CONWAY'S ZIP PROOF

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Surfaces arise naturally in many different forms, in branches of mathematics ranging from complex analysis to dynamical systems. The Classification Theorem, known since the 1860's, asserts that all closed surfaces, despite their diverse origins and seemingly diverse forms, are topologically equivalent to spheres with some number of handles or crosscaps (to be defined below). The proofs found in most modern textbooks follow that of Seifert and Threlfall [5]. Seifert and Threlfall's proof, while satisfyingly constructive, requires that a given surface be brought into a somewhat artificial standard form. Here we present a completely new proof, discovered by John H. Conway in about 1992, which retains the constructive nature of [5] while eliminating the irrelevancies of the standard form. Conway calls it his Zero Irrelevancy Proof, or "ZIP proof", and asks that it always be called by this name, remarking that "otherwise there's a real danger that its origin would be lost, since everyone who hears it immediately regards it as the obvious proof". We trust that Conway's ingenious proof will replace the customary textbook repetition of Seifert-Threlfall in favor of a lighter, fat-free *nouvelle cuisine* approach that retains all the classical flavor of elementary topology.

We work in the realm of topology, where surfaces may be freely stretched and deformed. For example, a sphere and an ellipsoid are topologically equivalent, because one may be smoothly deformed into the other. But a sphere and a doughnut surface are topologically different, because no such deformation is possible. All the figures in the present article depict deformations of surfaces. For example, the square with two holes in Figure 1A is topologically equivalent to the square with two tubes (1B), because one may be deformed to the other. More generally, two surfaces are considered equivalent, or *homeomorphic*, if and only if one may be mapped onto the other in a continuous, one-to-one fashion. That is, it's the final equivalence that counts, whether or not it was obtained via a deformation.

