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WS 10
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6. Übung Differentialgeometrie II: Mannigfaltigkeiten

(Frobenius theorem)

Hausaufgaben

1. Aufgabe (5 Punkte)

Apply the Frobenius theorem to prove the folowing statement:
Let $U_1 \subset \mathbb{R}^2$ and $U_2 \subset \mathbb{R}^n$ be open sets, $A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) : U_1 \times U_2 \to \mathbb{R}^n$ smooth maps, $(x_0, y_0) \in U_1$, and $p_0 \in U_2$. Then the following first order system

$$\frac{\partial}{\partial x} u = A(x, y, u(x, y)),$$
$$\frac{\partial}{\partial y} u = B(x, y, u(x, y)),$$
$$u(x_0, y_0) = p_0,$$

has a smooth solution for $u : \tilde{U}_1 \subset U_1 \to U_2$, $(x_0, y_0) \in \tilde{U}_1$, if and only if

$$(A_i)_y + \sum_{1 \le j \le n} (\frac{\partial}{\partial u_j} A_i) B_j = (B_i)_x + \sum_{1 \le j \le n} (\frac{\partial}{\partial u_j} B_i) A_j, \quad 1 \le i \le n.$$

2. Aufgabe (5 Punkte)

Prove the following statement directly or apply the previous exercise.
Let U_0 be an open subset of $\mathbb{R}^2$, $(x_0, y_0) \in U_0$, $C \in gl(n)$ the space of all $n \times n$ matrices, and $P, Q : U_0 \to gl(n)$ smooth maps. Then the the following initial value problem

$$\frac{\partial}{\partial x} g = gP,$$
$$\frac{\partial}{\partial y} g = gQ,$$
$$g(x_0, y_0) = C,$$

has a $gl(n)$-valued smooth solution g in a neighborhood of (x_0, y_0) if and only if

$$P_y - Q_x = [P, Q] := PQ - QP.$$

(This is known as the Maurer- Cartan Lemma).

3. Aufgabe (5 Punkte)

On $\mathbb{R}^3$ we consider the 1-form $\omega := xdy + dz$. Show

$$E^2 = \{v \in \mathbb{R}^3 : \omega(v) = 0\}$$

defines a two dimensional distribution which is not involutive.

4. Aufgabe (5 Punkte)

Determine the subset of $\mathbb{R}^2$ on which $\sigma^1 = x^1\,dx^1 + x^2\,dx^2$ and $\sigma^2 = x^2\,dx^1 + x^1\,dx^2$ are linearly independent and find a frame field dual to σ^1, σ^2 over this set.

Gesamtpunktzahl: 20