Technische Universität Berlin
Institut für Mathematik
Sullivan / Knöppel
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Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 6

(Frobenius theorem, tangent covectors)

due 30.11.2011

Exercise 1 — 5 points

Denote by $gl\,(n)$ the space of all $n \times n$ matrices. Let U_0 be an open subset of $\mathbb{R}^2$, $(x_0, y_0) \in U_0$, $C \in gl\,(n)$, and $P, Q\colon U_0 \to gl\,(n)$ smooth maps. Then the the initial value problem

$$\frac{\partial g}{\partial x} = gP, \quad \frac{\partial g}{\partial y} = gQ, \quad g\,(x_0, y_0) = C$$

has a $gl\,(n)$–valued smooth solution g in a neighborhood of (x_0, y_0) if and only if

$$P_y - Q_x = [P, Q] := PQ - QP.$$

(This is known as the Maurer–Cartan lemma.)

Exercise 2 — 5 points

On $\mathbb{R}^3$ we consider the 1–form $\omega := xdy + dz$. Show that

$$E^2 = \{X \in \mathbb{R}^3 \mid \omega\,(X) = 0\}$$

is a 2–dimsional distribution which is not involutive.

Exercise 3 — 5 points

Determine the subset of $\mathbb{R}^2$ on which $\sigma_1 = x_1 dx_1 + x_2 dx_2$, $\sigma_2 = x_2 dx_1 + x_1 dx_2$ are linearly independent and find a frame field dual to σ_1, σ_2 over this set.