Multivariate Shearlet Transform, Shearlet Coorbit Spaces and their Structural Properties

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Abstract This chapter is devoted to the generalization of the continuous shearlet transform to higher dimensions as well as to the construction of associated smoothness spaces and to the analysis of their structural properties, respectively. To construct canonical scales of smoothness spaces , so-called shearlet coorbit spaces , and associated atomic decompositions and Banach frames we prove that the general coorbit space theory of Feichtinger and Gröchenig is applicable for the proposed shearlet setting. For the two-dimensional case we show that for large classes of weights, variants of Sobolev embeddings exist. Furthermore, we prove that for natural subclasses of shearlet coorbit spaces which in a certain sense correspond to 'cone-adapted shearlets' there exist embeddings into homogeneous Besov spaces . Moreover, the traces of the same subclasses onto the coordinate axis can again be identified with homogeneous Besov spaces. These results are based on the characterization of Besov spaces by atomic decompositions and rely on the fact that shearlets with compact support can serve as analyzing vectors for shearlet coorbit spaces . Finally, we demonstrate that the proposed multivariate shearlet transform can be used to characterize certain singularities.

Key words: Atomic decompositions, Banach frames, coorbit theory, embeddings, multivariate shearlet transform, singularity analysis, smoothness spaces, traces.

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1 Introduction

In the context of *directional* signal analysis and information retrieval several approaches have been suggested such as ridgelets [3] , curvelets [4], contourlets [17], shearlets [29] and many others. Among all these approaches, the shearlet transform stands out because it is related to group theory, i.e., this transform can be derived from a square-integrable representation $\pi : \mathbb{S} \to \mathscr{U}(L_2(\mathbb{R}^2))$ of a certain group $\mathbb{S}$, the so-called shearlet group , see [10]. An admissible function with respect to this group is called a *shearlet* . Therefore, in the context of the shearlet transform , all the powerful tools of group representation theory can be exploited.

For analyzing data in $\mathbb{R}^d$, $d \geq 3$, we have to generalize the shearlet transform to higher dimensions. The first step towards a higher-dimensional shearlet transform is the identification of a suitable shear matrix. Given an d-dimensional vector space V and a k-dimensional subspace W of V, a reasonable model reads as follows: the shear should fix the space W and translate all vectors parallel to W. That is, for $V = W \oplus W'$ and $v = w + w'$, the shear operation S can be described as $S(v) = w + (w' + s(w'))$ where s is a linear mapping from W' to W. Then, with respect to an appropriate basis of V, the shear operation S corresponds to a block matrix of the form

$$S = \begin{pmatrix} I_k & s^{\mathrm{T}} \\ 0 & I_{d-k} \end{pmatrix}, \qquad s \in \mathbb{R}^{d-k,k}.$$

Then we are faced with the problem how to choose the block s. Since we want to end up with a square integrable group representation, one has to be careful. Usually, the number of parameters has to fit together with the space dimension, for otherwise the resulting group would be either too large or too small. Since we have d degrees of freedom related with the translates and one degree of freedom related with the dilation, $d-1$ degrees of freedom for the shear component would be optimal. Therefore one natural choice would be $s \in \mathbb{R}^{d-1,1}$, i.e., $k = 1$. Indeed, we show that with this choice the associated multivariate shearlet transform can be interpreted as a square integrable group representation of a $(2d)$-parameter group, the full shearlet group. It is a remarkable fact that this choice is in some sense a canonical one, other $(d-1)$-parameter choices might lead to nice group structures, but the representation will usually not be square integrable. Another approach, which we do not discuss in this chapter, involves shear matrices of Toeplitz type. We refer the interested reader to [9, 14].

With a square integrable group representation at hand, there is a very natural link to another useful concept, namely the coorbit space theory introduced by Feichtinger and Gröchenig in a series of papers [18, 19, 20, 21, 24]. By means of the coorbit space theory, it is possible to derive in a very natural way scales of smoothness spaces associated with the group representation. In this setting, the smoothness of functions is measured by the decay of the associated voice transform. Moreover, by a tricky discretization of the representation, it is possible to obtain (Banach) frames for these smoothness spaces. Fortunately, it turns out that for our multivariate continuous shearlet transform , all the necessary conditions for the application of the coorbit space theory can be established, so that we end up with new canonical smoothness spaces, the multivariate shearlet coorbit spaces , together with their atomic decompositions and Banach frames for these spaces.

Once these new smoothness spaces are established some natural questions arise. How do these spaces really look like? Are there 'nice' sets of functions that are dense in these spaces? What are the relations to classical smoothness spaces such as Besov spaces? Do there exist embeddings into Besov spaces ? And do there exist generalized versions of Sobolev embedding theorems for shearlet coorbit spaces ? Moreover, can the associated trace spaces be identified? We shall provide some first answers to these questions. We concentrate on the two-dimensional case where we show that for natural subclasses of shearlet coorbit spaces which correspond to 'shearlets on the cone', there exist embeddings into homogeneous Besov spaces and that for the same subclasses, the traces onto the coordinate axis can again be identified with homogeneous Besov spaces . The general d-dimensional scenario requires more sophisticated techniques than presented here and is the content of future work, see [8].

Finally, an interesting issue of the two-dimensional continuous shearlet transform is the fact that it can be used to analyze singularities. Indeed, as outlined in [32], see also [5] for curvelets , it turns out that the decay of the continuous shearlet transform exactly describes the location and orientation of certain singularities. By our approach these characterizations carry over to higher-dimensions.

2 Multivariate Continuous Shearlet Transform

In this section, we introduce the shearlet transform on $L_2(\mathbb{R}^d)$. This requires the generalization of the parabolic dilation matrix and of the shear matrix . We will start with a rather general definition of shearlet groups in Subsection 2.1 and then restrict ourself to those groups having square integrable representations in Subsection 2.2.

In the following, let I_d denote the (d,d)-identity matrix and 0_d, resp. 1_d the vectors with d entries 0, resp. 1.

2.1 Unitary Representations of the Shearlet Group

We define *dilation matrices* depending on one parameter $a \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ by

$$A_a := \operatorname{diag}(a_1(a), \ldots, a_d(a)),$$

where $a_1(a) := a$ and $a_j(a) = a^{\alpha_j}$ with $\alpha_j \in (0,1),\ j = 2, \ldots, d$. In order to have directional selectivity, the dilation factors at the diagonal of A_a should be chosen in an anisotropic way, i.e., $|a_k(a)|$, $k = 2, \ldots, d$ should increase less than linearly in a

as $a \to \infty$. Our favorite choice will be

$$A_a := \begin{pmatrix} a & 0_{d-1}^{\mathrm{T}} \\ 0_{d-1} & \operatorname{sgn}(a)|a|^{\frac{1}{d}} I_{d-1} \end{pmatrix}. \tag{1}$$

In Section 6, we will see that this choice leads to an increase of the shearlet transform at hyperplane singularities as $|a| \to 0$. Consequently, this enables us to detect special directional information. For fixed $k \in \{1, \ldots, d\}$, we define our *shear matrices* by

$$S_s = \begin{pmatrix} I_k & s^{\mathrm{T}} \\ 0_{d-k,k} & I_{d-k} \end{pmatrix}, \qquad s \in \mathbb{R}^{d-k,k}. \tag{2}$$

The shear matrices form a subgroup of $GL_d(\mathbb{R})$.

Remark 1. Shear matrices on $\mathbb{R}^d$ were also considered in [28], see also [34]. We want to show the relation of those matrices to our setting (2). The authors in [28] call $S \in \mathbb{R}^{d,d}$ a *general shear matrix* if

$$(I_d - S)^2 = 0_{d,d}. \tag{3}$$

Of course, our matrices in (2) fulfill this condition. Condition (3) is equivalent to the fact that S decomposes as

$$S = P^{-1} \operatorname{diag}(J_1, \ldots, J_r, 1_{d-2r}) P, \quad J_j := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad r \le d/2.$$

With $P := (p_1, \ldots, p_d)$ and $P^{-1} = (q_1, \ldots, q_d)^{\mathrm{T}}$ this can be written as

$$S = I_d + \sum_{j=1}^{r} q_{2j-1} p_{2j}^{\mathrm{T}}, \quad \text{with} \quad p_{2j}^{\mathrm{T}} q_{2i-1} = 0,\ i,j = 1, \ldots, r.$$

Matrices of the type $S_{qp} := I_d + q\,p^{\mathrm{T}}$ with $p^{\mathrm{T}} q = 0$ are called *elementary shear matrices*. The general shear matrices do not form a group. In particular, the product of two elementary shear matrices $S_{q_1 p_1}$ and $S_{q_2 p_2}$ is again a shear matrix if and only if the matrices commute which is the case if and only if $p_1^{\mathrm{T}} q_2 = p_2^{\mathrm{T}} q_1 = 0$. Then $S_{q_1 p_1} S_{q_2 p_2} = I_n + \sum_{j=1}^{2} q_j p_j^{\mathrm{T}}$ holds true. Hence we see that any general shear matrix is the product of elementary shear matrices . In [28] any subgroup of $GL_d(\mathbb{R})$ generated by *finitely* many pairwise commuting elementary matrices is called a shear group. A shear group is maximal if it is not a proper subgroup of any other shear group. It is not hard to show that maximal shear groups are those of the form

$$G := \left\{ I_d + \left(\sum_{i=1}^{k} c_i q_i \right) \left(\sum_{j=1}^{d-k} d_j p_j^{\mathrm{T}} \right) : c_i, d_j \in \mathbb{R} \right\}, \quad p_j^{\mathrm{T}} q_i = 0,$$

with linearly independent vectors q_i, $i = 1, \ldots, k$, resp., p_j, $j = 1, \ldots, k$. Let $\{\tilde{q}_i : i = 1, \ldots, k\}$ be the dual basis of $\{q_i : i = 1, \ldots, k\}$ in the linear space V spanned by these vectors and let $\{\tilde{p}_j : j = 1, \ldots, d-k\}$ be the dual basis of $\{p_j : j = 1, \ldots, d-k\}$

in $V^\perp$. Set $P := (q_1,\ldots,q_k,\tilde{p}_1,\ldots,\tilde{p}_{d-k})$ so that $P^{-1} = (\tilde{q}_1,\ldots,\tilde{q}_k,p_1,\ldots,p_{d-k})^{\mathrm{T}}$. Then we see that for all $S \in G$

$$P^{-1}SP = \begin{pmatrix} I_k & c\,d^{\mathrm{T}} \\ 0_{d-k,k} & I_{d-k} \end{pmatrix}, \quad c = (c_1,\ldots,c_k)^{\mathrm{T}}, d = (d_1,\ldots,d_{d-k})^{\mathrm{T}}.$$

In other words, up to a basis transform, the maximal shear groups G coincide with our block matrix groups in (2).

Note that admissible subgroups of the semidirect product of the Heisenberg group and the symplectic group were examined in [6]. Some important progress in the construction of multivariate directional systems has been achieved for the curvelet case in [2] and for surfacelets in [35].

