EXPLICIT ESTIMATES FOR BIVARIATE HIERARCHICAL BASES

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ABSTRACT: We construct two types of hierarchical tensor product bases with the aid of univariate C^1-Hermite interpolation operators on nested sequences of interpolation points. It is proved that for functions f in $C^1[0,1]^2$ with continuous partial derivatives f_{xy} the basis coefficients and the interpolation error are bounded by explicit constants and the modulus of continuity of partial derivatives of f for the mesh size. In the univariate case, we derive estimates of this type for functions in $C^m[0,1]$. In order to obtain explicit estimates, we had to develop techniques which are different from those known in the literature.

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1. INTRODUCTION

A fundamental principle in analysis is to represent functions as superposition of simple, well–understood basis functions. Classical examples are the Schauder basis of $C[0,1]$ consisting of linear B-splines (cf. [5, 6, 11]) and wavelet bases of $L_2(\mathbb{R})$ (cf. [3]).

We describe a general approach for constructing hierarchical bases of spaces of differentiable functions in one and two variables. The locally supported basis functions are defined by using the fundamental functions of Hermite interpolation schemes on nested sequences of interpolation points.

The aim of this paper is to give estimates of the basis coefficients and the interpolation error *with explicit constants.* Given a sufficiently differentiable function f on $[0,1]$ respectively $[0,1]^2$, by combining the values of f and its derivatives at the interpolation points in an appropriate way and by using reproduction properties of the interpolation operator, we are able to give estimates involving the modulus of continuity of derivatives of f for the mesh size and explicit constants (depending on the fundamental functions of the Hermite interpolation scheme).

In Section 2, we construct hierarchical bases of $C^m[0,1]$. It is shown that for $f \in C^m[0,1]$, the basis coefficients and the interpolation error are bounded by expressions of the type $C\,h^m\,\omega(f^{(m)};h)$, where h is the mesh size of the nested sequences of interpolation points and the constant C is the norm of sums of fundamental functions. Standard examples of hierarchical bases are obtained via interpolation operators onto spaces of polynomials and splines.

In Section 3, we investigate hierarchical tensor product bases of $C^{1,+}[0,1]^2 := \{f \in C^1[0,1]^2 : f_{xy} \in C[0,1]^2\}$. In contrast to the univariate case, there is some freedom in defining such hierarchical bases. We construct two types of hierarchical tensor product bases and prove that for $f \in C^{1,+}[0,1]^2$, the basis coefficients and the interpolation error are bounded by the product of univariate constants and the univariate modulus of continuity of partial derivatives of f. Our results show that the size of the basis coefficients of a given function f reflects the behavior of the derivatives of

f locally.
There are no explicit estimates in the literature of this type. The known estimates either involve the norm of higher derivatives of f [4] or if only the modulus of continuity is involved, then the constants are not given explicitly - with one exception [9], where special hierarchical bases in $C^1[0,1]$ were considered.
In a forthcoming paper, we will give explicit constants for hierarchical basis resulting from Powell-Sabin-splines on triangulations (cf. [10]).

2. UNIVARIATE HIERARCHICAL C^m–BASES

Let $\mathbf{C}^m[0,1]$ denote the space of m times continuously differentiable functions. Let $L_{k,r} \in \mathbf{C}^m[0,1]$ $(k=0,1;\ r=0,...,m)$ be fundamental functions on $[0,1]$, i.e.

$$L_{k,r}^{(s)}(l) = \delta_{k,l}\,\delta_{r,s} \quad ((l=0,1;\ s=0,...,m), \tag{0.1}$$

where $\delta_{k,l}$ denotes the Kronecker symbol. We assume that the functions $L_{k,r}$ satisfy the *reproducing properties*

$$x^r = r!\,L_{0,r}(x) + \sum_{s=0}^{r} \frac{r!}{(r-s)!}\,L_{1,s}(x) \quad (r=0,...,m)\,, \tag{0.2}$$

i.e. the linear space spanned by $L_{k,r}$ contains the space of polynomials of degree less or equal to m. For example the polynomials $L_{k,r} \in \Pi_{2m+1}$ defined by

$$\begin{aligned} L_{0,r}(x) &:= \frac{x^r(1-x)^{m+1}}{r!} \sum_{k=0}^{m-r} \binom{m+k}{k} x^k, \\ L_{1,r}(x) &:= \frac{(x-1)^r x^{m+1}}{r!} \sum_{k=0}^{m-r} \binom{m+k}{k} (1-x)^k \end{aligned} \tag{0.3}$$

fulfill (0.2).
As another example we can construct spline functions $L_{k,r} \in \{s \in \mathbf{C}^m[0,1] : s|_{[0,\xi]},\ s|_{[\xi,1]} \in \Pi_n\}$ which satisfy (0.2), where m is odd, $n := (3m+1)/2$ and $\xi \in (0,1)$. See for example [9].
Let

$$G_j := \{x_k^j : k=0,...,2^j\} \quad (j \in \mathbf{N}_0)$$

with

$$x_0^j = 0\,,\ \ x_{2^j}^j = 1\,,\ \ x_{2k}^{j+1} = x_k^j\,,\ \ x_k^j < x_{2k+1}^{j+1} < x_{k+1}^j$$

be nested sequences of points. Set

$$h_k^j := x_{k+1}^j - x_k^j\,,\ \ h_j := \max\{h_k^j : k=0,...,2^j-1\}.$$

By definition of $L_{k,r}$, the following functions

$$\varphi_{k,r}^j(x) := \begin{cases} (x_k^j - x_{k-1}^j)^r L_{1,r}\left(\frac{x-x_{k-1}^j}{x_k^j-x_{k-1}^j}\right) & x \in [x_{k-1}^j, x_k^j), \\ (x_{k+1}^j - x_k^j)^r L_{0,r}\left(\frac{x-x_k^j}{x_{k+1}^j-x_k^j}\right) & x \in [x_k^j, x_{k+1}^j), \\ 0 & \text{otherwise} \end{cases} \tag{0.4}$$

$(k=1,...,2^j-1)$ and

$$\varphi^j_{0,r}(x) := \begin{cases} (x^j_1)^r L_{0,r}\left(\frac{x}{x^j_1}\right) & x \in [0, x^j_1), \\ 0 & \text{otherwise}, \end{cases}$$

$$\varphi^j_{2^j,r}(x) := \begin{cases} (1-x^j_{2^j-1})^r L_{1,r}\left(\frac{x}{1-x^j_{2^j-1}}\right) & x \in (x^j_{2^j-1}, 1], \\ 0 & \text{otherwise} \end{cases} \tag{0.5}$$

are in $\mathbf{C}^m[0,1]$. Moreover, the functions $\varphi^j_{k,r}$ have support $[x^j_{k-1}, x^j_{k+1}]$ and satisfy

$$(\varphi^j_{k,r})^{(s)}(x^j_l) = \delta_{k,l}\delta_{r,s} \quad (k,l = 0,...2^j; r,s = 0,\dots,m) \tag{0.6}$$

by (0.1). Let

$$V_j := \text{span}\{\varphi^j_{k,r} : k = 0,...,2^j; r = 0,\dots,m\}$$

and let $P_j : \mathbf{C}^m[0,1] \to V_j$ be the projection with

$$(P_j f)^{(s)}(x^j_k) = f^{(s)}(x^j_k) \;\; (k = 0,...,2^j;\; s = 0,\dots,m). \tag{0.7}$$

Then we obtain by definition of $\varphi^j_{k,r}$ for $x \in [x^j_k, x^j_{k+1}]$ that

$$(P_j f)(x) = \sum_{r=0}^{m} (f^{(r)}(x^j_k)\varphi^j_{k,r}(x) + f^{(r)}(x^j_{k+1})\varphi^j_{k+1,r}(x)). \tag{0.8}$$

Since by (0.7)

$$(P_{j+1}f - P_j f)^{(s)}(x^j_k) = 0 \quad (k = 0,...,2^j),$$

we have by (0.8) for $x \in [x^j_k, x^j_{k+1}]$ that

$$\begin{aligned} (P_{j+1}f)(x) &= (P_{j+1}f - P_j f)(x) + P_j f(x) \\ &= \sum_{r=0}^{m} \alpha^{j+1}_{k,r}\varphi^{j+1}_{2k+1,r}(x) + P_j f(x). \end{aligned} \tag{0.9}$$

By (0.6) and (0.7), the coefficients $\alpha^j_{k,r}$ can be rewritten as

$$\alpha^{j+1}_{k,r} = f^{(r)}(x^{j+1}_{2k+1}) - (P_j f)^{(r)}(x^{j+1}_{2k+1}) \quad (r = 0,\dots,m). \tag{0.10}$$

Let

$$x^{j+1}_{2k+1} = x^j_k + \lambda^j_k h^j_k \quad (\lambda^j_k \in (0,1);\; k = 0,\dots,2^j-1).$$

Then we obtain by (0.8) and by definition of $\varphi^j_{k,r}$ that

$$\alpha^{j+1}_{k,r} = f^{(r)}(x^{j+1}_{2k+1}) - \sum_{s=0}^{m} (h^j_k)^{s-r}(f^{(s)}(x^j_k)L^{(r)}_{0,s}(\lambda^j_k) + f^{(s)}(x^j_{k+1})L^{(r)}_{1,s}(\lambda^j_k)). \tag{0.11}$$

Theorem 2.1. *For $f \in \mathbf{C}^m[0,1]$, the coefficients $\alpha^{j+1}_{k,r}$ $(k = 0,...,2^j-1;\; r = 0,\dots,m)$ in (0.9) can be estimated by*

$$|\alpha^{j+1}_{k,r}| \le c_r\,(h_j)^{m-r}\,\omega(f^{(m)}, h_j)\,,$$

where ω denotes the modulus of continuity and

$$c_r := \sup_{x\in(0,1)} \sum_{s=0}^{m} \left(\frac{x^{m-s}}{(m-s)!} |L_{0,s}^{(r)}(x)| + \frac{(1-x)^{m-s}}{(m-s)!} |\, L_{1,s}^{(r)}(x)| \right).$$

The approximation error of the projection P_j is given by

$$\|f^{(r)} - (P_j f)^{(r)}\|_\infty \le c_r\,(h_j)^{m-r}\,\omega(f^{(m)}, h_j),$$

where $\|f\|_\infty := \max_{x\in[0,1]} |f(x)|$.

