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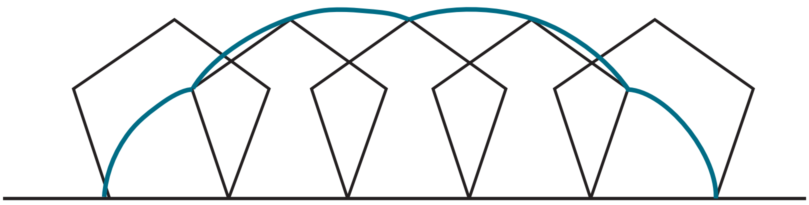
Geometry II
SS 10

http://www.math.tu-berlin.de/~sullivan/L/10S/Geo2/

Exercise Sheet 2

Exercise 1: Rolling n-gon. (6 pts)

When a regular n-gon P rolls without slipping along a straight line, a given vertex traces out a curve. This curve consists of a sequence of arches resting on the line as shown by the example of a rolling pentagon in the figure. Each arch is composed of $n-1$ circular arcs.



If S denotes the area of the region above the line and below one of these arches, then it can be shown that $S = S_P + 2S_D$, where S_P denotes the area of P and S_D is the area of the disk that circumscribes P.

1. Prove this formula in the case $n = 3$ (P is an equilateral triangle);
2. Prove this formula in the case $n = 4$ (P is a square).

Now consider the limit $n \to \infty$, where P approaches a circle and the curve above approaches a *cycloid*, the curve traced out by a given point on a rolling circle.

3. Use the formula above to find the area of the region above the line and below one arch of a cycloid.

Exercise 2: Fenchel's theorem. (4 pts)

Fenchel's theorem says that the total curvature of a closed curve is greater or equal to 2π. Prove that equality here holds only for a convex plane curve.
(Hint: first prove this for polygons, using our lemma about removing a single vertex.)

Turn over

Exercise 3: Convex n-gon. (4 pts)

Let P be a convex n-gon in the plane. The total curvature $TC(P)$ is always exactly 2π, but the alternate $TC_*(P)$ is smaller and depends on the angles of P. Prove that $TC_*(P)$ is maximized by a polygon with equal angles and is infimized in a limit of degenerating polygons.

(Hint: the sine function is concave.)