NOTES ON EXOTIC SMOOTH STRUCTURES

JOHN SULLIVAN – DIFFERENTIAL GEOMETRY II – TU BERLIN

Suppose N is a topological manifold with smooth structure (maximal atlas) $\mathcal{U}$. Suppose M is a topological manifold and f is a homeomorphism $M \to N$. We can use f to "pull back" the smooth structure $\mathcal{U}$ from N to M as follows: for each chart (U, ϕ) in $\mathcal{U}$ we have a chart $(f^{-1}(U), \phi \circ f)$ on M. These charts form a maximal smooth atlas $f^*(\mathcal{U})$ for M, that is a smooth structure on M. Almost by definition, the map $f\colon M \to N$ is a diffeomorphism from $(M, f^*(\mathcal{U}))$ to $(N, \mathcal{U})$.

Now consider the case $M = N$. If f is the identity map, then $f^*(\mathcal{U}) = \mathcal{U}$. More generally, this is true any time f is a diffeormorphism with respect to the smooth structure $\mathcal{U}$ (that is, a diffeomorphism from the smooth manifold $(M, \mathcal{U})$ to itself).

But suppose f is a homeomorphism which is *not* a diffeomorphism. Then $(N, f^*(\mathcal{U})) = (M, f^*(\mathcal{U}))$ is a *distinct* differentible structure on the same differentiable manifold M. For instance, a real-valued function $g\colon M \to \mathbb{R}$ is smooth w.r.t. $\mathcal{U}$ iff $g \circ f$ is smooth w.r.t. $f^*(\mathcal{U})$. But, as mentioned above, the map f is a diffeomorphism as a map $(M, f^*(\mathcal{U})) \to (M, \mathcal{U})$. In particular, the two smooth manifolds are diffeomorphic to each other, meaning they are really the same smooth manifold.

Thus, this simple construction never produces "exotic" smooth structures (two different – nondiffeomorphic – smooth manifolds that are homeomorphic to each other, meaning they share the same underlying topological manifold).

Facts:

(1) Up to diffeomorphism, there is a unique smooth structure on any topological manifold M^n in dimension $n \leq 3$. Up to diffeomorphism, there is a unique smooth structure on $\mathbb{R}^n$ for $n \neq 4$.

(2) The *Hauptvermutung* (with that name even in English) of geometric topology (formulated 100 years ago) suggested that every topological manifold should have a unique PL structure (given by a triangulation) and a unique smooth structure. This is now known to be false.

(3) Every smooth manifold has a triangulation. Not every topological M^4 has a triangulation.

(4) There are uncountably many smooth structures on $\mathbb{R}^4$. It is unknown if there is any exotic smooth structure on S^4.

(5) In higher dimensions, some things get easier. In dimensions $n \geq 7$, for instance, there are exotic spheres S^n, but these form a well-understood finite group (e.g., there are 28 for $n = 7$). In general, the differences between smooth and PL manifolds, and between PL and topological manifolds, can be analyzed for $n \geq 5$ by means of algebraic topology.