Proof. First, we prove by induction on s that the reproducing property (0.2) of the fundamental functions implies for $r, n = 0, \dots, m$ that

$$\sum_{s=0}^{n} \left(\frac{(-x)^{n-s}}{(n-s)!} L_{0,s}^{(r)}(x) + \frac{(1-x)^{n-s}}{(n-s)!} L_{1,s}^{(r)}(x) \right) = \delta_{r,n}. \qquad (0.12)$$

For $n = 0$, the assertion (0.12) follows immediately from (0.2).
In the following, let $n \ge 1$. Let $r = 0$. Then it follows by

$$0 = (1-1)^n = n! \sum_{s=0}^{n} \frac{(-1)^s}{(n-s)!s!}$$

that

$$0 = \sum_{s=0}^{n} \frac{x^{n-s}}{(n-s)!} \frac{(-1)^s}{s!} x^s.$$

Replacing x^{n-s} by (0.2), we obtain for $n \le m$

$$\begin{aligned} 0 &= \sum_{s=0}^{n} \left(L_{0,n-s}(x) + \sum_{r=0}^{n-s} \frac{1}{(n-s-r)!} L_{1,r}(x) \right) \frac{(-1)^s}{s!} x^s \\ &= \sum_{s=0}^{n} \frac{(-x)^{n-s}}{(n-s)!} L_{0,s}(x) + \sum_{s=0}^{n} \left(\sum_{r=0}^{n-s} \frac{(-1)^s x^s}{s!(n-s-r)!} L_{1,r}(x) \right) \\ &= \sum_{s=0}^{n} \frac{(-x)^{n-s}}{(n-s)!} L_{0,s}(x) + \sum_{r=0}^{n} \left(\sum_{s=0}^{n-r} \frac{(-1)^s x^s}{s!(n-s-r)!} L_{1,r}(x) \right) \\ &= \sum_{s=0}^{n} \frac{(-x)^{n-s}}{(n-s)!} L_{0,s}(x) + \sum_{k=0}^{n} \left(\frac{(1-x)^{n-k}}{(n-k)!} L_{1,k}(x) \right). \end{aligned}$$

Consequently, (0.12) holds for $r = 0$ and $n = 0, ..., m$.
Assume that (0.12) is true for $s < m$ and all $n = 0, ..., m$. Then

$$\begin{aligned} 0 &= \frac{d}{dx} \left(\sum_{s=0}^{n} \Big(\frac{(-x)^{n-s}}{(n-s)!} L_{0,s}^{(r)}(x) + \frac{(1-x)^{n-s}}{(n-s)!} L_{1,s}^{(r)}(x) \Big) \right) \\ &= -\sum_{s=0}^{n-1} \left(\frac{(-x)^{n-s-1}}{(n-s-1)!} L_{0,s}^{(r)}(x) + \frac{(1-x)^{n-s-1}}{(n-s-1)!} L_{1,s}^{(r)}(x) \right) \\ &\quad + \sum_{s=0}^{n} \left(\frac{(-x)^{n-s}}{(n-s)!} L_{0,s}^{(r+1)}(x) + \frac{(1-x)^{n-s}}{(n-s)!} L_{1,s}^{(r+1)}(x) \right) \end{aligned}$$

and by assumption

$$0 = -\delta_{r,n-1} + \sum_{s=0}^{n} \left(\frac{(-x)^{n-s}}{(n-s)!} L_{0,s}^{(r+1)}(x) + \frac{(1-x)^{n-s}}{(n-s)!} L_{1,s}^{(r+1)}(x) \right).$$

This completes the proof of (0.12).

To simplify the notation, we fix $j \in \mathbf{N}, k \in \{0, ..., 2^{j+1}\}$ and set $h := h_k^j$, $\lambda := \lambda_k^j$. Let

$$g(x) := f(x_k^j + x) \quad (x \in [0, h]).$$

Then we have by (0.11) with $x := \lambda h$ that

$$\alpha_{k,r}^{j+1} = g^{(r)}(\lambda h) - \sum_{s=0}^{m} h^{s-r} \left(g^{(s)}(0) L_{0,s}^{(r)}(\lambda) + g^{(s)}(h) L_{1,s}^{(r)}(\lambda) \right).$$

Using the Taylor expansions

$$\begin{aligned}
g^{(s)}(0) &= \sum_{l=0}^{m-1-s} g^{(l+s)}(\lambda h) \frac{h^l(-\lambda)^l}{l!} + \frac{h^{m-s}(-\lambda)^{m-s}}{(m-s)!} g^{(m)}(\xi_{0,s}), \\
g^{(s)}(h) &= \sum_{l=0}^{m-1-s} g^{(l+s)}(\lambda h) \frac{h^l(1-\lambda)^l}{l!} + \frac{h^{m-s}(1-\lambda)^{m-s}}{(m-s)!} g^{(m)}(\xi_{1,s})
\end{aligned}$$

with $\xi_{0,s} \in (0, \lambda h)$ and $\xi_{1,s} \in (\lambda h, h)$, this can be rewritten as

$$\begin{aligned}
\alpha_{k,r}^{j+1} &= g^{(r)}(\lambda h) \\
&\quad - \sum_{s=0}^{m-1} h^{s-r} \sum_{l=0}^{m-1-s} g^{(l+s)}(\lambda h)\, h^l \left(\frac{(-\lambda)^l}{l!} L_{0,s}^{(r)} + \frac{(1-\lambda)^l}{l!} L_{1,s}^{(r)} \right) \\
&\quad - h^{m-r} \sum_{s=0}^{m} \left(g^{(m)}(\xi_{0,s}) \frac{(-\lambda)^{m-s}}{(m-s)!} L_{0,s}^{(r)} + g^{(m)}(\xi_{1,s}) \frac{(1-\lambda)^{m-s}}{(m-s)!} L_{1,s}^{(r)} \right)
\end{aligned}$$

with $\xi_{0,m} := 0$, $\xi_{1,m} := h$ and $L_{i,s}^{(r)} := L_{i,s}^{(r)}(\lambda)$. Setting $n := l + s$, we obtain

$$\begin{aligned}
S_r &:= \sum_{s=0}^{m-1} h^{s-r} \sum_{l=0}^{m-1-s} g^{(l+s)}(\lambda h)\, h^l \left(\frac{(-\lambda)^l}{l!} L_{0,s}^{(r)} + \frac{(1-\lambda)^l}{l!} L_{1,s}^{(r)} \right) \\
&= \sum_{n=0}^{m-1} h^{n-r} g^{(n)}(\lambda h) \sum_{s=0}^{n} \left(\frac{(-\lambda)^{n-s}}{(n-s)!} L_{0,s}^{(r)} + \frac{(1-\lambda)^{n-s}}{(n-s)!} L_{1,s}^{(r)} \right)
\end{aligned}$$

and further by (0.12)

$$S_r = \begin{cases} g^{(r)}(\lambda h) & r = 0, ..., m-1, \\ 0 & r = m. \end{cases}$$

Thus, for $r = 0, \dots, m-1$,

$$\alpha_{k,r}^{j+1} = -h^{m-r} \sum_{s=0}^{m} \left(g^{(m)}(\xi_{0,s}) \frac{(-\lambda)^{m-s}}{(m-s)!} L_{0,s}^{(r)} + g^{(m)}(\xi_{1,s}) \frac{(1-\lambda)^{m-s}}{(m-s)!} L_{1,s}^{(r)} \right)$$

and

$$\alpha_{k,m}^{j+1} = g^{(m)}(\lambda h) - \sum_{s=0}^{m} \left(g^{(m)}(\xi_{0,s}) \frac{(-\lambda)^{m-s}}{(m-s)!} L_{0,s}^{(m)} + g^{(m)}(\xi_{1,s}) \frac{(1-\lambda)^{m-s}}{(m-s)!} L_{1,s}^{(m)} \right).$$

By (0.12), this can be rewritten for $r = 0, ..., m$ as

$$\begin{aligned} \alpha_{k,r}^{j+1} &= h^{m-r} \left(\sum_{s=0}^{m} \left(g^{(m)}(\lambda h) - g^{(m)}(\xi_{0,s})\right) \frac{(-\lambda)^{m-s}}{(m-s)!} L_{0,s}^{(r)} \right. \\ &\quad \left. + \sum_{s=0}^{m} \left(g^{(m)}(\lambda h) - g^{(m)}(\xi_{1,s})\right) \frac{(1-\lambda)^{m-s}}{(m-s)!} L_{1,s}^{(r)} \right). \end{aligned}$$

Since $h \leq h_j$ and

$$|g^{(m)}(\lambda h) - g^{(m)}(\xi_{i,s})| \leq \omega(f^{(m)}, h_j) \quad (i = 0, 1),$$

this implies that

$$|\alpha_{kr}^{j+1}| \leq c_r\, (h_j)^{m-r}\, \omega(f^{(m)}, h_j).$$

The second part of the assertion follows immediately by using (0.10). ∎

Let $h_j \to 0$ for $j \to \infty$. By (0.9), we obtain for $f \in C^m[0,1]$ the decomposition

$$f = \sum_{j=0}^{\infty} (P_{j+1} f - P_j f) + P_0 f\,,$$

$$f = \sum_{j=0}^{\infty} \left(\sum_{r=0}^{m} \sum_{k=0}^{2^j - 1} \alpha_{k,r}^{j+1} \varphi_{2k+1,r}^{j+1} \right) + \sum_{r=0}^{m} (\alpha_{0,r}^0 \varphi_{0,r}^0 + \alpha_{1,r}^0 \varphi_{1,r}^0). \qquad (0.13)$$

By Theorem 2.1 and since $h_j \to 0$ for $j \to \infty$, the sum on the right–hand side converges uniformly. Note that, following the lines of G. Faber [5, 6], we can conversely formulate a sufficient condition on a sequence $\{\alpha_{k,r}^{j+1}\}$ of real numbers such that the right–hand side of (0.13) converges uniformly to a function in $C^m[0,1]$.

