Martingale problems for some singular SPDEs

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SPDEs & Friends

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What are we doing?

- Formulate singular martingale problems for singular SPDEs, construct generator.
- Combine rough path ideas with martingale methods, Gaussian analysis, infinite-dimensional PDEs, functional analysis.
- Useful for scaling limits, ergodicity, functional inequalities, moments, fluctuations, . . .

Main example: conservative stochastic Burgers equation

Kardar-Parisi-Zhang '86: "universal" model for interface growth is $h = \partial_x^{-1} u$

Stochastic Burgers equation $u\colon \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$ (or $u\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$),

$$\partial_t u = \partial_{xx}^2 u + \partial_x u^2 + \partial_x \xi,$$

ξ: white noise (centered Gaussian, $\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y)$).

- Problem is singularity:
  law of white noise is invariant, $u^2 = ?$

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- Problem is singularity:
  law of white noise is invariant, $u^2 = ?$
- Hairer '13/Kupiainen-Marcozzi '17/Gubinelli-P. '17: solve w. pathwise arguments
- E.g. idea in regularity structures: freeze ω, write

$$u(y) = u(x) + \sum_\tau u^\tau(x)\Pi_x \tau(y) + O(|x-y|^\gamma)$$

for "trees" τ (generalized monomials).
$\Rightarrow$ u is "smooth".

Local subcriticality

Regularity structures work for locally subcritical equations as in Hairer '14.

The SPDE

$$\partial_t u = Au + f(u, \nabla u) + \xi$$

is locally subcritical if $\exists \alpha \in \mathbb{R}$, $\beta, \gamma > 0$ s.t.

$$u_\lambda(t,x) := \lambda^\alpha u(\lambda^\beta t, \lambda x)$$

solves

$$\partial_t u_\lambda = A u_\lambda + \lambda^\gamma f(u_\lambda, \nabla u_\lambda) + \xi_\lambda$$

with $\mathrm{law}(\xi_\lambda) = \mathrm{law}(\xi)$.

$\gamma = 0$: critical
$\gamma < 0$: supercritical

Analogy with SDEs

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s$$

Three types of solutions:

1. Strong Itô solution:
   Fix W, find $(\mathcal{F}_t^W)$-adapted X, interpret $\int_0^t \sigma(X_s)dW_s$ as Itô integral.
2. Weak Itô solution:
   Solution is couple (X, W), maybe $X_t \notin \mathcal{F}_t^W$.
   Equivalent to martingale problem.
3. Rough path solution: *freeze ω, rough path integral $\int_0^t \sigma(X_s(\omega))dW_s(\omega)$*

$$X_t(\omega) = x + \int_0^t b(X_s(\omega))ds + \int_0^t \sigma(X_s(\omega))dW_s(\omega)$$

Each has **different strengths and weaknesses.**
For singular SPDEs: mainly pathwise approach (3). What about (1), (2)?

The martingale problem

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s$$

- Itô's formula:

$$\varphi(X_t) - \varphi(x) = \int_0^t \underbrace{\left(b\partial_x\varphi + \frac{1}{2}\sigma^2\partial_{xx}\varphi\right)}_{=\mathcal{L}\varphi}(X_s)ds + \underbrace{\int_0^t (\partial_x\varphi\sigma)(X_s)dW_s}_{=M_t}$$

- Martingale problem:

$$\varphi(X_t) - \varphi(x) - \int_0^t \mathcal{L}\varphi(X_s)ds = \text{martingale}$$

- Describe evolution of law(X_t) via differential operator $\mathcal{L}$.
- Useful for: ergodic theory, scaling limits, fluctuations, stability analysis, ...

