Differentialgeometrie II

December 18 2008 Notes

1 Definitions of tensor fields

We met a generalisation of the concept of co-vectorfields: C^∞-covariant tensor fields of order r (Def. V.5.3).

These can be defined in the following three equivalent ways:

- a) a function Φ which assigns to each $p \in M$ an element $\Phi_p \in \mathcal{T}^r(T_pM)$ and which has the following property: for any local chart (U, ψ) with associated local frame E_i the functions $\alpha_{i_1,\dots,i_r} := \Phi(E_{i_1}, \dots, E_{i_r})$ (the local coordinates for Φ) belong to $C^\infty(U)$;
- b) a function Φ which assigns to each $p \in M$ an element $\Phi_p \in \mathcal{T}^r(T_pM)$ and which has the property that for any choice $X_1, \dots X_r$ of elements in $\mathcal{X}(M)$, $\Phi(X_1, \dots X_r) \in C^\infty(M)$;
- b bis) a function Φ which assigns to each $p \in M$ an element $\Phi_p \in \mathcal{T}^r(T_pM)$ and which has the property that for any open subset U of M and any choice $X_1, \dots X_r$ of elements in $\mathcal{X}(U)$, $\Phi(X_1, \dots X_r) \in C^\infty(U)$;
- c) a $C^\infty(M)$-linear function Φ from $\mathcal{X}(M) \times \dots \times \mathcal{X}(M)$ to $C^\infty(M)$.

Proof

a) $\Rightarrow$ *b)*. We must show that for any r-tuple r vector fields $X_1, \dots, X_r$, $\Phi(X_1, \dots X_r) \in C^\infty(M)$. It is sufficient to show this locally for any chart (U, ψ). Then the restrictions of the fields X_i on U are linear combinations $X_i = \sum_i \beta_i E_i$ with coefficients $\beta_i \in C^\infty(U)$. This observation plus *a)* imply that $\Phi(X_1, \dots X_r)_{|U} \in C^\infty(U)$. As this is true for any choice of (U, ψ), *b)* is proven.

b) $\Rightarrow$ *a)*. Let (U, ψ) be a local chart with local frame E_i. We will show that for any $p \in U$ the functions $\alpha_{i_1,\dots,i_r}$ are C^∞ in a neighbourhood of p. Choose an open neighbourhood $V \ni p$ and a function f such that $f = 1$ on V and $supp(f) \subset U$. Then $fE_i \in \mathcal{X}(M)$ are global vector fields, zero on $M \backslash U$ and coinciding with E_i on V. Thus $\alpha_{i_1,\dots,i_r|V} := \Phi(E_{i_1}, \dots, E_{i_r})_{|V} = \Phi(fE_{i_1}, \dots, fE_{i_r})_{|V}$. In the last expression all fields belong to $\mathcal{X}(M)$, therefore *b)* implies that $\alpha_{i_1,\dots,i_r}$ is C^∞ on V.

$a) \Rightarrow b\ bis)$ Exactly as in $a) \Rightarrow b)$.

$b\ bis) \Rightarrow a)$ Exactly as in $b) \Rightarrow a)$.

$b) \Rightarrow c)$. Follows immediately by observing that each Φ_p is $\mathbb{R}$-multilinear.

$c) \Rightarrow b)$. The only non trivial fact we must show is that Φ assigns an element Φ_p in $\mathcal{T}^r(T_pM)$ for each $p \in M$, as this is not part of the hypothesis in $c)$.

For any $X_{1p}, \dots X_{rp} \in T_p(M)$ choose vector fields $X_i \in \mathcal{X}(M)$ such that $X_{i|_p} = X_{ip}$, i.e. the value of the fields at the point p coincides with the above chosen vectors. It is always possible to find such vector fields.

We must show that Φ_p is well defined: suppose we choose another set of vector fields $Y_1, \dots, Y_r$ with the same property ($X_{ip} = Y_{i|_p}$), then we must have $\Phi(X_1, \dots, X_r)(p) = \Phi(Y_1, \dots, Y_r)(p)$. It is suffiecient to prove the equality for one variable at a time, i.e. $\Phi(X_1, \dots, X_i, \dots, X_r)(p) = \Phi(X_1, \dots, Y_i, \dots, X_r)(p)$.

As a first step, we notice that if $X_{i|_q} = Y_{i|_q}$ not only for $q = p$ but for any q in an open set U containing y, then the equality easily follows: we choose $f \in C^\infty(M)$ with $supp(f) \subset U$ and $f(p) = 1$. Then $fX_i = fY_i$. Thus $\Phi(X_1, \dots, fX_i, \dots X_r)(p) = \Phi(X_1, \dots, fY_i, \dots X_r)(p)$. As Φ is $C^\infty(M)$-linear by hypothesis, we have $f(p)\Phi(X_1, \dots, X_i, \dots, X_r)(p) = f(p)\Phi(X_1, \dots, Y_i, \dots X_r)(p)$. As $f(p) = 1$, the equality follows.