Remark. The norm $||\varphi_{k,r}^j||_\infty$ of our basis functions (0.4) and (0.5) depends on the refinement level j. Alternatively, we can of course choose the functions

$$\varphi_{k,r}^j(x) := \begin{cases} \left(\frac{x_k^j - x_{k-1}^j}{x_{k+1}^j - x_{k-1}^j} \right)^r L_{1,r} \left(\frac{x - x_{k-1}^j}{x_k^j - x_{k-1}^j} \right) & x \in [x_{k-1}^j, x_k^j), \\ \left(\frac{x_{k+1}^j - x_k^j}{x_{k+1}^j - x_{k-1}^j} \right)^r L_{0,r} \left(\frac{x - x_k^j}{x_{k+1}^j - x_k^j} \right) & x \in [x_k^j, x_{k+1}^j), \\ 0 & \text{otherwise} \end{cases}$$

$(k = 1, ..., 2^j - 1)$ and

$$\varphi_{0,r}^j(x) := \begin{cases} L_{0,r}\left(\frac{x}{x_1^j}\right) & x \in [0, x_1^j), \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi_{2^j,r}^j(x) := \begin{cases} L_{1,r}\left(\frac{x}{1 - x_{2^j-1}^j}\right) & x \in (x_{2^j-1}^j, 1], \\ 0 & \text{otherwise} \end{cases}$$

as basis functions of V_j. Here the values $||\varphi^j_{k,r}||_\infty$ depend only on the distribution of the grid points of G_j with respect to the grid points of G_{j-1} but not on the refinement level. With respect to the above basis we have to replace the estimation of the coefficients α in Theorem 2.1 by

$$|\alpha^{j+1}_{k,r}| \leq c_r\,(h_j)^m\,\omega(f^{(m)},h_j) \quad (k=0,...,2^j-1;\ r=0,\dots,m).$$

In the following we consider the special case $m=1$. The constants c_r $(r=0,1)$ in Theorem 2.1 are in general not optimal. We are looking for smaller constants at least in the polynomial case (0.3). Let the fundamental functions $L_{k,r}$ $(k,r=0,1)$ be given by (0.3). Let

$$x_0 := x^j_k,\ x_1 := x^{j+1}_{2k+1},\ x_2 := x^j_{k+1},\ h := h^j_k,\ \lambda := \lambda^j_k. \tag{0.14}$$

Then we obtain by (0.10) and by the well–known error formula of polynomial interpolation [7, p. 23] that

$$\begin{aligned}\alpha^{j+1}_{k,0} = f(x_1) - P_j f(x_1) &= f[x_0,x_0,x_1,x_2,x_2](x_1-x_0)^2(x_1-x_2)^2 \\ &= f[x_0,x_0,x_1,x_2,x_2]\lambda^2(\lambda-1)^2h^4.\end{aligned}$$

Using properties of divided differences $f[\dots]$ this can be rewritten as

$$\begin{aligned}\alpha^{j+1}_{k,0} &= (f[x_1,x_2,x_2] - 2f[x_0,x_1,x_2] + f[x_0,x_0,x_1])\lambda^2(\lambda-1)^2h^2 \\ &= h\,(\,(f[x_2,x_2]-f[x_1,x_2])\lambda^2(\lambda-1) - 2(f[x_1,x_2]-f[x_0,x_1])\lambda^2(\lambda-1)^2 \\ &\quad + (f[x_0,x_0]-f[x_0,x_1])\lambda(\lambda-1)^2\,) \\ &= h\,(\,(f'(x_2)-f'(\xi_2))\lambda^2(\lambda-1) - 2(f'(\xi_2)-f'(\xi_0))\lambda^2(\lambda-1)^2 \\ &\quad + (f'(x_1)-f'(\xi_0))\lambda(\lambda-1)^2\,).\end{aligned}$$

By (0.3), this implies

$$|\alpha^{j+1}_{k,0}| \leq h\,\omega(f',h)\,(|L_{1,1}(\lambda)| + |L_{0,1}(\lambda) - \lambda L_{0,0}(\lambda)| + |L_{0,1}(\lambda)|).$$

Applying the approach of Theorem 2.1, we can prove that the above result is not restricted to polynomials $L_{k,r}$.

Corollary 2.2. *For $f \in C^1[0,1]$, the coefficients $\alpha^{j+1}_{k,r}$ $(k=0,...,2^j-1;\ r=0,1)$ in (0.9) can be estimated by*

$$|\alpha^{j+1}_{k,0}| \leq C_0\,h_j\,\omega(f',h_j)\,,\ |\alpha^{j+1}_{k,1}| \leq C_1\,\omega(f',h_j),$$

where

$$C_0 := \sup_{x\in(0,1)} \{|L_{0,1}(x)| + |L_{1,1}(x)| + |L_{0,1}(x) - xL_{0,0}(x)|\}, \tag{0.15}$$

$$C_1 := \sup_{x\in(0,1)} \{|L'_{1,0}(x)| + |L'_{1,1}(x)| + |L'_{0,1}(x)|\}. \tag{0.16}$$

Proof. For $m = 1$, the reproducing property (0.2) reads as

$$1 = L_{0,0}(x) + L_{1,0}(x), \tag{0.17}$$
$$x = L_{0,1}(x) + L_{1,0}(x) + L_{1,1}(x). \tag{0.18}$$

Combining both equations we get

$$0 = -xL_{0,0}(x) + (1-x)L_{1,0}(x) + L_{0,1}(x) + L_{1,1}(x). \tag{0.19}$$

By (0.11) we have that

$$\alpha_{k,0}^{j+1} = f(x_1) - (\, f(x_0)L_{0,0}(\lambda) + f(x_2)L_{1,0}(\lambda)\,) \; - \; h\;(\, f'(x_0)L_{0,1}(\lambda) + f'(x_2)L_{1,1}(\lambda)\,)$$

and by (0.17), (0.19) and Taylor's formula that

$$\begin{aligned} \alpha_{k,0}^{j+1} &= (f(x_1)-f(x_0))L_{0,0} + (f(x_1)-f(x_2))L_{1,0} - h\,(f'(x_0)L_{0,1} + f'(x_2)L_{1,1}) \\ &= h\,(f'(\xi_0)\lambda L_{0,0} - f'(\xi_2)(1-\lambda)L_{1,0} - f'(x_0)L_{0,1} - f'(x_2)L_{1,1}) \\ &= h\;(\,(f'(\xi_0)-f'(x_0))L_{0,1} + (f'(\xi_2)-f'(x_2))L_{1,1} \\ &\quad +(f'(\xi_0)-f'(\xi_2))(\lambda L_{0,0} - L_{0,1})\,) \qquad (\xi_0 \in (x_0,x_1), \xi_2(x_1,x_2)). \end{aligned}$$

Thus

$$|\alpha_{k,0}^{j+1}| \le h\,\omega(f',h)\;(|L_{0,1}(\lambda)| + |L_{1,1}(\lambda)| + |L_{0,1}(\lambda) - \lambda L_{0,0}(\lambda)|)\,.$$