Formal generator

Burgers eq. $\partial_t u = \partial_{xx} u + \partial_x u^2 + \partial_x \xi$

- Test function $\varphi(u) = u(f)$ $\Rightarrow$ formally:

$$d\varphi(u_t) = [u_t(\partial_{xx} f) - u_t^2(\partial_x f)]dt + dM_t(f)$$

so

$$\mathcal{L}\varphi(u) = \frac{d}{dt}\mathbb{E}[\varphi(u_t)|u_0 = u]|_{t=0} = u(\partial_{xx} f) - u^2(\partial_x f)$$

- But state space is $H^{-1/2-\varepsilon}(\mathbb{T})$
$\Rightarrow u^2$ and thus $\mathcal{L}F(u)$ ill-defined for general $u \in H^{-1/2-\varepsilon}(\mathbb{T})$.

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$\Rightarrow u^2$ and thus $\mathcal{L}F(u)$ ill-defined for general $u \in H^{-1/2-\varepsilon}(\mathbb{T})$.
- Construct $\mathcal{L}\varphi(u)$ "for almost all" u?
Q: Which reference measure?
A: u formally invariant under law of white noise.

Intermezzo: energy solutions

Burgers eq. $\partial_t u = \partial_{xx} u + \partial_x u^2 + \partial_x \xi$

Gonçalves-Jara '14, Gubinelli-Jara '13, Gubinelli-P. '18:

- $u \in C(\mathbb{R}_+, H^{-1/2-\varepsilon})$ s.t. law$(u_t) \ll$ law(white noise) for all $t \geq 0$ + technical conditions, then

$$\int_0^t \mathcal{L}\varphi(u_s) ds$$

  well defined for "cylinder functions" $\varphi(u) = F(u(f_1), \ldots, u(f_n))$.
- Use this to define "energy solution".
- (Strong) well-posedness of energy solutions via "Cole-Hopf transformation".

Intermezzo: energy solutions

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Remained open:

- Construct $\mathcal{L}\varphi(u_t)$? (Compare $\int_0^t \delta(B_s)ds = L_t$ vs $\delta(B_t)$.)
- What to do when Cole-Hopf does not work?

Back to the generator

Formal derivation of $\mathcal{L}\varphi$ for general test functions $\varphi\colon H^{-1/2-\varepsilon} \to \mathbb{R}$:

- Recall generator for SDE $dY_t = b(Y_t)dt + dW_t$:

$$\mathcal{L}\varphi(y) = \sum_k b_k(y)\partial_k\varphi(y) + \frac{1}{2}\Delta\varphi(y)$$

- Burgers eq. $\partial_t u = \partial_{xx} u + \partial_x u^2 + \partial_x \xi$, so should get

$$\mathcal{L}\varphi(u) = \sum_k \langle \partial_x u^2, e_k\rangle D_{e_k}\varphi(u) + \mathcal{L}_0\varphi(u) =: \mathcal{G}\varphi(u) + \mathcal{L}_0\varphi(u)$$

for "Malliavin derivative" D.

How to find a domain?

$$\mathcal{L}\varphi(u) = \sum_k \langle \partial_x u^2, e_k \rangle D_{e_k}\varphi(u) + \mathcal{L}_0\varphi(u)$$

- Idea (inspired by SDE results of Flandoli-Russo-Wolf '03, Delarue-Diel '16): solve resolvent equation

$$(\lambda - \mathcal{L})\varphi = \psi,$$

  then $\mathcal{L}\varphi = \lambda\varphi - \psi$ ok.
- ∞-dimensional PDE!
  equation on $H^{-1/2-\varepsilon}(\mathbb{T})$*;*
  reference measure? regularity? function spaces?

White noise and Fock space

- Gaussian structure:
  Invariant measure μ for Burgers eq. is white noise: centered Gaussian,

$$\int u(f)u(g)\mu(du) = \langle f, g\rangle_{L^2(\mathbb{T})}.$$

- Chaos decomposition:
  $L^2(\mu)$ *isomorphic (even isometry) to Fock space* $\bigoplus_{n=0}^{\infty} L^2_s(\mathbb{T}^n)$,

$$\langle \varphi, \psi\rangle = \sum_{n=0}^{\infty} n! \langle \varphi_n, \psi_n\rangle_{L^2_s(\mathbb{T}^n)}.$$

($L^2_s(\mathbb{T}^n)$: φ_n *symmetric under* $x_i \leftrightarrow x_j$)

Regularity and distributions on Fock space

Fock space $\bigoplus_{n=0}^{\infty} L^2_s(\mathbb{T}^n)$ with $\langle \varphi, \psi \rangle = \sum_{n=0}^{\infty} n! \langle \varphi_n, \psi_n \rangle_{L^2_s(\mathbb{T}^n)}$.