Also notice that it is sufficient to show the following statement: for any Y_i such that $Y_{i|_p} = 0$ it follows that $\Phi(X_1, \dots, Y_i, \dots, X_r)(p) = 0$. The general statement follows by applying (multi) linearity to the field $X_i - Y_i$.

Choose a local chart (U, ψ) containing p and choose a function h and an open set $V \subset U$ containing p such that $h = 1$ on V and $supp(h) \subset U$. Let Y_i be any vector field satisfying $Y_{i|_p} = 0$. In the open neighbourhood U the restriction of Y is equal to $\sum_i \beta_i E_i$, where E_i are the elements of the frame associated to the chart (U, ψ). The functions β_i satisfy $\beta_i(p) = 0$. Consider the field $h^2 Y$. We have $Y_{i|_V} = h^2 Y_{i|_V} = h^2 \sum_i \beta_i E_{i|_V}$. As $supp(h) \subset U$, we extend hE_i to vector fields defined on M, by letting them be 0 on $M \backslash U$. Thus we have $\Phi(X_1, \dots, Y_i, \dots, X_r)(p) = \Phi(X_1, \dots, h^2 Y_i, \dots, X_r)(p)$

$= \Phi(X_1, \dots, h^2 \sum_i \beta_i E_i, \dots, X_r)(p) = h(p) \sum_i \beta_i(p) \Phi(X_1, \dots, hE_i, \dots, X_r)(p)$.

These expressions are all zero, as we have $\beta_i(p) = 0$.

Remark. The equation $\Phi(X_1, \dots, h^2 Y_i, \dots, X_r)(p) = \Phi(X_1, \dots, Y_i, \dots, X_r)(p)$ follows from the partial result found above, as $h^2 Y_i$ and Y_i coincide not only on thepoint p but on an open set.

It would have been incorrect to say that as $Y_i = \sum_i \beta_i E_i$ and $\beta_i(p) = 0$ it follows that $\Phi(X_1, \dots, \sum_i \beta_i E_i, \dots, X_r)(p) = \sum_i \beta_i(p) \Phi(X_1, \dots, E_i, \dots, X_r)(p) = 0$, as the E_i and β_i are defined only on U. We know that Φ is $C^\infty(M)$-linear, therefore we had to reduce the problem to one involving globally defined fields and functions.

2 Some remarks on dual spaces

We saw that any finite dimensional (real) vector space V is non canonically isomorphic to its dual space V^*: it is easy to describe an isomorphism once we have chosen a basis for V, and this isomorphism depends on the basis.

We can also consider the dual of the dual space V^{**}, i.e. the space of linear functionals on V^*. In this case there is a canonical isomorphism between V and V^{**}. Let $v \in V$ and let $\bar{w} \in V^*$ be any element of the dual space. Define $\iota(v) \in V^{**}$ as the linear functional on V^* given by the expression $\iota(v)(\bar{w}) := \bar{w}(v)$.

In other words, we can identify V and $\mathcal{T}^1(V^*)$, and use Def. V.6.1 and Theorem V.6.2, simply by replacing V with V^*. Thus for $v, w \in V$ we can make sense of expressions like $v \otimes w$. And if e_i is a basis for V, then $\{e_{i_1} \otimes \cdots \otimes e_{i_r}\}$ is a basis for $\mathcal{T}^r(V^*)$.

Recalling Definition V.5.1, we see that $\mathcal{T}^r(V^*)$ (the space of r-times linear functionals on V^*) is the same as $\mathcal{T}_r(V)$ (the space of contravariant tensors of order r). More generally, a moment of thought shows that we have $T^r_s(V^*) = T^s_r(V)$.

We can extend definition V.6.1 and theorem V.6.2 for the case of mixed covariant/contravariant tensors in a straightforward way. For example, let again e_i be a basis for V and let $\bar{e}_j$ be its dual basis. Then $\{\bar{e}_{j_1} \otimes \cdots \otimes \bar{e}_{j_r} \otimes e_{i_1} \otimes \ldots e_{i_s}\}$ is a basis for $\mathcal{T}^r_s(V)$.

All this carries on locally to the case of tensor fields. Let M be a differentiable manifold and (U, ψ) a local chart. Let $\{E_i\}$ be the associated local frame of vector fields and $\{\omega_i\}$ the associated coframe. Let $\Phi \in \mathcal{T}^r_s(M)$ be a C^∞ r-covariant s-contravariant tensor field on M. Then we have the following local description of our tensor field:

$$\Phi_{|U} = \sum_{i_1,\ldots,i_r,j_1,\ldots,j_s} \alpha_{i_1,\ldots,i_r,j_1,\ldots,j_s} \omega_{i_1} \otimes \cdots \otimes \omega_{i_r} \otimes E_{j_1} \otimes \cdots \otimes E_{j_s}$$

where the $\alpha_{i_1,\ldots,i_r,j_1,\ldots,j_s}$ belong to $C^\infty(U)$.

