n} &= 0. \end{aligned} \tag{4.4}$$

These equations are completely symmetric in $\hat{f}$ and $\hat{n}$.

Proposition 4.3.2. *If the dual surface of a smooth regular parametrized surface $[\hat{f}]$ in $\mathbb{RP}^3$ is itself regular, then the dual surface of a the dual surface is $[\hat{f}]$.*

Remark 4.3.3. The primal surface is regular if it is locally not a curve. The dual surface is regular if the primal surface is locally not developable.

We have established in Section 2.7 that conjugate line parametrizations are a notion of projective geometry.

Theorem 4.3.4. *A smooth regular parametrized surface $[\hat{f}] : \mathbb{R}^2 \supset U \to \mathbb{RP}^3$ is a conjugate line parametrization if and only if its dual surface $[\hat{n}] : U \to (\mathbb{RP}^3)^*$ is a conjugate line parametrization.*

Proof. $[\hat{f}]$ is a conjugate line parametrization if $\hat{f}$ satisfies an equation in homogeneous coordinates of the form

$$\hat{f}_{uv} = \alpha \hat{f}_u + \beta \hat{f}_v + \gamma \hat{f},$$

which is equivalent to

$$\hat{f}_{uv} \cdot \hat{n} = 0.$$

From equations (4.3), or equivalently, equations (4.4), we find that this is equivalent to

$\hat{n}\cdot\hat{f}=0 \rightsquigarrow \hat{n}_u\cdot\hat{f}+\underbrace{\hat{n}\cdot\hat{f}_u}_{=0}=0 \rightsquigarrow \hat{n}_{uv}\cdot\hat{f}+\hat{n}_u\cdot\hat{f}_v=0$

either of the three equations

$$\begin{aligned} \hat{f}_u \cdot \hat{n}_v &= 0, \\ \hat{f}_v \cdot \hat{n}_u &= 0, \\ \hat{f} \cdot \hat{n}_{uv} &= 0, \end{aligned} \tag{4.5}$$

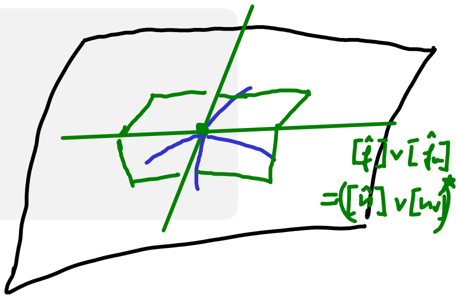
and thus in turn to

$$\hat{n}_{uv} = \tilde{\alpha}\hat{n}_u + \tilde{\beta}\hat{n}_v + \tilde{\gamma}\hat{n},$$

□

Remark 4.3.5. The first two equations of (4.5) state, respectively, that

$$[\hat{f}] \vee [\hat{f}_u] = ([\hat{n}] \vee [\hat{n}_v])^\star,$$
$$[\hat{f}] \vee [\hat{f}_v] = ([\hat{n}] \vee [\hat{n}_u])^\star.$$



11.6 Surfaces in Laguerre geometry

Let

$$f : \mathbb{R}^2 \supset U \to \mathbb{R}^3$$

be a smooth regular parametrized surface patch. Let

$$n : U \to \mathbb{R}^3$$

be an arbitrary smooth normal field of f such that at every point $(u, v) \in U$

$$n = \lambda(f_u \times f_v)$$

with some positive scalar $\lambda > 0$, and let

$$\sigma := \|n\| > 0$$

denote the norm of n. Furthermore, let h be such that

$$n \cdot f + h = 0.$$

Then the lift of f to the Blaschke cylinder is given by

$$\hat{f} := (n, \sigma, h).$$

$\langle \hat{f}, \hat{f} \rangle = n \cdot n - \sigma^2 = 0$

Recall that f is a curvature line parametrization if and only if f is orthogonal and conjugate. In Section 4.3 we have established that f is conjugate if and only if its dual surface $[n, h]$ is conjugate. Thus, to describe curvature line parametrizations in Laguerre geometry we should determine how to express the orthogonality in the homogeneous coordinates (n, σ, h).

Lemma 11.6.1. *For a parametrized surface f the lift to the Blaschke cylinder (n, σ, h) satisfies*

$$\begin{aligned} \sigma^2 &= n \cdot n, \\ \sigma\sigma_u &= n \cdot n_u, \\ \sigma\sigma_v &= n \cdot n_v, \\ \sigma\sigma_{uv} + \sigma_u\sigma_v &= n \cdot n_{uv} + n_u \cdot n_v. \end{aligned} \tag{11.5}$$

Lemma 11.6.2. Let f be a conjugate line parametrized surface. Then ~~*A parametrized surface*~~ *f is orthogonal if and only if its lift to the Blaschke cylinder (n, σ, h) satisfies*

$$\sigma\sigma_{uv} = n \cdot n_{uv}.$$

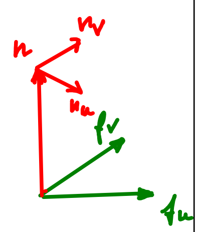
Proof. Since f_u is proportional to $n_v \times n$ and f_v is proportional to $n \times n_u$, the orthogonality condition

$$f_u \cdot f_v = 0$$

is equivalent to

$$\begin{aligned} & (n_v \times n) \cdot (n \times n_u) = 0 \\ \Leftrightarrow\ & (n \cdot n_u)(n \cdot n_v) = (n \cdot n)(n_u \cdot n_v) \\ \Leftrightarrow\ & \sigma_u \sigma_v = n_u \cdot n_v \\ \Leftrightarrow\ & \sigma \sigma_{uv} = n \cdot n_{uv}, \end{aligned}$$

where we used Lemma 11.6.1. □



f conjugate ⇔ $f_u \cdot n_v = 0$ ⇔ $f_v \cdot n_u = 0$

Theorem 11.6.3. *Let $f : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a parametrized surface and*

$$\hat{f} := (n, \sigma, h)$$

a lift to the Blaschke cylinder. Then f is a curvature line parametrization if and only if $[\hat{f}]$ is a conjugate parametrization.

Proof. f is a conjugate line parametrization if and only if $[n, h]$ is a conjugate line parametrization, i.e., if

$$\begin{aligned} n_{uv} &= \alpha n_u + \beta n_v + \gamma n, \\ h_{uv} &= \alpha h_u + \beta h_v + \gamma h \end{aligned}$$

.

with some functions $\alpha, \beta, \gamma : U \to \mathbb{R}$.

Now if f is orthogonal, then by Lemma 11.6.1 and Lemma 11.6.2

$$\sigma \sigma_{uv} = n_{uv} \cdot n = \alpha n_u \cdot n + \beta n_v \cdot n + \gamma n \cdot n = \alpha \sigma \sigma_u + \beta \sigma \sigma_v + \gamma \sigma^2$$

and thus

$$\sigma_{uv} = \alpha \sigma_u + \beta \sigma_v + \gamma \sigma.$$

Vice versa, if σ satisfies the previous equation, the argument may be reversed. □

Corollary: Curvature line parametrizations are Laguerre invariant.

Chapter 12

Lie geometry

The basic objects from Möbius geometry and Laguerre geometry may all be seen as special cases of oriented spheres.

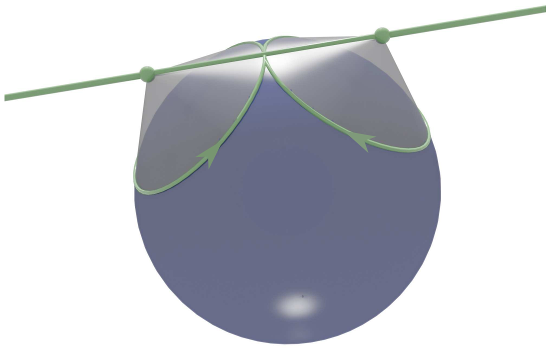


Figure 12.1. The Möbius quadric $\mathcal{S} \subset \mathbb{R}\mathrm{P}^{n+1}$ (depicted in the case $n = 2$) and two oriented hyperspheres in oriented contact.

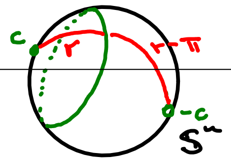
12.1 Oriented hyperspheres of $\mathbb{S}^n$

We first give an informal description of Lie (sphere) geometry as the geometry of oriented hyperspheres of the n-dimensional sphere $\mathbb{S}^n$ and their oriented contact.

Thus, let

$$\mathbb{S}^n = \left\{ y \in \mathbb{R}^{n+1} \;\middle|\; y \cdot y = 1 \right\} \subset \mathbb{R}^{n+1},$$

where $y \cdot y$ denotes the standard scalar product on $\mathbb{R}^{n+1}$. An oriented hypersphere of $\mathbb{S}^n$ can be represented by its center $c \in \mathbb{S}^n$ and its signed spherical radius $r \in \mathbb{R}$ (see Figure



12.1). Tuples $(c, r) \in \mathbb{S}^n \times \mathbb{R}$ represent the same oriented hypersphere if they are related by a sequence of the transformations

$$\rho_1 : (c, r) \mapsto (c, r + 2\pi), \quad \rho_2 : (c, r) \mapsto (-c, r - \pi). \tag{12.1}$$

The corresponding hypersphere as a set of points is given by

$$\{y \in \mathbb{S}^n \mid c \cdot y = \cos r\}, \tag{12.2}$$

while its orientation is obtained in the following way: The hypersphere separates the sphere $\mathbb{S}^n$ into two regions. For $r \in [0, \pi)$ consider the region which contains the center c to be the "inside" of the hypersphere, and endow the hypersphere with an orientation by assigning normal vectors pointing towards this region. The orientation of the hypersphere for other values of r is then obtained by (12.1).

Definition 12.1.1. We call

$$\vec{\mathscr{S}} := {(\mathbb{S}^n \times \mathbb{R})}\big/{\{\rho_1, \rho_2\}}.$$

the *space of oriented hyperspheres* of $\mathbb{S}^n$.

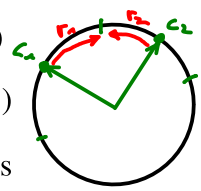
Remark 12.1.2. Orientation reversion defines an involution on $\vec{\mathscr{S}}$, which is given by

$$\rho : (c, r) \mapsto (c, -r).$$

Thus, the *space of (non-oriented) hyperspheres* of $\mathbb{S}^n$ may be represented by

$$\mathscr{S} := \vec{\mathscr{S}}\big/\rho = (\mathbb{S}^n \times \mathbb{R})\big/\{\rho, \rho_1, \rho_2\}.$$

Two oriented hyperspheres (c_1, r_1) and (c_2, r_2) are in *oriented contact* if (see Figure 12.1)



$$c_1 \cdot c_2 = \cos(r_1 - r_2), \tag{12.3}$$

which is a well-defined relation on $\vec{\mathscr{S}}$. Upon using the cosine addition formula, this is equivalent to

$$c_1 \cdot c_2 - \cos r_1 \cos r_2 - \sin r_1 \sin r_2 = 0, \tag{12.4}$$

which is a bilinear relation in $(c_i, \cos r_i, \sin r_i)$, $i = 1, 2$. This gives rise to a projective model of Lie geometry as described in the following.