- Heat equation generator:

$$(\mathcal{L}_0 \varphi)_n = \Delta \varphi_n = (\partial^2_{x_1 x_1} + \cdots + \partial^2_{x_n x_n}) \varphi_n$$

- Number operator (=Ornstein-Uhlenbeck operator):

$$(\mathcal{N} \varphi)_n = n \varphi_n$$

- Measure regularity with $\mathcal{L}_0$ and $\mathcal{N}$:
  E.g. φ is (Hida) distribution on $L^2(\mu)$ if

$$\|(1 + \mathcal{N})^{-1} (1 - \mathcal{L}_0)^{-2} \varphi\| < \infty.$$

  Can test $\langle \varphi, \psi \rangle$ for nice ψ, but
  $\varphi \notin L^2(\mu)$, not even a random variable.

$$\mathcal{H}^{\alpha}_{\beta} := \mathrm{dom}((1 + \mathcal{N})^{\beta} (1 - \mathcal{L}_0)^{\alpha}))$$

The Burgers generator

$$\partial_t u = \Delta u + \partial_x u^2 + \partial_x \xi \qquad \Rightarrow \qquad \mathcal{L} = \mathcal{L}_0 + \mathcal{G}.$$

- Same $\mathcal{L}_0$ as before.
- $\mathcal{G}\varphi$ only distribution – even for smooth φ:

$$\|\mathcal{G}\varphi\|_{\mathcal{H}^{\alpha-3/4}_{\beta-1}} \lesssim \|\varphi\|_{\mathcal{H}^{\alpha}_{\beta}} \qquad \underline{\text{if}}\ \alpha \in (\tfrac{1}{4}, \tfrac{1}{2}]$$

- Resolvent equation:

$$(\lambda - \mathcal{L})\varphi = \psi \qquad \Leftrightarrow \qquad \varphi = (\lambda - \mathcal{L}_0)^{-1}(\mathcal{G}\varphi + \psi).$$

- $\mathcal{G}$ loses $\mathcal{N}$-regularity:

$$\|\mathcal{G}\varphi\|_{\mathcal{H}^{\alpha-3/4}_{\beta-1}} \lesssim \|\varphi\|_{\mathcal{H}^{\alpha}_{\beta}};$$

- $(\lambda - \mathcal{L}_0)^{-1}$ only gains $\mathcal{L}_0$-regularity:

$$\|(\lambda - \mathcal{L}_0)^{-1}\varphi\|_{\mathcal{H}^{\alpha+1}_{\beta}} \lesssim \|\varphi\|_{\mathcal{H}^{\alpha}_{\beta}}.$$

Even have $(1 - \mathcal{L}_0)^{1/4}$ to spare; tradeoff $\mathcal{N}$ vs $\mathcal{L}_0$?

Tradeoff $\mathcal{N}$ vs $\mathcal{L}_0$: spectral calculus (=Fourier coordinates)

- Let

$$\Pi^{\succ} = \mathbb{1}_{1-\mathcal{L}_0 > (1+\mathcal{N})^4}$$

(just set $\mathcal{F}(\Pi^{\succ}\varphi)_n(k) = \mathbb{1}_{(1+|2\pi k|^2) > (1+n)^4} \mathcal{F}\varphi_n(k)$)

