Differentialgeometrie II
Notes 2

October 30, 2008

These are a few additional notes and remarks for the second tutorial.

1 $[-1,1]/\sim$

Consider $[-1,1]$ as a topological space with its usual topology τ as a subset of $\mathbb{R}$. Define on it the following equivalence relation $x \sim y$ if $|x| = |y| = 1$ or $x = y$. We want to show that the quotient space $[-1,1]/\sim$ is Hausdorff and second countable. The following is a (rather pedestrian) suggestion, any other proof you might think of is ok.

Notice that the quotient map $\pi : [-1,1] \to [-1,1]/\sim$ is not open, so we cannot use lemmas (III.2.3, III.2.4 - Boothby). In fact, consider for example the open subset $V := [-1,a) \subset [-1,1], -1 < a < 1$. Then $\pi(V) \subset [-1,1]/\sim$ is not open as $\pi^{-1} \circ \pi(V) = [-1,a) \cup \{1\}$.

All open sets of $V \subset [-1,1]$ of the form $V = \pi^{-1}(U)$ for some $U \subset [-1,1]/\sim$, i.e. all open sets which are pre-images of open sets in the quotient space are of two kinds: a) they contain neither 1 nor -1; b) they contain both 1 and -1. These constitute a subtopology τ' of τ, and the topology in the quotient space is (by definition of quotient topology) exactly $\pi(\tau')$.

One can consider the following countable family of open sets as a basis $\mathcal{B}$ for τ': open intervals (a,b) with $a,b \in \mathbb{Q}$ and $a,b \in (1,-1)$; open intervals $[-1,-1+\epsilon) \cup (1-\epsilon,1]$ with $\epsilon \in \mathbb{Q}, 0 < \epsilon < 1$. Then $\pi(\mathcal{B})$ is a countable basis for $\pi(\tau')$, the quotient topology of the quotient space.

The Hausdorff property is easy to check directly. Let $\alpha, \beta \in [-1,1]/\sim$, $\alpha \neq \beta$. We have two cases:

- If both $\alpha, \beta \neq \pi(-1) = \pi(1)$, then choose any two disjoint open intervals $\alpha \in I_\alpha \subset [-1,1]$, $\beta \in I_\beta \subset [-1,1]$ not containing 1 and -1. Then $\pi(I_\alpha)$ and $\pi(I_\beta)$ are both open, $\alpha \in \pi(I_\alpha)$ $\beta \in \pi(I_\beta)$ and $\pi(I_\alpha) \cap \pi(I_\beta) = \emptyset$

- If, say, $\beta = \pi(1) = \pi{-1}$, then choose I_α as above and $I_\beta := [-1, -1+\epsilon) \cup (1-\epsilon, 1]$ with ϵ such that $I_\beta \cap I_\alpha = \emptyset$. Then $\pi(I_\alpha)$ and $\pi(I_\beta)$ are open and will do the job.

Remark. $[-1, 1]$ is the one dimensional closed disc of radius one. In this week's problems you are asked to think about an analogous proof for the two dimensional case, i.e. the two dimensional closed unit disc with its border identified as a single point.

Remark. Consider the map $\phi : [-1, 0) \cup (0, 1] \to \mathbb{R}$ defined by$\phi(z) = \frac{\sqrt{1-z^2}}{z}$. Notice that $\phi(-1) = \phi(1)$, i.e. ϕ is constant on equivalence classes. Thus it induces a (continuous) map $\tilde{\phi} : \pi([-1, 0) \cup (0, 1]) \to \mathbb{R}$ assigning to z the value $\phi(x)$ for any $x \in \pi^{-1}(z)$.