Definition 12.1.3.

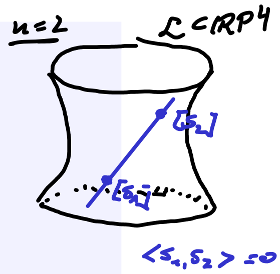
(i) The quadric

$$\mathcal{L} \subset \mathbb{RP}^{n+2}$$

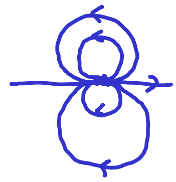
corresponding to the standard bilinear form of signature $(n+1,2)$

$$\langle x, y\rangle := \sum_{i=1}^{n+1} x_i y_i - x_{n+2}y_{n+2} - x_{n+3}y_{n+3}$$

for $x, y \in \mathbb{R}^{n+3}$, is called the *Lie quadric*.



(ii) Two points $[s_1], [s_2] \in \mathcal{L}$ on the Lie quadric are called *Lie orthogonal* if $\langle s_1, s_2\rangle = 0$, or equivalently if the line $[s_1] \vee [s_2]$ is isotropic, i.e. is contained in $\mathcal{L}$. An isotropic line is called a *contact element*.



(iii) The projective transformations of $\mathbb{RP}^{n+2}$ that preserve the Lie quadric $\mathcal{L}$

$$\mathrm{Lie} := \mathrm{PO}(n+1,2).$$

are called *Lie transformations*.

Proposition 12.1.4. *The set of oriented hyperspheres $\vec{\mathscr{S}}$ of $\mathbb{S}^n$ is in one-to-one correspondence with the Lie quadric $\mathcal{L}$ by the map*

$$\vec{\mathfrak{S}} : \vec{\mathscr{S}} \to \mathcal{L}, \quad (c, r) \mapsto (c, \cos r, \sin r)$$

such that two oriented hyperspheres are in oriented contact if and only if their corresponding points on the Lie quadric are Lie orthogonal.

for $[x] \in \mathcal{L}$: $x_1^2 + \dots + x_n^2 = x_{n+1}^2 + x_{n+2}^2 = 1$ (normalize)

Proof. A point $s \in \mathcal{L}$ can always be represented by $s = [c, \cos r, \sin r]$ with $c \in \mathbb{S}^n$, $r \in \mathbb{R}$. The transformations (12.1) act on $s = (c, \cos r, \sin r)$ as

$$\begin{aligned}(c, \cos r, \sin r) &\mapsto (c, \cos(r+2\pi), \sin(r+2\pi)) = (c, \cos r, \sin r),\\ (c, \cos r, \sin r) &\mapsto (-c, \cos(r-\pi), \sin(r-\pi)) = -(c, \cos r, \sin r).\end{aligned} \tag{12.5}$$

and the oriented contact becomes the bilinear relation (12.4). □

spherical geometry	**Lie geometry**
point $\hat{x} \in \mathbb{S}^n$	$[\hat{x}, 1, 0] \in \mathcal{L}$
oriented hypersphere with center $\hat{s} \in \mathbb{R}^n$ and signed radius $r \in \mathbb{R}$	$[\hat{s}, \cos r, \sin r] \in \mathcal{L}$

Table 12.1. Correspondence of hyperspheres of the n-sphere $\mathbb{S}^n$ and points on the Lie quadric

$$\mathcal{L} = \left\{ x = (x_1, \ldots, x_{n+3}) \in \mathbb{R}^{n+1,2} \;\middle|\; \langle x, x \rangle = 0 \right\} \subset \mathbb{RP}^{n+2}.$$

This correspondence leads to an embedding of $\mathbb{S}^n$ into the Lie quadric in the following way. Among all oriented hyperspheres the map $\vec{S}$ distinguishes the set of "points", or *null-spheres*, as the set of oriented hyperspheres with radius $r = 0$. It turns out that

$$\left\{ \vec{S}(c, 0) \;\middle|\; c \in \mathbb{S}^n \right\} = \{ \boldsymbol{x} \in \boldsymbol{\mathcal{L}} \mid x_{n+3} = 0 \} = \boldsymbol{\mathcal{L}} \cap \boldsymbol{p}^{\perp},$$

where

$$\boldsymbol{p} := [e_{n+3}] = [0, \ldots, 0, 1] \in \mathbb{RP}^{n+2}.$$

Week 9: Projective model of Lie geometry

12.2 The projective model of Lie geometry

Definition 12.2.1.

(i) The quadric

$$\mathcal{L} \subset \mathbb{R}\mathrm{P}^{n+2}$$

corresponding to the standard bilinear form of signature $(n+1, 2)$

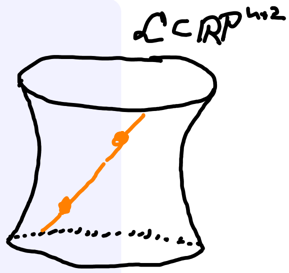
$$\langle x, y \rangle := \sum_{i=1}^{n+1} x_i y_i - x_{n+2} y_{n+2} - x_{n+3} y_{n+3}$$

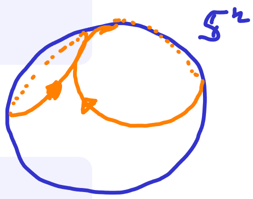
for $x, y \in \mathbb{R}^{n+3}$, is called the *Lie quadric*.

(ii) The projective transformations of $\mathbb{R}\mathrm{P}^{n+2}$ that preserve the Lie quadric $\mathcal{L}$

$$\mathrm{Lie} := \mathrm{PO}(n+1, 2).$$

are called *Lie transformations*.





Proposition 12.2.2. *The set of oriented hyperspheres $\vec{\mathscr{S}}$ of $\mathbb{S}^n$ is in one-to-one correspondence with the Lie quadric $\mathcal{L}$ by the map*

$$\Phi : \vec{\mathscr{S}} \to \mathcal{L}, \quad (c, r) \mapsto (c, \cos r, \sin r)$$

such that two oriented hyperspheres $(c_1, r_1), (c_2, r_2)$are in oriented contact if and only if their corresponding points on the Lie quadric satisfy

$$\langle \Phi(c_1, r_1), \Phi(c_2, r_2) \rangle = 0$$

Proof. A point $s \in \mathcal{L}$ can always be represented by $s = [c, \cos r, \sin r]$ with $c \in \mathbb{S}^n$, $r \in \mathbb{R}$. The transformations (12.1) act on $s = (c, \cos r, \sin r)$ as

$$\begin{aligned} (c, \cos r, \sin r) &\mapsto (c, \cos(r + 2\pi), \sin(r + 2\pi)) = (c, \cos r, \sin r), \\ (c, \cos r, \sin r) &\mapsto (-c, \cos(r - \pi), \sin(r - \pi)) = -(c, \cos r, \sin r). \end{aligned} \tag{12.5}$$

and the oriented contact becomes the bilinear relation (12.4). □

spherical geometry	Lie geometry
point $\hat{x} \in \mathbb{S}^n$	$[\hat{x}, 1, 0] \in \mathcal{L}$
oriented hypersphere with center $\hat{s} \in \mathbb{R}^n$ and signed radius $r \in \mathbb{R}$	$[\hat{s}, \cos r, \sin r] \in \mathcal{L}$

Table 12.1. Correspondence of hyperspheres of the n-sphere $\mathbb{S}^n$ and points on the Lie quadric

$$\mathcal{L} = \left\{[x] \in \mathbb{R}\mathrm{P}^{n+2} \,\middle|\, \langle x, x\rangle = 0\right\}.$$

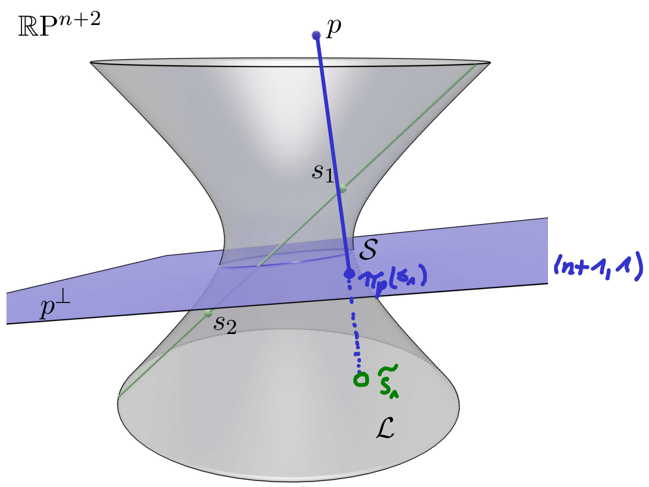


Figure 12.2. The Lie quadric $\mathcal{L} \subset \mathbb{R}\mathrm{P}^{n+2}$ and the Möbius quadric $\mathcal{S} = \mathcal{L} \cap p^\perp$ as a section.

This correspondence leads to an embedding of $\mathbb{S}^n$ into the Lie quadric in the following way. Among all oriented hyperspheres the map Φ distinguishes the set of "points", or *null-spheres*, as the set of oriented hyperspheres with radius $r = 0$. It turns out that

$$\{\Phi(c, 0) \mid c \in \mathbb{S}^n\} = \{x \in \mathcal{L} \mid x_{n+3} = 0\} = \mathcal{L} \cap p^\perp,$$

where

$$p := [e_{n+3}] = [0, \dots, 0, 1] \in \mathbb{R}\mathrm{P}^{n+2}.$$

The quadric $\mathcal{L} \cap p^\perp$ has signature $(n+1, 1)$ and may be identified with the Möbius quadric. In the projection

$$\pi_p : \mathcal{L} \to p^\perp, \quad [x_1, \dots, x_{n+3}] \mapsto [x_1, \dots, x_{n+2}, 0]$$

The points on the Lie quadric $\mathcal{L} \backslash p^\perp$ are mapped to the outside of $\mathcal{L} \cap p^\perp$.

Theorem 12.2.3. *Möbius geometry is a subgeometry of Lie geometry in the following sense: If the quadric*

$$\mathcal{S} := \mathcal{L} \cap p^{\perp}$$

is identified with the Möbius quadric, then the group of Lie transformations that preserve the hyperplane $p^{\perp}$ *(or equivalently that fixes the point* p*) acts on* $p^{\perp}$ *as the group of Möbius transformations.*

Proof. The Lie transformations that preserve the hyperplane $p^{\perp}$ build a group whose action can be restricted to $p^{\perp}$. In $p^{\perp}$ they preserve the quadric $\mathcal{L} \cap p^{\perp}$ and thus act as Möbius transformations.

Vice versa, a Möbius transformation on $p^{\perp}$ represented by

$$A \in \mathrm{O}(n+1,1)$$

can be lifted to a Lie transformation as

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & \pm 1 \end{array}\right].$$

□

Remark 12.2.4. The group of Lie transformations that fixes the point p is a double cover of the group of Möbius transformations.