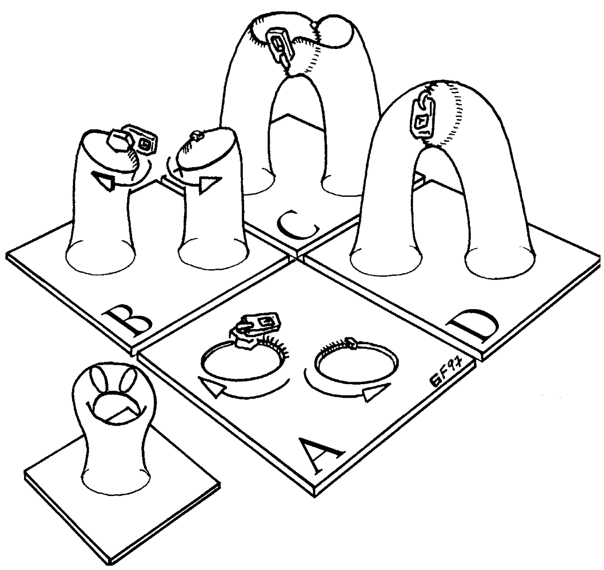

Figure 1. Handle

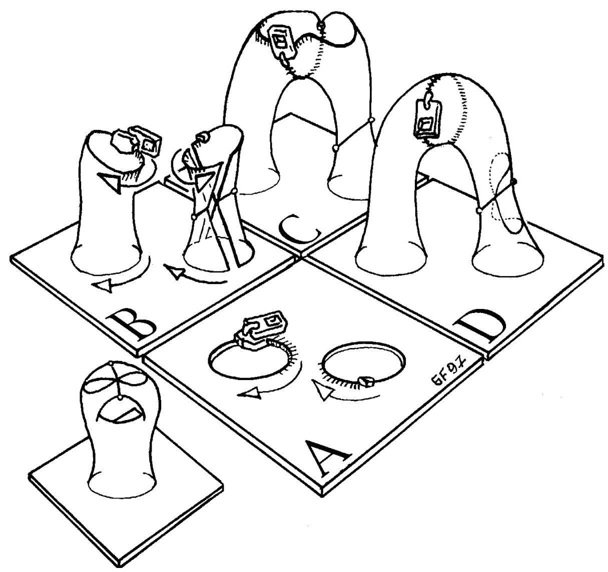

Figure 2. Crosshandle

Let us introduce the primitive topological features in terms of zippers or "zip-pairs", a zip being half a zipper. Figure 1A shows a surface with two boundary circles, each with a zip. Zip the zips, and the surface acquires a *handle* (1D). If we reverse the direction of one of the zips (2A), then one of the tubes must "pass through itself" (2B) to get the zip orientations to match. Figure 2B shows the self-intersecting tube with a vertical slice temporarily removed, so the reader may see its structure more easily. Zipping the zips (2C) yields a *crosshandle* (2D). This picture of a crosshandle contains a line of self-intersection. The self-intersection is an interesting feature of the surface's placement in 3-dimensional space, but has no effect on the intrinsic topology of the surface itself.

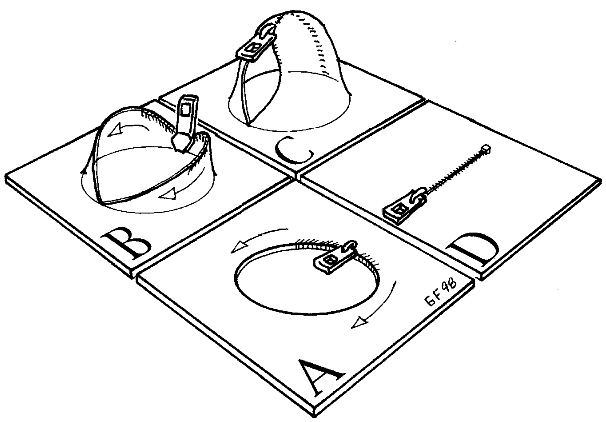

FIGURE 3. CAP

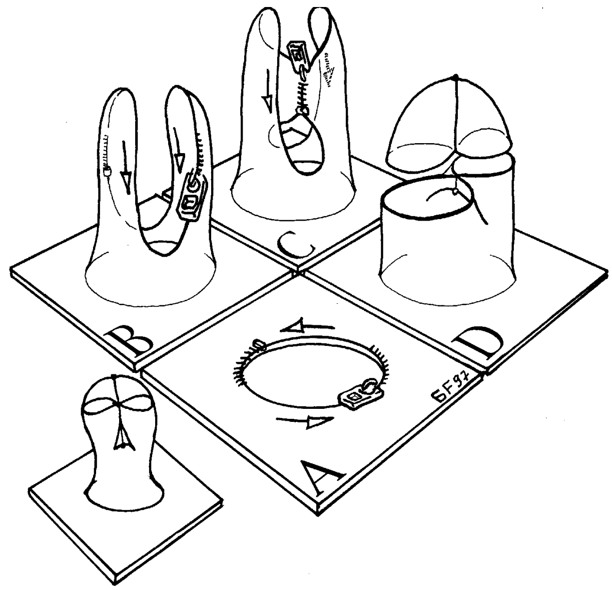

FIGURE 4. CROSSCAP

If the zips occupy two halves of a single boundary circle (Figure 3A), and their orientations are consistent, then we get a *cap* (3C), which is topologically trivial (3D) and won't be considered further. If the zip orientations are inconsistent (4A), the result is more interesting. We deform the surface so that corresponding points on the two zips lie opposite one another (4B), and begin zipping. At first the zipper head moves uneventfully upward (4C), but upon reaching the top it starts downward, zipping together the "other two sheets" and creating a line of self-intersection. As before, the self-intersection is merely an artifact of the model, and has no effect on the intrinsic topology of the surface. The result is a *crosscap* (4D), shown here with a cut-away view to make the self-intersections clearer.

The preceding constructions should make the concept of a surface clear to non-specialists. Specialists may note that our surfaces are compact, and may have boundary.

Comment. A surface is *not* assumed to be connected.

Comment. Figure 5 shows an example of a triangulated surface. All surfaces may be triangulated, but the proof [4] is difficult. Instead we may consider the Classification Theorem to be a statement about surfaces that have already been triangulated.

Definition. A *perforation* is what's left when you remove an open disk from a surface. For example, Figure 1A shows a portion of a surface with two perforations.

Definition. A surface is *ordinary* if it is homeomorphic to a finite collection of spheres, each with a finite number of handles, crosshandles, crosscaps, and perforations.