- $\Rightarrow$

$$\begin{aligned} \|\Pi^{\succ}\varphi\|_{\mathcal{H}^{\alpha}_{\beta+1}} &= \|(1-\mathcal{L}_0)^{\alpha}(1+\mathcal{N})^{\beta+1}\mathbb{1}_{1-\mathcal{L}_0>(1+\mathcal{N})^4}\varphi\| \\ &\leq \|(1-\mathcal{L}_0)^{\alpha}(1+\mathcal{N})^{\beta}(1-\mathcal{L}_0)^{1/4}\mathbb{1}_{1-\mathcal{L}_0>(1+\mathcal{N})^4}\varphi\| \\ &= \|\Pi^{\succ}\varphi\|_{\mathcal{H}^{\alpha+1/4}_{\beta}} \end{aligned}$$

Modified resolvent equation

Recall:

- $\|\mathcal{G}\varphi\|_{\mathcal{H}_{\beta-1}^{-1/4}} \lesssim \|\varphi\|_{\mathcal{H}_{\beta}^{1/2}}$;
- $\|\Pi^{\succ}\varphi\|_{\mathcal{H}_{\beta}^{-1/2}} \leq \|\Pi^{\succ}\varphi\|_{\mathcal{H}_{\beta-1}^{-1/4}}$

$\Rightarrow$ for $\mathcal{G}^{\succ} = \Pi^{\succ}\mathcal{G}$:

$$\|\mathcal{G}^{\succ}\varphi\|_{\mathcal{H}_{\beta}^{-1/2}} \lesssim \|\varphi\|_{\mathcal{H}_{\beta}^{1/2}}$$

and we can solve modified resolvent eq. for all $\lambda \geq 0$:

$$(\lambda - \mathcal{L}_0)\varphi = \mathcal{G}^{\succ}\varphi + \psi$$

Definition

φ is controlled if

$$\varphi = (-\mathcal{L}_0)^{-1}\mathcal{G}^{\succ}\varphi + \varphi^{\sharp}, \qquad \varphi^{\sharp} \text{ “nice”}.$$

Domain for the Burgers generator

φ controlled if with $\mathcal{G}^{\succ} = \Pi^{\succ}\mathcal{G}$:

$$\varphi = (-\mathcal{L}_0)^{-1}\mathcal{G}^{\succ}\varphi + \varphi^{\sharp}, \qquad \varphi^{\sharp} \text{ “nice”}.$$

Lemma

Controlled φ exist and are dense.

Proof: Picard iteration + (mildly) tedious estimates.

Lemma

$\mathcal{L}\varphi \in L^2(\mu)$ for all controlled φ.

Proof: direct computation via

$$\mathcal{L}\varphi = \mathcal{L}_0[(-\mathcal{L}_0)^{-1}\mathcal{G}^{\succ}\varphi + \varphi^{\sharp}] + \mathcal{G}\varphi = \mathcal{L}_0\varphi^{\sharp} + \mathcal{G}^{\prec}\varphi.$$

Martingale problem: existence

$$\mathrm{dom}(\mathcal{L}) = \{\varphi : \varphi = (-\mathcal{L}_0)^{-1}\mathcal{G}^{\succ}\varphi + \varphi^{\sharp}, \varphi^{\sharp} \text{ “nice”}\}$$

- Absolute continuity:
  $\mathrm{dom}(\mathcal{L}) \subset L^2(\mu)$, *but $\varphi(u)$ not defined $\forall\, u \in H^{-1/2-\varepsilon}(\mathbb{T})$;*
  $\Rightarrow$ *let (u_t) w/ trajectories in $C(\mathbb{R}_+, H^{-1/2-\varepsilon}(\mathbb{T}))$ s.t. $\mathrm{law}(u_t) \ll \mu \;\forall\; t \geq 0$.*
- Martingale problem: absolutely continuous u with

$$\varphi(u_t) - \varphi(u_0) - \int_0^t \mathcal{L}\varphi(u_s)ds = \mathrm{Mart.} \qquad \forall\varphi \in \mathrm{dom}(\mathcal{L}).$$

- Existence:
  via Galerkin approximation, easy.
- Uniqueness?
  $\rightarrow$ *duality of martingale problem and backward equation.*

Duality and uniqueness for the martingale problem

Kolmogorov backward equation

$$\partial_t \varphi = \mathcal{L}\varphi, \qquad \varphi(0) = \varphi_0 \in \mathrm{dom}(\mathcal{L})$$

Existence via energy estimates + controlled ansatz.