Similarly, we obtain by differentiation of (0.17) and (0.18) that

$$\begin{aligned} \alpha_{k,1}^{j+1} &= -\frac{1}{h}(f(x_0)L'_{0,0} + f(x_2)L'_{1,0}) + f'(x_1) - f'(x_0)L'_{0,1} - f'(x_2)L'_{1,1} \\ &= -f'(\xi)L'_{1,0} + f'(x_1) - f'(x_0)L'_{0,1} - f'(x_2)L'_{1,1} \\ &= (f'(x_1)-f'(\xi))L'_{1,0} + (f'(x_1)-f'(x_0))L'_{0,1} + (f'(x_1)-f'(x_2))L'_{1,1}, \end{aligned}$$

which leads to the estimate of $|\alpha_{k,1}^{j+1}|$. ∎

For the polynomial case (0.3), we have that $c_0 = \frac{3}{4}$ while $C_0 = \frac{3}{8}$ and that $c_1 = C_1 = 2$, where the supremum in c_i and C_i $(i = 0,1)$ over $(0,1)$ is always reached for $x = \frac{1}{2}$. Finally, it is useful to formulate the stepwise transition of $P_{j+1}f$ from the basis

$$\{\varphi_{k,r}^{j+1} \,:\, k = 0,\ldots,2^{j+1};\, r = 0,1\}$$

of V_{j+1} to the basis

$$\{\varphi_{0,0}, \varphi_{1,0}, \varphi_{0,1}, \varphi_{1,1}\} \cup \bigcup_{l=0}^{j} \{\varphi_{2k+1,r}^{l+1} \,:\, k = 0,\ldots,2^l - 1;\, r = 0,1\}$$

in matrix–vector notation. Having [11] in mind, we call the first basis a *nodal basis* and the second one a *hierarchical basis*. Again, by the support of the involved basis function, we can fix an arbitrary $k \in \{0,...,2^j\}$ and use the notation (0.14). By (0.11) we obtain that

$$\begin{pmatrix} f(x_0) \\ f(x_2) \\ f'(x_0) \\ f'(x_2) \\ \alpha_{k,0}^{j} \\ \alpha_{k,1}^{j} \end{pmatrix} = \begin{pmatrix} G \\ H_\lambda \end{pmatrix} \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f'(x_0) \\ f'(x_1) \\ f'(x_2) \end{pmatrix} \tag{0.20}$$

with

$$G := \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right), \tag{0.21}$$

$$H_\lambda := \left(\begin{array}{ccc|ccc} -L_{0,0}(\lambda) & 1 & -L_{1,0}(\lambda) & -hL_{0,1}(\lambda) & 0 & -hL_{1,1}(\lambda) \\ -\frac{1}{h}L'_{0,0}(\lambda) & 0 & -\frac{1}{h}L'_{1,0}(\lambda) & -L'_{0,1}(\lambda) & 1 & -L'_{1,1}(\lambda) \end{array} \right) \tag{0.22}$$

or equivalently for $x \in [x_k^j, x_{k+1}^j]$

$$\Phi^T = \Psi^T \left(\begin{array}{c} G \\ H_\lambda \end{array} \right) \tag{0.23}$$

with

$$\begin{aligned} \Phi^T &:= (\varphi_{2k,0}^{j+1}, \varphi_{2k+1,0}^{j+1}, \varphi_{2k+2,0}^{j+1}, \varphi_{2k,1}^{j+1}, \varphi_{2k+1,1}^{j+1}, \varphi_{2k+2,1}^{j+1}), \\ \Psi^T &:= (\varphi_{k,0}^{j}, \varphi_{k+1,0}^{j}, \varphi_{k,1}^{j}, \varphi_{k+1,1}^{j}, \varphi_{2k+1,0}^{j+1}, \varphi_{2k+1,1}^{j+1}). \end{aligned} \tag{0.24}$$

While (0.20) describes the stepwise *decomposition* of $P_{j+1}f$ with respect to the hierarchical basis, conversely the *reconstruction* is determined by

$$\left(\begin{array}{c} f(x_0) \\ f(x_1) \\ f(x_2) \\ f'(x_0) \\ f'(x_1) \\ f'(x_2) \end{array} \right) = \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ L_{0,0}(\lambda) & L_{1,0}(\lambda) & hL_{0,1}(\lambda) & hL_{1,1}(\lambda) & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{h}L'_{0,0}(\lambda) & \frac{1}{h}L'_{1,0}(\lambda) & L'_{0,1}(\lambda) & L'_{1,1}(\lambda) & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right) \left(\begin{array}{c} f(x_0) \\ f(x_2) \\ f'(x_0) \\ f'(x_2) \\ \alpha_{k,0}^j \\ \alpha_{k,1}^j \end{array} \right)$$

Finally, note that the matrix–vector representation in the case $\mathbf{C}^m[0,1]$ ($m \in \mathbf{N}$) follows from (0.11) in a completely similar way.

3. BIVARIATE HIERARCHICAL BASES

Only for notational reasons we restrict our attention in the bivariate setting to $m = 1$. By the previous section we can deduce similar results for arbitrary $m \in \mathbf{N}$.
For the bivariate nested sequences of points

$$G_j := \{(x_k^j, y_l^j) : k, l = 0, ..., 2^j\}$$

with

$$x_0^j = y_0^j = 0\,,\; x_{2^j}^j = y_{2^j}^j = 1\,,\; x_{2k}^{j+1} = x_k^j\,,\; y_{2l}^{j+1} = y_l^j,$$
$$x_k^j < x_{2k+1}^{j+1} < x_{k+1}^j\,,\; y_l^j < y_{2l+1}^{j+1} < y_{l+1}^j$$

we consider the tensor product spaces

$$V_j := \text{span}\{\varphi_{k,r}^j(x)\ \varphi_{l,s}^j(y)\ : k, l = 0, ..., 2^j;\ r, s = 0, 1\}.$$

Set

$$h^j_{kx} := x^j_{k+1} - x^j_k \quad , \quad h_{xj} = \max\{h^j_{kx} : k = 0, ..., 2^j\},$$
$$h^j_{ly} := y^j_{l+1} - y^j_l \quad , \quad h_{yj} = \max\{h^j_{ly} : l = 0, ..., 2^j\}.$$

Let $P_j : C^{1,+}[0,1]^2 \to V_j$ be the projection with

$$(P_j f)_{x^{r_1}, y^{r_2}}(x^j_k, y^j_l) = f_{x^{r_1}, y^{r_2}}(x^j_k, y^j_l) \;\; (k, l = 0, ..., 2^j; \;\; r_1, r_2 = 0, 1). \tag{0.25}$$

As in the univariate case we want to decompose functions $P_{j+1}f \in V_{j+1}$ in the form

$$P_{j+1}f = P_j f + (P_{j+1} - P_j)f.$$

Let

$$\{\psi^{j+1}_{2k,r}\, \psi^{j+1}_{2l+1,s}, \psi^{j+1}_{2k+2,r}\, \psi^{j+1}_{2l+1,s}, \psi^{j+1}_{2k+1,r}\, \psi^{j+1}_{2l,s}, \psi^{j+1}_{2k+1,r}\, \psi^{j+1}_{2l+2,s}, \psi^{j+1}_{2k+1,r}\, \psi^{j+1}_{2l+1,s} : \\ k, l = 0, ..., 2^j - 1;\; r, s = 0, 1\}$$

be a basis of $W_j := V_{j+1} - V_j$, where for fixed k, l and $r, s = 0, 1$ the above 20 functions are the only basis functions which do not completely vanish on $[x^j_k, x^j_{k+1}] \times [y^j_l, y^j_{l+1}]$. Popular examples of such basis functions are

$$B^{j+1}_N = \begin{array}{l} \{\varphi^j_{k,r}\, \varphi^{j+1}_{2l+1,s}, \varphi^j_{k+1,r}\, \varphi^{j+1}_{2l+1,s}, \varphi^{j+1}_{2k+1,r}\, \varphi^j_{l,s}, \varphi^{j+1}_{2k+1,r}\, \varphi^j_{l+1,s}, \varphi^{j+1}_{2k+1,r}\, \varphi^{j+1}_{2l+1,s} : \\ k, l = 0, ..., 2^j - 1;\; r, s = 0, 1\} \end{array}$$

and

$$B^{j+1}_M = \begin{array}{l} \{\varphi^{j+1}_{2k,r}\, \varphi^{j+1}_{2l+1,s}, \varphi^{j+1}_{2k+2,r}\, \varphi^{j+1}_{2l+1,s}, \varphi^{j+1}_{2k+1,r}\, \varphi^{j+1}_{2l,s}, \varphi^{j+1}_{2k+1,r}\, \varphi^{j+1}_{2l+2,s}, \varphi^{j+1}_{2k+1,r}\, \varphi^{j+1}_{2l+1,s} : \\ k, l = 0, ..., 2^j - 1;\; r, s = 0, 1\}. \end{array}$$

The functions in B^{j+1}_N correspond to the "**n**onstandard decomposition" of $C^{1,+}[0,1]^2$ known from wavelet context [1], while the functions in B^{j+1}_M play a role in **m**ultigrid decompositions [2]. In the following, we restrict our attention to the above bases. By the same arguments as in the univariate case, we obtain for $(x, y) \in [x^j_k, x^j_{k+1}] \times [y^j_l, y^j_{l+1}]$

$$\begin{aligned} (P_{j+1}f)(x, y) \;\; = \;\; & (P_j f)(x, y) + \sum_{r_1, r_2 = 0}^{1} \Big(\alpha^{j+1}_{(2k, 2l+1), (r_1, r_2)} \psi^{j+1}_{2k, r_1}(x) \psi^{j+1}_{2l+1, r_2}(y) \\ & + \alpha^{j+1}_{(2k+2, 2l+1), (r_1, r_2)} \psi^{j+1}_{2k+2, r_1}(x) \psi^{j+1}_{2l+1, r_2}(y) \\ & + \alpha^{j+1}_{(2k+1, 2l), (r_1, r_2)} \psi^{j+1}_{2k+1, r_1}(x) \psi^{j+1}_{2l, r_2}(y) \\ & + \alpha^{j+1}_{(2k+1, 2l+2), (r_1, r_2)} \psi^{j+1}_{2k+1, r_1}(x) \psi^{j+1}_{2l+2, r_2}(y) \\ & + \beta^{j+1}_{(2k+1, 2l+1), (r_1, r_2)} \psi^{j+1}_{2k+1, r_1}(x) \psi^{j+1}_{2l+1, r_2}(y) \Big) \end{aligned} \tag{0.26}$$

and further by (0.25) and (0.6) that with respect to both bases

$$\begin{aligned} \alpha^{j+1}_{(2k, 2l+1), (r_1, r_2)} &= f_{x^{r_1}, y^{r_2}}(x^{j+1}_{2k}, x^{j+1}_{2l+1}) - (P_j f)_{x^{r_1}, y^{r_2}}(x^{j+1}_{2k}, x^{j+1}_{2l+1}), \\ \alpha^{j+1}_{(2k+2, 2l+1), (r_1, r_2)} &= f_{x^{r_1}, y^{r_2}}(x^{j+1}_{2k+2}, x^{j+1}_{2l+1}) - (P_j f)_{x^{r_1}, y^{r_2}}(x^{j+1}_{2k+2}, x^{j+1}_{2l+1}), \\ \alpha^{j+1}_{(2k+1, 2l), (r_1, r_2)} &= f_{x^{r_1}, y^{r_2}}(x^{j+1}_{2k+1}, x^{j+1}_{2l}) - (P_j f)_{x^{r_1}, y^{r_2}}(x^{j+1}_{2k+1}, x^{j+1}_{2l}), \\ \alpha^{j+1}_{(2k+1, 2l+2), (r_1, r_2)} &= f_{x^{r_1}, y^{r_2}}(x^{j+1}_{2k+1}, x^{j+1}_{2l+2}) - (P_j f)_{x^{r_1}, y^{r_2}}(x^{j+1}_{2k+1}, x^{j+1}_{2l+2}). \end{aligned} \tag{0.27}$$