A relation between the Euclidean and the projective model of Möbius geometry has been established by stereographic projection, or equivalently, by introducing

$$e_0 := \tfrac{1}{2}(e_{n+2} - e_{n+1}), \quad e_\infty := \tfrac{1}{2}(e_{n+2} + e_{n+1}),$$

which satisfy

$$\langle e_0, e_0 \rangle = \langle e_\infty, e_\infty \rangle = 0, \quad \langle e_0, e_\infty \rangle = -\tfrac{1}{2}.$$

Theorem 12.2.5. *The identification of points on the Lie quadric $\mathcal{L}$ and oriented hyperspheres (including points and oriented hyperplanes) of (to one point compactification of) the n-dimensional Euclidean space $\mathbb{R}^n \cup \{\infty\}$*

$$\hat{s} = c + e_0 + (\|c\|^2 - r^2)e_\infty + re_{n+3} \quad \leftrightarrow \quad \text{oriented hypersphere with center } c \in \mathbb{R}^n \text{ and signed radius } r > 0,$$

$$\hat{s} = \nu + 2(\nu \cdot a)e_\infty + e_{n+3} \quad \leftrightarrow \quad \text{hyperplane through } a \in \mathbb{R}^n \text{ with normal vector } \nu \in \mathbb{S}^n.$$

$$\hat{s} = x + e_0 + \|x\|^2 e_\infty \quad \leftrightarrow \quad \text{point } x \in \mathbb{R}^n$$

$$\hat{s} = e_\infty \quad \leftrightarrow \quad \text{point at infinity } \infty$$

is one-to-one and such that for $[\hat{s}_1], [\hat{s}_2] \in \mathcal{L}$

$$\langle \hat{s}_1, \hat{s}_2 \rangle = 0$$

if and only the oriented hyperspheres corresponding to $[s_1]$ and $[s_2]$ are in oriented contact.

Proof. We first check that a point

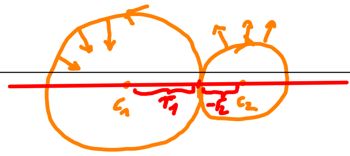
$$\hat{s} = c + e_0 + (\|c\|^2 - r^2)e_\infty + re_{n+3}$$

lies on the Lie quadric:

$$\langle \hat{s}, \hat{s} \rangle = \|c\|^2 - (\|c\|^2 - r^2) - r^2 = 0.$$

Now for two points

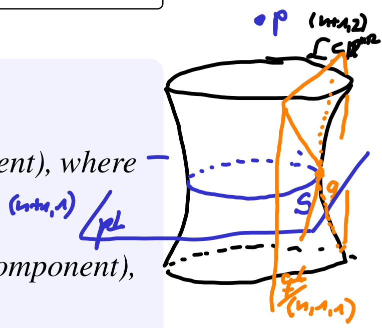
$$\hat{s}_i = c_i + e_0 + (\|c_i\|^2 - r_i^2)e_\infty + r_i e_{n+3}, \quad i = 1, 2$$



we find

$$\begin{aligned}\langle \hat{s}_1, \hat{s}_2 \rangle &= c_1 \cdot c_2 - \tfrac{1}{2}(\|c_1\|^2 - r_1^2) - \tfrac{1}{2}(\|c_2\|^2 - r_2^2) - r_1 r_2 \\ &= -\tfrac{1}{2}\|c_1 - c_2\|^2 + \tfrac{1}{2}(r_1 - r_2)^2 \\ &= 0 \\ \Leftrightarrow \quad \|c_1 - c_2\|^2 &= (r_1 - r_2)^2.\end{aligned}$$

The remaining claims are left to the reader. □



Proposition 12.2.6. *In the correspondence of Thereom 12.2.5:*

(i) $[\hat{s}] \in \mathcal{L}$ *corresponds to a point if and only if* $[\hat{s}] \in p^\perp$ *(no* e_{n+3}*-component), where* $p = [e_{n+3}]$.

(ii) $[\hat{s}] \in \mathcal{L}$ *corresponds to a hyperplane if and only if* $[\hat{s}] \in q^\perp$ *(no* e_0*-component), where* $q = [e_\infty]$.

Theorem 12.2.7. *Laguerre geometry is a subgeometry of Lie geometry in the following sense: If the quadric*

$$\mathcal{Z} := \mathcal{L} \cap q^\perp$$

is identified with the Blaschke cylinder, then the group of Lie transformations that preserve the hyperplane $q^\perp$ *(or equivalently that fixes the point* q*) acts on* $q^\perp$ *as the group of Laguerre transformations.*

Proof. The hyperplane $q^\perp$ can be spanned by

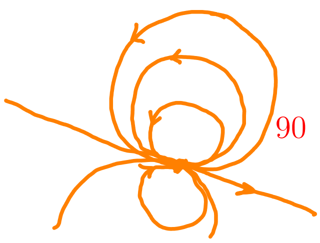
$$q^\perp = \operatorname{span}\{e_1, \dots, e_n, e_\infty, e_{n+3}\},$$

and thus is a hyperplane of signature $(n+1, 1, 1)$.

Now show that every Laguerre transformation on $q^\perp$ can be lifted to a Lie transformation. □

Two oriented hyperspheres which are in oriented contact span an isotropic line in the Lie quadric.

Definition 12.2.8. The one-parameter family of oriented hyperspheres corresponding to an isotropic line in the Lie quadric is called a *contact element*.



Each isotropic line (not contained in $q^\perp$) intersects $p^\perp$ and $q^\perp$ in exactly one point respectively. Thus, a contact element can always be thought of being spanned by a point and an oriented hyperplane through this point.

Week 10: Sphere complexes and signed inversive distance

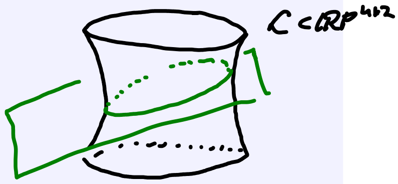
12.3 Sphere complexes and signed inversive distance

Any hyperplane in $\mathbb{RP}^{n+2}$ can equivalently be described by its polar point with respect to the Lie quadric.

Definition 12.3.1. For a point $z \in \mathbb{RP}^{n+2}$ the set of points

$$\mathcal{L} \cap z^{\perp}$$

on the Lie quadric as well as the n-parameter family of oriented hyperspheres corresponding to these points is called a *(linear) sphere complex*. A sphere complex is further called



- *elliptic* if $\langle \hat{z}, \hat{z} \rangle > 0$,
- *hyperbolic* if $\langle \hat{z}, \hat{z} \rangle < 0$,
- *parabolic* if $\langle \hat{z}, \hat{z} \rangle = 0$,

where $z = [\hat{z}]$

Two points in $\mathbb{RP}^{n+2}$ can be mapped to each other by a Lie transformation if and only if they have the same signature. Thus, any two sphere complexes of the same signature are Lie equivalent.

Example 12.3.2.

(i) The point $p = [e_{n+3}]$ defines a hyperbolic sphere complex, which contains all null-spheres and is called the *point complex*.

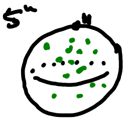
(ii) The point $[0, -\sin R, \cos R]$ defines a hyperbolic sphere complex, which contains all oriented hyperspheres of $\mathbb{S}^n$ with spherical radius R.

(iii) The point $[-2Re_{\infty} + e_{n+3}]$ defines a hyperbolic sphere complex, which contains all oriented hyperspheres of $\mathbb{R}^n$ with (Euclidean) radius R.

(iv) The point $[\nu - 2he_{\infty}]$ defines an elliptic sphere complex, which contains all oriented hyperspheres of $\mathbb{R}^n$ which are orthogonal to the hyperplane $\nu \cdot x + h = 0$.

(v) The point $q = [e_{\infty}]$ defines a parabolic sphere complex, which contains all oriented hyperplanes of $\mathbb{R}^n$ and is called the *plane complex*.

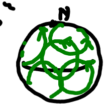
(i)

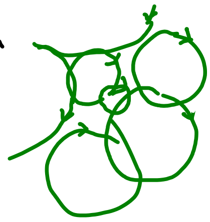


$\langle e_{n+2}, e_{n+3} \rangle = -1 < 0$

(ii) $\langle (0,-\sin R, \cos R),(0,-\sin R, \cos R) \rangle = -\sin^2 R - \cos^2 R = -1 < 0$

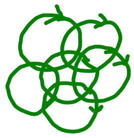
$\langle (0,-\sin R, \cos R),(c, \cos r, \sin r) \rangle = \sin R \cos r - \cos R \sin r = 0 \iff \tan R = \tan r$
$\iff R = r \pmod{\pi}$





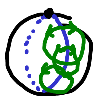
(iii) $\langle -2R e_\infty + e_{n+3}, -2R e_\infty + e_{n+3} \rangle = -1 < 0$

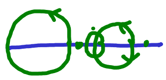
$\langle -2R e_\infty + e_{n+3}, c + e_0 + (|c|^2 - r^2) e_\infty + r e_{n+3} \rangle = R - r = 0 \iff R = r$



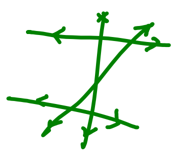
(iv) $\langle v - 2h e_\infty, v - 2h e_\infty \rangle = |v|^2 = 1 > 0$

$\langle v - 2h e_\infty, c + e_0 + (|c|^2 - r^2) e_\infty + r e_{n+3} \rangle = v \cdot c + h = 0 \iff c$ lies on $v \cdot x + h = 0$

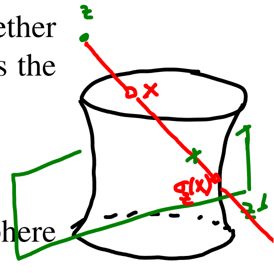




(v) $\langle e_\infty, e_\infty \rangle = 0$



A point $z \in \mathbb{RP}^{n+2} \backslash \mathcal{L}$ not on the Lie quadric and its polar hyperplane $z^\perp$ together induce an involution $\sigma_z \in \mathrm{Lie}$ that fixes the point z, every point on $z^\perp$, and preserves the Lie quadric $\mathcal{L}$:



$$\sigma_z(x) := \left[\hat{x} - 2\frac{\langle \hat{z}, \hat{x}\rangle}{\langle \hat{z}, \hat{z}\rangle}\hat{z}\right].$$

Thus, every non-parabolic sphere complex comes with an involution that fixes the sphere complex.

Example 12.3.3. For the point complex defined by the point p, the corresponding involution

$$[x_1, \ldots, x_{n+2}, x_{n+3}] \mapsto [x_1, \ldots, x_{n+2}, -x_{n+3}]$$

describes the orientation reversion of hyperspheres. Note that it preserves the plane complex.

Furthermore, a non-parabolic sphere complex induces an invariant for pairs of oriented hyperspheres. This invariant will eventually allow for a more general geometric description of the different types of sphere complexes.

Definition 12.3.4. Let $z = [\hat{z}] \in \mathbb{RP}^{n+2} \backslash \mathcal{L}$. Then we define

$$I_z(x, y) := 1 - \frac{\langle \hat{x}, \hat{y}\rangle \langle \hat{z}, \hat{z}\rangle}{\langle \hat{x}, \hat{z}\rangle \langle \hat{y}, \hat{z}\rangle}$$

for any two points $x = [\hat{x}], y = [\hat{y}] \in \mathcal{L}$.