3 Example: Boothby, exercise V.5.1

Let $\{e_i\}$ be a basis for V ans $\{\bar{e}_i\}$ the dual basis for V^*.. For $r = 2$ $\mathcal{T}^2(V)$, $\{\bar{e}_i \otimes \bar{e}_j$ is a basis (theorem V.6.2). Show that the following family of vectors off $\mathcal{T}^2(V)$ is a basis as well: $\{\bar{e}_i \otimes \bar{e}_j + \bar{e}_j \otimes \bar{e}_i\}$ with $i \leq j$ and $\{\bar{e}_i \wedge \bar{e}_j\}$ with $i < j$. The first group of $\frac{n}{2}(n+1)$ vectors is a basis for $\Sigma^2(V)$ and the second group of $\frac{n(n-1)}{2}$ vectors is a basis for $\bigwedge^2(V)$.

Alternative proof Each element of $\mathcal{T}^2(V)$ is sum of elements of the form $\bar{v} \otimes \bar{w}$, with $\bar{v}, \bar{w} \in \mathcal{T}^1(V)$. But $\bar{v} \otimes \bar{w} = \frac{1}{2}(\bar{v} \otimes \bar{w} - \bar{w} \otimes \bar{v}) + \frac{1}{2}(\bar{v} \otimes \bar{w} + \bar{w} \otimes \bar{v})$, i.e. a sum of an element of $\bigwedge^2(V)$ and an elemnt of $\Sigma^2(V)$. In other words, $\bar{v} \otimes \bar{w} \in \bigwedge^2(V) \oplus \bigwedge^2(V)$. This proves the assertion for $r = 2$.

Now consider the case $r > 2$. Consider the symmetrizing mapping $\mathcal{S} : \mathcal{T}^r(V) \to \mathcal{T}^r(V)$ and the alternating mapping $\mathcal{A} : \mathcal{T}^r(V) \to \mathcal{T}^r(V)$ introduced in Definition V.5.6. Check that their product is zero: $\mathcal{S}\mathcal{A} = \mathcal{A}\mathcal{S} = 0$.

Suppose that $\mathcal{T}^r(V) = \Sigma^r(V) \oplus \bigwedge^r(V)$. This means that any $\phi \in \mathcal{T}^r(V)$ is of the form $\phi^+ + \phi^-$ with $\phi^+ \in \Sigma^r(V)$ (equivalently, $\mathcal{S}\phi^+ = \phi^+$) and $\phi^- \in \bigwedge^r(V)$ (equivalently, $\mathcal{A}\phi^- = \phi^-$.)

Consider the tensor $\phi := \sum_\sigma \bar{e}_1 \wedge \bar{e}_2 \otimes \bar{e}_{\sigma(3)} \otimes \cdots \otimes \bar{e}_{\sigma(r)}$, where the sum is over all permutations σ involving only the last $r-2$ terms and leaving the first two fixed. Check that $\mathcal{S}\phi = 0$, which implies $\phi^+ = 0$. Likeweise, check that $\mathcal{A}\phi = 0$, which in turn imples $\phi^- = 0$. But ϕ is not zero, and we have just proven that $\phi \notin \Sigma^r(V) \oplus \bigwedge^r(V)$.

4 Boothby, exercise V.5.2

We have seen that for any $\phi \in \mathcal{T}^r(V)$ the following are equivalent:

- *a*) ϕ is antisymmetric;
- *b*) $\phi(v_1, \ldots, v_r) = 0$ whenever $v_i = v_j$ for $i \neq j$;
- *c*) $\phi(v_1, \ldots, v_r) = 0$ whenever $v_1, \ldots, v_r$ are linearly dependent.

a) $\Rightarrow$ *b*) follows form $\phi(v_1, \ldots, v_i, \ldots, v_i, \ldots, v_r) = -\phi(v_1, \ldots, v_i, \ldots, v_i, \ldots, v_r)$ (where we have permutated v_i and $v_j = v_i$.)

b) $\Rightarrow$ *c*). Suppose that $v_1, \ldots, v_r$ are linearly dependent. Then, for example, $v_i = \sum_{j \neq i} \beta_j v_j$.

