Differentialgeometrie II

Uebungsblatt 8

Due on Wednesday 21.01.2009

1 Aufgabe

Boothby problem V.8.2.

2 Aufgabe

Let V be an n-dimensional real vector space and let $\{e_i\}$ be a basis for V. Let V^* be its dual vector space with dual basis $\{\bar{e}_j\}$.

Consider, for $r = 1, 2, \dots n$, the map $\rho : \bigwedge^r(V) \to \bigwedge^r(V^*)$ defined as follows:

for

$$\phi = \sum_{1 \le i_1 \le \dots \le i_r \le n} a^{i_1,\dots,i_r} \bar{e}_{i_1} \wedge \dots \wedge \bar{e}_{i_r} \in \bigwedge^r(V), \quad a^{i_1,\dots,i_r} \in \mathbb{R},$$

$$\rho(\phi) := \sum_{1 \le i_1 \le \dots \le i_r \le n} a^{i_1,\dots,i_r} e_{i_1} \wedge \dots \wedge e_{i_r} \in \bigwedge^r(V^*).$$

The spaces $\bigwedge^r(V)$ and $\bigwedge^r(V^*)$ are in duality: for $\phi = \bar{w}_1 \wedge \dots \wedge \bar{w}_r \in \bigwedge^r(V)$, and $\hat{\psi} = \frac{1}{r!} v_1 \wedge \dots \wedge v_r \in \bigwedge^r(V^*)$, the pairing is defined as $\hat{\psi}(\phi) := \bar{w}_1(v_1)\bar{w}_2(v_2)\dots\bar{w}_r(v_r) \in \mathbb{R}$. For generic elements, this definition is extend by linearity.

Consider, for $x \in V$, the map $\iota_x : \bigwedge^r(V) \to \bigwedge^{r-1}(V)$ defined as

$$\iota_x(\phi)(v_1, \dots, v_{r-1}) := \phi(x, v_1, \dots, v_{r-1}), \quad \forall \phi \in \bigwedge^r(V), \forall v_1, \dots, v_{r-1} \in V.$$

Show that, for $\phi \in \bigwedge^{r+1}(V), \psi \in \bigwedge^r(V), x \in V$, the following holds:

$$\iota_x(\phi)(\rho(\psi)) = \frac{1}{r+1}\phi(\rho(\omega_x \wedge \psi)),$$

where $\omega_x \in \bigwedge^1(V) = V^*$ is (the unique dual vector) such that $\rho(\omega_x) = x$.

3 Aufgabe

Boothby, problem V.8.6

4 Aufgabe

Boothby, problem V.8.7.

Let $\theta : W \to M$ be the local flow corresponding to the vector field $X \in \mathcal{X}(M)$. Let $\phi \in \bigwedge^r(M), r \ge 0$.

Show that $L_X\phi = \frac{d}{dt}_{|t=0}\theta_t^*\phi$, or, in other words, $(L_X\phi)_p = \lim_{t\to 0}\frac{1}{t}(\theta_t^*\phi_{\theta_t(p)} - \phi_p)$, $\forall p \in M$.