TECHNISCHE UNIVERSITÄT BERLIN WS 10
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http://www.math.tu-berlin.de/~sullivan/L/10W/DG2

3. Übung Differentialgeometrie II: Mannigfaltigkeiten

(vector fields, tangent bundle)

Hausaufgaben

1. Aufgabe (5 Punkte)

On $\mathbb{S}^2 = \{(x_0, x_1, x_2)\colon x_0^2 + x_1^2 + x_2^2 = 1\}$ we consider coordinates given by the stereographic projection from the north pole

$$y_1 = \frac{x_0}{1 - x_2}, \qquad y_2 = \frac{x_1}{1 - x_2}.$$

Let X, Y be the vector fields defined on $\mathbb{S}^2 \backslash \{0, 0, 1\}$ which are given in these coordinates by

$$X = y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2}, \qquad Y = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}.$$

Express these two vector fields in coordinates corresponding to the sterographic projection from the south pole.

2. Aufgabe (5 Punkte)

Let M^n be a smooth manifold. For a chart

$$\phi = (\phi^1, \dots, \phi^n)\colon U \to \mathbb{R}^n$$

we define

$$\Phi\colon TU = \bigcup_{p \in U} T_p M \to \mathbb{R}^{2n},$$
$$X_p \mapsto (\phi^1(p), \dots, \phi^n(p), X_p(\phi^1), \dots, X_p(\phi^n)),$$

where $X_p \in T_p M$

1. Show that Φ is injective and $\Phi[TU] = \phi[U] \times \mathbb{R}^n$.

2. Given another chart $\psi\colon V \to \mathbb{R}^n$ and $\Psi\colon TV \to \mathbb{R}^{2n}$ as above, show that

$$\Psi \circ \Phi^{-1}\colon \Phi[T(U \cap V)] \to \Psi[T(U \cap V)]$$

is a diffeomorphism.

After defining a Hausdorff topology on TM such that for every chart $\phi\colon U \to \mathbb{R}^n$ of M the corresponding map Φ is a homeomorphism of an open subset of TM onto its image (which we will now assume has been done), the functions Φ define a smooth structure on TM.
From now on , we will consider TM as a smooth $2m$-manifold with this structure.

3. Aufgabe (5 Punkte)

Prove that the tangent bundle of a product of manifolds is diffeomorphic to the product of the tangent bundles of the manifolds. Deduce that the tangent bundle of a torus $\mathbb{S}^1 \times \mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^2$.

Gesamtpunktzahl: 15