For our shearlet transform we have to combine dilation matrices and shear matrices . Let $A_{a,1} := \operatorname{diag}(a_1,\ldots,a_k)$ and $A_{a,2} := \operatorname{diag}(a_{k+1},\ldots,a_d)$. We will use the relations

$$S_s^{-1} = \begin{pmatrix} I_k & -s^{\mathrm{T}} \\ 0_{d-k,k} & I_{d-k} \end{pmatrix} \quad \text{and} \quad S_s A_a S_{s'} A_{a'} = S_{s+A_{a,2}^{-1}s'A_{a,1}} A_{aa'}. \tag{4}$$

For the special setting in (1), the last relation simplifies to

$$S_s A_a S_{s'} A_{a'} = S_{s+|a|^{1-\frac{1}{d}}s'} A_{aa'}.$$

Lemma 1. *The set $\mathbb{R}^* \times \mathbb{R}^{d-k,k} \times \mathbb{R}^d$ endowed with the operation*

$$(a,s,t) \circ (a',s',t') = (aa', s + A_{a,2}^{-1} s' A_{a,1}, t + S_s A_a t')$$

is a locally compact group $\mathbb{S}$. The left and right Haar measures on $\mathbb{S}$ are given by

$$d\mu_l(a,s,t) = \frac{|\det A_{a,2}|^{k-1}}{|a||\det A_{a,1}|^{d-k+1}}\, da\,ds\,dt \quad \text{and} \quad d\mu_r(a,s,t) = \frac{1}{|a|}\, da\,ds\,dt.$$

Proof. By the left relation in (4) it follows that $e := (1, 0_{d-k,k}, 0_d)$ is the neutral element in $\mathbb{S}$ and that the inverse of $(a,s,t) \in \mathbb{R}^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d$ is given by

$$(a,s,t)^{-1} = \big(a^{-1}, -A_{a,2}\, s A_{a,1}^{-1}, -A_a^{-1} S_s^{-1} t\big).$$

By straightforward computation it can be checked that the multiplication is associative.

Further, we have for a function F on $\mathbb{S}$ that

$$\int_{\mathbb{S}} F\left((a',s',t') \circ (a,s,t)\right) d\mu_l(a,s,t)$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}^{k(d-k)}} \int_{\mathbb{R}^d} F(a'a, s' + A_{a',2}^{-1} s A_{a',1}, t' + S_{s'} A_{a'} t) \, d\mu_l(a,s,t).$$

By substituting $\tilde{t} := t' + S_{s'} A_{a'} t$, i.e., $d\tilde{t} = |\det A_{a'}| \, dt$ and $\tilde{s} := s' + A_{a',2}^{-1} s A_{a',1}$, i.e., $d\tilde{s} = |\det A_{a',1}|^{d-k} / |\det A_{a',2}|^k \, ds$ and $\tilde{a} := a'a$ this can be rewritten as

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^{k(d-k)}} \int_{\mathbb{R}} F(\tilde{a}, \tilde{s}, \tilde{t}) \, \frac{1}{|\det A_{a'}|} \, \frac{|\det A_{a',2}|^k}{|\det A_{a',1}|^{d-k}} \, \frac{1}{|a'|} \, \frac{|a'| |\det A_{a',1}|^{d-k+1}}{|\det A_{a',2}|^{k-1}} \, d\mu_l(\tilde{a}, \tilde{s}, \tilde{t})$$

so that $d\mu_l$ is indeed the left Haar measure on $\mathbb{S}$. Similarly we can verify that $d\mu_r$ is the right Haar measure on $\mathbb{S}$. □

In the following, we use only the left Haar measure and the abbreviation $d\mu = d\mu_l$. For $f \in L_2(\mathbb{R}^d)$ we define

$$\pi(a,s,t) f(x) = f_{a,s,t}(x) := |\det A_a|^{-\frac{1}{2}} f(A_a^{-1} S_s^{-1} (x - t)). \tag{5}$$

It is easy to check that $\pi : \mathbb{S} \to \mathscr{U}(L_2(\mathbb{R}^d))$ is a mapping from $\mathbb{S}$ into the group $\mathscr{U}(L_2(\mathbb{R}^d))$ of unitary operators on $L_2(\mathbb{R}^d)$. Recall that a *unitary representation* of a locally compact group G with the left Haar measure μ on a Hilbert space $\mathscr{H}$ is a homomorphism π from G into the group of unitary operators $\mathscr{U}(\mathscr{H})$ on $\mathscr{H}$ which is continuous with respect to the strong operator topology.

Lemma 2. *The mapping π defined by* (5) *is a unitary representation of* $\mathbb{S}$.

Proof. We verify that π is a homomorphism. Let $\psi \in L_2(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, and $(a,s,t), (a',s',t') \in \mathbb{S}$. Using (4) we obtain

$$\begin{aligned}
\pi(a,s,t)(\pi(a',s',t')\psi)(x) &= |\det A_a|^{-\frac{1}{2}} \left(\pi(a',s',t')\psi\right) (A_a^{-1} S_s^{-1} (x-t)) \\
&= |\det A_{aa'}|^{-\frac{1}{2}} \psi(A_{a'}^{-1} S_{s'}^{-1} (A_a^{-1} S_s^{-1} (x-t) - t')) \\
&= |\det A_{aa'}|^{-\frac{1}{2}} \psi(A_{a'}^{-1} S_{s'}^{-1} A_a^{-1} S_s^{-1} (x - (t + S_s A_a t'))) \\
&= |\det A_{aa'}|^{-\frac{1}{2}} \psi(A_{aa'}^{-1} S_{s+A_{a,2}^{-1} s' A_{a,1}}^{-1} (x - (t + S_s A_a t'))) \\
&= \pi((a,s,t) \circ (a',s',t')) \psi(x).
\end{aligned}$$

□

2.2 Square Integrable Representations of the Shearlet Group

A nontrivial function $\psi \in L_2(\mathbb{R}^d)$ is called *admissible* , if

$$\int_{\mathbb{S}} |\langle \psi, \pi(a,s,t)\psi \rangle|^2 d\mu(a,s,t) < \infty.$$

If π is irreducible and there exits at least one admissible function $\psi \in L_2(\mathbb{R}^d)$, then π is called *square integrable* . By the following remark we will only consider a special setting in the rest of this paper.

Remark 2. Assume that our shear matrix has the form (2) with $s^{\mathrm{T}} = (s_{ij})_{i,j=1}^{k,d-k} \in \mathbb{R}^{k,d-k}$. Let s contain N different entries (variables). We assume that $N \geq d-1$ since we have one dilation parameter and otherwise the group becomes too small. Then we obtain instead of (12)

$$\int_{\mathbb{S}} |\langle f, \psi_{a,s,t}\rangle|^2 \, d\mu(a,s,t) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^N} |\hat{f}(\omega)|^2 |\det A_a| \, |\hat{\psi}(A_a \begin{pmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 + s\tilde{\omega}_1 \end{pmatrix})|^2 \, d\mu(a,s,t) \tag{6}$$

where $\tilde{\omega}_1 := (\omega_1, \ldots, \omega_k)^{\mathrm{T}}$, $\tilde{\omega}_2 := (\omega_{k+1}, \ldots, \omega_d)^{\mathrm{T}}$ and the *Fourier transform* $\mathscr{F} f_{a,s,t} = \hat{f}_{a,s,t}$ of $f_{a,s,t}$ is given by

$$\begin{aligned} \hat{f}_{a,s,t}(\omega) &= \int_{\mathbb{R}^d} f_{a,s,t}(x) e^{-2\pi i \langle x, \omega\rangle} \, dx \\ &= |\det A_a|^{\frac{1}{2}} \, e^{-2\pi i \langle t, \omega\rangle} \, \hat{f}(A_a^{\mathrm{T}} S_s^{\mathrm{T}} \omega) \\ &= |\det A_a|^{\frac{1}{2}} \, e^{-2\pi i \langle t, \omega\rangle} \, \hat{f}\left(A_a \begin{pmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 + s\tilde{\omega}_1 \end{pmatrix}\right). \end{aligned} \tag{7}$$

Now we can use the following substitution procedure:

$$\xi_{k+1} := (\omega_{k+1} + s_{11}\omega_1 + \ldots + s_{1k}\omega_k), \tag{8}$$

i.e., $d\xi_{k+1} = |\omega_1| ds_{11}$ and with corresponding modifications if some of the s_{1j}, $j > 1$ are the same as s_{11}. Then we replace s_{11} in the other rows of $\tilde{\omega}_2 + s\tilde{\omega}_1$ where it appears by (8). Next we continue to substitute the second row if it contains an integration variable from s ($\neq s_{11}$). Continuing this substitution process up to the final row we have at the end replaced the lower $d-k$ values in $\hat{\psi}$ by $d-r$, $r \leq k$ variables $\xi_1 = \xi_{j_1}, \ldots, \xi_{j_{d-r}}$ and some functions depending only on $a, \omega, \xi_{j_1}, \ldots, \xi_{j_{d-r}}$. Consequently, the integrand depends only on these variables. However, we have to integrate over $a, \omega, \xi_{j_1}, \ldots, \xi_{j_{d-r}}$ and over the remaining $N-(d-r)$ variables from s. But then the integral in (6) becomes infinity unless $N = d-r$. Since $d-1 \leq N$ this implies $r = k = 1$, i.e., our choice of S_s with (9).

By Remark 2 we will deal only with shear matrices (2) with $k = 1$, i.e., with

$$S = \begin{pmatrix} 1 & s^{\mathrm{T}} \\ 0_{d-1} & I_{d-1} \end{pmatrix}, \qquad s \in \mathbb{R}^{d-1}. \tag{9}$$

and with dilation matrices of the form (1). Then we have that

$$d\mu(a,s,t) = \frac{1}{|a|^{d+1}} \, da \, ds \, dt.$$

Then the following result shows that the unitary representation π defined in (5) is square integrable .

Theorem 1. *A function $\psi \in L_2(\mathbb{R}^d)$ is admissible if and only if it fulfills the* admissibility condition

$$C_\psi := \int_{\mathbb{R}^d} \frac{|\hat{\psi}(\omega)|^2}{|\omega_1|^d} \, d\omega < \infty. \tag{10}$$

If ψ is admissible, then, for any $f \in L_2(\mathbb{R}^d)$, the following equality holds true:

$$\int_{\mathbb{S}} |\langle f, \psi_{a,s,t} \rangle|^2 \, d\mu(a,s,t) = C_\psi \, \|f\|^2_{L_2(\mathbb{R}^d)} \, . \tag{11}$$

In particular, the unitary representation π is irreducible and hence square integrable.

Proof. Employing the Plancherel theorem and (7), we obtain

$$\begin{aligned}
&\int_{\mathbb{S}} |\langle f, \psi_{a,s,t} \rangle|^2 \, d\mu(a,s,t) = \int_{\mathbb{S}} |f * \psi^*_{a,s,0}(t)|^2 \, dt \, ds \, \frac{da}{|a|^{d+1}} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 |\hat{\psi}^*{}_{a,s,0}(\omega)|^2 \, d\omega \, ds \, \frac{da}{|a|^{d+1}} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 |\det A_a| |\hat{\psi}(A_a^{\mathrm{T}} S_s^{\mathrm{T}} \omega)|^2 \, d\omega \, ds \, \frac{da}{|a|^{d+1}} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-1}} |\hat{f}(\omega)|^2 \frac{|\det A_{a,2}|}{|a|^d} |\hat{\psi} \begin{pmatrix} a\omega_1 \\ A_{a,2}(\tilde{\omega} + \omega_1 s) \end{pmatrix} |^2 \, ds \, d\omega \, da,
\end{aligned} \tag{12}$$

where $\psi^*_{a,s,0}(x) = \overline{\psi_{a,s,0}(-x)}$. Substituting $\tilde{\xi} := A_{a,2}(\tilde{\omega} + \omega_1 s)$, i.e., $|\det A_{a,2}| \, |\omega_1|^{d-1} \, ds = d\tilde{\xi}$, we obtain

$$\int_{\mathbb{S}} |\langle f, \psi_{a,s,t} \rangle|^2 \, d\mu = |a|^{-d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-1}} |\hat{f}(\omega)|^2 \, |\omega_1|^{-(d-1)} \, |\hat{\psi} \begin{pmatrix} a\omega_1 \\ \tilde{\xi} \end{pmatrix} |^2 \, d\tilde{\xi} \, d\omega \, da.$$

Next, we substitute $\xi_1 := a\omega_1$, i.e., $\omega_1 \, da = d\xi_1$ which results in

$$\begin{aligned}
\int_{\mathbb{S}} |\langle f, \psi_{a,s,t} \rangle|^2 \, d\mu &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-1}} |\hat{f}(\omega)|^2 \, \frac{|\omega_1|^d}{|\xi_1|^d |\omega_1|^d} \, |\hat{\psi} \begin{pmatrix} \xi_1 \\ \tilde{\xi} \end{pmatrix} |^2 \, d\tilde{\xi} \, d\omega \, d\xi_1 \\
&= C_\psi \, \|f\|^2_{L_2(\mathbb{R}^d)}.
\end{aligned}$$

Setting $f := \psi$, we see that ψ is admissible if and only if C_ψ is finite.

