Differentialgeometrie II
Problems

Remark "Covector field" and "1-form" are synonymous.

1 Boothby, Ex. V.3.1

Using spherical coordinates (θ, ϕ) on the unit sphere $\rho = 1$ in $\mathbb{R}^3$, determine the components $(g_{i,j})$ of the Riemannian metric on the domain of the coordinates.

Sketch of solution We view $\mathbb{R}^3$ as a Remannian manifold with with its usual coordinates $\{x_1, x_2, x_3\}$, associated frame $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\}$ and coframe $\{dx_1, dx_2, dx_3\}$ and $\Psi_*^{-1}(\frac{\partial}{\partial \theta}$ its standard metric $\Phi(\frac{\partial}{\partial x_i}_{|_p}, \frac{\partial}{\partial x_j}_{|_p}) = g_{i,j} = \delta_{i,j}$.

We consider spherical coordinates on $\mathbb{R}^3$, that is, a chart (U, Ψ), where U is $\mathbb{R}^3$ minus the x_3-axis and $\Psi : U \to \mathbb{R}^+ \times (-\pi/2, \pi/2) \times [0, 2\pi)$ is given by the coordinate functions $\{\theta, \phi, \rho\}$. We give the expression for Ψ^{-1}, which is what we will use in the following:

$$x_1 = \rho \cos(\theta) \cos(\phi),$$

$$x_2 = \rho \cos(\theta) \sin(\phi),$$

$$x_3 = \rho \sin(\theta).$$

We now consider,on its domain of definition, the frame associated to these coordinates. It would be correct (but uncomfortable) to denote these fields as $\Psi_*^{-1}(\frac{\partial}{\partial \theta}), \Psi_*^{-1}(\frac{\partial}{\partial \phi}), \Psi_*^{-1}(\frac{\partial}{\partial \rho})$. We will instead, as costumary, simply write $\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \rho}$.

Describe now $\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \rho}$ as $C^\infty(\mathbb{R}^3)$-linear combinations of the $\frac{\partial}{\partial x_i}$. Remember that for an arbitrary vector field X, its $i-$th component with respect to the standard frame is given by $dx_i(X)$.

Thus, for example, $dx_1(\frac{\partial}{\partial \theta}) = \frac{\partial}{\partial \theta} x_1 = \frac{\partial}{\partial \theta}(\rho \cos(\theta) \cos(\phi) = -\sin(\theta) \cos(\phi) \rho$.

The same for the other components gives $\frac{\partial}{\partial \theta} = -\sin(\theta)\cos(\phi)\rho(\frac{\partial}{\partial x_1}) - \sin(\theta)\sin(\phi)\rho(\frac{\partial}{\partial x_2}) + \cos(\theta)\rho(\frac{\partial}{\partial x_3})$.

Compute $\Phi(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \rho})$ as well as the analogous expression for the remaining choices of pairs of fields.Remember that Φ is $C^\infty(\mathbb{R}^3)$-bilinear, so you can reduce the computation to combinations of the $\Phi(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$.

This will give you the expression of the (same) Remannian metric with repspect to the new coordinates.

2 Boothby, Ex. V.2.3

Let $f_1, \ldots, f_r$, $r \leq n$, be C^∞ functions on an open set of an n-dimensional manifold M. Prove that there are coordinates (V, ψ) in a neighbourhood of p such that $f_1, \ldots, f_r$ are among the coordinate functions if and only if $df_1, \ldots, df_r$ are linearly independent at p.

Sketch of the proof. "If " part. If the 1-forms are linearly independent at p, they will be linearly independent on an open neighbourhood W of p as well. Consider the set of vector fields in $\{X \in \mathcal{X}(W) \mid df_i(X) = 0\}$, i.e. all those vector fields on which the df_i take value zero. Show that it is a distribution of dimension $n - r$ and that it is involutive, i.e. if $df_i(X) = 0, df_i(Y) = 0$ then it follows that $df_i([X, Y]) = 0$. Use Frobenius' theorem to conclude that there is a coordinate chart (V, ϕ) with coordinates $y_1, \ldots, y_n$ such that the last $n - r$ coordinates describe integral manifolds for the distribution. The f_i are constant on these integral manifolds, i.e. $df_i(\frac{\partial}{\partial y_i}) = 0$ for $i = n - r + 1, \ldots, n$. Thus the f_i depend only on the first r variables : $f_i(y_1, \ldots, y_r)$. By assumption the df_i are linearly independent. With respect to the coframe $\psi^{-1*}(dy_j)$ their coordinates are given by the Jacobi matrix $\frac{\partial f_i}{\partial y_j}, i, j \in (1, \ldots, r)$, which thus must be invertible on V. So the f_i determine a diffeomorphism (from a subset of $\mathbb{R}^r$ described by $(y_1, \ldots, y_r)$ to a subset of $\mathbb{R}^r$). The requested (V, ψ) is described by coordinates $(f_1, \ldots, f_r, y_{r+1}, \ldots, y_n)$.

