Weighted norms and a priori estimates for rough paths

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Weighted norms in ODEs/SDEs

Consider a Lipschitz map $b : \mathbb{R}^d \to \mathbb{R}^d$ and the standard ODE in $\mathbb{R}^d$

$$Y_t = Y_0 + \int_0^t b(Y_s)\,\mathrm{d}s.$$

Classical approach: fixed point method in the Banach space $C([0,T];\mathbb{R}^d)$.

If $C([0,T];\mathbb{R}^d)$ is endowed with the sup-norm $\|f\|_\infty$, then T has to be small in order to have a contraction.

An alternative method is to use a weighted norm

$$\|f\|_{\infty,\tau} := \sup_{t\in[0,T]} \exp\left(-\frac{t}{\tau}\right)|f_t|, \qquad \tau > 0.$$

Now we obtain a contraction if $\tau > 0$ small enough (uniformly in T).

The same method works for SDEs.

Controlled equations

Let us consider $X : [0,T] \to \mathbb{R}^d$ of class C^1, and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ smooth.

We consider the controlled equation

$$Y_t = Y_0 + \int_0^t \sigma(Y_s)\, \dot{X}_s \, \mathrm{d}s.$$

One can see that the following Taylor expansion in time holds

$$Y_t - Y_s = \sigma(Y_s)\, \mathbb{X}^1_{st} + \nabla\sigma(Y_s)\sigma(Y_s)\, \mathbb{X}^2_{st} + O(|t-s|^3)$$

where

$$\mathbb{X}^1_{st} = X_t - X_s, \qquad \mathbb{X}^2_{st} = \int_s^t (X_u - X_s) \otimes \dot{X}_u \, \mathrm{d}u.$$

We can also see that

$$\mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut}, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^2.$$

Rough equations

We now consider a $X : [0, T] \to \mathbb{R}^d$ which is only of class C^α with $\alpha \in]0, 1[$, let us say $\alpha \in]1/3, 1/2]$.

A rough path over X is a pair $(\mathbb{X}^1, \mathbb{X}^2)$, with $\mathbb{X}^1 : [0, T]^2 \to \mathbb{R}^d$ and $\mathbb{X}^1 : [0, T]^2 \to \mathbb{R}^d \otimes \mathbb{R}^d$, such that

$$\mathbb{X}^1_{st} = X_t - X_s, \qquad \mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut}, \qquad |\mathbb{X}^2_{st}| \lesssim |t - s|^{2\alpha}.$$

Then we look for $Y : [0, T] \to \mathbb{R}^d$ such that

$$Y_t - Y_s = \sigma(Y_s)\, \mathbb{X}^1_{st} + \nabla\sigma(Y_s)\sigma(Y_s)\, \mathbb{X}^2_{st} + O(|t - s|^{3\alpha}).$$

Note that $3\alpha > 1$ by assumption.

Norms

Also in the context of rough equations one uses naturally fixed point methods.

The relevant space is that of controlled paths, namely $\mathbf{y} = (y, y^1) : [0, T] \to \mathbb{R}^d \times \mathbb{R}^d$ s.t.

$$|y_t - y_s| + |y_t^1 - y_s^1| \lesssim |t-s|^\alpha, \qquad |y_{st}^2| \lesssim |t-s|^{2\alpha}, \qquad y_{st}^2 := y_t - y_s - y_s^1\, \mathbb{X}_{st}^1,$$

with norm

$$\|\mathbf{y}\| = |y_0| + |y_0^1| + \sup_{s \neq t} \frac{|y_t^1 - y_s^1|}{|t-s|^\alpha} + \sup_{s \neq t} \frac{|y_{st}^2|}{|t-s|^{2\alpha}}.$$

If Y is a solution to the above equation, then $(Y, \sigma(Y))$ is a controlled path.

Estimates

Let us introduce some notations: for $F : [0,T]^2 \to \mathbb{R}^d$ and $\eta > 0$

$$\|F\|_\eta := \sup_{s \neq t} \frac{|F_{st}|}{|t-s|^\eta},$$

so that setting also $\delta y_{st} := y_t - y_s$

$$\|\mathbf{y}\| = |y_0| + |y_0^1| + \|\delta y^1\|_\alpha + \|y^2\|_{2\alpha}.$$

There are two main estimates in this context: for $f : [0,T] \to \mathbb{R}^d$ and $F : [0,T]^2 \to \mathbb{R}^d$

$$\|f\|_\infty \leq |f_0| + T^\eta \|\delta f\|_\eta, \qquad \|F\|_\eta \leq T^{\eta'-\eta} \|F\|_{\eta'}, \quad \eta' > \eta > 0.$$

When $T > 0$ is small enough, then these estimates allow to obtain a contraction in the space of controlled paths.

Local versus global

The drawback of the above setting is that most of the arguments are local.

The transition from local to global requires additional arguments which are usually tedious and unelegant.

Moreover a priori estimates must be global and this approach does not seem to be optimal.