The irreducibility of π follows from (11) in the same way as in [11]. □

2.3 Continuous Shearlet Transform

A function $\psi \in L_2(\mathbb{R}^d)$ fulfilling the admissibility condition (10) is called a *continuous shearlet* , the transform $\mathcal{SH}_\psi : L_2(\mathbb{R}^d) \to L_2(\mathbb{S})$,

$$\mathcal{SH}_\psi f(a,s,t) := \langle f, \psi_{a,s,t} \rangle = (f * \psi^*_{a,s,0})(t),$$

continuous shearlet transform and $\mathbb{S}$ defined in Lemma 1 with (9) a *shearlet group* .

Remark 3. An example of a continuous shearlet can be constructed as follows: Let ψ_1 be an admissable wavelet with $\hat{\psi}_1 \in C^\infty(\mathbb{R})$ and supp $\hat{\psi}_1 \subseteq [-2,-\frac{1}{2}] \cup [\frac{1}{2},2]$, and let ψ_2 be such that $\hat{\psi}_2 \in C^\infty(\mathbb{R}^{d-1})$ and supp $\hat{\psi}_2 \subseteq [-1,1]^{d-1}$. Then the function $\psi \in L^2(\mathbb{R}^d)$ defined by

$$\hat{\psi}(\omega) = \hat{\psi}(\omega_1, \tilde{\omega}) = \hat{\psi}_1(\omega_1)\, \hat{\psi}_2\left(\frac{1}{\omega_1}\,\tilde{\omega}\right)$$

is a continuous shearlet. The support of $\hat{\psi}$ is depicted for $\omega_1 \geq 0$ in Fig. 1.

Remark 4. In [34] the authors consider admissible subgroups G of $GL_d(\mathbb{R})$, i.e., those subgroups for which the semidirect product with the translation group gives rise to a square integrable representation $\pi(g,t)f(x) = |\det g|^{-\frac{1}{2}} f(g^{-1}(x-t))$. Let $\triangle$ denotes the modular function on G, i.e., $d\mu_l(g) = \triangle(g)\, d\mu_r(g)$ and write $\triangle \equiv |\det|$ to mean that $\triangle(g) = |\det g|$ for all $g \in G$. Then [34] contains the following result:

i) If G is admissible, then $\triangle \not\equiv |\det|$ and $G_x^0 := \{g \in G : gx = x\}$ is compact for a.e. $x \in \mathbb{R}^d$.
ii) If $\triangle \not\equiv |\det|$ and for a.e. $x \in \mathbb{R}^d$ there exits $\varepsilon(x) > 0$ such that $G_x^\varepsilon : \{g \in G : |gx - x| \leq \varepsilon(x)\}$ is compact, then G is admissible.

Unfortunately, the above conditions ‘just fail’ to be a characterization of admissibility by the ”ε-gap” in the compactness condition. In our case we have that $\triangle \not\equiv |\det|$ since $|a|^{-d} \neq |a|\,|a|^{\alpha_2+\dots+\alpha_d}$ for $|a| \neq 1$. Further, $G_x^0 = (1, 0_{d-1})$ and $G_x^\varepsilon = \{(a,s) : |a| \in [1-\varepsilon_1, 1+\varepsilon_1],\ s_j \in [-\varepsilon_j, \varepsilon_j],\ j = 2,\dots,d\}$ for some small ε_j, so that the necessary condition i) and the sufficient condition ii) are fulfilled.

3 General Concept of Coorbit Space Theory

In this section, we want to briefly recall the basic facts concerning the coorbit theory as developed by Feichtinger and Gröchenig in a series of papers [18, 19, 20, 21]. This theory is based on square-integrable group representations and has the following important advantages:

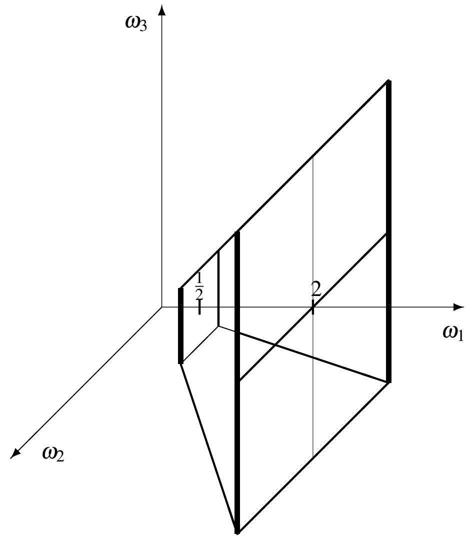

Fig. 1 Support of the shearlet $\hat{\psi}$ in Remark 3 for $\omega_1 \geq 0$.

- The theory is universal in the following sense: Given a Hilbert space $\mathscr{H}$, a square-integrable representation of a group G and a non-empty set of so-called analyzing functions , the whole abstract machinery can be applied.
- The approach provides us with natural families of smoothness spaces, the coorbit spaces . They are defined as the collection of all elements in the Hilbert space $\mathscr{H}$ for which the voice transform associated with the group representation has a certain decay. In many cases, e.g., for the affine group and the Weyl-Heisenberg group , these coorbit spaces coincide with classical smoothness spaces such as Besov and modulation spaces , respectively.
- The Feichtinger-Gröchenig theory does not only give rise to Hilbert frames in $\mathscr{H}$, but also to frames in scales of the associated coorbit spaces . Moreover, not only Hilbert spaces, but also Banach spaces can be handled.
- The discretization process that produces the frame does not take place in $\mathscr{H}$ (which might look ugly and complicated), but on the topological group at hand (which is usually a more handy object), and is transported to $\mathscr{H}$ by the group representation.

First of all, in Subsection 3.1, we explain how the coorbit spaces can be established. Then, in Subsection 3.2, we discuss the discretization problem, i.e., we outline the basic steps to construct Banach frames for these spaces. The facts are mainly taken from [24].

3.1 General Coorbit Spaces

Fix an irreducible, unitary, continuous representation π of a σ-compact group G in a Hilbert space $\mathscr{H}$. Let w be real-valued, continuous, sub-multiplicative weight on $\mathbb{S}$, i.e., $w(gh) \le w(g)w(h)$ for all $g,h \in \mathbb{S}$. Furthermore, we will always assume that the weight function w satisfies all the coorbit-theory conditions as stated in [24, Section 2.2]. To define our coorbit spaces we need the set

$$\mathscr{A}_w := \{\psi \in L_2(\mathbb{R}^d) : V_\psi(\psi) = \langle \psi, \pi(\cdot)\psi\rangle \in L_{1,w}\}.$$

of *analyzing vectors* . In particular, we assume that our weight is symmetric with respect to the modular function , i.e., $w(g) = w(g^{-1})\triangle(g^{-1})$. Starting with an ordinary weight function w, its symmetric version can be obtained by $w^{\#}(g) := w(g)+w(g^{-1})\triangle(g^{-1})$. It was proved in Lemma 2.4 of [18] that $\mathscr{A}_w = \mathscr{A}_{w^{\#}}$.

For an analyzing vector ψ we can consider the space

$$\mathscr{H}_{1,w} := \{f \in L_2(\mathbb{R}^d) : V_\psi(f) = \langle f, \pi(\cdot)\psi\rangle \in L_{1,w}(G)\},$$

with norm $\|f\|_{\mathscr{H}_{1,w}} := \|V_\psi(f)\|_{L_{1,w}(G)}$ and its anti-dual $\mathscr{H}_{1,w}^{\sim}$, the space of all continuous conjugate-linear functionals on $\mathscr{H}_{1,w}$. The spaces $\mathscr{H}_{1,w}$ and $\mathscr{H}_{1,w}^{\sim}$ are π-invariant Banach spaces with continuous embeddings $\mathscr{H}_{1,w} \hookrightarrow \mathscr{H} \hookrightarrow \mathscr{H}_{1,w}^{\sim}$. Then the inner product on $L_2(\mathbb{R}^d) \times L_2(\mathbb{R}^d)$ extends to a sesquilinear form on $\mathscr{H}_{1,w}^{\sim} \times \mathscr{H}_{1,w}$, therefore for $\psi \in \mathscr{H}_{1,w}$ and $f \in \mathscr{H}_{1,w}^{\sim}$ the *extended representation coefficients*

$$V_\psi(f)(g) := \langle f, \pi(g)\psi\rangle_{\mathscr{H}_{1,w}^{\sim} \times \mathscr{H}_{1,w}}$$

are well-defined.

Let m be a w-moderate weight on G which means that $m(xyz) \le w(x)m(y)w(z)$ for all $x,y,z \in G$ and moreover, for $1 \le p \le \infty$, let

$$L_{p,m}(G) := \{F \text{ measurable} : Fm \in L_p(G)\}.$$

We can now define Banach spaces which are called *coorbit spaces* by

$$\mathscr{H}_{p,m} := \{f \in \mathscr{H}_{1,w}^{\sim} : V_\psi(f) \in L_{p,m}(G)\}, \quad \|f\|_{\mathscr{H}_{p,m}} := \|V_\psi f\|_{L_{p,m}(G)}.$$

Note that the definition of $\mathscr{H}_{p,m}$ is independent of the analyzing vector ψ and of the weight w in the sense that $\tilde{w}$ with $w(g) \le C\tilde{w}(g)$ for all $g \in G$ and $\mathscr{A}_{\tilde{w}} \ne \{0\}$ give rise to the same space, see [18, Theorem 4.2]. In applications, one may start with some sub-multiplicative weight m and use the symmetric weight $w := m^{\#}$ for the definition of $\mathscr{A}_w$. Obviously, we have that m is w-moderate.