Classification Theorem (preliminary version) *Every surface is ordinary.*

Proof: Begin with an arbitrary triangulated surface. Imagine it as a patchwork quilt, only instead of imagining traditional square patches of material held together with stitching, imagine triangular patches held together with zip-pairs (Figure 5). Unzip all the zip-pairs, and the surface falls into a collection of triangles with zips along their edges. This collection of triangles is an ordinary surface, because each triangle is homeomorphic to a sphere with a single perforation. Now re-zip one zip to its original mate. It's not hard to show that the resulting surface must again be ordinary, but for clarity we postpone the details to Lemma 1. Continue re-zipping the zips to their original mates, one pair at a time, noting that at each step Lemma 1 ensures that the surface remains ordinary. When the last zip-pair is zipped, the original surface is restored, and is seen to be ordinary. □

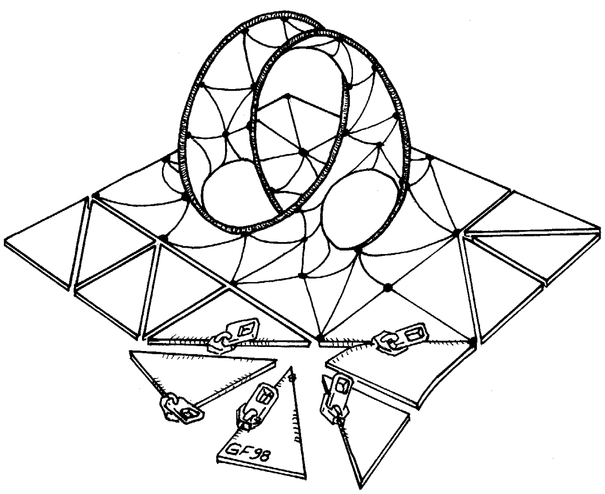

Figure 5. Install a zip-pair along each edge of the triangulation, unzip them all, and then re-zip them one at a time.

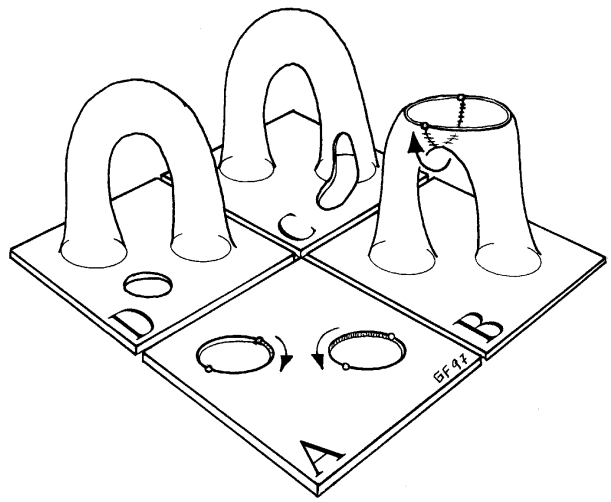

Figure 6. These zips only partially occupy the boundary circles, so zipping them yields a handle with a puncture.

Lemma 1. *Consider a surface with two zips attached to portions of its boundary. If the surface is ordinary before the zips are zipped together, it is ordinary afterwards as well.*

Proof: First consider the case that each of the two zips completely occupies a boundary circle. If the two boundary circles lie on the same connected component of the surface, then the surface may be deformed so that the boundary circles are adjacent to one another, and zipping them together converts them into either a handle (Figure 1) or a crosshandle (Figure 2), according to their relative orientation. If the two boundary circles lie on different connected components, then zipping them together joins the two components into one.

Next consider the case that the two zips share a single boundary circle, which they occupy completely. Zipping them together creates either a cap (Figure 3) or a crosscap (Figure 4), according to their relative orientation.

