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http://page.math.tu-berlin.de/~sullivan/L/11W/DG2/ WS 11

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 9

(Lie derivative of forms, manifolds with boundary, Stokes' theorem)

due 11.1.2012

Exercise 1 **5 points**

A smooth vector field X on a manifold M determines two operations on differential forms,

$$i_X \colon \Lambda^k(M) \to \Lambda^{k-1}(M), \quad L_X \colon \Lambda^k(M) \to \Lambda^k(M),$$

as follows: If $\omega \in \Lambda^k(M)$, then

$$(i_X\omega)(X_1, \ldots, X_{k-1}) = \omega(X, X_1, \ldots, X_{k-1}),$$

and, as we already know,

$$L_X\omega = \left.\frac{d}{dt}\right|_{t=0} \Phi_t^*\omega,$$

where Φ_t denotes the local flow of X. Show that for $k \geq 1$

$$L_X\omega = i_X d\omega + d(i_X\omega).$$

Exercise 2 **5 points**

Show that all line integrals of $\omega = P\,dx + Q\,dy + R\,dz$ in $\mathbb{R}^3$ are independent of the path if and only if the value of the integral over any closed (piecewise C^1) path is zero. Use this and Stokes' theorem to obtain a condition on P, Q, R which is sufficient to show independence of path. (Assume ω is defined on all of $\mathbb{R}^3$.)

Exercise 3 **5 points**

Let $\big(M; g\big)$ be a Riemannian manifold of dimension n and Ω the corresponding volume form. For a vector field X on M we define its divergence by

$$L_X\Omega =: (\operatorname{div} X)\Omega.$$

Further we consider the inward pointing unit normal vector field N on ∂M and define a volume form ω on ∂M by

$$\omega\big(X_1, \ldots, X_{n-1}\big) := i_N\Omega\big(X_1, \ldots, X_{n-1}\big).$$

Prove the divergence theorem of Gauß

$$\int_M \operatorname{div}\big(X\big)\Omega = \int_{\partial M} g\big(X, N\big)\omega.$$