The invariant I_z is projectively well-defined, in the sense that it does not depend on the choice of homogeneous coordinate vectors for the points x, y, and z, and it is invariant under Lie transformations that fix the point z.

Remark 12.3.5. Although we are interested in this invariant for points on the Lie quadric, it can be extended to all of $\mathbb{RP}^{n+2} \backslash z^\perp$. It then satisfies

$$\frac{(1 - I_z(x, y))^2}{(1 - I_z(x, x))(1 - I_z(y, y))} = \frac{\langle \hat{x}, \hat{y}\rangle^2}{\langle \hat{x}, \hat{x}\rangle \langle \hat{y}, \hat{y}\rangle}$$

for $x = [\hat{x}], y = [\hat{y}] \in \mathbb{RP}^{n+2} \backslash (\mathcal{L} \cup z^\perp)$.

Applying the involution σ_z to one of the arguments of I_z results in a change of sign.

Proposition 12.3.6. *Let $z \in \mathbb{RP}^{n+2} \backslash \mathcal{L}$. Then*

$$I_z(\sigma_z(x), y) = I_z(x, \sigma_z(y)) = -I_z(x, y).$$

for all $x, y \in \mathcal{L}$.

Proof. We compute

$$\begin{aligned} I_z(\sigma_z(x), y) &= 1 - \frac{\langle \hat{x}, \hat{y} \rangle \langle \hat{z}, \hat{z} \rangle - 2 \langle \hat{x}, \hat{z} \rangle \langle \hat{y}, \hat{z} \rangle}{-\langle \hat{x}, \hat{z} \rangle \langle \hat{y}, \hat{z} \rangle} \\ &= \frac{\langle \hat{x}, \hat{y} \rangle \langle \hat{z}, \hat{z} \rangle}{\langle \hat{x}, \hat{z} \rangle \langle \hat{y}, \hat{z} \rangle} - 1 = -I_z(x, y). \end{aligned}$$

□

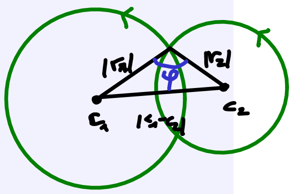
We now consider the specific example of the invariant I_p corresponding to the point complex. It is invariant under all Lie transformations that fix the point p, i.e., all Möbius transformations, and turns out to be a signed version of the inversive distance from Möbius geometry.

Theorem 12.3.7. *For two oriented hyperspheres represented by*

$$\hat{s}_i = c_i + e_0 + (|c_i|^2 - r_i^2) e_\infty + r_i e_{n+3}, \qquad i = 1, 2$$

with Euclidean centers $c_1, c_2 \in \mathbb{R}^n$ and signed radii $r_1, r_2 \neq 0$ the point complex invariant is given by

$$I_p([\hat{s}_1], [\hat{s}_2]) = \frac{r_1^2 + r_2^2 - |c_1 - c_2|^2}{2 r_1 r_2}.$$



It further satisfies:

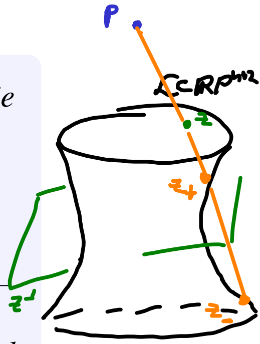
- $I_p \in (-1, 1) \Leftrightarrow$ *the two oriented hyperspheres intersect. In this case $I_p = \cos \varphi$ where $\varphi \in [0, \pi]$ is the angle between the two oriented hyperspheres.*
- $I_p = 1 \Leftrightarrow$ *the two oriented hyperspheres touch with matching orientation (oriented contact).*
- $I_p = -1 \Leftrightarrow$ *the two oriented hyperspheres touch with opposite orientation.*
- $I_p \in (\infty, -1) \cup (1, \infty) \Leftrightarrow$ *the two oriented hyperspheres are disjoint.*

Proof. With the given representation of the hyperspheres we find

$$I_p([\hat{s}_1], [\hat{s}_2]) = 1 - \frac{\langle \hat{s}_1, \hat{s}_2 \rangle \langle e_{n+3}, e_{n+3} \rangle}{\langle \hat{s}_1, e_{n+3} \rangle \langle \hat{s}_2, e_{n+3} \rangle} = \frac{r_1^2 + r_2^2 - |c_1 - c_2|^2}{2 r_1 r_2}.$$

□

We now use the inversive distance to give a geometric interpretation for most sphere complexes in Lie geometry.



Theorem 12.3.8. *Let $z \in \mathbb{RP}^{n+2}$, $z \neq p$ such that the line $p \vee z$ intersects the Lie quadric in two points, i.e. has signature $(+-)$. Denote by*

$$\{z_+, z_-\} := (p \vee z) \cap \mathcal{L}$$

the two intersection points (the two oriented hyperspheres corresponding to z_+ and z_- only differ by their orientation).

Then the sphere complex corresponding to the point z is given by the set of oriented hyperspheres that have some fixed constant inversive distance I_p to the oriented hypersphere corresponding to z_+, or equivalently, fixed constant inversive distance $-I_p$ to the oriented hypersphere corresponding to z_-.

Furthermore, in this case the sphere complex is

- elliptic *if $I_p \in (-1, 1)$,*
- hyperbolic *if $I_p \in (-\infty, -1) \cup (1, \infty)$,*
- parabolic *if $I_p \in \{-1, 1\}$.*

Proof. The two points $z_\pm$ may be represented by

$$\hat{z}_\pm = \tilde{z} + e_0 + \left(|\tilde{z}|^2 - R^2\right) e_\infty \pm R e_{n+3},$$

with some $R \neq 0$, where we assumed that the e_0-component of $\hat{z}$ does not vanish. The case with $\langle \hat{z}, e_\infty \rangle = 0$, which corresponds to $z_\pm$ being planes, may be treated analogously.

Now the point z can be represented by

$$\hat{z} = \tilde{z} + e_0 + \left(|\tilde{z}|^2 - R^2\right) e_\infty + \kappa e_{n+3}$$

with some $\kappa \in \mathbb{R}$, and for any point $s \in \mathcal{L}$ represented by

$$\hat{s} = \tilde{s} + e_0 + \left(|\tilde{s}|^2 - r^2\right) e_\infty + r e_{n+3},$$

we find that the condition for it to lie on the sphere complex is given by

$$\langle \hat{z}, \hat{s} \rangle = 0 \quad \Leftrightarrow \quad \langle \hat{z}, \hat{s} \rangle_p = r\kappa,$$

where $\langle \cdot, \cdot \rangle_p$ is the Lie scalar product of the projection from p. Thus, the signed inversive distance of z_+ and s is given by

$$I_p(z_+, s) = 1 - \frac{\langle \hat{z}_+, \hat{s} \rangle \langle e_{n+3}, e_{n+3} \rangle}{\langle \hat{z}_+, e_{n+3} \rangle \langle \hat{s}, e_{n+3} \rangle} = \frac{\langle \hat{s}, \hat{z} \rangle_p}{rR} = \frac{\kappa}{R}.$$

The change $z_+ \to z_-$ is equivalent to $R \to -R$ which leads to $I \to -I$.

The distinction of the three types of sphere complexes in terms of the value of the inversive distance is obtained by observing that

$$\begin{aligned} \langle \hat{z}, \hat{z} \rangle &> 0, \quad \text{if } \kappa^2 < R^2, \\ \langle \hat{z}, \hat{z} \rangle &< 0, \quad \text{if } \kappa^2 > R^2, \\ \langle \hat{z}, \hat{z} \rangle &= 0, \quad \text{if } \kappa^2 = R^2. \end{aligned}$$

□

Examples 12.3.9.

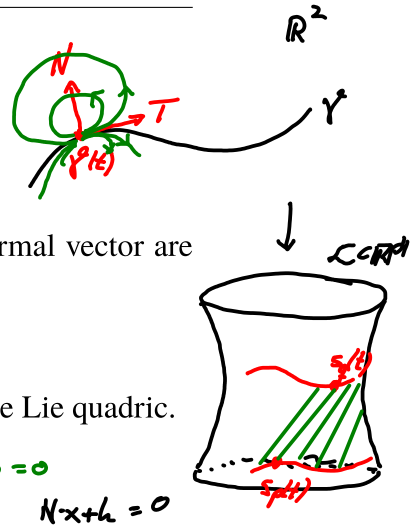
(i) For an elliptic sphere complex the line $p \vee z$ always has signature $(+-)$. Furthermore, in this case we have $I_p \in (-1, 1)$. Thus, any elliptic sphere complex is given by all oriented hyperspheres with constant angle to some fixed oriented hypersphere.

(ii) For hyperbolic sphere complexes the line $p \vee z$ can have signature $(+-)$, $(--)$, or (-0). Only the first case is captured by Theorem 12.3.8. In Example 12.3.2 (ii), the line has signature $(--)$, while in Example 12.3.2 (iii) it has signature (-0).

(iii) Parabolic sphere complexes are captured by Theorem 12.3.8 if and only if $z \notin p^\perp$.

Week 11: Curves and surfaces in Lie geometry

12.4 Planar curves in Lie geometry



Let

$$\gamma : [a,b] \to \mathbb{R}^2$$

be a smooth regular curve in the Euclidean plane. Its unit tangent and normal vector are given by

$$T(t) := \frac{\dot\gamma}{\|\dot\gamma\|}, \qquad N(t) := JT(t), \quad \text{where } J := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right).$$

We can lift the points of the curve as well as the oriented tangent lines to the Lie quadric.

$$\begin{aligned} s_p(t) &:= \gamma(t) + e_0 + \|\gamma(t)\|^2 e_\infty, \\ s_q(t) &:= N(t) - 2h(t)e_\infty + e_5. \end{aligned}$$

Neither a point nor an oriented line are Lie invariant objects, yet together they span a contact element (an isotropic line in the Lie quadric). Thus, we can lift the curve γ to a one-parameter family of lines (a ruled surface) in the Lie quadric:

$$\ell(t) := [s_p(t)] \vee [s_q(t)]$$

The condition for the oriented lines to be the tangent lines of the curve becomes

$$\langle \dot s_p, s_q \rangle = \dot\gamma \cdot N = 0, \tag{12.6}$$

or equivalently

$$\langle s_p, \dot s_q \rangle = \gamma \cdot \dot N + \dot h = 0.$$

Proposition 12.4.1. *Let*

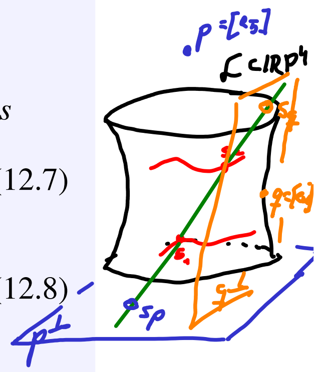
$$\ell(t) := [s_1(t)] \vee [s_2(t)] \subset \mathcal{L} \subset \mathbb{R}\mathrm{P}^4$$

be a smooth regular one-parameter family of lines in the Lie quadric that satisfies

$$\langle \dot s_1, s_2 \rangle = 0, \tag{12.7}$$

or equivalently,

$$\langle s_1, \dot s_2 \rangle = 0. \tag{12.8}$$



Then its sections with the point complex and plane (line) complex

$$\begin{aligned} [s_p(t)] &:= \ell(t) \cap p^\perp, \\ [s_q(t)] &:= \ell(t) \cap q^\perp \end{aligned}$$

define a planar curve in the Euclidean plane together with its oriented tangent lines.