Finally, consider the various cases in which the zips needn't completely occupy their boundary circle(s), but may leave gaps. For example, zipping together the zips in Figure 6A converts two perforations into a handle with a perforation on top (6B). The perforation may then be slid free of the handle (6C,6D). Returning to the general case of two zips that needn't completely occupy their boundary circle(s), imagine that those two zips retain their normal size, while all other zips shrink to a size so small that we can't see them with our eyeglasses off. This reduces us (with our eyeglasses still off!) to the case of two zips that *do* completely occupy their boundary circle(s), so we zip them and obtain a handle, crosshandle, cap, or crosscap, as illustrated in Figures 1–4. When we put our eyeglasses back on, we notice that the surface has small perforations as well, which we now restore to their original size. □

The following two lemmas express the relationships among handles, crosshandles, and crosscaps.

Lemma 2. *A crosshandle is homeomorphic to two crosscaps.*

Proof: Consider a surface with a "Klein perforation" (Figure 7A). If the parallel zips (shown with black arrows in 7A) are zipped first, the perforation splits in two (7B). Zipping the remaining zips yields a crosshandle (7C).

If, on the other hand, the antiparallel zips (shown with white arrows in Figure 7A) are zipped first, we get a perforation with a "Möbius bridge" (7D). Raising its boundary to a constant height, while letting the surface droop below it, yields the bottom half of a crosscap (7E). Temporarily fill in the top half of the crosscap with an "invisible disk" (7F), slide the disk free of the crosscap's line of self-intersection (7G), and then remove the temporary disk. Slide the perforation off the crosscap (7H) and zip the remaining zip-pair (shown with black arrows) to create a second crosscap (7I).

The intrinsic topology of the surface does not depend on which zip-pair is zipped first, so we conclude that the crosshandle (7C) is homeomorphic to two crosscaps (7I). □

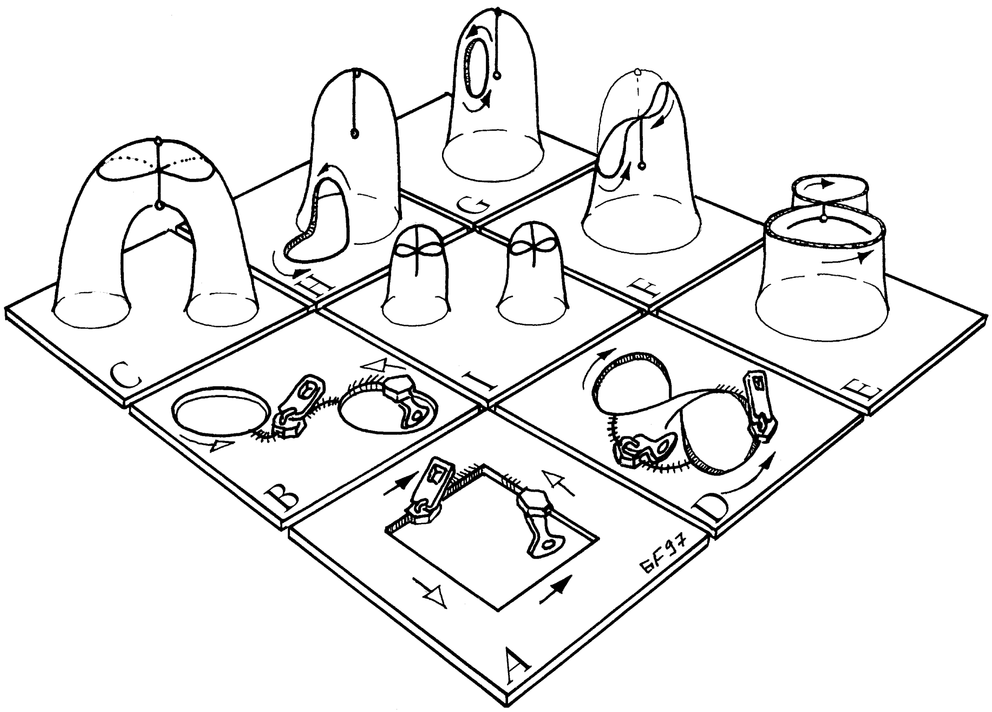

FIGURE 7. A CROSSHANDLE IS HOMEOMORPHIC TO TWO CROSSCAPS.