$\phi(v_1, \ldots, v_i, \ldots, v_r) = \sum_{j \neq i} \beta_j \phi(v_1, \ldots, v_j, \ldots, v_r) = 0$, as each v_j appears (at least) twice.

c) $\Rightarrow$ *b*) obvious.

b) $\Rightarrow$ *a*). We know that $\phi(v_1, \ldots, v_i + v_j, \ldots, v_i + v_j, \ldots, v_r) = 0$. It follows, by r-linearity, that $\phi(v_1, \ldots, v_i, \ldots, v_i, \ldots, v_r) + \phi(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_r) + \phi(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_r) + \phi(v_1, \ldots, v_j, \ldots, v_j, \ldots, v_r) = \phi(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_r) + \phi(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_r) = 0$. Thus ϕ is antisymmetric.

Now, when $r > dimV$, it is obvious that any r-tuple of vectors is linearly dependent, so any antisymmetric $\phi \in \bigwedge^r(V)$ must always take value zero.

5 An example using partitions of unity (Boothby, problem V.4.2)

Let N be an n-dimensional closed regular submanifold of an m-dimensional manifold M. Show that a C^∞ vector field X on N can be extend to a C^∞-vector field on M.

As a first step, let's prove the local version of this statement. Choose a point $p \in N$ and a preferred (in the sense of Boothby Definition III.5.1)coordinate neighbourhood (U, ϕ) with coordinate functions $x_1, \ldots, x_m$, and with $E_1, \ldots, E_m$ the associated coordinate frame, such that $q \in N \cap U$ iff the last $m - n$ coordinate functions take value zero on q. Denote by $(U \cap N, \tilde{\phi})$ the local chart for N given by the restriction $\tilde{x}_1, \ldots$ on $U \cap N$ of first n coordinate functions $x_1, \ldots, x_n$ (cfr. Boothby Lemma III.5.2.) Analogously, denote by $\tilde{E}_1, \ldots, \tilde{E}_n$ the restrictions of the $E_1, \ldots, E_n$.

Any smooth vector field X on $U \cap N$ is of the form $\sum_i \tilde{\alpha}_i \tilde{E}_i$, with $\tilde{\alpha}_i \in C^\infty(U \cap N)$. In local coordinates these functions take the form $\tilde{\alpha}_i \circ \tilde{\phi}^{-1} = \tilde{\alpha}_i(x_1, \ldots, x_n)$.

Consider the field $X_U := \sum_i \alpha_i E_i$, where $\alpha_i(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m)$

$:= \tilde{\alpha}(x_1, \ldots, x_n)$. It is defined on all U and it is an extension of X. This extension depends on the local chart.

Now let's suppose we have a smooth field X defined on the whole of N.

Do the same procedure as above for every point $p \in N$. This gives a covering of N. Add to this covering the set $M \backslash N$, which is open as N is closed. Thus we obtain an open covering of M. From Boothby Lemma V.4.1. we know that we can find a countable locally finite refinement (U_i, V_i, ϕ_i) which is regular (in the sense of remark 4.2). From Theorem V.4.4 we know that we can construct a smooth partition of unit f_i subordinate to this locally finite refinement. The idea is to paste together the local extensions X_{U_i} into a global extension using the functions f_i, i.e. multiplying each X_{U_i} by f_i and extending its domain of definition to M by defyining it to be zero out of U_i and finally considering the sum $\sum_i f_i X_{U_i}$.

We have only a small problem. To pursue the above construction we need each (U_i, V_i, ϕ_i) to be a preferred neighbourhhod, which is something which is not assured from Lemma V.4.1. To be precise, we have two kinds of neighbourhoods: those containing points of N (derived from the covering of N) and those which don't (derived from $M \backslash N$). For these second kind of neighbourhoods sinply define X_{U_i} to be zero.

We have two choices: go through the proof of the lemma, and see that when the original covering consists of subspaces associated to preffered charts, then it is possible to choose the locally finite refinement U_i, V_i, ϕ_i with ϕ preferred as well (I will not do this here, but it is not difficult). Second choice: remember that each U_i having non trivial intersection with N is contained in some U corresponding to a preferred chart U, ϕ of the original covering. Simply forget the ϕ_i given by the lemma, and define a new ϕ_i as the restriction of the original ϕ on U_i. We are actually only interested in the existence of the f_i, and we don't need all the properties of a regular covering. Now we can pursue the construction stated above. Remark: the extension we found depends stronlgy on everything we chose: local charts, refinement, partitions of unity.

A counterexample which shows why we asked N to be closed. Consider $M := \mathbb{R}^3$, $N := \{x, y, z \in \mathbb{R} \mid x^2 + y^2 = z^2\} \backslash \{(0, 0, 0)\}$, i.e. two cones without vertices embedded in $\mathbb{R}^3$. Let X be, for example, $\frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ on the upper cone and zero on the lower one. This field cannot be extended, as we would have a singularity at the origin. In this example N is not closed, so $M\backslash N$ is not part of an open covering of M and the proof we gave above breaks down.