1 Solution to problem 3 of the 2^{nd} Übungsblatt

(a) Show that the relation $\sim$ is an equivalence.

Reflexivity ($x_\alpha \sim x_\alpha$) and symmetry ($x_\alpha \sim x_\beta \Rightarrow x_\beta \sim x_\alpha$) are obvious. Let $x_\alpha \sim x_\beta$, $x_\beta \sim x_\gamma$ with $x_\alpha \in V_\alpha$, $x_\beta \in V_\beta$, $x_\gamma \in V_\gamma$. Then $\exists x_1 \in V_\alpha \cap V_\beta \subset M|\ \psi_\alpha(x_1) = x_\alpha, \psi_\beta(x_1) = x_\beta$ and $\exists x_2 \in V_\beta \cap V_\gamma \subset M|\ \psi_\beta(x_2) = x_\beta, \psi_\gamma(x_2) = x_\gamma$. But $x_1 = x_2$, as we have both $\psi_\beta(x_1) = x_\beta$ and $\psi_\beta(x_2) = x_\beta$ and ψ_β is bijective.

So it follows that $x_1 = x_2 \in V_\alpha \cap V_\beta \cap V_\gamma \subset V_\alpha \cap V_\gamma$ and $\psi_\alpha(x_1) = x_\alpha \in V_\alpha$, $\psi_\gamma(x_1) = x_\gamma \in V_\gamma$. Thus transitivity is verified.

Interlude Now we consider the disjoint union $V := \coprod_{\alpha\in I} V_\alpha$. As mentioned in the text, one should denote elements of this set as $(x, \alpha) \in \coprod_{\alpha\in I} V_\alpha$, where $x \in V_\alpha$, but often one simple writes x_α instead. This simplifies the notation, but may create some confusion. For example, it might happen that $V_\alpha \cap V_\beta \neq \emptyset$ as subsets of $\mathbb{R}^n$ and that $x_\alpha = x_\beta$, where we are thinking of x_α, x_β as elements of $\mathbb{R}^n$. But when viewed as elements in the disjoint uniion, they are distinct, i.e. $(x_\alpha, \alpha) \neq (x_\beta, \beta)$ even if $x_\alpha = x_\beta$.

We could denote by $\iota_\alpha : V_\alpha \to V$ the injection sending the space V_α into V, then, for example, $x_\alpha \in V_\alpha$ and $\iota_\alpha(x_\alpha) \in V$.

The topology of V is briefly described in the text: first consder $I \times \mathbb{R}^n$ with the product topology (discrete topology for I, i.e. any subset of I is open, and the usual topology for $\mathbb{R}^n$). Then V inherits the subset topology, as $V \subset I \times \mathbb{R}^n$.

Equivalently, a subset $O \subset V$ is open, iff $\iota_\alpha^{-1}(O) \subset V_\alpha$ is open for each ι_α. To make a short story even shorter, a set $O \subset V$ is open if and only if it is of the form $\coprod_{\alpha\in I} O_\alpha$, with each O_α open in V_α.

(b) consider $N := V/\sim$ (the quotient space with quotient topology). We have to show that the map $\phi : V \to M$ descends to a map $\bar\phi : N \to M$. This just amounts to checking that ϕ is constant on equivalence classes, i.e. $\forall x_\alpha \in V_\alpha$, $x_\beta \in V_\beta$ such that $x_\alpha \sim x_\beta$, $\phi(x_\alpha) = \phi(x_\beta)$. But this property is obvious, it is a restatement of the equivalence condition given above: $x_\alpha \sim x_\beta$ precisely when $\exists x \in M|\ \psi_\alpha^{-1}(x_\alpha) = x = \psi_\beta^{-1}(x_\beta)$.

Thus we can define $\bar\phi([x_\alpha]) : \phi(x_\alpha)$, as the result does not depend on the choice of representative x_α of the class $[x_\alpha]$ (x_α viewed as element of V here!)

Remark. Let (U_β, ψ_β) be a chart of the atlas. Then the inverse $\bar\phi^{-1}$ does not restrict to a homeomorphism between U_β and V_β! In fact, $\bar\phi^{-1}(U_\beta) = \pi(\coprod_{\alpha\in I} \psi_\alpha(U_\alpha \cap U_\beta))$, where $\pi : V \to N$ is, as usual, the quotient map.

(c) $\bar\phi$ is a homeomorphism:

- Surjectivity. For any point $x \in M$ we can find a chart (U_α, ψ_α) with $x \in U$. Then $\psi_\alpha(x) \in V$ and $\bar\phi([\psi_\alpha(x)]) = x$.
- Injectivity. Let $[x_\alpha], [x_\beta] \in N$ and suppose $\bar\phi([x_\alpha]) = \bar\phi([x_\beta])$. This means $\psi_\alpha^{-1}(x_\alpha) = \psi_\beta^{-1}(x_\beta) = x$ for some $x \in M$. But this is precisley the equivalence relation.
- $\bar\phi$ is continuous. Let $O \subset M$ be an open subset. Then $\bar\phi^{-1}(O) = \pi(\coprod_{\alpha\in I} \psi_\alpha(U_\alpha \cap O))$. $\bar\phi^{-1}(O)$ os open in N iff $\pi^{-1}(\bar\phi^{-1}(O))$ is open in V. $\pi^{-1}(\bar\phi^{-1}(O)) = \coprod_{\alpha\in I} \psi_\alpha(U_\alpha \cap O)$ and this is an open set (in V), as each $\psi_\alpha(U_\alpha \cap O) \subset V_\alpha$ is open.
- $\bar\phi$ is open. Let $E \subset N$ be open. Then $\pi^{-1}(E) \subset V$ is open, i.e. $\pi^{-1}(E) = \coprod_{\alpha\in I} E_\alpha$, with each $E_\alpha \subset V_\alpha$ open. Then $\bar\phi(E) = \cup_{\alpha\in I} \phi_\alpha(E_\alpha) \subset M$ which, as a union of open sets, is open.