3.2 Atomic Decompositions and Banach Frames

The Feichtinger-Gröchenig theory also provides us with a machinery to construct atomic decompositions and Banach frames for the coorbit spaces introduced above. To this end, the subset $\mathscr{B}_w$ of $\mathscr{A}_w$ has to be non-empty:

$$\mathscr{B}_w := \{\psi \in L_2(\mathbb{R}^d) : V_\psi(\psi) \in \mathscr{W}(C_0, L_{1,w})\},$$

where $\mathscr{W}(C_0, L_{1,w})$ is the *Wiener-Amalgam space*

$$\mathscr{W}(C_0, L_{1,w}) := \{F : \|(L_x\chi_{\mathscr{Q}})F\|_\infty \in L_{1,w}\}, \quad \|(L_x\chi_{\mathscr{Q}})F\|_\infty = \sup_{y\in x\mathscr{Q}} |F(y)|$$

$L_x f(y) := f(x^{-1}y)$ is the *left translation* and $\mathscr{Q}$ is a relatively compact neighborhood of the identity element in G, see [24]. Note that in general $\mathscr{B}_w$ is defined with respect to the right version $\mathscr{W}^R(C_0, L_{1,w}) := \{F : \|(R_x\chi_Q)F\|_\infty = \sup_{y\in\mathscr{Q}x^{-1}} |F(y)| \in L_{1,w}\}$ of the Wiener-Amalgam space, where $R_x f(y) := f(yx)$ denotes the *right translation* . Since $V_\psi(\psi)(g) = V_\psi(\psi)(g^{-1})$ and assuming that $\mathscr{Q} = \mathscr{Q}^{-1}$ both definitions of $\mathscr{B}_w$ coincide. It follows that $\mathscr{B}_w \subset \mathscr{H}_{1,w}$.

Moreover, a (countable) family $X = (g_\lambda)_{\lambda\in\Lambda}$ in G is said to be *U-dense* if $\cup_{\lambda\in\Lambda} g_\lambda U = G$, and *separated* if for some compact neighborhood Q of e we have $g_i Q \cap g_j Q = \emptyset, i \neq j$, and *relatively separated* if X is a finite union of separated sets.

Then, the following decomposition theorem, which was proved in a general setting in [18, 19, 20], says that discretizing the representation by means of an U-dense set produces an atomic decomposition for $\mathscr{H}_{p,m}$. Furthermore, given such an atomic decomposition, the theorem provides conditions under which a function f is completely determined by its moments $\langle f, \pi(g_\lambda)\psi\rangle$ and how f can be reconstructed from these moments.

Theorem 2. *Let $1 \le p \le \infty$ and $\psi \in \mathscr{B}_w$, $\psi \neq 0$. Then there exists a (sufficiently small) neighborhood U of e so that for any U-dense and relatively separated set $X = (g_\lambda)_{\lambda\in\Lambda}$ the set $\{\pi(g_\lambda)\psi : \lambda \in \Lambda\}$ provides an atomic decomposition and a Banach frame for $\mathscr{H}_{p,m}$:*
Atomic Decompositions: *If $f \in \mathscr{H}_{p,m}$, then*

$$f = \sum_{\lambda\in\Lambda} c_\lambda(f)\pi(g_\lambda)\psi$$

where the sequence of coefficients depends linearly on f and satisfies

$$\|(c_\lambda(f))_{\lambda\in\Lambda}\|_{\ell_{p,m}} \le C\|f\|_{\mathscr{H}_{p,m}}$$

with a constant C depending only on ψ and with $\ell_{p,m}$ being defined by

$$\ell_{p,m} := \{c = (c_\lambda)_{\lambda\in\Lambda} : \|c\|_{\ell_{p,m}} := \|cm\|_{\ell_p} < \infty\},$$

where $m = (m(g_\lambda))_{\lambda\in\Lambda}$. Conversely, if $(c_\lambda(f))_{\lambda\in\Lambda} \in \ell_{p,m}$, then $f = \sum_{\lambda\in\Lambda} c_\lambda \pi(g_\lambda)\psi$ is in $\mathscr{H}_{p,m}$ and

$$\|f\|_{\mathscr{H}_{p,m}} \le C' \|(c_\lambda(f))_{\lambda\in\Lambda}\|_{\ell_{p,m}}.$$

Banach Frames: *The set* $\{\pi(g_\lambda)\psi : \lambda \in \Lambda\}$ *is a Banach frame for* $\mathscr{H}_{p,m}$ *which means that*

i) $f \in \mathscr{H}_{p,m}$ *if and only if* $(\langle f, \pi(g_\lambda)\psi\rangle_{\mathscr{H}_{1,m}^{\sim}\times\mathscr{H}_{1,m}})_{\lambda\in\Lambda} \in \ell_{p,m}$;
ii) *there exist two constants* $0 < D \le D' < \infty$ *such that*

$$D\|f\|_{\mathscr{H}_{p,m}} \le \|(\langle f, \pi(g_\lambda)\psi\rangle_{\mathscr{H}_{1,m}^{\sim}\times\mathscr{H}_{1,m}})_{\lambda\in\Lambda}\|_{\ell_{p,m}} \le D'\|f\|_{\mathscr{H}_{p,m}};$$

iii) *there exists a bounded, linear reconstruction operator* $\mathscr{R}$ *from* $\ell_{p,m}$ *to* $\mathscr{H}_{p,m}$ *such that* $\mathscr{R}\left((\langle f, \pi(g_\lambda)\psi\rangle_{\mathscr{H}_{1,m}^{\sim}\times\mathscr{H}_{1,m}})_{\lambda\in\Lambda}\right) = f$.

4 Multivariate Shearlet Coorbit Theory

In this section we want to establish the coorbit theory based on the square integrable representation (5) of the shearlet group $\mathbb{S}$ defined with (1) and (9). We mainly follow the lines of [11] and [12].

4.1 Shearlet Coorbit Spaces

We consider weight functions $w(a,s,t) = w(a,s)$ that are locally integrable with respect to a and s, i.e., $w \in L_1^{loc}(\mathbb{R}^d)$ and fulfill the requirements made at the beginning of Section 3.1.

In order to construct the coorbit spaces related to the shearlet group we have to ensure that there exists a function $\psi \in L_2(\mathbb{R}^d)$ such that

$$\mathscr{SH}_\psi(\psi) = \langle \psi, \pi(a,s,t)\psi\rangle \in L_{1,w}(\mathbb{S}). \tag{13}$$

Concerning the integrability of group representations we also mention [26]. To this end, we need a preliminary lemma on the support of ψ which is shown in [12].

Lemma 3. *Let* $a_1 > a_0 \ge \alpha > 0$ *and* $b = (b_1,\dots,b_{d-1})^T$ *be a vector with positive components. Suppose that* $\operatorname{supp}\hat\psi \subseteq ([-a_1,-a_0]\cup[a_0,a_1]) \times Q_b$, *where* $Q_b := [-b_1,b_1]\times\cdots\times[-b_{d-1},b_{d-1}]$. *Then* $\hat\psi\hat\psi_{a,s,0} \not\equiv 0$ *implies* $a \in \left[-\frac{a_1}{a_0}, -\frac{a_0}{a_1}\right] \cup \left[\frac{a_0}{a_1}, \frac{a_1}{a_0}\right]$ *and* $s \in Q_c$, *where* $c := \frac{1+(a_1/a_0)^{1/d}}{a_0}\, b$.

Now we can prove the required property (13) of $\mathscr{SH}_\psi(\psi)$.

Theorem 3. *Let* ψ *be a Schwartz function such that* $\operatorname{supp}\hat\psi \subseteq ([-a_1,-a_0]\cup[a_0,a_1]) \times Q_b$. *Then we have that* $\mathscr{SH}_\psi(\psi) \in L_{1,w}(\mathbb{S})$, *i.e.,*

$$\|\langle\psi, \pi(\cdot)\psi\rangle\|_{L_{1,w}(\mathbb{S})} = \int_{\mathbb{S}} |\mathscr{SH}_\psi(\psi)(a,s,t)|\, w(a,s,t)\, d\mu(a,s,t) < \infty.$$

Proof. Straightforward computation gives

$$\begin{aligned}
\|\langle\psi,\pi(\cdot)\psi\rangle\|_{L_{1,w}(\mathbb{S})} &= \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d} |\langle\psi,\psi_{a,s,t}\rangle|\, w(a,s)\, dt\, ds \frac{da}{|a|^{d+1}} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d} |\psi * \psi^*_{a,s,0}(t)|\, w(a,s)\, dt\, ds \frac{da}{|a|^{d+1}} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d} |\mathscr{F}^{-1}\mathscr{F}\left(\psi * \psi^*_{a,s,0}\right)(t)|\, dt\, w(a,s)\, ds \frac{da}{|a|^{d+1}} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}} \|\mathscr{F}\left(\psi * \psi^*_{a,s,0}\right)\|_{\mathscr{F}^{-1}L_1}\, w(a,s)\, ds \frac{da}{|a|^{d+1}} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}} \|\hat{\psi}\, \bar{\hat{\psi}}_{a,s,0}\|_{\mathscr{F}^{-1}L_1}\, w(a,s)\, ds \frac{da}{|a|^{d+1}},
\end{aligned}$$

where $\|f\|_{\mathscr{F}^{-1}L_1(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |\mathscr{F}^{-1}f(x)|\, dx$. By Lemma 3 this can be rewritten as

$$\|\langle\psi,\pi(\cdot)\psi\rangle\|_{L_{1,w}(\mathbb{S})} = \left(\int_{-a_1/a_0}^{-a_0/a_1} + \int_{a_0/a_1}^{a_1/a_0}\right)\int_{Q_c} \|\hat{\psi}\, \hat{\psi}^*_{a,s,0}\|_{\mathscr{F}^{-1}L_1(\mathbb{R}^d)}\, w(a,s)\, ds \frac{da}{|a|^{d+1}},$$

which is obviously finite. □

For ψ satisfying (13) we can therefore consider the space

$$\mathscr{H}_{1,w} = \{f \in L_2(\mathbb{R}^d) : \mathscr{SH}_\psi(f) \in L_{1,w}(\mathbb{S})\}$$

and its anti-dual $\mathscr{H}^\sim_{1,w}$. By the reasoning of Section 3.1, the *extended representation coefficients*

$$\mathscr{SH}_\psi(f)(a,s,t) = \langle f, \pi(a,s,t)\psi\rangle_{\mathscr{H}^\sim_{1,w}\times\mathscr{H}_{1,w}}$$

are well-defined and, for $1 \le p \le \infty$, we can define Banach spaces which we call *shearlet coorbit spaces* by

$$\mathscr{SC}_{p,m} := \{f \in \mathscr{H}^\sim_{1,w} : \mathscr{SH}_\psi(f) \in L_{p,m}(\mathbb{S})\}$$

with norms $\|f\|_{\mathscr{SC}_{p,m}} := \|\mathscr{SH}_\psi f\|_{L_{p,m}(\mathbb{S})}$.

4.2 Shearlet Atomic Decompositions and Shearlet Banach Frames

In a first step, we have to determine, for a compact neighborhood U of $e \in \mathbb{S}$ with non-void interior, U–dense sets, see Subsection 3.2.

Lemma 4. *Let U be a neighborhood of the identity in $\mathbb{S}$, and let $\alpha > 1$ and $\beta, \tau > 0$ be defined such that*

$$[\alpha^{\frac{1}{d}-1}, \alpha^{\frac{1}{d}}) \times [-\tfrac{\beta}{2}, \tfrac{\beta}{2})^{d-1} \times [-\tfrac{\tau}{2}, \tfrac{\tau}{2})^d \subseteq U.$$

Then the sequence

$$\{(\varepsilon\alpha^j, \beta\alpha^{j(1-\frac{1}{d})}k, S_{\beta\alpha^{j(1-\frac{1}{d})}k} A_{\alpha^j} \tau m) : j \in \mathbb{Z}, k \in \mathbb{Z}^{d-1}, m \in \mathbb{Z}^d, \varepsilon \in \{-1,1\}\} \quad (14)$$

is U-dense and relatively separated.

For a proof we refer to [12]. Then, as shown in [12], the following decomposition theorem says that discretizing the representation by means of this U-dense set produces an atomic decomposition for $\mathscr{SC}_{p,m}$.