Therefore only the coefficients β^{j+1} of $(P_{j+1}-P_j)f$ differ with respect to B_N^{j+1} and B_M^{j+1}, respectively. Since the different coefficients $\alpha^{j+1}_{(\cdot,\cdot)(r_1,r_2)}$ can be estimated in the same way, we restrict our attention to $\alpha^{j+1}_{(2k,2l+1)(r_1,r_2)}$.

For the basis B_M^{j+1}, we have by (0.26) as in the univariate case that

$$\beta^{j+1}_{(2k+1,2l+1),(r_1,r_2)} = f_{x^{r_1},y^{r_2}}(x^{j+1}_{2k+1}, y^{j+1}_{2l+1}) - (P_j f)_{x^{r_1},y^{r_2}}(x^{j+1}_{2k+1}, y^{j+1}_{2l+1}) \qquad (0.28)$$

for $r_1, r_2 = 0, 1$. Thus, estimations of α^{j+1} and β^{j+1} in this case lead to estimations of the approximation error.
For the basis B_N^{j+1}, the representation of β^{j+1} which follows from (0.26) contains various summands with coefficients α^{j+1}. To estimate β^{j+1} in this case, we can use the more simple matrix–vector approach. For the "nonstandard decomposition", we obtain the following estimates of the coefficients:

Theorem 3.1. *Let $f \in C^{1,+}[0,1]^2$. Let the projectors P_j be given by (0.25). Then the coefficients of $(P_{j+1}-P_j)f$ in (0.26) with respect to the hierarchical basis B_N^{j+1} can be estimated for $k = 0, ..., 2^j$ and $l = 0, ..., 2^j - 1$ by*

$$\begin{aligned}
|\alpha^{j+1}_{(2k,2l+1)(0,0)}| &\leq C_0\, h_{yj}\, \omega_y(f_y, h_{yj}),\\
|\alpha^{j+1}_{(2k,2l+1)(1,0)}| &\leq C_0\, h_{yj}\, \omega_y(f_{xy}, h_{yj}),\\
|\alpha^{j+1}_{(2k,2l+1)(0,1)}| &\leq C_1\, \omega_y(f_y, h_{yj}),\\
|\alpha^{j+1}_{(2k,2l+1)(1,1)}| &\leq C_1\, \omega_y(f_{xy}, h_{yj})
\end{aligned}$$

and for $k, l = 0, ..., 2^j - 1$ by

$$\begin{aligned}
|\beta^{j+1}_{(2k+1,2l+1)(0,0)}| &\leq C_0^2\, h_{xj}h_{yj}\, (\omega_x(f_{xy}, h_{xj}) + \omega_y(f_{xy}, h_{yj})),\\
|\beta^{j+1}_{(2k+1,2l+1)(1,0)}| &\leq C_0C_1\, h_{yj}\, (\omega_x(f_{xy}, h_{xj}) + \omega_y(f_{xy}, h_{yj})),\\
|\beta^{j+1}_{(2k+1,2l+1)(0,1)}| &\leq C_0C_1\, h_{xj}\, (\omega_x(f_{xy}, h_{xj}) + \omega_y(f_{xy}, h_{yj})),\\
|\beta^{j+1}_{(2k+1,2l+1)(1,1)}| &\leq C_1^2\, (\omega_x(f_{xy}, h_{xj}) + \omega_y(f_{xy}, h_{yj})),
\end{aligned}$$

where C_0 and C_1 are given by (0.15) and (0.16), respectively and where $\omega_y(f,t)$ denotes the univariate modulus of continuity

$$\omega_y(f,t) := \sup_{|h|\leq t}\ \sup_{x\in[0,1]} |f(x, y+h) - f(x,y)|.$$

Proof. As in the univariate case, we fix arbitrary $j \in \mathbf{N}$; $k, l \in \{0, ..., 2^j - 1\}$ and simplify the notation as follows

$$x_0 := x_k^j,\ x_1 := x^{j+1}_{2k+1},\ x_2 := x^j_{k+1}\,,\ h_x := h^j_{kx},$$

$$y_0 := x_l^j,\ y_1 := y^{j+1}_{2l+1},\ y_2 := y^j_{l+1}\,,\ h_y := h^j_{ly}.$$

Let $\lambda_x, \lambda_y \in (0,1)$ be defined by

$$x_1 = x_0 + \lambda_x h_x,\ \ y_1 = y_0 + \lambda_y h_y$$

and let

$$\alpha_{r_1 r_2} := \alpha^{j+1}_{(2k,2l+1),(r_1,r_2)}\,,\ \beta_{r_1 r_2} := \beta^{j+1}_{(2k+1,2l+1),(r_1,r_2)}.$$

Set

$$\begin{aligned}
\mathbf{F} &= \begin{pmatrix} \mathbf{f} & \mathbf{f}_y \\ \mathbf{f}_x & \mathbf{f}_{xy} \end{pmatrix} \\
&:= \left(\begin{array}{ccc|ccc}
f(x_0,y_0) & f(x_0,y_1) & f(x_0,y_2) & f_y(x_0,y_0) & f_y(x_0,y_1) & f_y(x_0,y_2) \\
f(x_1,y_0) & f(x_1,y_1) & f(x_1,y_2) & f_y(x_1,y_0) & f_y(x_1,y_1) & f_y(x_1,y_2) \\
f(x_2,y_0) & f(x_2,y_1) & f(x_2,y_2) & f_y(x_2,y_0) & f_y(x_2,y_1) & f_y(x_2,y_2) \\ \hline
f_x(x_0,y_0) & f_x(x_0,y_1) & f_x(x_0,y_2) & f_{xy}(x_0,y_0) & f_{xy}(x_0,y_1) & f_{xy}(x_0,y_2) \\
f_x(x_1,y_0) & f_x(x_1,y_1) & f_x(x_1,y_2) & f_{xy}(x_1,y_0) & f_{xy}(x_1,y_1) & f_{xy}(x_1,y_2) \\
f_x(x_2,y_0) & f_x(x_2,y_1) & f_x(x_2,y_2) & f_{xy}(x_2,y_0) & f_{xy}(x_2,y_1) & f_{xy}(x_2,y_2)
\end{array}\right).
\end{aligned}$$

Using the notation (0.24), we obtain for $(x,y) \in [x_0,x_2] \times [y_0,y_2]$ by (0.8) that

$$(P_{j+1}f)(x,y) = \Phi(x)^T\, \mathbf{F}\, \Phi(y)$$

and further by (0.23) that

$$\begin{aligned}
(P_{j+1}f)(x,y) &= \Psi(x)^T \begin{pmatrix} \mathbf{G} \\ \mathbf{H}_{\lambda_x} \end{pmatrix} \mathbf{F}\,(\mathbf{G}^T, \mathbf{H}^T_{\lambda_y})\, \Psi(y) \\
&= \Psi(x)^T \left(\begin{array}{c|c} \mathbf{G}\,\mathbf{F}\,\mathbf{G}^T & \mathbf{G}\,\mathbf{F}\,\mathbf{H}^T_{\lambda_y} \\ \hline \mathbf{H}_{\lambda_x}\,\mathbf{F}\,\mathbf{G}^T & \mathbf{H}_{\lambda_x}\,\mathbf{F}\,\mathbf{H}^T_{\lambda_y} \end{array}\right) \Psi(y). \qquad (0.29)
\end{aligned}$$