Lemma 3 (Dyck's Theorem [1]). *Handles and crosshandles are equivalent in the presence of a crosscap.*

Proof: Consider a pair of perforations with zips installed as in Figure 8A. If, on the one hand, the black arrows are zipped first (8B), we get a handle along with instructions for a crosscap. If, on the other hand, one tube crosses though itself (8C, recall also Figure 2B) and the white arrows are zipped first, we get a crosshandle with instructions for a crosscap (8D). In both cases, of course, the crosscap may be slid free of the handle or crosshandle, just as the perforation was slid free of the handle in Figure 6BCD. Thus a handle-with-crosscap is homeomorphic to a crosshandle-with-crosscap. □

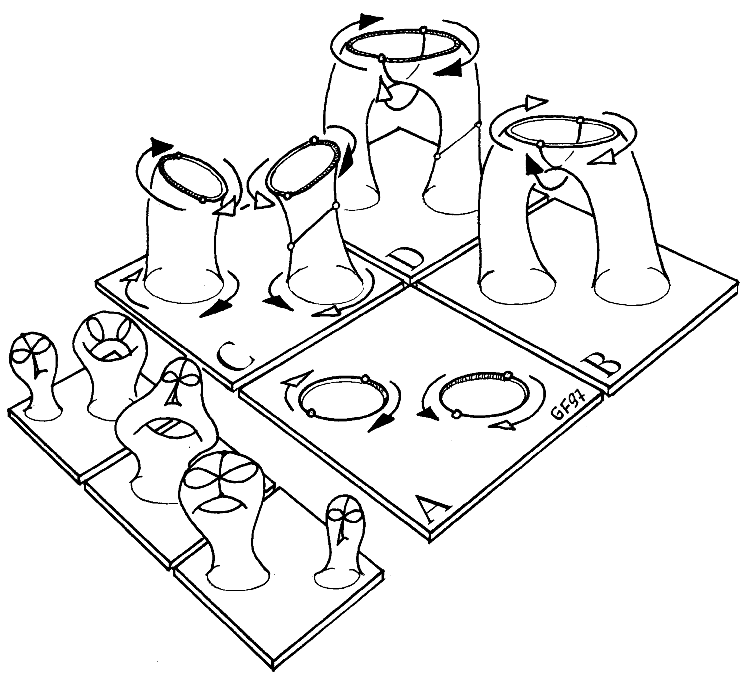

FIGURE 8. THE PRESENCE OF A CROSSCAP MAKES A HANDLE CROSS.

Classification Theorem *Every connected closed surface is homeomorphic to either a sphere with crosscaps or a sphere with handles.*

Proof: By the preliminary version of the Classification Theorem, a connected closed surface is homeomorphic to a sphere with handles, crosshandles, and crosscaps.

Case 1: At least one crosshandle or crosscap is present. Each crosshandle is homeomorphic to two crosscaps (Lemma 2), so the surface as a whole is homeomorphic to a sphere with crosscaps and handles only. At least one crosscap is present, so each handle is equivalent to a crosshandle (Lemma 3), which is in turn homeomorphic to two crosscaps (Lemma 2), resulting in a sphere with crosscaps only.

Case 2: No crosshandle or crosscap is present. The surface is a sphere with handles only.

We have shown that every connected closed surface is homeomorphic to either a sphere with crosscaps or a sphere with handles. □

Comment. The surfaces named in the Classification Theorem are all topologically distinct, and may be recognized by their orientability and Euler number. A sphere with n handles is orientable with Euler number $2 - 2n$, while a sphere with n crosscaps is nonorientable with Euler number $2 - n$. Most topology books provide details; elementary introductions appear in [6] and [2].

Nomenclature. A sphere with one handle is a *torus,* a sphere with two handles is a *double torus,* with three handles a *triple torus,* and so on. A sphere with one crosscap has traditionally been called a real projective plane. That name is appropriate in the study of projective geometry, when an affine structure is present, but is inappropriate for a purely topological object. Instead, Conway proposes that a sphere with one crosscap be called a *cross surface.* The name cross surface evokes not only the crosscap, but also the surface's elegant alternative construction as a sphere with antipodal points identified. A sphere with two crosscaps then becomes a *double cross surface,* with three crosscaps a *triple cross surface,* and so on. As special cases, the double cross surface is often called a *Klein bottle,* and the triple cross surface *Dyck's surface* [3].

References

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