Theorem 4. *Let $1 \le p \le \infty$ and $\psi \in \mathscr{B}_w$, $\psi \neq 0$. Then there exists a (sufficiently small) neighborhood U of e so that for any U-dense and relatively separated set $X = ((a,s,t)_\lambda)_{\lambda\in\Lambda}$ the set $\{\pi(g_\lambda)\psi : \lambda \in \Lambda\}$ provides an atomic decomposition and a Banach frame for $\mathscr{SC}_{p,m}$:*
Atomic Decompositions*: If $f \in \mathscr{SC}_{p,m}$, then*

$$f = \sum_{\lambda\in\Lambda} c_\lambda(f)\pi((a,s,t)_\lambda)\psi \quad (15)$$

where the sequence of coefficients depends linearly on f and satisfies

$$\|(c_\lambda(f))_{\lambda\in\Lambda}\|_{\ell_{p,m}} \le C\|f\|_{\mathscr{SC}_{p,m}}.$$

Conversely, if $(c_\lambda(f))_{\lambda\in\Lambda} \in \ell_{p,m}$, then
$f = \sum_{\lambda\in\Lambda} c_\lambda \pi((a,s,t)_\lambda)\psi$ is in $\mathscr{SC}_{p,m}$ and

$$\|f\|_{\mathscr{SC}_{p,m}} \le C'\|(c_\lambda(f))_{\lambda\in\Lambda}\|_{\ell_{p,m}}.$$

Banach Frames: *The set $\{\pi(g_\lambda)\psi : \lambda \in \Lambda\}$ is a Banach frame for $\mathscr{SC}_{p,m}$ which means that*

i) *$f \in \mathscr{SC}_{p,m}$ if and only if $(\langle f, \pi((a,s,t)_\lambda)\psi\rangle_{\mathscr{H}^{\sim}_{1,w}\times\mathscr{H}_{1,w}})_{\lambda\in\Lambda} \in \ell_{p,m}$;*
ii) *there exist two constants $0 < D \le D' < \infty$ such that*

$$D\|f\|_{\mathscr{SC}_{p,m}} \le \|(\langle f, \pi((a,s,t)_\lambda)\psi\rangle_{\mathscr{H}^{\sim}_{1,w}\times\mathscr{H}_{1,w}})_{\lambda\in\Lambda}\|_{\ell_{p,m}} \le D'\|f\|_{\mathscr{SC}_{p,m}};$$

iii) *there exists a bounded, linear reconstruction operator $\mathscr{R}$ from $\ell_{p,m}$ to $\mathscr{SC}_{p,m}$ such that $\mathscr{R}\Big((\langle f, \pi((a,s,t)_\lambda)\psi\rangle_{\mathscr{H}^{\sim}_{1,w}\times\mathscr{H}_{1,w}})_{\lambda\in\Lambda}\Big) = f$.*

4.3 Non-Linear Approximation

In Section 4.2 we established atomic decompositions of functions from the shearlet coorbit spaces $\mathscr{SC}_{p,m}$ by means of special discretized shearlet systems $(\psi_\lambda)_{\lambda\in\Lambda}$, $\Lambda \subset \mathbb{S}$. From the computational point of view, this naturally leads us to the question of the quality of approximation schemes in $\mathscr{SC}_{p,m}$ using only a finite number of elements from $(\psi_\lambda)_{\lambda\in\Lambda}$.

In this section we will focus on the non-linear approximation scheme of *best N–term approximation*, i.e., of approximating functions f of $\mathscr{SC}_{p,m}$ in an "optimal" way by a linear combination of precisely N elements from $(\psi_\lambda)_{\lambda\in\Lambda}$. In order to study the quality of best N–term approximation we will prove estimates for the asymptotic behavior of the approximation error.

Let us now delve more into the specific setting we are considering here. Let U be a properly chosen small neighborhood of e in $\mathbb{S}$. Further, let $\Lambda \subset \mathbb{S}$ be a relatively separated, U-dense sequence, which exists by Lemma 4. Then the associated shearlet system

$$\{\psi_\lambda = \psi_{a,s,t} : \lambda = (a,s,t) \in \Lambda\} \tag{16}$$

can be employed for atomic decompositions of elements from $\mathscr{SC}_{p,m}$, where $1 \le p < \infty$. Indeed, by Theorem 4, for any $f \in \mathscr{SC}_{p,m}$, we have

$$f = \sum_{\lambda\in\Lambda} c_\lambda \psi_\lambda$$

with $(c_\lambda)_{\lambda\in\Lambda}$ depending linearly on f, and

$$C_1 \|f\|_{\mathscr{SC}_{p,m}} \le \|(c_\lambda)_{\lambda\in\Lambda}\|_{\ell_{p,m}} \le C_2 \|f\|_{\mathscr{SC}_{p,m}}$$

with constants C_1, C_2 being independent of f. We intend to approximate functions f from the shearlet coorbit spaces $\mathscr{SC}_{p,m}$ by elements from the nonlinear manifolds Σ_n, $n \in \mathbb{N}$, which consist of all functions $S \in \mathscr{SC}_{p,m}$ whose expansions with respect to the shearlet system $(\psi_\lambda)_{\lambda\in\Lambda}$ from (16) have at most n nonzero coefficients, i.e.,

$$\Sigma_n := \left\{ S \in \mathscr{SC}_{p,m} : S = \sum_{\lambda\in\Gamma} d_\lambda \psi_\lambda, \Gamma \subseteq \Lambda, \#\Gamma \le n \right\}.$$

Then we are interested in the asymptotic behavior of the error

$$E_n(f)_{\mathscr{SC}_{p,m}} := \inf_{S\in\Sigma_n} \|f - S\|_{\mathscr{SC}_{p,m}}.$$

Usually, the order of approximation which can be achieved depends on the regularity of the approximated function as measured in some associated smoothness space. For instance, for nonlinear wavelet approximation, the order of convergence is determined by the regularity as measured in a specific scale of Besov spaces , see [15]. For nonlinear approximation based on Gabor frames, it has been shown in [27] that the 'right' smoothness spaces are given by a specific scale of modulation spaces. An extension of these relations to systems arising from the Weyl-Heisenberg group and α-modulation spaces has been studied in [7].

In our case it turns out that a result from [27], i.e., an estimate in one direction, carries over. The basic ingredient in the proof of the theorem is the following lemma which has been shown in [27], see also [16].

Lemma 5. *Let $0 < p < q \le \infty$. Then there exists a constant $D_p > 0$ independent of q such that, for all decreasing sequences of positive numbers $a = (a_i)_{i=1}^\infty$, we have*

$$2^{-1/p}\|a\|_{\ell_p} \leq \left(\sum_{n=1}^{\infty} \frac{1}{n} (n^{1/p-1/q} E_{n,q}(a))^p \right)^{1/p} \leq D_p \|a\|_{\ell_p},$$

where $E_{n,q}(a) := \left(\sum_{i=n}^{\infty} a_i^q\right)^{1/q}$.

The following theorem, which provides an upper estimate for the asymptotic behavior of $E_n(f)_{\mathscr{S}\mathscr{C}_{p,m}}$ was proved in [11].

Theorem 5. *Let* $(\psi_\lambda)_{\lambda\in\Lambda}$ *be a discrete shearlet system as in* (16)*, and let* $1 \leq p < q < \infty$*. Then there exists a constant* $C = C(p,q) < \infty$ *such that, for all* $f \in \mathscr{S}\mathscr{C}_{p,m}$*, we have*

$$\left(\sum_{n=1}^{\infty} \frac{1}{n} \left(n^{1/p-1/q} E_n(f)_{\mathscr{S}\mathscr{C}_{q,m}} \right)^p \right)^{1/p} \leq C \|f\|_{\mathscr{S}\mathscr{C}_{p,m}}.$$

5 Structure of Shearlet Coorbit Spaces

In this section we provide some first structural properties of the shearlet coorbit spaces . We use the notation $f \lesssim g$ for the relation $f \leq C\, g$ with some generic constant $C \geq 0$, and the notation '$\sim$' stands for equivalence up to constants which are independent of the involved parameters.

The subsequent analysis is limited to the *two-dimensional* case (more general concepts are provided in [8]), and we show that

- for large classes of weights, variants of Sobolev embeddings exist;
- for natural subclasses which in a certain sense correspond to the 'cone adapted shearlets' [32], there exist embeddings into (homogeneous) Besov spaces;
- for the same subclass, the traces onto the coordinate axis can again be identified with homogeneous Besov spaces .

The two-dimensional approach heavily relies on atomic decomposition techniques. We have seen that the coorbit space theory naturally gives rise to Banach frames , and therefore, by using the associated norm equivalences, all the tasks outlined above can be studied by means of weighted sequence norms. In particular, based on the general analysis in [30], quite recently this technique has been applied to derive new embedding and trace results for Besov spaces [36].

To make this approach really powerful, it is very convenient and sometimes even necessary to work with *compactly supported* building blocks. In the shearlet case, this is a nontrivial problem, since usually the analyzing shearlets are band-limited functions, see Theorem 3. For the specific case of 'cone adapted shearlets', quite recently a first solution has been provided in [31]. We refer to the overview article [33] for a detailed discussion. As the 'cone adapted shearlets' do not really fit into the group theoretical setting, we provide a new construction of families of compactly supported shearlets . We show that indeed a compactly supported function with sufficient smoothness and enough vanishing moments can serve as an analyzing vector for shearlet coorbit spaces , i.e., we show that $\mathscr{A}_w$ contains shearlets with compact support. To this end, we need the following auxiliary lemma which is a modification of Lemma 11.1.1 in [25].

Lemma 6. *For $r > 1$ and $\alpha > 0$, the following estimate holds true*

$$I(x) := \int_{\mathbb{R}} (1+|t|)^{-r} (1+\alpha|x-t|)^{-r} \, dt \le C \left(\frac{1}{\alpha}(1+|x|)^{-r} + (1+\alpha|x|)^{-r} \right).$$

Proof. Let

$$\mathscr{N}_x := \left\{ t \in \mathbb{R} : |t-x| \le \frac{|x|}{2} \right\}, \quad \mathscr{N}_x^c := \left\{ t \in \mathbb{R} : |t-x| > \frac{|x|}{2} \right\}.$$

Then we obtain for $t \in \mathscr{N}_x$ by $|x| - |t| \le |t-x| \le |x|/2$ that $|t| \ge |x|/2$ and consequently

$$(1+|t|)^{-r} \le \left(1 + \frac{|x|}{2}\right)^{-r} \le 2^r (1+|x|)^{-r}.$$

Now the above integral can be estimated as follows:

$$\begin{aligned} I(x) &= \int_{\mathscr{N}_x} (1+|t|)^{-r}(1+\alpha|x-t|)^{-r} \, dt + \int_{\mathscr{N}_x^c} (1+|t|)^{-r}(1+\alpha|x-t|)^{-r} \, dt \\ &\le 2^r (1+|x|)^{-r} \int_{\mathscr{N}_x} (1+\alpha|x-t|)^{-r} \, dt + \left(1 + \alpha \frac{|x|}{2}\right)^{-r} \int_{\mathscr{N}_x^c} (1+|t|)^{-r} \, dt \\ &\le 2^r \frac{1}{\alpha} (1+|x|)^{-r} \int_{\mathbb{R}} (1+|u|)^{-r} \, du + 2^r (1+\alpha|x|)^{-r} \int_{\mathbb{R}} (1+|t|)^{-r} \, dt. \end{aligned}$$