On the other hand, we have by (0.26) that

$$(P_{j+1}f)(x,y) = \Psi(x)^T \left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right) \Psi(y) \qquad (0.30)$$

with

$$\begin{aligned}
\mathbf{A} &:= \left(\begin{array}{cc|cc}
f(x_0,y_0) & f(x_0,y_2) & f_y(x_0,y_0) & f_y(x_0,y_2) \\
f(x_2,y_0) & f(x_2,y_2) & f_y(x_2,y_0) & f_y(x_2,y_2) \\ \hline
f_x(x_0,y_0) & f_x(x_0,y_2) & f_{xy}(x_0,y_0) & f_{xy}(x_0,y_2) \\
f_x(x_2,y_0) & f_x(x_2,y_2) & f_{xy}(x_2,y_0) & f_{xy}(x_2,y_2)
\end{array}\right), \\
\mathbf{B} &:= \begin{pmatrix}
\alpha^{j+1}_{(2k,2l+1)(0,0)} & \alpha^{j+1}_{(2k,2l+1)(0,1)} \\
\alpha^{j+1}_{(2k+2,2l+1)(0,0)} & \alpha^{j+1}_{(2k+2,2l+1)(0,1)} \\
\alpha^{j+1}_{(2k,2l+1)(1,0)} & \alpha^{j+1}_{(2k,2l+1)(1,1)} \\
\alpha^{j+1}_{(2k+2,2l+1)(1,0)} & \alpha^{j+1}_{(2k+2,2l+1)(1,1)}
\end{pmatrix} = \begin{pmatrix}
\alpha_{0,0} & \alpha_{0,1} \\
\cdot & \cdot \\
\alpha_{1,0} & \alpha_{1,1} \\
\cdot & \cdot
\end{pmatrix}, \\
\mathbf{D} &:= \begin{pmatrix} \beta_{0,0} & \beta_{0,1} \\ \beta_{1,0} & \beta_{1,1} \end{pmatrix}.
\end{aligned}$$

To avoid confusion, let $L_{kr} = L_{k,r}$. Comparing (0.29) and (0.30), we get by (0.21) and (0.22) that

$$
\begin{aligned}
\alpha_{0,0} = f(x_0,y_1) &- (\, f(x_0,y_0)L_{00}(\lambda_y) + f(x_0,y_2)L_{10}(\lambda_y)\,) \\
&- h\,(\, f_y(x_0,y_0)L_{01}(\lambda_y) + f_y(x_0,y_2)L_{11}(\lambda_y)\,)\,, \\
\alpha_{1,0} = f_x(x_0,y_1) &- (\, f_x(x_0,y_0)L_{00}(\lambda_y) + f_x(x_0,y_0)L_{10}(\lambda_y)\,) \\
&- h\,(\, f_{xy}(x_0,y_0)L_{01}(\lambda_y) + f_{xy}(x_0,y_2)L_{11}(\lambda_y)\,)\,, \\
\alpha_{0,1} = f_y(x_0,y_1) &- \frac{1}{h}(\, f(x_0,y_0)L'_{00}(\lambda_y) + f(x_0,y_2)L'_{10}(\lambda_y)\,) \\
&- (\, f_y(x_0,y_0)L'_{01}(\lambda_y) + f_y(x_0,y_2)L'_{11}(\lambda_y)\,)\,, \\
\alpha_{1,1} = f_{xy}(x_0,y_1) &- \frac{1}{h}(\, f_x(x_0,y_0)L'_{00}(\lambda_y) + f_x(x_0,y_2)L'_{10}(\lambda_y)\,) \\
&- (\, f_{xy}(x_0,y_0)L'_{01}(\lambda_y) + f_{xy}(x_0,y_2)L'_{11}(\lambda_y)\,)\,.
\end{aligned}
$$

and

$$
\begin{aligned}
\beta_{0,0} &= (L_{00},-1,L_{10})\mathbf{f}(L_{00},-1,L_{10})^T + h_x(L_{01},0,L_{11})\mathbf{f}_x(L_{00},-1,L_{10})^T \\
&\quad + h_y(L_{00},-1,L_{10})\mathbf{f}_y(L_{01},0,L_{11})^T + h_xh_y(L_{01},0,L_{11})\mathbf{f}_{xy}(L_{01},0,L_{11})^T, \\
\beta_{1,0} &= \frac{1}{h_x}(L'_{00},0,L'_{10})\mathbf{f}(L_{00},-1,L_{10})^T + (L'_{01},-1,L'_{11})\mathbf{f}_x(L_{00},-1,L_{10})^T \\
&\quad + \frac{h_y}{h_x}(L'_{00},0,L'_{10})\mathbf{f}_y(L_{01},0,L_{11})^T + h_y(L'_{01},-1,L'_{11})\mathbf{f}_{xy}(L_{01},0,L_{11})^T, \\
\beta_{1,1} &= \frac{1}{h_xh_y}(L'_{00},0,L'_{10})\mathbf{f}(L'_{00},0,L'_{10})^T + \frac{1}{h_y}(L'_{01},-1,L'_{11})\mathbf{f}_x(L'_{00},0,L'_{10})^T \\
&\quad + \frac{1}{h_x}(L'_{00},0,L'_{10})\mathbf{f}_y(L'_{01},-1,L'_{11})^T + (L'_{01},-1,L'_{11})\mathbf{f}_{xy}(L'_{01},-1,L'_{11})^T.
\end{aligned}
$$

The estimates for α follow as in the proof of Corollary 2.2.
Next we estimate the coefficients β. We use the short notation

$$L_{rs}\, f\, L_{kl} := L_{rs}(\lambda_x)\, f\, L_{kl}(\lambda_y).$$

i) *Estimation of* $\beta_{0,0}$

First, we see that

$$
\begin{aligned}
&(L_{00},-1,L_{10})\,\mathbf{f}\,(L_{00},-1,L_{10})^T + h_x\,(L_{01},0,L_{11})\,\mathbf{f}_x\,(L_{00},-1,L_{10})^T \\
&= (\, L_{00}f(x_0,y_0) - f(x_1,y_0) + L_{10}f(x_2,y_0) + h_x(L_{01}f_x(x_0,y_0) + L_{11}f_x(x_2,y_0))\,)\, L_{00} \\
&\;- (\, L_{00}f(x_0,y_1) - f(x_1,y_1) + L_{10}f(x_2,y_1) + h_x(L_{01}f_x(x_0,y_1) + L_{11}f_x(x_2,y_1))\,) \\
&\;+ (\, L_{00}f(x_0,y_2) - f(x_1,y_2) + L_{10}f(x_2,y_2) + h_x(L_{01}f_x(x_0,y_2) + L_{11}f_x(x_2,y_2))\,)\, L_{10} \\
&= g(y_0)L_{00} - g(y_1) + g(y_2)L_{10}
\end{aligned}
$$

with

$$g(y) := L_{00}f(x_0,y) - f(x_1,y) + L_{10}f(x_2,y) + h_x\,(L_{01}f_x(x_0,y) + L_{11}f_x(x_2,y)).$$

Thus, by (0.17)

$$\begin{array}{rll}(L_{00},-1,L_{10})\,\mathbf{f}\,(L_{00},-1,L_{10})^T + h_x\,(L_{01},0,L_{11})\,\mathbf{f}_x\,(L_{00},-1,L_{10})^T & & \\ = \quad h_y & (L_{00} f_y(x_0,\eta_1) - f_y(x_1,\eta_1) + L_{10} f_y(x_2,\eta_1))\, L_{00}\,(-\lambda_y) & \\ + \quad h_x h_y & (L_{01} f_{xy}(x_0,\eta_1) + L_{11} f_{xy}(x_2,\eta_1))\, L_{00}\,(-\lambda_y) & \\ + \quad h_y & (L_{00} f_y(x_0,\eta_2) - f_y(x_1,\eta_2) + L_{10} f_y(x_2,\eta_2))\, L_{10}\,(1-\lambda_y) & \\ + \quad h_x h_y & (L_{01} f_{xy}(x_0,\eta_2) + L_{11} f_{xy}(x_2,\eta_2))\, L_{10}\,(1-\lambda_y). & \end{array}$$

Further, we obtain by addition of $(L_{00},-1,L_{10})\,\mathbf{f}_y\,(L_{01},0,L_{11})^T$ to the above summands containing f_y

$$\begin{array}{ll} L_{00} & (-f_y(x_0,\eta_1)L_{00}\,\lambda_y - f_y(x_0,\eta_2)L_{10}\,(\lambda_y-1) + f_y(x_0,y_0)L_{01} + f_y(x_0,y_2)L_{11}) \\ - & (-f_y(x_1,\eta_1)L_{00}\,\lambda_y - f_y(x_1,\eta_2)L_{10}\,(\lambda_y-1) + f_y(x_1,y_0)L_{01} + f_y(x_1,y_2)L_{11}) \\ +\, L_{10} & (-f_y(x_2,\eta_1)L_{00}\,\lambda_y - f_y(x_2,\eta_2)L_{10}\,(\lambda_y-1) + f_y(x_2,y_0)L_{01} + f_y(x_2,y_2)L_{11}) \\ =\, h_x h_y\, (& \lambda_x L_{00}\,(\, f_{xy}(\xi_1,\eta_1)L_{00}\,\lambda_y + f_{xy}(\xi_1,\eta_2)L_{10}\,(\lambda_y-1) - f_{xy}(\xi_1,y_0)L_{01} \\ & -f_{xy}(\xi_1,y_2)L_{11}\ ) \\ & +\,(\lambda_x-1)L_{10}\,(\, f_{xy}(\xi_2,\eta_1)L_{00}\lambda_y + f_{xy}(\xi_2,\eta_2)L_{10}(\lambda_y-1) - f_{xy}(\xi_2,y_0)L_{01} \\ & -f_{xy}(\xi_2,y_2)L_{11})\ )\ )\,. \end{array}$$

