TECHNISCHE UNIVERSITÄT BERLIN WS 10
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Sullivan / Chubelaschwili
http://www.math.tu-berlin.de/~sullivan/L/10W/DG2

11. Übung Differentialgeometrie II: Mannigfaltigkeiten

(Stokes)

Hausaufgaben

1. Aufgabe (5 Punkte)

Let $M := \{(x, y, z) \in \mathbb{R}^3 : R - \sqrt{r^2 - x^2} \leq \sqrt{y^2 + z^2} \leq R + \sqrt{r^2 - x^2}\}$, with $0 < r < R$. Show that M is a manifold with boundary. Consider $\omega = x dy \wedge dz$ and compute $\int_{\partial M} \omega$ first by using the Stokes theorem and then by computing the integral directly.

2. Aufgabe (10 Punkte)

Let ω be the volume form on a riemannian manifold (M, g) and V a vector field on M. We define the divergence of a vector field V implicitly by

$$L_V \omega =: (\operatorname{div} V)\omega.$$

Further we consider a unit normal vector field N on ∂M and define a volume form $i_N \omega$ on ∂M (tutorial).

1. Prove the divergence theorem of Gauss

$$\int_M (\operatorname{div} V)\omega = \int_{\partial M} g(V, N) i_N \omega.$$

2. Fix a point $x \in M$ and consider a small neighborhood $U_\epsilon(x) := \{y : d(x, y) < \epsilon\} \subset M$ for some $\epsilon > 0$. Show

$$\int_{U_\epsilon(x)} (\operatorname{div} V)\omega = \frac{d}{dt}(vol(\Phi^t(U_\epsilon(x))))_{t=0},$$

where Φ^t is the flow of V.

3. Conclude

$$\operatorname{div} V(x) = \lim_{\epsilon \to 0}\left(\frac{1}{vol(U_\epsilon(x))}\frac{d}{dt}(vol(\Phi^t(U_\epsilon(x))))_{t=0}\right)$$

and give an interpretation for this formula. (Hint: you will need to use the intermediate value theorem.)

4. Prove that $\operatorname{div} V(x) = 0$ for all $x \in M$ if and only if $vol(\Phi^t(A))$ is constant in t for any given (measurable) subset A of M.

Gesamtpunktzahl: 15