Differentialgeometrie II
Übungsblatt 3

Due 12 November 2008

1 Aufgabe

(Boothby, problem IV.1.4) Let M be a smooth n-dimensional manifold. Let $p \in M$ and let $\mathcal{C}$ denote the collection of all coordinate neighborhoods containing p, i.e., all charts (U, ϕ) such that $p \in U$. Assign to each chart $(U, \phi) \in \mathcal{C}$ a copy of $\mathbb{R}^n$ and consider their disjoint union. Denote by $w_{(U,\phi)}$ a typical element of this disjoint union: an element of $\mathbb{R}^n$ labeled by its corresponding chart.

Two elements $w_{(U,\phi)}$ and $v_{(V,\psi)}$ are said to be *equivalent* if $w_{(U,\phi)} = D(\phi \circ \psi^{-1})v_{(V,\psi)}$, where $D(\phi \circ \psi^{-1})$ is the Jacobi matrix of the diffeomorphism $\phi \circ \psi^{-1}$. (Written in a coordinate notation like Boothby's, this equation becomes:

$$w^i = \sum_j \frac{\partial x^i}{\partial \tilde{x}^j} v^j,$$

where $\{x^i : U \to \mathbb{R}\}$ and $\{\tilde{x}^j : V \to \mathbb{R}\}$ are, respectively, coordinate functions for (U, ϕ) and (V, ψ) and $(\frac{\partial x^i}{\partial \tilde{x}^j}) = D(\phi \circ \psi^{-1})$.)

Show that this relation is indeed an equivalence relation.

Show that the set of equivalence classes is a vector space.

Show that this vector space is naturally isomorphic to $T_p(M)$ (Hint: choose one chart in $\mathcal{C}$ and use its natural basis for $T_p(M)$ to construct an isomorphism with the set of equivalence classes.)

2 Aufgabe

(Boothby, problem IV.2.1) Let M be a smooth manifold, and suppose X is a function assigning to each $p \in M$ an element of $T_p(M)$. Show that X is C^∞ (in the sense that its coordinate expression in any fixed frame is C^∞ — Boothby, Def. IV.2.1) if and only if whenever $f \in C^\infty(W_f)$ (i.e., whenever f is C^∞ on an open set $W_f \subset M$), then the function Xf defined by $Xf(p) := X_p f$ is C^∞ on W_f.

3 Aufgabe

(Boothby, problem IV.2.5) Let M be a smooth manifold. Let (U, ϕ) be any chart together with its local coordinate functions x^i and associated frames $E_i = \phi_*^{-1}(\frac{\partial}{\partial x^i})$.

Define $(\tilde{U}, \tilde{\phi})$ as follows:

- $\tilde{U} := \pi^{-1}(U)$, the union of all the tangent spaces associated to points $p \in U$.

- $\tilde{\phi} : \tilde{U} \to \mathbb{R}^n \times \mathbb{R}^n$ with $\tilde{\phi}(X_p) = (x^1(p), x^2(p), \ldots, x^n(p), w^1, w^2, \ldots, w^n)$ for any X_p with $p \in U, X_p \in T_p(M)$ and $X_p = \sum_i w^i E_i$.

Show that the set of all $(\tilde{U}, \tilde{\phi})$ is a smooth structure on $T(M)$.

4 Aufgabe

Consider the smooth manifold $\mathbb{R}^2$ together with its natural frame $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}$ of smooth, everywhere-defined vector fields.

Let Y, Z be two smooth fields defined (on all of M) by

$$Y = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2}; \quad Z = z^1 \frac{\partial}{\partial x^1} + z^2 \frac{\partial}{\partial x^2}$$

where $y^i, z^j \in C^\infty(\mathbb{R}^2, \mathbb{R})$.

Then, for example, if $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$, we have $Y_p f = y^1 \frac{\partial f}{\partial x^1}|_p + y^2 \frac{\partial f}{\partial x^2}|_p$.

We have seen in problem 2 that given f, we can consider the new smooth function Yf. We can iterate the operation and obtain $Z(Yf)$.

Show that the linear functional assigning to each $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ the value $Z(Yf)(p)$ for $p \in \mathbb{R}^2$, is not (in general) given by a tangent vector. (Hint: is $\frac{\partial^2}{\partial x^1 \partial x^2}|_p$ a tangent vector? Does it satisfy the Lebniz rule?)

Show, however, that the linear functional assigning to each $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ the value $Z(Yf)(p) - Y(Zf)(p)$ is given by a tangent vector.

Find its coordinates with respect to the basis $\frac{\partial}{\partial x^1}|_p, \frac{\partial}{\partial x^2}|_p$.