In summary, we get

$$\begin{array}{rlll} \frac{\beta_{0,0}}{h_x h_y} & = & -L_{01} & (f_{xy}(x_0,\eta_1)L_{00}\lambda_y + f_{xy}(x_0,\eta_2)L_{10}(\lambda_y-1) - f_{xy}(x_0,y_0)L_{01} \\ & & & -f_{xy}(x_0,y_2)L_{11}) \\ & & -L_{11} & (f_{xy}(x_2,\eta_1)L_{00}\lambda_y + f_{xy}(x_2,\eta_2)L_{10}(\lambda_y-1) - f_{xy}(x_2,y_0)L_{01} \\ & & & -f_{xy}(x_2,y_2)L_{11}) \\ & & +\lambda_x L_{00} & (f_{xy}(\xi_1,\eta_1)L_{00}\lambda_y + f_{xy}(\xi_1,\eta_2)L_{10}(\lambda_y-1) - f_{xy}(\xi_1,y_0)L_{01} \\ & & & -f_{xy}(\xi_1,y_2)L_{11}) \\ & & +(\lambda_x-1)L_{10} & (f_{xy}(\xi_2,\eta_1)L_{00}\lambda_y + f_{xy}(\xi_2,\eta_2)L_{10}(\lambda_y-1) - f_{xy}(\xi_2,y_0)L_{01} \\ & & & -f_{xy}(\xi_2,y_2)L_{11}) \end{array}$$

and further by (0.19) with

$$\begin{array}{l} d_1(x) := f_{xy}(x,\eta_1) - f_{xy}(x,y_0)\,,\ d_2(x) := f_{xy}(x,\eta_2) - f_{xy}(x,y_2), \\ d_3(x) := f_{xy}(x,\eta_1) - f_{xy}(x,\eta_2) \end{array}$$

that

$$\begin{array}{rll} \frac{\beta_{0,0}}{h_x h_y} & = & -L_{01}\,(d_1(x_0)L_{01} + d_2(x_0)L_{11} + d_3(x_0)(\lambda_y L_{00} - L_{01})) \\ & & -L_{11}\,(d_1(x_2)L_{01} + d_2(x_2)L_{11} + d_3(x_2)(\lambda_y L_{00} - L_{01})) \\ & & +\lambda_x L_{00}\,(d_1(\xi_1)L_{01} + d_2(\xi_1)L_{11} + d_3(\xi_1)(\lambda_y L_{00} - L_{01})) \\ & & +(\lambda_x-1)L_{10}\,(d_1(\xi_2)L_{01} + d_2(\xi_2)L_{11} + d_3(\xi_2)(\lambda_y L_{00} - L_{01})) \end{array}$$

$$\begin{array}{rll} \frac{\beta_{0,0}}{h_x h_y} & = & L_{01}((d_1(\xi_1) - d_1(x_0))L_{01} + (d_2(\xi_1) - d_2(x_0))L_{11} \\ & & \quad +(d_3(\xi_1) - d_3(x_0))(\lambda_y L_{00} - L_{01})) \\ & & +L_{11}((d_1(\xi_2) - d_1(x_2))L_{01} + (d_2(\xi_2) - d_2(x_2))L_{11} \\ & & \quad +(d_3(\xi_2) - d_3(x_2))(\lambda_y L_{00} - L_{01})) \\ & & +(\lambda_x L_{00} - L_{01})((d_1(\xi_1) - d_1(\xi_2))L_{01} + (d_2(\xi_1) - d_2(\xi_2))L_{11} \\ & & \qquad +(d_3(\xi_1) - d_3(\xi_2))(\lambda_y L_{00} - L_{01})). \end{array}$$

Finally, this implies

$$|\beta_{0,0}| \le h_x h_y \, C_0^2 \, (\omega_x(f_{xy}, h_x) + \omega_y(f_{xy}, h_y)).$$

ii) *Estimation of* $\beta_{1,0}$

By (0.17), we obtain

$$\begin{aligned} &(L'_{00}, 0, L'_{10}) \, \mathbf{f} \, (L_{00}, -1, L_{10})^T \; + \; h_y \, (L'_{00}, 0, L'_{10}) \, \mathbf{f}_y \, (L_{01}, 0, L_{11})^T \\ &= \; L'_{00}(f(x_0,y_0)L_{00} - f(x_0,y_1) + f(x_0,y_2)L_{10} + h_y(f_y(x_0,y_0)L_{01} + f_y(x_0,y_2)L_{11})) \\ &\quad +L'_{10}(f(x_2,y_0)L_{00} - f(x_2,y_1) + f(x_2,y_2)L_{10} + h_y(f_y(x_2,y_0)L_{01} + f_y(x_2,y_2)L_{11})) \\ &= \; h_x L'_{10}(f_x(\xi,y_0)L_{00} - f_x(\xi,y_1) + f_x(\xi,y_2)L_{10} + h_y(f_{xy}(\xi,y_0)L_{01} + f_{xy}(\xi,y_2)L_{11})). \end{aligned}$$

Next, we have that

$$\begin{aligned} &L'_{10} f_x(\xi,y_0)L_{00} - L'_{10} f_x(\xi,y_1) + L'_{10} f_x(\xi,y_2)L_{10} \; + \; (L'_{01}, -1, L'_{11}) \, \mathbf{f}_x \, (L_{00}, -1, L_{10})^T \\ &= \; (L'_{10} f_x(\xi,y_0) + L'_{01} f_x(x_0,y_0) - f_x(x_1,y_0) + L'_{11} f_x(x_2,y_0))L_{00} \\ &\quad -(L'_{10} f_x(\xi,y_1) + L'_{01} f_x(x_0,y_1) - f_x(x_1,y_1) + L'_{11} f_x(x_2,y_1)) \\ &\quad +(L'_{10} f_x(\xi,y_2) + L'_{01} f_x(x_0,y_2) - f_x(x_1,y_2) + L'_{11} f_x(x_2,y_2))L_{10} \\ &= \; h_y(L'_{10} f_{xy}(\xi,\eta_1) + L'_{01} f_{xy}(x_0,\eta_1) - f_{xy}(x_1,\eta_1) + L'_{11} f_{xy}(x_2,\eta_1))L_{00}(-\lambda_y) \\ &\quad + h_y(L'_{10} f_{xy}(\xi,\eta_2) + L'_{01} f_{xy}(x_0,\eta_2) - f_{xy}(x_1,\eta_2) + L'_{11} f_{xy}(x_2,\eta_2))L_{10}(1-\lambda_y). \end{aligned}$$

In summary, we get

$$\begin{aligned} \frac{\beta_{1,0}}{h_y} \; &= \; (L'_{10} f_{xy}(\xi,\eta_1) + L'_{01} f_{xy}(x_0,\eta_1) - f_{xy}(x_1,\eta_1) + L'_{11} f_{xy}(x_2,\eta_1))L_{00}(-\lambda_y) \\ &\quad +(L'_{10} f_{xy}(\xi,\eta_2) + L'_{01} f_{xy}(x_0,\eta_2) - f_{xy}(x_1,\eta_2) + L'_{11} f_{xy}(x_2,\eta_2))L_{10}(1-\lambda_y) \\ &\quad +(L'_{10} f_{xy}(\xi,y_0) + L'_{01} f_{xy}(x_0,y_0) - f_{xy}(x_1,y_0) + L'_{11} f_{xy}(x_2,y_0))L_{01} \\ &\quad +(L'_{10} f_{xy}(\xi,y_2) + L'_{01} f_{xy}(x_0,y_2) - f_{xy}(x_1,y_2) + L'_{11} f_{xy}(x_2,y_2))L_{11} \end{aligned}$$

and finally

$$|\beta_{10}| \le h_y \, C_0 C_1 \, (\omega_x(f_{xy}, h_x) + \omega_y(f_{xy}, h_y)).$$

iii) *Estimation of* $\beta_{1,1}$

First, we have by (0.17) that

$$\begin{aligned} &\tfrac{1}{h_x} \, (L'_{00}, 0, L'_{10}) \, \mathbf{f} \, (L'_{00}, 0, L'_{10})^T \; + \; (L'_{01}, -1, L'_{11}) \, \mathbf{f}_x \, (L'_{00} \; 0 \; L'_{10})^T \\ &= \; (\tfrac{1}{h_x}(L'_{00} f(x_0,y_0) + L'_{10} f(x_2,y_0)) + L'_{01} f_x(x_0,y_0) - f_x(x_1,y_0) + L'_{11} f_x(x_2,y_0))L'_{00} \\ &\quad +(\tfrac{1}{h_x}(L'_{00} f(x_0,y_2) + L'_{10} f(x_2,y_2)) + L'_{01} f_x(x_0,y_2) - f_x(x_1,y_2) + L'_{11} f_x(x_2,y_2))L'_{10} \\ &= \; h_y(\tfrac{1}{h_x}(L'_{00} f_y(x_0,\eta) + L'_{10} f_y(x_2,\eta)) + L'_{01} f_{xy}(x_0,\eta) - f_{xy}(x_1,\eta) + L'_{11} f_{xy}(x_2,\eta))L'_{10} \end{aligned}$$

and further

$$\begin{aligned} &(L'_{00}, 0, L'_{10}) \, \mathbf{f}_y \, (L'_{01}, -1, L'_{11})^T \; + \; L'_{00} f_y(x_0,\eta) L'_{10} + L'_{10} f_y(x_2,\eta) L'_{10} \\ &= \; L'_{00}(f_y(x_0,\eta)L'_{10} + f_y(x_0,y_0)L'_{01} - f_y(x_0,y_1) + f_y(x_0,y_2)L'_{11}) \\ &\quad +L'_{10}(f_y(x_2,\eta)L'_{10} + f_y(x_2,y_0)L'_{01} - f_y(x_2,y_1) + f_y(x_2,y_2)L'_{11}) \\ &= \; h_x \, L'_{10}(f_{xy}(\xi,\eta)L'_{10} + f_{xy}(\xi,y_0)L'_{01} - f_{xy}(\xi,y_1) + f_{xy}(\xi,y_2)L'_{11}) \end{aligned}$$

In summary, we obtain

$$\begin{aligned}
\beta_{1,1} &= L'_{01} f_{xy}(x_0,\eta) L'_{10} - f_{xy}(x_1,\eta) L'_{10} + L'_{11} f_{xy}(x_2,\eta) L'_{10} \\
&\quad + L'_{10} f_{xy}(\xi, y_0) L'_{01} - L'_{10} f_{xy}(\xi, y_1) + L'_{10} f_{xy}(\xi, y_2) L'_{11} + L'_{10} f_{xy}(\xi,\eta) L'_{10} \\
&\quad + L'_{01} f_{xy}(x_0, y_0) L'_{01} + L'_{11} f_{xy}(x_2, y_0) L'_{01} + L'_{01} f_{xy}(x_0, y_2) L'_{11} + L'_{11} f_{xy}(x_2, y_2) L'_{11} \\
&\quad + f_{xy}(x_1, y_1) - L'_{01} f_{xy}(x_0, y_1) - L'_{11} f_{xy}(x_2, y_1) - f_{xy}(x_1, y_0) L'_{01} - f_{xy}(x_1, y_2) L'_{11}.
\end{aligned}$$

See Figure 1.