Proof. Firstly, note that equations (12.7)and (12.8) are equivalent since

$$\langle s_1, s_2 \rangle = 0$$

implies

$$\langle \dot{s}_1, s_2 \rangle + \langle s_1, \dot{s}_2 \rangle = 0.$$

Secondly, this condition is invariant under a change of choice of points spanning the lines ℓ. Indeed, for

$$\tilde{s}_1 := \lambda_1 s_1 + \lambda_2 s_2,$$
$$\tilde{s}_2 := \mu_1 s_1 + \mu_2 s_2$$

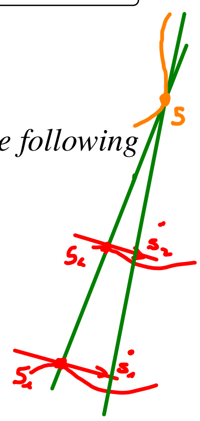
with smooth $\lambda_1, \lambda_2, \mu_1, \mu_2$, we find

$$\left\langle \dot{\tilde{s}}_1, \tilde{s}_2 \right\rangle = \left\langle \dot{\lambda}_1 s_1 + \lambda_1 \dot{s}_1 + \dot{\lambda}_2 s_2 + \lambda_2 \dot{s}_2, \mu_1 s_1 + \mu_2 s_2 \right\rangle = 0.$$

Thus, in particular

$$\langle \dot{s}_p, s_q \rangle = 0$$

which by (12.6) is equivalent to the claimed tangency condition. □



Lemma 12.4.2. *Let*

$$\ell(t) := [s_1(t)] \vee [s_2(t)] \subset \mathcal{L} \subset \mathbb{R}\mathrm{P}^4$$

be a smooth regular one-parameter family of lines in the Lie quadric. Then the following are equivalent:

(i) $\langle \dot{s}_1, s_2 \rangle = 0.$

(ii) $\langle s_1, \dot{s}_2 \rangle = 0.$

(iii) $[s_1], [s_2], [\dot{s}_1], [\dot{s}_2]$ *span a plane.*

(iv) There exists a unique curve $s(t) = \lambda_1(t) s_1(t) + \lambda_2(t) s_2(t)$ *such that*

$$[s] \vee [\dot{s}] = \ell.$$

The curve $[s]$ *is called the* edge of regression *of* ℓ.

Remark 12.4.3.

- Conditions (iii) and (iv) are also equivalent for a general ruled surface in a projective space (not necessarily contained in a quadric). A ruled surface satisfying condition (iii) or (iv) is called a developable surface. The edge of regression is a curve tangent to its rulings.

- The equivalence (iii) and (iv) to (i) and (ii) only holds for a ruled surface contained in the Lie quadric in $\mathbb{RP}^4$.

Proposition 12.4.4. *Let γ a smooth regular curve in $\mathbb{R}^2$. Let*

$$\ell(t) := [s_p(t)] \vee [s_q(t)]$$

with

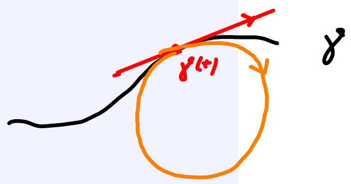
$$s_p(t) := \gamma(t) + e_0 + \|\gamma(t)\|^2 e_\infty,$$
$$s_q(t) := N(t) - 2h(t)e_\infty + e_5.$$

be its lift to the Lie quadric $\mathcal{L} \subset \mathbb{RP}^4$, and let

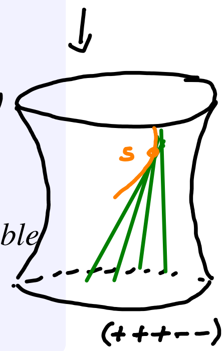
$$s(t) := c(t) + e_0 + (\|c(t)\|^2 - r(t)^2)e_\infty + r(t)e_5$$

be the lift of its osculating circles. Then $[s(t)]$ is the edge of regression of the developable surface $\ell(t)$, i.e.

$$[s] \vee [\dot{s}] = \ell.$$



$c(t) = \gamma(t) + r(t)N(t)$, $r(t) = \frac{1}{\kappa(t)}$



Proof. We first check that

$$s = s_p + r s_q$$

and thus $[s] \in \ell$.

As a linear combination of s_p and s_q the curve s satisfies

$$\langle \dot{s}, s_p \rangle = \langle \dot{s}, s_q \rangle = 0,$$

sgn $\ell = (0\,0)$
sgn $\ell^\perp = (+\,0\,0)$, $[s_p], [s_q] \in \ell^\perp$
$\ell^\perp \cap \mathcal{L} = \ell$

and thus $[\dot{s}] \in \ell^\perp$. We check that furthermore, $[\dot{s}] \in \mathcal{L}$, and thus $[\dot{s}] \in \ell$. Indeed, with

$$\dot{s} = \dot{c} + 2(\dot{c} \cdot c - \dot{r}r)e_\infty + \dot{r}e_5$$

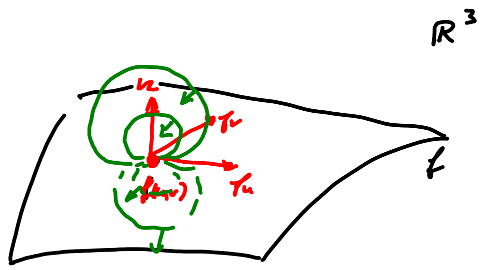
we find

$$\langle \dot{s}, \dot{s} \rangle = \|\dot{c}\|^2 - (\dot{r})^2 = \left\|\dot{\gamma} + \dot{r}N + r\dot{N}\right\|^2 - (\dot{r})^2 = 0,$$

where we used $\|N\|^2 = 1$ and $\dot{\gamma} = -r\dot{N}$. □

Corollary 12.4.5. *The osculating circles of a planar curve are Lie invariant.*

12.5 Surfaces in Lie geometry



Let

$$f : \mathbb{R}^2 \supset U \to \mathbb{R}^3$$

be a smooth regular parametrized surface patch. Let

$$n : U \to \mathbb{S}^2$$

be the unit normal field of f such that at every point $(u, v) \in U$

$$n = \frac{f_u \times f_v}{\|f_u \times f_v\|}.$$

Furthermore, let h be such that

$$n \cdot f + h = 0.$$

At each point of the surface this point together with the oriented tangent plane defines a contact element. The lift of f to the Lie quadric is given by the two-parameter family of isotropic lines representing these contact elements:

$$\ell(u, v) := [s_p(u, v)] \vee [s_q(u, v)]$$

where

$$s_p(u, v) := f(u, v) + e_0 + \|f(u, v)\|^2 e_\infty,$$
$$s_q(u, v) := n(u, v) - 2h(u, v)e_\infty + e_6.$$

The conditions for oriented planes to be tangent planes of the surface becomes

$$\langle \partial_u s_p, s_q \rangle = f_u \cdot n = 0,$$
$$\langle \partial_v s_p, s_q \rangle = f_v \cdot n = 0,$$

or equivalently,

$$\langle s_p, \partial_u s_q \rangle = \langle s_p, \partial_v s_q \rangle = 0.$$

Proposition 12.5.1. *Let*

$$\ell(u, v) := [s_1(u, v)] \vee [s_2(u, v)] \subset \mathcal{L} \subset \mathbb{R}\mathrm{P}^5$$

be a smooth regular two-parameter family of lines in the Lie quadric that satisfies

$$\langle \partial_u s_1, s_2 \rangle = \langle \partial_v s_1, s_2 \rangle = 0, \tag{12.9}$$

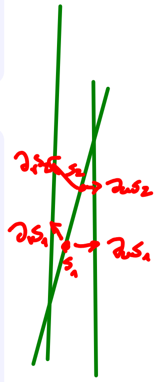
or equivalently,

$$\langle s_1, \partial_u s_2 \rangle = \langle s_1, \partial_v s_2 \rangle = 0, \tag{12.10}$$

Then its sections with the point complex and plane complex

$$[s_p(u,v)] := \ell(u,v) \cap p^\perp,$$
$$[s_q(u,v)] := \ell(u,v) \cap q^\perp$$

define a smooth regular surface in Euclidean space $\mathbb{R}^3$ *together with its oriented tangent planes.*



Definition 12.5.2. Let

$$\ell(u,v) := [s_1(u,v)] \vee [s_2(u,v)] \subset \mathbb{R}\mathrm{P}^n$$

be a smooth regular two-parameter family of lines in a projective space $\mathbb{R}\mathrm{P}^n$. Then ℓ is called a *(torsal) line congruence* if the two ruled surfaces given by $u \mapsto \ell(u,v)$ and $v \mapsto \ell(u,v)$ are developable, i.e.,

$$[s_1], [s_2], [\partial_u s_1], [\partial_u s_2] \text{ span a plane, and}$$
$$[s_1], [s_2], [\partial_v s_1], [\partial_v s_2] \text{ span a plane.}$$

Theorem 12.5.3. *Let* $f : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ *be a parametrized surface and*

$$\ell(u,v) := [s_p(u,v)] \vee [s_q(u,v)]$$

be its lift to the Lie quadric $\mathcal{L} \subset \mathbb{R}\mathrm{P}^5$, *where*

$$s_p(u,v) := f(u,v) + e_0 + \|f(u,v)\|^2 e_\infty,$$
$$s_q(u,v) := n(u,v) - 2h(u,v)e_\infty + e_6.$$

If f *is a curvature line parametrization then* ℓ *is a (torsal) line congruence.*
Vice versa, let

$$\ell(u,v) := [s_1(u,v)] \vee [s_2(u,v)] \subset \mathcal{L} \subset \mathbb{R}\mathrm{P}^5$$

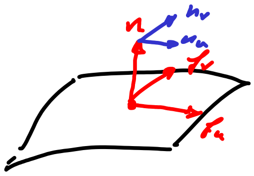
be a (torsal) line congruence in the Lie quadric. Then its section with the point complex

$$[s_p(u,v)] := \ell(u,v) \cap p^\perp$$

is a curvature line parametrization.

Proof. Let f be a parametrized surface. Then

$$\partial_u s_p = f_u + 2(f_u \cdot f)e_\infty,$$
$$\partial_v s_q = n_u - 2h_u e_\infty = n_u + 2(f \cdot n_u)e_\infty.$$



$n_u \cdot f_v = 0$
$f_u \cdot f_v = 0$
$\to n_u = \kappa_1 f_u$

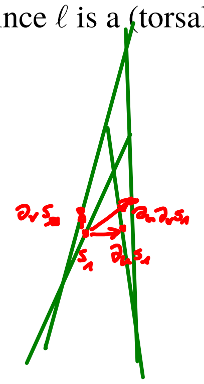
Thus,

$$0 \cdot s_p + 0 \cdot s_q + \kappa_1 \partial_u s_p - \partial_v s_q = 0,$$

where we used $n_u = \kappa_1 f_u$ for some κ_1, since f is a curvature line parametrization. Similarly, for the v direction.

