TECHNISCHE UNIVERSITÄT BERLIN WS 10
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http://www.math.tu-berlin.de/~sullivan/L/10W/DG2

14. Übung Differentialgeometrie II: Mannigfaltigkeiten

(Product metric, Ricci curvature, Einstein manifold)

Hausaufgaben

1. Aufgabe (5 Punkte)

Let $(M_1, g^1), (M_2, g^2)$ be Riemannian manifolds and consider the cartesian product $M_1 \times M_2$ with the product structure. Let $\pi^1 : M_1 \times M_2 \to M_1$ and $\pi^2 : M_1 \times M_2 \to M_2$ be the natural projections.

1. Show that
$$g_{(p,q)}(u, v) := g^1_p(\pi^1_*(u), \pi^1_*(v)) + g^2_q(\pi^2_*(u), \pi^2_*(v));$$
for all $(p, q) \in M_1 \times M_2$, $(u, v) \in T_{(p,q)}(M_1 \times M_2)$ defines a Riemannian metric on $M_1 \times M_2$.

2. Show that the Riemannian connection ∇ of $M_1 \times M_2$ is given by
$$\nabla_X Y = \nabla^1_{\pi^1_*(X)} \pi^1_*(Y) + \nabla^2_{\pi^2_*(X)} \pi^2_*(Y),$$
with X, Y being vector fields of $M_1 \times M_2$.

3. Let $span(X, Y) \subset T_{(p,q)}(M_1 \times M_2)$ be a plane such that $\pi^2_*(X) = \pi^1_*(Y) = 0$. Show that the sectional curvature $K(span(X, Y)) = 0$.

2. Aufgabe (5 Punkte)

Show that $\mathbb{S}^2 \times \mathbb{S}^2$ is a Einstein manifold but does not have constant sectional curvature.

3. Aufgabe (5 Punkte)

Prove that the scalar curvature $S(p)$ at $p \in M^n$ is given by
$$S(p) = \frac{1}{vol(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} Ric_p(x, x)\omega,$$
with the canonical volume form ω on $\mathbb{S}^{n-1}$.
Hint: Use a eigenbasis of the Ricci-Tensor.

Gesamtpunktzahl: 15