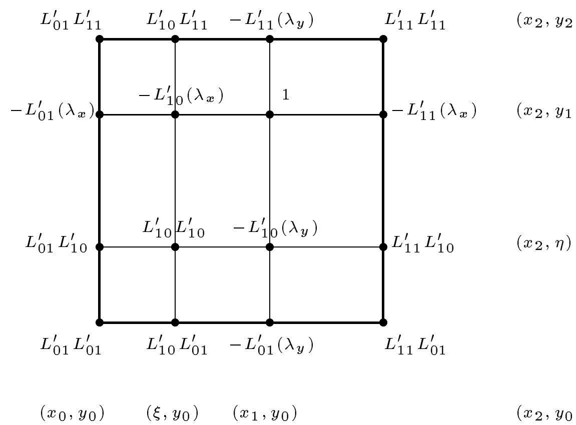

Figure 1. Coefficients of f_{xy} in $\beta_{1,1}$.

This can be rewritten as

$$\begin{aligned}
\beta_{1,1} &= L'_{01}((f_{xy}(x_0,y_0) - f_{xy}(x_0,y_1) - f_{xy}(x_1,y_0) + f_{xy}(x_1,y_1))L'_{01} \\
&\quad +(f_{xy}(x_0,\eta) - f_{xy}(x_0,y_1) - f_{xy}(x_1,\eta) + f_{xy}(x_1,y_1))L'_{10} \\
&\quad +(f_{xy}(x_0,y_2) - f_{xy}(x_0,y_1) - f_{xy}(x_1,y_2) + f_{xy}(x_1,y_1))L'_{11}) \\
&\quad +L'_{10}((f_{xy}(\xi,y_0) - f_{xy}(\xi,y_1) - f_{xy}(x_1,y_0) + f_{xy}(x_1,y_1)))L'_{01} \\
&\quad +(f_{xy}(\xi,\eta) - f_{xy}(\xi,y_1) - f_{xy}(x_1,\eta) + f_{xy}(x_1,y_1))L'_{10} \\
&\quad +(f_{xy}(\xi,y_2) - f_{xy}(\xi,y_1) - f_{xy}(x_1,y_2) + f_{xy}(x_1,y_1))L'_{11}) \\
&\quad +L'_{11}((f_{xy}(x_2,y_0) - f_{xy}(x_2,y_1) - f_{xy}(x_1,y_0) + f_{xy}(x_1,y_1))L'_{01} \\
&\quad +(f_{xy}(x_2,\eta) - f_{xy}(x_2,y_1) - f_{xy}(x_1,\eta) + f_{xy}(x_1,y_1))L'_{10} \\
&\quad +(f_{xy}(x_2,y_2) - f_{xy}(x_2,y_1) - f_{xy}(x_1,y_2) + f_{xy}(x_1,y_1))L'_{11}).
\end{aligned}$$

Thus,

$$|\beta_{1,1}| \leq C_1^2\,(\omega_y(f_{xy},h_y) + \omega_x(f_{xy},h_x)).$$

This finishes the proof. ∎

Now we consider the representation of $(P_{j+1} - P_j)f$ with respect to the basis B_M^{j+1}.

Theorem 3.2. *Let $f \in C^{1,+}[0,1]^2$. Let the projectors P_j be given by (0.25). Then the coefficients α of $(P_{j+1} - P_j)f$ in (0.26) with respect to the hierarchical basis B_M^{j+1}*

can be estimated as in Theorem 3.1 and the coefficients β by

$$\begin{aligned}
|\beta^{j+1}_{(2k+1,2l+1)(0,0)}| &\leq C_0\,(h_{xj}\omega_x(f_x,h_{xj})+h_{yj}\omega_y(f_y h_{yj})) \\
&\quad + C_0^2\,h_{xj}h_{yj}\,(\omega_x(f_{xy},h_{xj})+\omega_y(f_{xy}h_{yj})), \\
|\beta^{j+1}_{(2k+1,2l+1)(1,0)}| &\leq C_1\,\omega_x(f_x,h_{xj})+C_0C_1\,h_{yj}\,\omega_y(f_{xy},h_{yj}), \\
|\beta^{j+1}_{(2k+1,2l+1)(0,1)}| &\leq C_1\,\omega_y(f_y,h_{yj})+C_0C_1\,h_{xj}\,\omega_x(f_{xy},h_{xj}), \\
|\beta^{j+1}_{(2k+1,2l+1)(1,1)}| &\leq C_1^2\,(\omega_x(f_{xy},h_{xj})+\omega_y(f_{xy},h_{yj})).
\end{aligned}$$

The technical proof of Theorem 3.2 uses similar ideas as the proof of Theorem 3.1. Therefore we only refer to [8].
By (0.27), (0.28) and Theorem 3.2, we obtain

Corollary 3.3. *Let $f \in C^{1,+}[0,1]^2$. Let the projectors P_j be given by (0.25). Then the approximation error of the interpolatory projection can be estimated by*

$$\begin{aligned}
\|f-P_jf\|_\infty &\leq C_0\,(h_{xj}\,\omega_x(f_x,h_{xj})+h_{yj}\,\omega_y(f_y,h_{yj})) \\
&\quad + C_0^2\,h_{xj}h_{yj}\,(\omega_x(f_{xy},h_{xj})+\omega_y(f_{xy},h_{yj})) \\
\|f_x-P_jf_x\|_\infty &\leq \max\{C_0\,h_y\,\omega_y(f_{xy},h_{yj}),\ C_1\,\omega_x(f_x,h_{xj})\ +\ C_0C_1\,h_{yj}\,\omega_y(f_{xy},h_{yj})\} \\
\|f_y-P_jf_y\|_\infty &\leq \max\{C_0\,h_x\,\omega_x(f_{xy},h_{xj}),\ C_1\omega_y(f_y,h_{yj})\ +\ C_0C_1\,h_{xj}\,\omega_y(f_{xy},h_{xj})\} \\
\|f_{xy}-P_jf_{xy}\|_\infty &\leq \max\{C_1^2\,(\omega_x(f_{xy},h_x^j)+\omega_y(f_{xy},h_y^j)),\ C_1\,\omega_x(f_{xy},h_x^j),\ C_1\,\omega_y(f_{xy},h_y^j)\}
\end{aligned}$$

with $||f||_\infty := \max_{(x,y)\in[0,1]^2} |f(x,y)|$.

Remark. We briefly discuss the case when instead of a univariate or bivariate function f only data (i.e. the values of f at the grid points) are given.
In this case, we compute the derivatives of f approximatively as follows:
For data, given on an interval, we interpolate three consecutive data by a quadratic polynomial and replace the derivatives of f by the derivative of the polynomial at the data points. Then we apply our methods described in Section 2.
In the tensor product case, we consider constellations of nine neighboring data points forming a rectangle. The approximate values of f_x, f_y and f_{xy} at the data points can be computed by using univariate interpolation:
As above, the approximate values of f_x and f_y are obtained by univariate interpolation by quadratic polynomials at each set of three data points in the x-direction and the y-direction, respectively. Moreover, we interpolate the approximate values of f_x at each set of three data points in the y-direction by a univariate quadratic polynomial q. By the uniqueness of the tensor product polynomial Q which interpolates the nine data, the values q_y coincide with the values Q_{xy} at the three data points. We replace the values of f_{xy} at the data points by values of y-derivatives of these polynomials q and apply our methods described in Section 3.
The derivative of the polynomial pieces approximate the derivative of f at the data points with an error bounded by $\tilde{C}\omega_x(f_x;h_x)$, respectively $\tilde{C}\omega_y(f_y;h_y)$, where h_x, respectively h_y is the mesh size of the grid and the constant $\tilde{C}$ is the norm of sums of fundamental (Lagrange) functions. By combining the nine values of f in an appropriate

way and by using reproducing properties of the tensor product interpolation operator, we can see that the derivatives in y-direction of the polynomial pieces q approximate f_{xy} at the data points with an error bounded by $\tilde{C}(\omega_x(f_{xy}; h_x) + \omega_y(f_{xy}; h_y))$. This can be shown by applying similar arguments as in the above sections for Lagrange instead of Hermite interpolation. Therefore, we omit the details.

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