Now let ℓ be a (torsal) line congruence. We first need to check that conditions (12.9) are satisfied, so that the ℓ actually defines a surface. Indeed, since ℓ is a (torsal) line congruence there exist α, β, γ such that

$$\partial_u s_2 = \alpha s_1 + \beta s_2 + \gamma \partial_u s_1.$$



Thus,

$$\langle s_1, \partial_u s_2 \rangle = 0.$$

Similarly,

$$\langle s_1, \partial_v s_2 \rangle = 0.$$

By Lemma 12.5.4 the points

$$[s_1], [s_2], [\partial_u s_1], [\partial_u s_2], [\partial_v s_1], [\partial_v s_2], [\partial_u \partial_v s_1], [\partial_u \partial_v s_2]$$

lie in a 3-dimensional space Π, which here is given by

$$\Pi = ([s_1] \vee [s_2])^\perp.$$

Thus, the four points

$$[s], [\partial_u s], [\partial_v s], [\partial_u \partial_v s] \in \Pi$$

lie in Π for any linear combination $s = \lambda_1 s_1 + \lambda_2 s_2$ such as s_p. On the other hand $[s_p]$ lies in the hyperplane $p^\perp$. The intersection $\Pi \cap p^\perp$ is 2-dimensional. Thus, the four points

$$[s_p], [\partial_u s_p], [\partial_v s_p], [\partial_u \partial_v s_p] \in \Pi \cap p^\perp$$

lie in a plane, i.e., the parametrization $[s_p]$ is conjugate. But a conjugate parametrization in the Möbius quadric represents a curvature line parametrization in $\mathbb{R}^3$. □

Lemma 12.5.4. *Let*

$$\ell(u, v) := [s_1(u, v)] \vee [s_2(u, v)] \subset \mathbb{RP}^n$$

be a (torsal) line congruence. Then

$$[s_1], [s_2], [\partial_u s_1], [\partial_u s_2], [\partial_v s_1], [\partial_v s_2], [\partial_u \partial_v s_1], [\partial_u \partial_v s_2]$$

span a 3-dimensional subspace.

Proof. By the condition for a (torsal) line congruence the points

$$[s_1], [s_2], [\partial_u s_1], [\partial_u s_2], [\partial_v s_1], [\partial_v s_2]$$

lie in a 3-dimensional subspace Π. Thus, we need to show $[\partial_u\partial_v s_1], [\partial_u\partial_v s_1] \in \Pi$.

There exist $\alpha, \beta, \gamma, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ such that

$$\partial_u s_2 = \alpha s_1 \beta s_2 \gamma \partial_u s_1,$$
$$\partial_v s_2 = \alpha s_1 \beta s_2 \gamma \partial_v s_1.$$

Cross-differentiation leads to

$$\partial_u\partial_v s_2 = \partial_v \alpha s_1 + \partial_v \beta s_2 + \alpha \partial_v s_1 + \beta \partial_v s_2 + \partial_v \gamma \partial_u s_1 + \gamma \partial_u \partial_v s_1,$$
$$\partial_u\partial_v s_2 = \partial_u \tilde{\alpha} s_1 + \partial_u \tilde{\beta} s_2 + \tilde{\alpha} \partial_u s_1 + \tilde{\beta} \partial_u s_2 + \partial_u \tilde{\gamma} \partial_v s_1 + \tilde{\gamma} \partial_u \partial_v s_1,$$

which shows that $[\partial_u\partial_v s_1] \in \Pi$. Similarly, $[\partial_u\partial_v s_2] \in \Pi$. □

Corollary 12.5.5. *Curvature line parametrizations are Lie invariant.*

Week 12: Plücker geometry, exterior calculus

Chapter 13

Plücker geometry

Plücker geometry is the geometry of lines in the 3-dimensional real projective space $\mathbb{R}\mathrm{P}^3$ and their incidences. From the fundamental theorem of projective geometry we know that the bijective transformations of $\mathbb{R}\mathrm{P}^3$ that map lines to lines and preserve their incidences are the projective transformations. Yet Plücker geometry comes with a different model of projective geometry in which the lines instead of the points (or by duality the planes) are the fundamental objects. This model is based on exterior calculus, which we introduce in arbitrary dimensions, and which may be used to generally describe k-dimensional projective subspaces in an n-dimensional projective space.

13.1 Exterior calculus

A vector $a \in V$ (an element of a vector space V) may be thought of representing a weighted version of the 1-dimensional linear subspace that it spans. The weight can be interpreted as a length on that line compared to some unit length. Then a vector space contains some weighted 1-dimensional linear subspaces and all of its linear combinations.

The exterior product $a \wedge b$ of two linearly independent vectors $a, b \in V$ may be thought of a weighted version of the 2-dimensional linear subspace they span. The weight can be interpreted as an area in that plane compared to some unit area. The exterior products of all vectors of a vector space together with its linear combinations constitute a vector space themselves $\bigwedge^k V$.

This construction can be formalized by the following definition. As in the case of vector spaces, more important than the definition of the exterior powers of a vector space are its properties, which allow for the given interpretation.

Definition 13.1.1. Let V be a vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, then a multilinear map

$$m : \underbrace{V \times \cdots \times V}_{k} \to \mathbb{F}$$

that satisfies

$$m(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -m(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

for all $v_1, \ldots, v_k \in V$ is called an *alternating multilinear form* of degree k (or an *alternating k-form*) on V.

Example 13.1.2. On the vector space $\mathbb{F}^n$ the determinant

$$\det(v_1 \cdots v_n)$$

is an alternating n-form.

The set of alternating multilinear forms of degree k on V is a vector space of dimension $\binom{n}{k}$. If $b_1, \ldots, b_n$ is a basis of V, then an alternating multilinear form m is uniquely determined by the values

$$m(b_{i_1}, \ldots, b_{i_k}), \qquad \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} \text{ with } i_1 < i_2 < \cdots < i_k.$$

Alternating 1-forms are just linear forms and thus constitute the dual vector space of V. Alternating 0-forms may be identified with elements from the field $\mathbb{F}$.

Definition 13.1.3. Let V be a finite dimensional vector space. Then the dual space of the vector space of alternating k-forms on V is called the *k-th exterior power of V* and denoted by

$$\textstyle\bigwedge^k V.$$

Elements of $\bigwedge^k V$ are called *k-vectors*.

In particular $\bigwedge^0 V = \mathbb{F}$, $\bigwedge^1 V = V$, and

$$\dim \textstyle\bigwedge^k V = \binom{n}{k}.$$

Definition 13.1.4. Let $v_1, \cdots, v_k \in V$. Then their *exterior product* (or *wedge product*) $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V$ is defined by

$$(v_1 \wedge \cdots \wedge v_k)(m) := m(v_1, \ldots, v_k)$$

for any alternating k-form m on V.

The exterior product has the following characterizing properties. The map

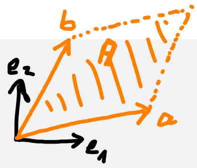
$$\underbrace{V \times \cdots \times V}_{k} \to \textstyle\bigwedge^k V, \qquad (v_1, \ldots, v_k) \mapsto v_1 \wedge \cdots \wedge v_k$$

is linear in every variable, and alternating.

Furthermore, if $b_1, \ldots, b_n$ is a basis of V, then the set of

$$b_{i_1} \wedge \cdots \wedge b_{i_k}$$

for $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ with $i_1 < \cdots < i_k$ is a basis of $\bigwedge^k V$.



Example 13.1.5.

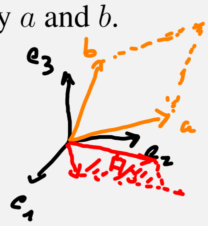
(i) $V = \mathbb{R}^2$. Let $e_1, e_2 \in \mathbb{R}^2$ be the canonical basis.

- $\bigwedge^0 \mathbb{R}^2 = \mathbb{R}$ with basis 1.
- $\bigwedge^1 \mathbb{R}^2 = \mathbb{R}^2$ with basis e_1, e_2.
- $\dim \bigwedge^2 \mathbb{R}^2 = 1$ with basis $e_1 \wedge e_2 = -e_2 \wedge e_1$. $a \wedge a = -a \wedge a \Rightarrow a \wedge a = 0$

Let $a = a_1 e_1 + a_2 e_2, b = b_1 e_1 + b_2 e_2 \in \mathbb{R}^2$ be two vectors. Then

$$\begin{aligned} a \wedge b &= (a_1 e_1 + a_2 e_2) \wedge (b_1 e_1 + b_2 e_2) \\ &= a_1 b_1 e_1 \wedge e_1 + a_1 b_2 e_1 \wedge e_2 + a_2 b_1 e_2 \wedge e_1 + a_2 b_2 e_2 \wedge e_2 \\ &= (a_1 b_2 - a_2 b_1) e_1 \wedge e_2 \\ &= \underbrace{\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}}_{A} e_1 \wedge e_2. \end{aligned}$$

The coefficient are the area of the parallelogram spanned by a and b.



(ii) $V = \mathbb{R}^3$. Let $e_1, e_2, e_3 \in \mathbb{R}^3$ be the canonical basis.

- $\bigwedge^0 \mathbb{R}^3 = \mathbb{R}$ with basis 1.
- $\bigwedge^1 \mathbb{R}^3 = \mathbb{R}^3$ with basis e_1, e_2, e_3.
- $\dim \bigwedge^2 \mathbb{R}^3 = 3$ with basis $e_1 \wedge e_2, e_2 \wedge e_3, e_3 \wedge e_1$
- $\dim \bigwedge^3 \mathbb{R}^3 = 1$ with basis $e_1 \wedge e_2 \wedge e_3$.

Let $a = a_1 e_1 + a_2 e_2 + a_3 e_3, b = b_1 e_1 + b_2 e_2 + b_3 e_3 \in \mathbb{R}^3$ be two vectors. Then

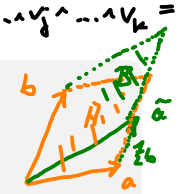
$$\begin{aligned} a \wedge b &= (a_1 e_1 + a_2 e_2 + a_3 e_3) \wedge (b_1 e_1 + b_2 e_2 + b_3 e_3) \\ &= \underbrace{(a_1 b_2 - a_2 b_1)}_{A_3} e_1 \wedge e_2 + (a_2 b_3 - a_3 b_2) e_2 \wedge e_3 + (a_3 b_1 - a_1 b_3) e_3 \wedge e_1, \end{aligned}$$

The coefficients are areas of the projections of the parallelogram spanned by a and b to the coordinate planes. With $c = c_1e_1 + c_2e_2 + c_3e_3 \in \mathbb{R}^3$ we obtain

$$a \wedge b \wedge c = \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} e_1 \wedge e_2 \wedge e_3$$

The coefficient is the volume of the parallelepiped spanned by a, b, and c.

The exterior product vanishes if any two entries are the same.