Weighted norms for controlled paths

For $\tau > 0$ we set for $f : [0,T] \to \mathbb{R}^d$, $F : [0,T]^2 \to \mathbb{R}^d$ and $\eta, \tau > 0$

$$\|f\|_{\infty,\tau} := \sup_{t\in[0,T]} \exp\left(-\frac{t}{\tau}\right) |f_t|,$$

$$\|F\|_{\eta,\tau} := \sup_{s\neq t} \mathbb{1}_{(|t-s|\leq\tau)} \exp\left(-\frac{t\vee s}{\tau}\right) \frac{|F_{st}|}{|t-s|^\eta}.$$

Now the relevant estimates are for $\eta' > \eta > 0$

$$\|f\|_{\infty,\tau} \leq |f_0| + 3(\tau\wedge T)^\eta \|\delta f\|_{\eta,\tau}, \qquad \|F\|_{\eta,\tau} \leq (\tau\wedge T)^{\eta'-\eta} \|F\|_{\eta',\tau}.$$

Note that

$$\lim_{\tau\to+\infty} \|f\|_{\infty,\tau} = \|f\|_\infty, \qquad \lim_{\tau\to+\infty} \|F\|_{\eta,\tau} = \|F\|_\eta.$$

The Sewing Lemma

The Sewing Lemma holds with these norms (with constant uniform in $\tau > 0$).

For $G : [0,T]^3 \to \mathbb{R}^d$ and $\eta, \tau > 0$ we set

$$\|G\|_{\eta,\tau} := \sup_{s,u,t\in[0,T]} \mathbb{1}_{(0<|t-u|\vee|u-s|\leq\tau)}\, e^{-\frac{\max\{s,u,t\}}{\tau}} \frac{|G_{sut}|}{(|t-u|\vee|u-s|)^{\eta}}.$$

Lemma

Let $\eta > 1$ and $\tau > 0$. For all $B \in C_3^{\eta} \cap \delta C_2$

$$\|\Lambda B\|_{\eta,\tau} \leq K_{\eta} \|B\|_{\eta,\tau} \tag{1}$$

where $\Lambda : B \in C_3^{\eta} \cap \delta C_2 \to C_2^{\eta}$ is the sewing map.

Results: uniqueness

Recall that we study solutions to

$$Y_t - Y_s = \sigma(Y_s)\,\mathbb{X}^1_{st} + \sigma_2(Y_s)\,\mathbb{X}^2_{st} + O(|t-s|^{3\alpha}), \qquad s,t \in [0,T], \tag{2}$$

with $3\alpha > 1$ and arbitrary $T > 0$, where we set

$$\sigma_2(y) := \nabla\sigma(y)\sigma(y).$$

Theorem

Suppose $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is of class C^3 (without boundedness assumptions). Let Y, Y' two solutions to (2). Then for $\tau > 0$ small enough

$$\|Y - Y'\|_{\infty,\tau} \leq 2|Y_0 - Y'_0|.$$

In particular, if $Y_0 = Y'_0$ then $Y \equiv Y'$.

A priori estimate

Suppose that σ and $\sigma_2 = \nabla\sigma\,\sigma$ are of class C^1 and globally Lipschitz, namely $\|\nabla\sigma\|_\infty + \|\nabla\sigma_2\|_\infty < +\infty$. We fix

$$L \geq \|\nabla\sigma\|_\infty + \|\nabla\sigma_2\|_\infty.$$

Let us also define

$$\rho_\alpha(\mathbb{X}) := \|\mathbb{X}^1\|_\alpha + \|\mathbb{X}^2\|_{2\alpha}.$$

Theorem

Let $M > 0$. There exists $\varepsilon_{M,L} > 0$ such that if $\rho_\alpha(\mathbb{X}) \leq M$ and $(\tau \wedge T)^\alpha \leq \varepsilon_{M,L}$, then any solution to (2) satisfies

$$\|\delta Y\|_{\alpha,\tau} + \|Y^2\|_{2\alpha,\tau} \leq 4\,\rho_\alpha(\mathbb{X})\left(|\sigma(Y_0)| + |\sigma_2(Y_0)|\right). \tag{3}$$

Continuity of the Itô-Lyons map

Suppose that σ is of class C^3, with $\|\nabla\sigma\|_\infty + \|\nabla^2\sigma\|_\infty + \|\nabla^3\sigma\|_\infty < +\infty$. We fix

$$L \geq \|\nabla\sigma\|_\infty + \|\nabla^2\sigma\|_\infty + \|\nabla^3\sigma\|_\infty.$$

Theorem

Let $M > 0$. Then for every $T > 0$ there exist $\varepsilon_{M,L,T}, C_{M,L,T} > 0$ such that if $\tau \in]0, \varepsilon_{M,L,T}]$, then any pair of solutions $(\mathbb{X}, Y), (\bar{\mathbb{X}}, \bar{Y})$ to (2) with $\max\{|Y_0|, \rho_\alpha(\mathbb{X}), |\bar{Y}_0|, \rho_\alpha(\bar{\mathbb{X}})\} \leq M$ satisfy

$$\|\delta Y - \delta\bar{Y}\|_{\alpha,\tau} + \|Y^2 - \bar{Y}^2\|_{2\alpha,\tau} \leq C_{M,L,T}\left(|Y_0 - \bar{Y}_0| + \rho_\alpha(\mathbb{X} - \bar{\mathbb{X}})\right).$$

See [Lejay 2012] for results were σ is of class C^2 and $\sigma_2 = \nabla\sigma\,\sigma$ is of class $C^{1+\gamma}$.