This implies the assertion. □

Theorem 6. *For some $D > 0$, let $Q_D := [-D,D] \times [-D,D]$ and let $\psi(x) \in L_2(\mathbb{R}^2)$ fulfill* $\operatorname{supp}\psi \in Q_D$. *Suppose that the weight function satisfies $w(a,s,t) = w(a) \le |a|^{-\rho_1} + |a|^{\rho_2}$ for $\rho_1, \rho_2 \ge 0$ and that*

$$|\hat{\psi}(\omega_1, \omega_2)| \lesssim \frac{|\omega_1|^n}{(1+|\omega_1|)^r} \frac{1}{(1+|\omega_2|)^r}$$

with $n \ge \max(\frac{1}{4} + \rho_2, \frac{9}{4} + \rho_1)$ and $r > n + \max(\frac{7}{4} + \rho_2, \frac{9}{4} + \rho_1)$. Then we have that $\mathscr{SH}_\psi(\psi) \in L_{1,w}(\mathbb{S})$, i.e.,

$$I := \int_{\mathbb{S}} |\mathscr{SH}_\psi(\psi)(g)| \, w(g) \, d\mu(g) < \infty.$$

Proof. First we have by the support property of ψ that $\mathscr{SH}_\psi(\psi) = \langle \psi, \psi_{a,s,t} \rangle \ne 0$ requires $(x_1, x_2) \in Q_D$ and

$$-D \le \frac{\operatorname{sgn} a}{\sqrt{|a|}}\,(x_2 - t_2) \;\le\; D,$$
$$-D \le \frac{1}{a}\,(x_1 - t_1 - s(x_2 - t_2)) \;\le\; D.$$

Hence $\langle \psi, \psi_{a,s,t} \rangle \neq 0$ implies that

$$-D(1+\sqrt{|a|}) \le t_2 \;\le\; D(1+\sqrt{|a|}),$$
$$-D\left(1+|a|+|s|(2+\sqrt{|a|})\right) \le t_1 \;\le\; D\left(1+|a|+|s|(2+\sqrt{|a|})\right).$$

Using this relation we obtain that

$$I \le \int_{\mathbb{R}^*}\int_{\mathbb{R}} 4D^2(1+\sqrt{|a|})\left(1+|a|+|s|(2+\sqrt{|a|})\right)|\langle \psi, \psi_{a,s,t}\rangle|\,ds\,w(a)\frac{da}{|a|^3}.$$

Next, Plancherel's equality together with (7) and the decay assumptions on $\hat{\psi}$ yield

$$I \le C\int_{\mathbb{R}^*}\int_{\mathbb{R}} (1+\sqrt{|a|})\left(1+|a|+|s|(2+\sqrt{|a|})\right)|\langle \hat{\psi}, \hat{\psi}_{a,s,t}\rangle|\,ds\,w(a)\frac{da}{|a|^3}$$
$$\le C\int_{\mathbb{R}^*}\int_{\mathbb{R}} \left(\underbrace{1+|a|^{\frac{1}{2}}+|a|+|a|^{\frac{3}{2}}}_{=:p_3(|a|^{\frac{1}{2}})} + |s|(\underbrace{2+3|a|^{\frac{1}{2}}+a}_{=:p_2(|a|^{\frac{1}{2}})})\right) J(a,s)\,ds\,w(a)\frac{da}{|a|^3}$$

where $p_k \in \Pi_k$ are polynomials of degree $\le k$, $|\mathscr{SH}_\psi \psi(a,s,t)| \le J(a,s)$ and

$$J(a,s) := |a|^{\frac{3}{4}}$$
$$\times \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\omega_1|^n}{(1+|\omega_1|)^r}\frac{1}{(1+|\omega_2|)^r}\frac{|a\omega_1|^n}{(1+|a\omega_1|)^r}\frac{1}{(1+\sqrt{|a|}\,|s\omega_1+\omega_2|)^r}\,d\omega_2 d\omega_1$$
$$= \int_{\mathbb{R}} \frac{|\omega_1|^n}{(1+|\omega_1|)^r}\frac{|a\omega_1|^n}{(1+|a\omega_1|)^r}\int_{\mathbb{R}}\frac{1}{(1+|\omega_2|)^r}\frac{1}{(1+\sqrt{|a|}\,|s\omega_1+\omega_2|)^r}\,d\omega_2 d\omega_1.$$

The inner integral can be estimated by Lemma 6 which results in

$$J(a,s) \le C\,|a|^{n+\frac{3}{4}}$$
$$\times \int_{\mathbb{R}} \frac{|\omega_1|^n}{(1+|\omega_1|)^r}\frac{|\omega_1|^n}{(1+|a\omega_1|)^r}\left(\frac{1}{\sqrt{|a|}\,(1+|s\omega_1|)^r}+\frac{1}{(1+\sqrt{|a|}\,|s\omega_1|)^r}\right)d\omega_1.$$

Now we obtain

$$I \le C\left(\int_{\mathbb{R}^*}\int_{\mathbb{R}}\int_{\mathbb{R}} |a|^{n-\frac{11}{4}}\,(p_3+|s|p_2)\,\frac{|\omega_1|^{2n}}{(1+|\omega_1|)^r(1+|a\omega_1|)^r}\,\frac{w(a)}{(1+|s\omega_1|)^r}\,ds\,d\omega_1\,da\right.$$
$$\left.+\int_{\mathbb{R}^*}\int_{\mathbb{R}}\int_{\mathbb{R}} |a|^{n-\frac{9}{4}}\,(p_3+|s|p_2)\,\frac{|\omega_1|^{2n}}{(1+|\omega_1|)^r(1+|a\omega_1|)^r}\,\frac{w(a)}{(1+\sqrt{|a|}\,|s\omega_1|)^r}\,ds\,d\omega_1\,da\right).$$

Since the integrand is even in ω_1, s and a this can be further simplified as

$$I \le C\left(\int_0^\infty a^{n-\frac{11}{4}}\,p_3(\sqrt{a})\int_0^\infty \frac{\omega_1^{2n}}{(1+\omega_1)^r(1+a\omega_1)^r}\int_0^\infty \frac{w(a)}{(1+s\omega_1)^r}\,ds\,d\omega_1\,da\right.$$
$$+\int_0^\infty a^{n-\frac{11}{4}}\,p_2(\sqrt{a})\int_0^\infty \frac{\omega_1^{2n}}{(1+\omega_1)^r(1+a\omega_1)^r}\int_0^\infty \frac{w(a)s}{(1+s\omega_1)^r}\,ds\,d\omega_1\,da$$
$$+\int_0^\infty a^{n-\frac{9}{4}}\,p_3(\sqrt{a})\int_0^\infty \frac{\omega_1^{2n}}{(1+\omega_1)^r(1+a\omega_1)^r}\int_0^\infty \frac{w(a)}{(1+\sqrt{a}s\omega_1)^r}\,ds\,d\omega_1\,da$$
$$\left.+\int_0^\infty a^{n-\frac{9}{4}}\,p_2(\sqrt{a})\int_0^\infty \frac{\omega_1^{2n}}{(1+\omega_1)^r(1+a\omega_1)^r}\int_0^\infty \frac{w(a)s}{(1+\sqrt{a}s\omega_1)^r}\,ds\,d\omega_1\,da\right).$$

Substituting $t := s\omega_1$ with $dt = \omega_1\,ds$ in the first two integrals and $t := \sqrt{a}s\omega_1$ with $dt = \sqrt{a}\omega_1\,ds$ in the last two integrals, we obtain for $r > 2$ that

$$I \le C\left(\int_0^\infty \frac{\omega_1^{2n-1}}{(1+\omega_1)^r}\int_0^\infty a^{n-\frac{11}{4}}\,p_3(\sqrt{a})\frac{w(a)}{(1+a\omega_1)^r}\,da\,d\omega_1\right.$$
$$+\int_0^\infty \frac{\omega_1^{2n-2}}{(1+\omega_1)^r}\int_0^\infty a^{n-\frac{11}{4}}\,p_2(\sqrt{a})\frac{w(a)}{(1+a\omega_1)^r}\,da\,d\omega_1$$
$$+\int_0^\infty \frac{\omega_1^{2n-1}}{(1+\omega_1)^r}\int_0^\infty a^{n-\frac{11}{4}}\,p_3(\sqrt{a})\frac{w(a)}{(1+a\omega_1)^r}\,da\,d\omega_1$$
$$\left.+\int_0^\infty \frac{\omega_1^{2n-2}}{(1+\omega_1)^r}\int_0^\infty a^{n-\frac{13}{4}}\,p_2(\sqrt{a})\frac{w(a)}{(1+a\omega_1)^r}\,da\,d\omega_1\right).$$

Substituting $b := a\omega_1$ with $db = \omega_1 da$ and bounding w accordingly we conclude further that

$$
\begin{aligned}
I \le C \Bigg(& \int_0^\infty \frac{\omega_1^{n+\frac{3}{4}+\rho_1}}{(1+\omega_1)^r} \int_0^\infty p_3\left(\sqrt{\frac{b}{\omega_1}}\right) \frac{b^{n-\frac{11}{4}-\rho_1}}{(1+b)^r}\, db\, d\omega_1 \\
&+ \int_0^\infty \frac{\omega_1^{n-\frac{1}{4}+\rho_1}}{(1+\omega_1)^r} \int_0^\infty p_2\left(\sqrt{\frac{b}{\omega_1}}\right) \frac{b^{n-\frac{11}{4}-\rho_1}}{(1+b)^r}\, db\, d\omega_1 \\
&+ \int_0^\infty \frac{\omega_1^{n+\frac{1}{4}+\rho_1}}{(1+\omega_1)^r} \int_0^\infty p_2\left(\sqrt{\frac{b}{\omega_1}}\right) \frac{b^{n-\frac{13}{4}-\rho_1}}{(1+b)^r}\, db\, d\omega_1 \\
&+ \int_0^\infty \frac{\omega_1^{n+\frac{3}{4}-\rho_2}}{(1+\omega_1)^r} \int_0^\infty p_3\left(\sqrt{\frac{b}{\omega_1}}\right) \frac{b^{n-\frac{11}{4}+\rho_2}}{(1+b)^r}\, db\, d\omega_1 \\
&+ \int_0^\infty \frac{\omega_1^{n-\frac{1}{4}-\rho_2}}{(1+\omega_1)^r} \int_0^\infty p_2\left(\sqrt{\frac{b}{\omega_1}}\right) \frac{b^{n-\frac{11}{4}+\rho_2}}{(1+b)^r}\, db\, d\omega_1 \\
&+ \int_0^\infty \frac{\omega_1^{n+\frac{1}{4}-\rho_2}}{(1+\omega_1)^r} \int_0^\infty p_2\left(\sqrt{\frac{b}{\omega_1}}\right) \frac{b^{n-\frac{13}{4}+\rho_2}}{(1+b)^r}\, db\, d\omega_1 \Bigg).
\end{aligned}
$$

Since $p_k \in \Pi_k$, $k = 2,3$ we see that the integrals are finite if $n \ge \max(\frac{1}{4}+\rho_2, \frac{9}{4}+\rho_1)$ and $r > n + \max(\frac{7}{4}+\rho_2, \frac{9}{4}+\rho_1)$. This finishes the proof. □

By the help of the following corollary which was proved in [13] we additionally establish $\psi \in \mathscr{B}_w$ and therewith the existence of atomic decompositions and Banach frames for $\mathscr{S}\mathscr{C}_{p,m}$.