Example 13.1.6. Let $a, b \in \mathbb{R}^n$, and $\tilde{a} := a + \frac{1}{2}b$.

$$\tilde{a} \wedge b = (a + \tfrac{1}{2}b) \wedge b = a \wedge b + \tfrac{1}{2}b \wedge b = a \wedge b.$$

Note that the two parallelgrams spanned by a, b and by $\tilde{a}, b$ have the same area.

Thus, we can always add a linear combination of, say $v_2, \ldots, v_k$ to v_1 without changing the exterior product:

$$(v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k) \wedge v_2 \wedge \cdots \wedge v_k = v_1 \wedge v_2 \wedge \cdots \wedge v_k.$$

More generally:

Proposition 13.1.7. *Let* $v_1, \ldots, v_k \in V$. *Then*

$$v_1 \wedge \cdots \wedge v_k = 0 \quad \Leftrightarrow \quad v_1, \ldots, v_k \text{ linearly dependent.}$$

Proof. Let $v_1, \ldots, v_k$ be linearly dependent. Then there exists an $i \in \{1, \ldots, k\}$ such that

$$v_i = \sum_{j \neq i} \lambda_j v_j,$$

and thus,

$$v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k = \sum_{j \neq i} \lambda_j \underbrace{v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_j \wedge \cdots v_k}_{=0} = 0.$$

Vice versa, let $v_1, \ldots, v_k$ be linearly independent. Then they can be extended to a basis of V, and thus

$$v_1 \wedge \cdots \wedge v_k$$

is a basis vector of $\bigwedge^k V$, which cannot be 0. □

Definition 13.1.8. A k-vector $a \in \bigwedge^k V$ that can be expressed as the wedge product of k 1-vectors, i.e., there exists $v_1, \dots, v_k \in V$ such that

$$a = v_1 \wedge \cdots \wedge v_k$$

is called *decomposable*, or a *k-blade*.

Not every k-vector $a \in \bigwedge^k V$ is decomposable. If $a \in \bigwedge^k V$ is decomposable, then certainly

$$a \wedge a = 0.$$

Starting at dimension $n = \dim V = 4$ we can create a 2-vector which does not satisfy this property:

Example 13.1.9. Let $v_1, v_2, v_3, v_4 \in V$ be linearly independent. Then

$$a := v_1 \wedge v_2 + v_3 \wedge v_4$$

$$a \wedge a = (v_1 \wedge v_2 + v_3 \wedge v_4) \wedge (v_1 \wedge v_2 + v_3 \wedge v_4) = \underbrace{v_1 \wedge v_2 \wedge v_1 \wedge v_2}_{=0} + v_1 \wedge v_2 \wedge v_3 \wedge v_4 + v_3 \wedge v_4 \wedge v_1 \wedge v_2 + \underbrace{v_3 \wedge v_4 \wedge v_3 \wedge v_4}_{=0}$$

satisfies

$$a \wedge a = 2 v_1 \wedge v_2 \wedge v_3 \wedge v_4 \neq 0,$$

and therefore is not decomposable.

We can easily extend the exterior product to multi-vectors which are decomposable: For a p-blade $a = v_1 \wedge \cdots \wedge v_p \in \bigwedge^p V$ and a q-blade $b = w_1 \wedge \cdots \wedge w_q \in \bigwedge^q V$ the exterior product $a \wedge b \in \bigwedge^{p+q} V$ is defined in the obvious way

$$a \wedge b := v_1 \wedge \cdots v_p \wedge w_1 \wedge \cdots \wedge w_q.$$

From there it can be extended to arbitrary multi-vectors $a \in \bigwedge^p V, b \in \bigwedge^q V$ by linearity. If $b_1, \dots, b_n \in V$ is a basis, then

$$a = \sum_{i_1 < \cdots < i_p} \lambda_{i_1 \dots i_p} b_{i_1} \wedge \cdots \wedge b_{i_p}$$

for some $\lambda_{i_1 \dots i_p} \in \mathbb{F}$ and

$$b = \sum_{j_1 < \cdots < j_q} \mu_{j_1 \dots j_q} b_{j_1} \wedge \cdots \wedge b_{j_q}$$

for some $\mu_{j_1 \dots j_q} \in \mathbb{F}$. We define $a \in \bigwedge^{p+q} V$ by

$$a \wedge b := \sum_{i_1 < \cdots < i_p} \sum_{j_1 < \cdots < q_q} \lambda_{i_1 \dots i_p} \mu_{j_1 \dots j_q} b_{i_1} \wedge \cdots b_{i_p} \wedge b_{j_1} \wedge \cdots \wedge b_{j_q},$$

which does not depend on the chosen basis. The resulting general exterior product is still linear by definition.

For 1-vectors the exterior product is alternating. Thus, for a p-blade $a = v_1 \wedge \cdots \wedge v_p \in \bigwedge^p V$ and a q-blade $b = w_1 \wedge \cdots \wedge w_q \in \bigwedge^q V$ this leads to

$$\begin{aligned} a \wedge b &= v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q \\ &= (-1)^p \, w_1 \wedge v_1 \wedge \cdots \wedge v_p \wedge w_2 \wedge \cdots \wedge w_q \\ &\;\vdots \\ &= (-1)^{pq} \, w_1 \wedge \cdots \wedge w_q \wedge v_1 \wedge \cdots \wedge v_p \\ &= (-1)^{pq} \, b \wedge a, \end{aligned}$$

which again extends to general multi-vectors by linearity.

Thus, we have obtained the following general properties of the exterior product:

Proposition 13.1.10. *The exterior product*

$$\wedge : \textstyle\bigwedge^p V \times \bigwedge^q V \to \bigwedge^{p+q} V$$

satisfies the following properties:

(i) For $a \in \bigwedge^p V$, $b \in \bigwedge^q V$, $b \in \bigwedge^r V$

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c.$$

(ii) For $a \in \bigwedge^p V$, $b, c \in \bigwedge^q V$

$$a \wedge (b + c) = a \wedge b + a \wedge c$$

.

(iii) For $a \in \bigwedge^p V$, $b, c \in \bigwedge^q V$

$$a \wedge b = (-1)^{pq} \, b \wedge a.$$

Example 13.1.11 (Cramer's rule)**.** Let $a, b \in \mathbb{R}^2$ linearly independent, and

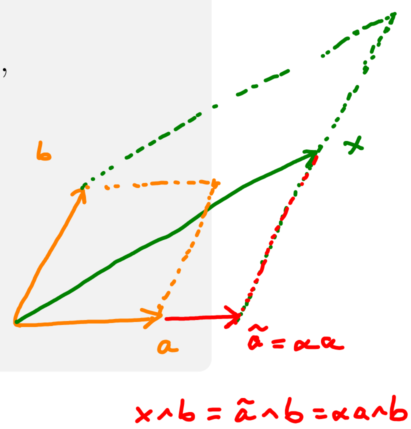
$$x = \alpha a + \beta b \in \mathbb{R}^2.$$

To determine the coefficients we take the exterior products

$$x \wedge a = \beta b \wedge a$$
$$x \wedge b = \alpha a \wedge b.$$

Since the 2-blades appearing on both sides are linearly dependent their quotient is well defined, and we obtain

$$\alpha = \frac{x \wedge b}{a \wedge b} = \frac{\det \begin{pmatrix} x_1 & b_1 \\ x_2 & b_2 \end{pmatrix}}{\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}}, \qquad \beta = \frac{a \wedge x}{a \wedge b} = \frac{\det \begin{pmatrix} a_1 & x_1 \\ a_2 & x_2 \end{pmatrix}}{\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}},$$



which is Cramer's rule for solving the linear system

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and, thus, leads to a geometric interpretation of it.

Remark 13.1.12. The direct sum

$$\textstyle\bigwedge^0 V \oplus \bigwedge^1 V \oplus \cdots \oplus \bigwedge^n V$$

is called the *Grassmann algebra* of V and constitutes a vector space of dimension 2^n.

Week 13: Grassmannians and the Plücker embedding

13.2 Grassmannians and the Plücker embedding

The projective subspaces of a projective space $\mathrm{P}(V)$ are represented by the linear subspaces of the underlying vector space V.

Definition 13.2.1. Let V be a vector space, $k \in \mathbb{N}$. Then the *Grassmannian* $\mathbf{Gr}(k, V)$ is the set of all k-dimensional linear subspaces of V.

In particular,

$$\mathbf{Gr}(1, V) = \mathrm{P}(V).$$

We can now take the decomposable vectors of th k-th exterior power of V to represent k-dimensional linear subspaces of V: Let $U \in \mathbf{Gr}(k, V)$ be a k-dimensional linear subspace of V Then there always exist k vectors $v_1, \ldots, v_k \in V$ such that

$$U = \operatorname{span}\{v_1, \ldots, v_k\}.$$

Furthermore, $v_1, \ldots, v_k$ are linearly independent. Now let $\tilde{v}_1, \ldots, \tilde{v}_k$ be another k vectors such that

$$U = \operatorname{span}\{\tilde{v}_1, \ldots, \tilde{v}_k\}.$$

Since $v_1, \ldots, v_k$ are a basis in U, we have

$$\tilde{v}_i = \sum_{j=1}^{k} \alpha_{ij} v_j$$

for some $\alpha_{ij} \in \mathbb{R}$. With $A := (\alpha_{ij})_{i,j=1,\ldots k}$ we find

$$0 \neq \tilde{v}_1 \wedge \cdots \wedge \tilde{v}_k = \det A \, v_1 \wedge \cdots \wedge v_k$$

where $\det A \neq 0$ since $\tilde{v}_1 \wedge \cdots \wedge \tilde{v}_k \neq 0$. Thus,

$$[\tilde{v}_1 \wedge \cdots \wedge \tilde{v}_k] = [v_1 \wedge \cdots \wedge v_k] \in \mathrm{P}(\textstyle\bigwedge^k V),$$

and the following map is well defined.

Definition 13.2.2. The map

$$\iota : \mathbf{Gr}(k, V) \to \mathrm{P}(\textstyle\bigwedge^k V), \qquad \operatorname{span}\{v_1, \ldots, v_k\} \mapsto [v_1 \wedge \cdots \wedge v_k]$$

is called the *Plücker embedding*.

Proposition 13.2.3. *The Plücker embedding is injective, and surjective onto the subset represented by decomposable k-vectors.*

13.2.1 Decomposable 2-vectors

Let V be a finite dimensional vector space. We have seen that a necessary condition for a k-vector $a \in \bigwedge^k V$ to be decomposable is

$a = v_1 \wedge \dots \wedge v_k$, $v_1, \dots, v_k \in V$

$$a \wedge a = 0 \in \bigwedge^{2k} V.$$

In the case $k = 2$, this condition is also sufficient.

We start with the case $\dim V = 3$, in which all 2-vectors are decomposable.