3 Boothby, Ex. V.2.2

Sketch of solution We view $G = GL(n, \mathbb{R})$ as a subset of $\mathbb{R}^{n^2}$ with coordinates $x_{i,j}$ and associated coframe $dx_{i,j}$. Thus a point $X \in G$ is a matrix $(x_{i,j})$. $X^{-1} = Y = (y_{i,j})$. $R_A : G \to G$ is the diffeomorphism of G given by $(x_{i,j}) \to (\sum_k x_{i,k} a_{k,j})$, where $a_{i.j}$ are the coefficients of the invertible matrix A. Thus R_A is determined by n^2 real valued functions $f_{i,j}(x_{\alpha,\beta}) = \sum_k x_{i,k} a_{k,j}$.

The value of the form $\sigma_{i,j}$ at the point X of the manifold is given by ${\sigma_{i,j}}_X = \sum_k y_{i,k} {dx_{k,j}}_X$, while the value at $R_A(X)$ is given by ${\sigma_{ij}}_{R_A(X)} = \sum_{\alpha,k} (a^{-1})_{i\alpha} y_{\alpha,k} {dx_{kj}}_{R_A(X)}$, where $\sum_\alpha (a^{-1})_{i,\alpha} y_{\alpha k}$ is the inverse of XA.

We must show that $R_A^*(\sigma_{i,j}) = \sigma_{i,j}$, i.e. $R_A^*({\sigma_{i,j}}_{R_A(X)}) = {\sigma_{i.j}}_X$.

You may want to confront the formulas given in *Boothby, Th. V.1.6, Cor. V.1.7.* In our case $M = N = G$, $F = R_A$. As coordinates we have $x_{i,j}$ (instead of x_i) and the $f_{i,j}$ introduced above play the role of the y_i in the book. The role of the ω_i and $\tilde{\omega}_i$ are played by the $dx_{i,j}$ in our case.

So we can write $R_A^*({dx_{i,j}}_{R_A(X)}) = \sum_{\alpha,\beta} \frac{\partial f_{i,j}}{\partial x_{\alpha,\beta}}_{|X} {dx_{\alpha,\beta}}_X$, with $\frac{\partial f_{i,j}}{\partial x_{\alpha,\beta}}_{|X} = a_{\beta,j} \delta_{\alpha,i}$.

Compute and check ...

4 Boothby, Ex. V.1.6

Sketch of solution The first question should sound pretty obvious. The second question is almost, but not exactly, what is proven in *Boothby V.1.2.* That is, let σ be a $C^\infty(M)$-linear mapping from $\mathcal{X}(M)$ to $C^\infty(M)$. It is not assumed that σ assigns an element of $T_p^* M$ to each $p \in M$. This can be done a posteriori: define $\sigma_p(X_P) := \sigma(X)(p)$ for any $X \in \mathcal{X}(M)$ which takes the value X_p at p. You must show that this definition is consistent, i.e. that for any other $X' \in \mathcal{X}(M)$ with $X'_p = X_p$ it follows that $\sigma(X)(p) = \sigma(X')(p)$, as well as all the other properties which characterise a C^∞ covectorfield.

5 Question

Let M be an m-dimensional manifold and let N be an n-dimensional regular submanifold. Choose a point $p \in N$ and a regular coordinate neighbourhood (U, ϕ) of p with coordinates $x_1, \ldots, x_m$ such that $U \cap N$ is described by $(x_1, \ldots, x_n, 0, \ldots)$. Consider the associated frame $E_i = \phi_*^{-1}(\frac{\partial}{\partial x_i}),\ i \in (1, \ldots, m)$. So the generic element of $X_p \in T_P M$ can be described as $X_p = \sum_{i=1}^m \alpha_i E_{i_p}$ and the generic vector Y_p in $T_p N$ as $\sum_{i=1}^n \beta_i E_{i_p}$. Consider the following linear map $\pi : T_p M \to T_p N$ defined by $\pi(X_p) =$

$\pi(\sum_{i=1}^{m} \alpha_i E_{i_P}) = \sum_{i=1}^{n} \alpha_i E_{i_p}$, i.e. the last $m - n$ components of the vector are cut out, leaving a vector in T_pN. Does this map depend on the choice of the regular neighbourhood?