Corollary 1. *Let* $\psi(x) \in L_2(\mathbb{R}^2)$ *fulfill* $\operatorname{supp}\psi \in Q_D$. *Suppose that the weight function satisfies* $w(a,s,t) = w(a) \le |a|^{-\rho_1} + |a|^{\rho_2}$ *for* $\rho_1, \rho_2 \ge 0$ *and that*

$$
|\hat{\psi}(\omega_1, \omega_2)| \lesssim \frac{|\omega_1|^n}{(1+|\omega_1|)^r} \frac{1}{(1+|\omega_2|)^r}
$$

for sufficiently large n and r. Then we have that $\psi \in \mathscr{B}_w$.

5.1 Atomic Decomposition of Besov Spaces

Let us recall the characterization of homogeneous Besov spaces $B^\sigma_{p,q}$ from [23], see also [30, 37]. For inhomogeneous Besov spaces we refer to [36]. For $\alpha > 1$, $D > 1$ and $K \in \mathbb{N}_0$, a K times differentiable function a on $\mathbb{R}^d$ is called a *K-atom* if the following two conditions are fulfilled:

A1) $\operatorname{supp} a \subset DQ_{j,m}(\mathbb{R}^d)$ for some $m \in \mathbb{R}^d$, where $Q_{j,m}(\mathbb{R}^d)$ denotes the cube in $\mathbb{R}^d$ centered at $\alpha^{-j}m$ with sides parallel to the coordinate axes and side length $2\alpha^{-j}$.
A2) $|D^\gamma a(x)| \le \alpha^{|\gamma| j}$ for $|\gamma| \le K$.

Now the homogeneous Besov spaces can be characterized as follows.

Theorem 7. *Let $D > 1$, $\sigma > 0$ and $K \in \mathbb{N}_0$ with $K \geq 1 + \lfloor \sigma \rfloor$ be fixed. Let $1 \leq p \leq \infty$. Then $f \in B^{\sigma}_{p,q}$ if and only if it can be represented* [1] *as*

$$f(x) = \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^d} \lambda(j,l) a_{j,l}(x), \tag{17}$$

where the $a_{j,l}$ are K-atoms with $\operatorname{supp} a_{j,l} \subset DQ_{j,l}(\mathbb{R}^d)$ *and*

$$\|f\|_{B^{\sigma}_{p,q}} \sim \inf\Big(\sum_{j \in \mathbb{Z}} \alpha^{j(\sigma - \frac{d}{p})q} \big(\sum_{l \in \mathbb{Z}^d} |\lambda(j,l)|^p \big)^{\frac{q}{p}} \Big)^{\frac{1}{q}}$$

where the infimum is taken over all admissible representations (17).

In this section, we are mainly interested in weights

$$m(a,s,t) = m(a) := |a|^{-r}, r \geq 0$$

and use the abbreviation

$$\mathscr{SC}_{p,r} := \mathscr{SC}_{p,m}.$$

For simplicity, we further assume in the following that we can use $\beta = \tau = 1$ in the U-dense, relatively separated set (14) and restrict ourselves to the case $\varepsilon = 1$. In other words, we assume that $f \in \mathscr{SC}_{p,r}$ can be written as

$$\begin{aligned} f(x) &= \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^2} c(j,k,l) \pi(\alpha^{-j}, \beta\alpha^{-j/2}k, S_{\alpha^{-j/2}k} A_{\alpha^{-j}} l) \psi(x) \\ &= \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^2} c(j,k,l) \alpha^{\frac{3}{4}j} \psi(\alpha^j x_1 - \alpha^{j/2} k x_2 - l_1, \alpha^{j/2} x_2 - l_2). \end{aligned} \tag{18}$$

To derive reasonable trace and embedding theorems, it is necessary to introduce the following subspaces of $\mathscr{SC}_{p,r}$. For fixed $\psi \in B_w$ we denote by $\mathscr{SCC}_{p,r}$ the closed subspace of $\mathscr{SC}_{p,r}$ consisting of those functions which are representable as in (18) but with integers $|k| \leq \alpha^{j/2}$. As we shall see in the sequel for each of these ψ the resulting spaces $\mathscr{SCC}_{p,r}$ embed in the same scale of Besov spaces, and the same holds true for the trace theorems.

5.2 A Density Result

In most of the classical smoothness spaces like Sobolev and Besov spaces dense subsets of ‘nice’ functions can be identified. Typically, the set of Schwartz functions S serves as such a dense subset. We refer to [1] and any book of Hans Triebel for further information. By the following theorem the same is true for our shearlet coorbit spaces.

[1] In the sense of distributions, a-posteriori implying norm convergence for $p < \infty$.

Theorem 8. *Let*

$$\mathrm{S}_0 := \left\{ f \in \mathrm{S} : |\hat{f}(\omega)| \le \frac{\omega_1^{2\alpha}}{(1+\|\omega\|^2)^{2\alpha}} \ \forall\, \alpha > 0 \right\}$$

and $m(a,s,t) = m(a,s) := |a|^r(\frac{1}{|a|} + |a| + |s|)^n$ *for some* $r \in \mathbb{R}, n \ge 0$. *Then the set of Schwartz functions forms a dense subset of the shearlet coorbit space* $\mathscr{S}\mathscr{C}_{p,m}$.

Proof. As in [11, Theorem 4.7] it can be shown that S_0 is at least contained in $\mathscr{S}\mathscr{C}_{p,m}$. (Note that in [11] the weight $(\frac{1}{|a|} + |a|)^r(\frac{1}{|a|} + |a| + |s|)^n$, $r,n > 0$ which is not smaller than 1 was considered.) It remains to show the density. To this end, we observe from Theorem 3 that certain band-limited Schwartz functions can be used as analyzing shearlets. Now let us recall that the atomic decomposition in (15) has to be understood as a limit of *finite* linear combinations with respect to the shearlet coorbit norm. However, every finite linear combination of Schwartz functions is again a Schwartz function, hence (15) implies that we have found for any $f \in \mathscr{S}\mathscr{C}_{p,m}$ a sequence of Schwartz functions which converges to f. □

5.3 Traces on the Real Axes

In this subsection which is based on [13], we investigate the traces of functions lying in certain subspaces of $\mathscr{S}\mathscr{C}_{p,r}$ with respect to the horizontal and vertical axes, respectively. With larger technical effort it is also possible to prove trace theorems with respect to more general lines.

Theorem 9. *Let* $Tr_h f$ *denote the restriction of* f *to the (horizontal)* x_1*-axis, i.e.,* $(Tr_h f)(x_1) := f(x_1, 0)$. *Then* $Tr_h(\mathscr{S}\mathscr{C}\mathscr{C}_{p,r}) \subset B^{\sigma_1}_{p,p}(\mathbb{R}) + B^{\sigma_2}_{p,p}(\mathbb{R})$*, where*

$$B^{\sigma_1}_{p,p}(\mathbb{R}) + B^{\sigma_2}_{p,p}(\mathbb{R}) := \{h \mid h = h_1 + h_2, h_1 \in B^{\sigma_1}_{p,p}(\mathbb{R}), h_2 \in B^{\sigma_2}_{p,p}(\mathbb{R})\}$$

and the parameters σ_1 *and* σ_2 *satisfy the conditions*

$$\sigma_1 = r - \frac{5}{4} + \frac{3}{2p}, \quad \sigma_2 = r - \frac{3}{4} + \frac{1}{p}.$$

Note that $\sigma_1 \le \sigma_2$ for $p \ge 2$.

Proof. Using (18) we split f into $f = f_1 + f_2$, where

$$f_1(x_1,x_2) := \sum_{j\ge 0} \sum_{|k| \le \alpha^{j/2}} \sum_{l \in \mathbb{Z}^2} c(j,k,l) \alpha^{\frac{3}{4}j} \psi(\alpha^j x_1 - \alpha^{j/2} k x_2 - l_1, \alpha^{j/2} x_2 - l_2), \tag{19}$$

$$f_2(x_1,x_2) := \sum_{j<0} \sum_{l \in \mathbb{Z}^2} c(j,0,l) \alpha^{\frac{3}{4}j} \psi(\alpha^j x_1 - l_1, \alpha^{j/2} x_2 - l_2). \tag{20}$$

By Corollary 1 we can choose ψ compactly supported in $[-D,D] \times [-D,D]$ for some $D > 1$. Moreover, we can assume that $|D_1^\gamma \psi| \le 1$ for $0 \le \gamma \le K := \max\{K_1, K_2\}$, where $K_1 := 1 + \lfloor \sigma_1 \rfloor$, $K_2 := 1 + \lfloor \sigma_2 \rfloor$ and where $D_1 \psi$ denotes the derivative with respect to the first component of ψ. Now $Tr_h f$ can be written as

$$\begin{aligned}
Tr_h f(x_1) = f(x_1, 0) &= \sum_{j \in \mathbb{Z}} \sum_{|k| \le \alpha^{j/2}} \sum_{l \in \mathbb{Z}^2} c(j,k,l) \alpha^{\frac{3}{4} j} \psi(\alpha^j x_1 - l_1, -l_2) \\
&= \sum_{j \in \mathbb{Z}} \sum_{l_1 \in \mathbb{Z}} \sum_{|k| \le \alpha^{j/2}} \sum_{|l_2| \le D} c(j,k,l_1,l_2) \alpha^{\frac{3}{4} j} \psi(\alpha^j x_1 - l_1, -l_2) \\
&= \sum_{j \ge 0} \sum_{l_1 \in \mathbb{Z}} \lambda(j,l_1) a_{j,l_1}(x_1) + \sum_{j < 0} \sum_{l_1 \in \mathbb{Z}} \lambda(j,l_1) a_{j,l_1}(x_1) \\
&= Tr_h f_1(x_1) + Tr_h f_2(x_1),
\end{aligned}$$

where for $j \ge 0$,

$$a_{j,l_1}(x_1) := \begin{cases} \lambda(j,l_1)^{-1} \alpha^{\frac{3}{4} j} \sum\limits_{|k| \le \alpha^{j/2}} \sum\limits_{|l_2| \le D} c(j,k,l_1,l_2) \psi(\alpha^j x_1 - l_1, -l_2) & \text{if } \lambda(j,l_1) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\lambda(j,l_1) := \alpha^{\frac{3}{4} j} \sum_{|k| \le \alpha^{j/2}} \sum_{|l_2| \le D} |c(j,k,l_1,l_2)|,$$

and for $j < 0$

$$a_{j,l_1}(x_1) := \begin{cases} \lambda(j,l_1)^{-1} \alpha^{\frac{3}{4} j} \sum\limits_{|l_2| \le D} c(j,0,l_1,l_2) \psi(\alpha^j x_1 - l_1, -l_2) & \text{if } \lambda(j,l_1) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\lambda(j,l_1) := \alpha^{\frac{3}{4} j} \sum_{|l_2| \le D} |c(j,0,l_1,l_2)|.$$

We have that $\operatorname{supp} \psi(\alpha^j x_1 - l_1, -l_2) \subset DQ_{j,l_1}(\mathbb{R})$ which is also true for all a_{j,l_1} and by construction we know further that $|D^\gamma a_{j,l_1}| \le \alpha^{j\gamma}$, $0 \le \gamma \le K$. Thus, the a_{j,l_1} are K_1-atoms on $\mathbb{R}$. Next, we consider

$$\begin{aligned}
\|Tr_h f_1\|_{B^{\sigma_1}_{p,p}} &\lesssim \Big(\sum_{j \ge 0} \alpha^{j(\sigma_1 - \frac{1}{p})p} \sum_{l_1 \in \mathbb{Z}} |\lambda(j,l_1)|^p \Big)^{\frac{1}{p}} \\
&= \Big(\sum_{j \ge 0} \alpha^{jp(\sigma_1 + \frac{3}{4} - \frac{1}{p})} \sum_{l_1 \in \mathbb{Z}} \big(\sum_{|k| \le \alpha^{j/2}} \sum_{|l_2| \le D} |c(j,k,l_1,l_2)| \big)^p \Big)^{\frac{1}{p}}.
\end{aligned}$$