- Duality:
  u solution to martingale problem, φ solution to backward eq., then

$$\begin{aligned}\mathbb{E}[\varphi_0(u_t)] &= \mathbb{E}[\varphi(t-t, u_t)] \\ &= \mathbb{E}\Big[\varphi(t-0, u_0) + \int_0^t (\partial_s + \mathcal{L})\varphi(t-s, u_s)ds\Big] \\ &= \mathbb{E}[\varphi(t, u_0)].\end{aligned}$$

- Existence for martingale problem $\Rightarrow$ uniqueness for backward eq.
- Existence for backward eq. $\Rightarrow$ uniqueness of martingale problem
  (one-dim margins, then inductively for finite-dim margins).

A better idea by L. Gräfner

Consider "nice" approximation $\mathcal{G}^m$ of $\mathcal{G}$ and

$$(1 - \mathcal{L}_0 - \mathcal{G}^m)\varphi^m = \psi.$$

Test against φ^m:

$$\begin{aligned}\langle \varphi^m, \psi \rangle &= \langle \varphi^m, (1 - \mathcal{L}_0 - \mathcal{G}^m)\varphi^m \rangle \\ &= \|(1 - \mathcal{L}_0)^{1/2}\varphi^m\|^2 - \underbrace{\langle \varphi^m, \mathcal{G}^m \varphi^m \rangle}_{\overset{\text{antisymm.}}{=} \langle -\mathcal{G}^m\varphi^m, \varphi^m\rangle = 0} \\ &= \|\varphi^m\|^2_{\mathcal{H}_0^{1/2}}\end{aligned}$$

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By Young's inequality:

$$\|\varphi^m\|^2_{\mathcal{H}_0^{1/2}} = \langle \varphi^m, \psi \rangle \leq \frac{1}{2}\|\varphi^m\|^2 + \frac{1}{2}\|\psi\|^2$$
$$\Rightarrow \|\varphi^m\|^2_{\mathcal{H}_0^{1/2}} \leq \|\psi\|^2$$

Gräfner's idea continued

$$(1 - \underbrace{\mathcal{L}_0 - \mathcal{G}^m}_{=:\mathcal{L}^m})\varphi^m = \psi \qquad \Rightarrow \qquad \|\varphi^m\|^2_{\mathcal{H}_0^{1/2}} \leq \|\psi\|^2$$

- Subsequence $(\varphi^{m_k})_k$ converges in $\mathcal{H}_{-\varepsilon}^{1/2-\varepsilon}$ to some φ.
- Recall: $\mathcal{G}, \mathcal{L}_0 \colon \mathcal{H}_\beta^{1/3} \to \mathcal{H}_{\beta-1}^{-2/3}$ are bounded

$$\Rightarrow \qquad (1 - \mathcal{L})\varphi = \lim_k \underbrace{(1 - \mathcal{L}^{m_k})\varphi^{m_k}}_{\text{conv. in } \mathcal{H}_{-1-\varepsilon}^{-2/3}} = \psi$$

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- Recall: $\mathcal{G}, \mathcal{L}_0 \colon \mathcal{H}^{1/3}_{\beta} \to \mathcal{H}^{-2/3}_{\beta-1}$ are bounded

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- Some more work: Find one subsequence (m_k) s.t. $\mathcal{A}\psi = \lim_k (1-\mathcal{L}^{m_k})^{-1}\psi$ exists for all $\psi \in L^2 = \mathcal{H}^0_0$

A domain for $\mathcal{L}$

$\mathrm{dom}(\mathcal{L}) := \mathcal{A}L^2$ with

$$\mathcal{L}\mathcal{A}\psi = -(1-\mathcal{L})\mathcal{A}\psi + \mathcal{A}\psi = -\psi + \mathcal{A}\psi \in L^2$$

The semigroup

Lumer-Philips Theorem

Operator L on Hilbert space which is dissipative ($\langle h, Lh\rangle_H \leq 0$) and $(1 - L)$ is surjective.
Then L is closed, $\mathrm{dom}(L)$ is dense and L generates a contraction semigroup $T_t = e^{tL}$, $t \geq 0$.