Lemma 13.2.4. *Let $\dim V = 3$. Then every 2-vector $a \in \bigwedge^2 V$ is decomposable.*

Proof. Let $a \in \bigwedge^2 V$. Consider the linear map

$$A : V \to \bigwedge^3 V, \qquad v \mapsto a \wedge v.$$

Since $\dim \bigwedge^3 V = 1$, we have $\dim \ker A \geqslant 2$. Let $v_1, v_2 \in \ker A$ linearly independent and extend them to a basis $v_1, v_2, v_3 \in V$. Then

$$a = a_1 v_2 \wedge v_3 + a_2 v_3 \wedge v_1 + a_3 v_1 \wedge v_2.$$

Now

$$0 = A(v_1) = a_1 \underbrace{v_1 \wedge v_2 \wedge v_3}_{\neq 0},$$

and thus $a_1 = 0$. Similarly, $a_2 = 0$. Therefore,

$$a = a_3 v_1 \wedge v_2,$$

which is decomposable. ∎

Theorem 13.2.5.

$$a \in \bigwedge^2 V \text{ decomposable} \quad \Leftrightarrow \quad a \wedge a = 0 \in \bigwedge^4 V.$$

Proof.

(⇒) Let $a \in \bigwedge^2 V$ be decomposable, i.e.,

$$a = v_1 \wedge v_2$$

with $v_1, v_2 \in V$. Then

$$a \wedge a = v_1 \wedge v_2 \wedge v_1 \wedge v_2 = 0.$$

(⇐) Let $a \in \bigwedge^2 V$ with $a \wedge a = 0$.

In the cases $\dim V = 0$ and $\dim V = 1$, we have $\dim \bigwedge^2 V = 0$.

In the case $\dim V = 2$, we have $\dim \bigwedge^2 V = 1$. If $v_1, v_2 \in V$ is a basis, then $0 \neq v_1 \wedge v_2 \in \bigwedge^2 V$ and thus all 2-vectores are demposable.

The case $\dim V = 3$ has been treated separately in Lemma 13.2.4.

We continue by induction in the dimension of V. Assume the statement is true for all dimensions $\dim V \leqslant n$, and consider the case $\dim V = n+1$. Let $v_1, \dots, v_{n+1} \in V$ be a basis. Then

$$\begin{aligned} a &= \sum_{1 \leqslant i < j \leqslant n+1} a_{ij} v_i \wedge v_j \\ &= \underbrace{\left(\sum_{i=1}^{n} a_{i,n+1} v_i \right)}_{=:u} \wedge v_{n+1} + \underbrace{\sum_{i \leqslant i < j \leqslant n} a_{ij} v_i \wedge v_j}_{\tilde{a}} \\ &= u \wedge v_{n+1} + \tilde{a} \end{aligned}$$

where $u \in U$ and $\tilde{a} \in \bigwedge^2 U$ with $U := \operatorname{span}\{v_1, \dots, v_n\}$, $\dim U = n$.

Now

$$\begin{aligned} 0 &= a \wedge a \\ &= (u \wedge v_{n+1} + \tilde{a}) \wedge (u \wedge v_{n+1} + \tilde{a}) \\ &= \underbrace{u \wedge v_{n+1} \wedge u \wedge v_{n+1}}_{=0} + 2\tilde{a} \wedge u \wedge v_{n+1} + \tilde{a} \wedge \tilde{a} \end{aligned}$$

The vector v_{n+1} does neither appear in the expansion of $\tilde{a} \wedge u$ nor $\tilde{a} \wedge \tilde{a}$, thus we obtain

$$\tilde{a} \wedge u = 0, \qquad \tilde{a} \wedge \tilde{a} = 0.$$

By induction $\tilde{a} \wedge \tilde{a} = 0$ implies

$$\tilde{a} = u_1 \wedge u_2$$

with some $u_1, u_2 \in V$. Then the first equation becomes

$$u_1 \wedge u_2 \wedge u = 0.$$

Thus, by Proposition 13.1.7, u_1, u_2, u are linearly dependent, i.e.,

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda u = 0$$

with some $\lambda_1, \lambda_2, \lambda \in \mathbb{F}$ which are not all 0.

If $\lambda = 0$, then u_1, u_2 are linearly dependent, and thus $\tilde{a} = u_1 \wedge u_2 = 0$. Then

$$a = u \wedge v_{n+1},$$

which is decomposable.

If $\lambda \neq 0$ we can write

$$u = \mu_1 u_1 + \mu_2 u_2$$

and thus

$$a = \mu_1 u_1 \wedge v_{n+1} + \mu_2 u_2 \wedge v_{n+1} + u_1 \wedge u_2.$$

This is the 3-dimensional case, which by induction, or by Lemma 13.2.4, is always decomposable.

□

13.3 The Klein-Plücker quadric

We now look at the Plücker embedding in the case $V = \mathbb{R}^4$. A line $\ell \subset \mathbb{R}\mathrm{P}^3$ is represented by a 2-dimensional linear subspace $U \in \mathbf{Gr}(2, \mathbb{R}^4)$, $\ell = \mathrm{P}(U)$. By means of the Plücker embedding, this subspace, in turn, is represented by a decomposable 2-vector $a \in \bigwedge^2 \mathbb{R}^4$.

Let $e_1, e_2, e_3, e_4 \in \mathbb{R}^4$ be the canonical basis. Then

$$a = \lambda_{12} e_1 \wedge e_2 + \lambda_{13} e_1 \wedge e_3 + \lambda_{14} e_1 \wedge e_4 + \lambda_{34} e_3 \wedge e_4 + \lambda_{42} e_4 \wedge e_2 + \lambda_{23} e_2 \wedge e_3,$$

and

$$a \wedge a = \underbrace{2(\lambda_{12}\lambda_{34} + \lambda_{13}\lambda_{42} + \lambda_{14}\lambda_{23})}_{=:\langle a,a \rangle} e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

where $\langle \cdot, \cdot \rangle$ is a quadratic form on the 6-dimensional vector space $\bigwedge^2 \mathbb{R}^4$. Thus, the decomposable 2-vectors of $\mathbb{R}^4$ are given by the kernel of the quadratic form $\langle \cdot, \cdot \rangle$:

$$a \wedge a = 0 \quad \Leftrightarrow \quad \langle a, a \rangle = 0.$$

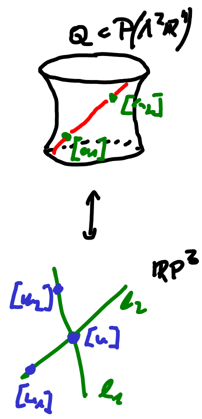
Definition 13.3.1. The *Klein-Plücker quadric* is the quadric

$$\mathcal{Q} := \left\{ [a] \in \mathrm{P}(\Lambda^2\mathbb{R}^4) \;\middle|\; \langle a, a\rangle = 0 \right\} \subset \mathrm{P}(\Lambda^2\mathbb{R}^4) \cong \mathbb{R}\mathrm{P}^5.$$

In the homogeneous coordinates $[\lambda_{12}, \lambda_{34}, \lambda_{13}, \lambda_{42}, \lambda_{14}, \lambda_{23}]$ the Gram-matrix of the Plücker quadric takes the form

$$\begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}$$

from which we see that $\mathcal{Q}$ is a quadric of neutral signature $(+++---)$, and thus contains isotropic lines and planes.



Theorem 13.3.2. *By the Plücker embedding, lines in $\mathbb{R}\mathrm{P}^3$ are in one-to-one correspondence with points on the Plücker quadric $\mathcal{Q} \subset \mathrm{P}(\Lambda^2\mathbb{R}^4)$, such that two lines $\ell_1, \ell_2 \subset \mathbb{R}\mathrm{P}^3$ intersect if and only if the line through their two corresponding points $[a_1], [a_2] \in \mathcal{Q}$ is isotropic. i.e.,*

$$\langle a_1, a_2\rangle = 0.$$

Proof. The one-to-one correspondence follows from the previous discussions and Proposition 13.2.3.

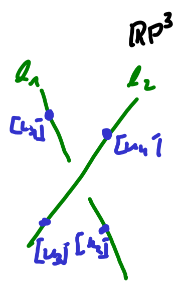
Let $\ell_1, \ell_2 \subset \mathbb{R}\mathrm{P}^3$ be two lines that intersect in the point $[u] \in \mathbb{R}\mathrm{P}^3$. Let $[u_1], [u_2] \in \mathbb{R}\mathrm{P}^3$ such that

$$\ell_1 = [u] \vee [u_1], \qquad \ell_2 = [u] \vee [u_2].$$

Then the line $[a_1] \vee [a_2] \subset \mathrm{P}(\Lambda^2\mathbb{R}^4)$ is spanned by $u \wedge u_1$ and $u \wedge u_2$. Thus, a point $[x] \in [a_1] \vee [a_2]$ is of the form

$$x = \lambda_1 u \wedge u_1 + \lambda_2 u \wedge u_2 = u \wedge (\lambda_1 u_1 + \lambda_2 u_2),$$

which is decomposable. Therefore, $[x] \in \mathcal{Q}$.



Let $\ell_1, \ell_2 \subset \mathbb{R}\mathrm{P}^3$ be two lines that do not intersect. Then $\mathbb{R}\mathrm{P}^3 = \ell_1 \vee \ell_2$ and there exists a basis $u_1, u_2, u_3, u_4 \in \mathbb{R}^4$ such that

$$\ell_1 = [u_1] \vee [u_2], \qquad \ell_2 = [u_3] \vee [u_4].$$

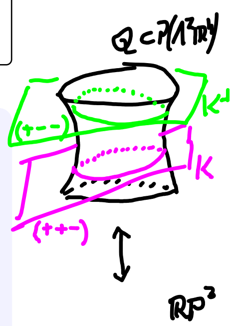
Now a point $[x] \in [a_1] \vee [a_2]$ is represented by

$$x = \lambda u_1 \wedge u_2 + \mu u_3 \wedge u_4,$$

and thus

$$x \wedge x = 2\lambda\mu \underbrace{u_1 \wedge u_2 \wedge u_3 \wedge u_4}_{\neq 0}.$$

This only vanishes for $\lambda = 0$ or $\mu = 0$. Therefore, the line $[a_1] \vee [a_2]$ intersects quadric $\mathcal{Q}$ in exactly two points, and thus, is not isotropic. □

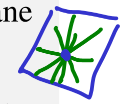


Corollary 13.3.3. *A non-degenerate non-empty planar section of the Plücker quadric corresponds to one of the two one-parameter families of rulings of a one-sheeted hyperboloid in* $\mathbb{R}\mathrm{P}^3$.

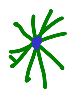
The intersection with its polar plane corresponds to the other one-parameter family of rulings.

Remark 13.3.4.

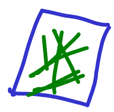
- All lines in $\mathbb{R}\mathrm{P}^3$ that lie in a common plane and intersect in a common plane correspond to an isotropic line in the Plücker quadric.



- All lines in $\mathbb{R}\mathrm{P}^3$ through a common point correspond to an isotropic plane in the Plücker quadric. Such isotropic plane are called α-planes. Two α-planes always intersect in a point.



- By duality, all lines in $\mathbb{R}\mathrm{P}^3$ through that lie in a common plane correspond also correspond to an isotropic plane in the Plücker quadric. Such isotropic plane are called β-planes. Two β-planes always intersect in a point.



- Each isotropic plane in the Plücker quadric is either an α-plane or a β-plane. Generically, an α-plane and a β-plane intersect. The special case in which they intersect (which is always in an isotropic line), occurs when the point that corresponds to the α-plane lies in the plane corresponding to the β-plane.