Differentialgeometrie II
Übungsblatt 10
Due on Wednesday 4.02.2009

1 Aufgabe (6 points)

- Give a sketch of the proof of the following sentence: "Let M be a compact manifold. Let $\phi \in \bigwedge^1(M)$ be closed. If for any curve $\gamma : [0,1] \to M$ the value of the integral $\int_\gamma \phi$ depends only on the endpoints of the curve, then ϕ is exact."
- Let ω denote a normalized volume form for the unit sphere S^1, i.e. $\omega \in \bigwedge^1(S^1)$ and $\int_{S^1} \omega = 1$. Let ϕ be any closed 1-form on S^1. Show that $\phi - (\int_{S^1} \phi)\omega$ is exact.
- Describe $H^i(S^1),\ i = 0, 1, 2, \ldots$.

2 Aufgabe (6 points)

Let $M = \mathbb{R}^{n+1} \backslash \{(0, \ldots, 0)\}$ and let $F : M \to M$ denote the map sending x to $\frac{x}{\|x\|}$ (where $x = (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1}$).
Consider the n-form $\omega := \sum_{i=0}^{n+1} (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1}$.

- Show that $F^*\omega$ is closed.

Hint: explicit computation is not necessary.

3 Aufgabe (8 points)

Let $M = R^2$ with coordinates (p, q). Let $\omega = dp \wedge dq$ be the volume form. For any 1-form η, denote by $X_\eta \in \mathcal{X}(M)$ the unique vector field such that $\iota_{X_\eta}\omega = \eta$. Let $H(p, q) \in \bigwedge^0(M)$ be a function (the Hamiltonian if you like).

- Describe X_{dH} in terms of the basis frame $\frac{\partial}{\partial p}, \frac{\partial}{\partial q}$.

Let θ_t denote the associated flow, which we assume to be global. Let $\gamma : [0, 1] \to M$ be a parametrized curve.

- Show that $\int_\gamma \omega(X_{dH},\ .\ ) = H(\gamma(1)) - H(\gamma(0))$.

Let now be γ a closed curve, i.e. $\gamma(1) = \gamma(0)$ and $\gamma = \partial\Sigma$ for some compact surface $\Sigma \subset M$. Let $\sigma \in [0, 1]$ denote its parameter.
Consider the surface $[0, 1] \times \gamma$ with parametrization (t, σ) and the 2-form $\theta^*\omega \in \bigwedge^2([0, 1] \times M)$.

- Show that $\int_{[0,1]\times\gamma} \theta^*\omega = 0$
- Show that $\theta^*\omega$ is closed
- Use Stokes' theorem to conclude that $\int_\Sigma \omega = \int_{\theta_1(\Sigma)} \omega$, i.e. the flow leaves the volume form invariant.

4 Aufgabe (extra points)

(*Boothby*, problem VI.7.4) Let M be a manifold and N a submanifold. Let $F : M \to N$ be a C^∞ map such that its restriction on N is the identity map, i.e. $F(p) = p, \forall p \in N$.

- Show that $F^* : H^i(N) \to H^i(M), (i \in \{0, 1, \ldots\})$ is well defined;
- Show that F^* is injective.