For us:

- $(1 - \mathcal{L})\mathcal{A}\psi = \psi$ for all $\psi \in L^2 \Rightarrow (1 - \mathcal{L})$ is surjective.
- $\langle \varphi, \mathcal{L}\varphi\rangle = -\|(-\mathcal{L}_0)^{1/2}\varphi\|^2 + \underbrace{\langle \varphi, \mathcal{G}\varphi\rangle}_{=0} \leq 0.$

Remarks

- Non-perturbative approach. Works for

$$\partial_t u = -(-\Delta)^\alpha u + \partial_x u^2 + (-\Delta)^{\alpha/2}\xi$$

  if $\alpha > 1/2$. Note: equ. is critical for $\alpha = 3/4$.
- Uniqueness of the semigroup $\leftrightarrow$ uniqueness of dom(L).
- Get uniqueness and duality with martingale problem if we find dense

$$D \subset \mathrm{dom}(\mathcal{L}) \cap \mathcal{H}_2^{1/2}.$$

- L. Gräfner: achieve this by treating $\mathcal{L}^m$ as operator in $\mathcal{H}_2^0$.
  Then $\mathcal{G}^m$ no longer antisymmetric, but "close to antisymmetric" (perturbative argument).
  Works if $\alpha > 3/4$ or $\alpha = 3/4$ + smallness condition.

Extensions and limitations

Approach works also for:

- Multi-component Burgers eq. of Funaki-Hoshino '17, Kupiainen-Marcozzi '17

$$\partial_t u^i = \Delta u^i + \sum_{j,k} \Gamma^i_{jk} \partial_x (u^j u^k) + \partial_x \xi^i$$

  under "trilinear condition" of Funaki-Hoshino '17: $\Gamma^i_{jk} = \Gamma^i_{kj} = \Gamma^j_{ki}$.
- 2d Navier-Stokes type eq. Gubinelli-Turra '20.
- Modified surface quasi-geostrophic eq. Luo-Zhu '20.
- (Critical) surface quasi-geostrophic eq. Gräfner '21.
- Limitations:
  need Gaussian invariant measure (quasi-invariance might work),
  need antisymmetry of $\mathcal{G}$.

Some easy consequences

- Weak universality for fractional Burgers Sethuraman '16, Gonçalves-Jara '18 and multi-component Burgers Bernardin-Funaki-Sethuraman '19.
- Burgers on $\mathbb{R}$ is ergodic.
- Burgers on $\mathbb{T}$ is exponentially ergodic:

$$\int \left| \mathbb{E}_u[\varphi(u_t)] - \int \varphi \, d\mu \right|^2 d\mu(u) \leq e^{-8\pi^2 t} \int \varphi^2 \, d\mu.$$

Ergodicity on $\mathbb{T}$ known by Hairer-Mattingly '18.

- (Multi-component / fractional) KPZ on $\mathbb{T}$ has Gaussian fluctuations.
  Tracy-Widom fluctuations on $\mathbb{R}$ Sasamoto-Spohn '10, Amir-Corwin-Quastel '11.
  (Usual) KPZ on $\mathbb{T}$ has Gaussian fluctuations by Gu-Komorowski '21.

Summary

- Probabilistic theory for singular SPDEs
  $\leftrightarrow \infty$-dim singular operator $\mathcal{L} = \mathcal{L}_0 + \mathcal{G}$.
- Under Gaussian (invariant) measure:
  use chaos decomposition $\to$ work on Fock space
- Construct $\mathrm{dom}(\mathcal{L})$ via ideas from paracontrolled distributions.
- Duality gives uniqueness for martingale prob. and backward eq.
- With new input from Gräfner: works for some critical equations.
- Need Gaussian measure. Beyond: unclear.

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Thank you!