Since $\big(\sum_{i=1}^N |z_i| \big)^p \le N^{p-1} \sum_{i=1}^N |z_i|^p$ and the set $\{k \in \mathbb{Z} : |k| \le \alpha^{j/2}\}$ contains $C\alpha^{j/2}$ elements we can estimate

$$\begin{aligned}
\|Tr_h f_1\|_{B^{\sigma_1}_{p,p}} &\lesssim \Big(\sum_{j \ge 0} \alpha^{jp(\sigma_1 + \frac{5}{4} - \frac{3}{2p})} \sum_{|k| \le \alpha^{j/2}} \sum_{l \in \mathbb{R}^2} |c(j,k,l)|^p \Big)^{\frac{1}{p}} \\
&\lesssim \Big(\sum_{j \in \mathbb{Z}} \alpha^{jpr} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{R}^2} |c(j,k,l)|^p \Big)^{\frac{1}{p}} \lesssim \|f\|_{\mathscr{SC}_{p,r}}
\end{aligned}$$

with $r = \sigma_1 + \frac{5}{4} - \frac{3}{2p}$. In the same way we obtain that

$$\|Tr_h f_2\|_{B^{\sigma_2}_{p,p}} \lesssim \Big(\sum_{j<0} \alpha^{jp(\sigma_2 + \frac{3}{4} - \frac{1}{p})} \sum_{l \in \mathbb{R}^2} |c(j,0,l)|^p \Big)^{\frac{1}{p}}$$
$$\lesssim \Big(\sum_{j \in \mathbb{Z}} \alpha^{jpr} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{R}^2} |c(j,k,l)|^p \Big)^{\frac{1}{p}} \lesssim \|f\|_{\mathscr{S}\mathscr{C}_{p,r}}$$

with $r = \sigma_2 + \frac{3}{4} - \frac{1}{p}$. This completes the proof. □

By the following corollary the restriction to $\mathscr{S}\mathscr{C}\mathscr{C}_{p,r}$ is not necessary for $p = 1$.

Corollary 2. *For $p = 1$, the embedding $Tr_h(\mathscr{S}\mathscr{C}_{1,r}) \subset B^{\sigma}_{1,1}(\mathbb{R})$ with $\sigma = r - \frac{3}{4} + \frac{1}{p}$ holds true.*

Proof. Following the lines of the previous proof, where the summation with respect to k is over $\mathbb{Z}$, we obtain

$$\|Tr_h f\|_{B^{\sigma}_{1,1}} \lesssim \sum_{j \in \mathbb{Z}} \alpha^{j((\sigma + \frac{3}{4})p - 1)} \sum_{l_1 \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \sum_{|l_2| \le D} |c(j,k,l_1,l_2)| \le C \|f\|_{\mathscr{S}\mathscr{C}_{1,r}}$$

with $r = \sigma + \frac{3}{4} - \frac{1}{p}$ and we are done. □

Let us turn to traces on the vertical axis.

Theorem 10. *Let $Tr_v f$ denote the restriction of f to the (vertical) x_2-axis, i.e., $(Tr_v f)(x_2) := f(0,x_2)$. Then the embedding $Tr_v(\mathscr{S}\mathscr{C}\mathscr{C}_{p,r}) \subset B^{\sigma_1}_{p,p}(\mathbb{R}) + B^{\sigma_2}_{p,p}(\mathbb{R})$, holds true, where σ_1 is the largest number such that*

$$\sigma_1 + \lfloor \sigma_1 \rfloor \le 2r - \frac{9}{2} + \frac{3}{p}, \quad \text{and} \quad \sigma_2 = 2r - \frac{3}{2} + \frac{1}{p}.$$

Proof. As in (19) and (20) we split f into $f = f_1 + f_2$, where we can choose ψ compactly supported in $[-D,D] \times [-D,D]$ for some $D > 1$ and normalized such that the derivatives of order $0 \le \gamma \le K$ with $K := \max\{K_1, K_2\}$, where $K_1 := 1 + \lfloor \sigma_1 \rfloor$, $K_2 := 1 + \lfloor \sigma_2 \rfloor$ are not larger than 1. By the support assumption on ψ we have that

$$\alpha^{-j/2}(l_2 - D) \le x_2 \le \alpha^{-j/2}(l_2 + D),$$
$$-kl_2 - D(1 + |k|) \le l_1 \le -kl_2 + D(1 + |k|).$$

Let $I_{k,l_2} := \{r \in \mathbb{Z} : |r + kl_2| \le D(1 + |k|)\}$. Now we obtain that

$$Tr_v f(x_2) = f(0,x_2) = \sum_{j \in \mathbb{Z}} \sum_{|k| \le \alpha^{j/2}} \sum_{l \in \mathbb{Z}^2} c(j,k,l) \alpha^{\frac{3}{4}j} \psi(-\alpha^{j/2} k x_2 - l_1, \alpha^{j/2} x_2 - l_2).$$

This can be rewritten as

$$
\begin{aligned}
f(0,x_2) &= \sum_{j\geq 0}\sum_{l_2\in\mathbb{Z}} \lambda(j,l_2) a_{j,l_2}(x_2) + \sum_{j<0}\sum_{l_2\in\mathbb{Z}} \lambda(j,l_2) a_{j,l_2}(x_2) \\
&= Tr_v f_1(x_2) + Tr_v f_2(x_2),
\end{aligned}
$$

where for $j \geq 0$,

$$
a_{j,l_2}(x_2) := \lambda(j,l_2)^{-1} \alpha^{\frac{3+2K_1}{4} j} \sum_{|k|\leq \alpha^{j/2}} \sum_{l_1 \in I_{k,l_2}} c(j,k,l_1,l_2) \alpha^{-K_1 j/2} \psi(-\alpha^{j/2} k x_2 - l_1, \alpha^{j/2} x_2 - l_2)
$$

if $\lambda(j,l_2) \neq 0$ and $a_{j,l_2}(x_2) = 0$ otherwise and

$$
\lambda(j,l_2) := \alpha^{\frac{3+2K_1}{4} j} \sum_{|k|\leq \alpha^{j/2}} \sum_{l_1 \in I_{k,l_2}} |c(j,k,l_1,l_2)|
$$

and for $j < 0$,

$$
a_{j,l_2}(x_2) := \lambda(j,l_2)^{-1} \alpha^{\frac{3}{4} j} \sum_{|l_1|\leq D} c(j,0,l_1,l_2) \psi(-l_1, \alpha^{j/2} x_2 - l_2)
$$

if $\lambda(j,l_2) \neq 0$ and $a_{j,l_2}(x_2) = 0$ otherwise and

$$
\lambda(j,l_2) := \alpha^{\frac{3}{4} j} \sum_{|l_1|\leq D} |c(j,0,l_1,l_2)|.
$$

We have that $\operatorname{supp} \psi(-\alpha^{j/2} k x_2 - l_1, \alpha^{j/2} x_2 - l_2) \subset DQ_{j,l_2}(\mathbb{R})$, where the cube is considered with respect to $\sqrt{\alpha}$ now. This is also true for a_{j,l_2}. For $j \geq 0$ we conclude by $|k| \leq \alpha^{j/2}$ that $\alpha^{-Kj/2} |D^\gamma \psi(-\alpha^{j/2} k x_2 - l_1, \alpha^{j/2} x_2 - l_2)| \leq \alpha^{\frac{j}{2}\gamma}$ and consequently $|D^\gamma a_{j,l_2}| \leq \alpha^{\frac{j}{2}\gamma}$, $\gamma \leq K_1$. For $j < 0$ we also have that $|D^\gamma a_{j,l_2}| \leq \alpha^{\frac{j}{2}\gamma}$. Thus a_{j,l_2} are K_1-atoms. We get

$$
\begin{aligned}
\|Tr_v f_1\|_{B^{\sigma_1}_{p,p}} &\lesssim \Big(\sum_{j\in\mathbb{Z}} \alpha^{\frac{j}{2}(\sigma_1 - \frac{1}{p})p} \sum_{l_2\in\mathbb{Z}} |\lambda(j,l_2)|^p \Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_{j\geq 0} \alpha^{\frac{j}{2}(\sigma_1 - \frac{1}{p})p} \alpha^{\frac{j}{2}(\frac{3+2K_1}{2})p} \alpha^{\frac{j}{2}(2-\frac{2}{p})p} \sum_{|k|\leq\alpha^{j/2}} \sum_{l\in\mathbb{R}^2} |c(j,k,l)|^p \Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_{j\in\mathbb{Z}} \alpha^{\frac{j}{2}(\sigma_1 + \frac{7}{2} + K_1 - \frac{3}{p})p} \sum_{|k|\leq\alpha^{j/2}} \sum_{l\in\mathbb{R}^2} |c(j,k,l)|^p \Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_{j\in\mathbb{Z}} \alpha^{\frac{j}{2}(\sigma_1 + \frac{7}{2} + 1 + \lfloor\sigma_1\rfloor - \frac{3}{p})p} \sum_{|k|\leq\alpha^{j/2}} \sum_{l\in\mathbb{R}^2} |c(j,k,l)|^p \Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_{j\in\mathbb{Z}} \alpha^{jpr} \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{R}^2} |c(j,k,l)|^p \Big)^{\frac{1}{p}} \\
&\lesssim \|f\|_{\mathscr{SC}_{p,r}}
\end{aligned}
$$

with $r \geq \frac{1}{2}(\sigma_1 + \lfloor \sigma_1 \rfloor + \frac{9}{2} - \frac{3}{p})$. Analogously we can compute

$$\begin{aligned} \|Tr_v f_2\|_{B^{\sigma_2}_{p,p}} &\lesssim \Big(\sum_{j\in\mathbb{Z}} \alpha^{\frac{j}{2}(\sigma_2 - \frac{1}{p})p} \sum_{l_2\in\mathbb{Z}} |\lambda(j,l_2)|^p \Big)^{\frac{1}{p}} \\ &\leq \Big(\sum_{j<0} \alpha^{\frac{j}{2}(\sigma_2 - \frac{1}{p} + \frac{3}{2})p} \sum_{l\in\mathbb{R}^2} |c(j,0,l)|^p \Big)^{\frac{1}{p}} \\ &\leq \Big(\sum_{j\in\mathbb{Z}} \alpha^{jpr} \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{R}^2} |c(j,k,l)|^p \Big)^{\frac{1}{p}} \\ &\lesssim \|f\|_{\mathscr{SC}_{p,r}} \end{aligned}$$

with $r = \frac{1}{2}(\sigma_2 + \frac{3}{2} - \frac{1}{p})$ and we are done. □

Remark 5. An alternative way to obtain trace results would be first to apply the Besov embedding discussed in the next subsection and afterwards the classical trace theorem for homogeneous Besov spaces . Let us briefly discuss the relation between these different approaches. For simplicity we restrict ourselves to the positives scales and traces to the x_2-axis. Usually an application of trace theorems in Besov spaces leads to a loss of smoothness of order $1/p$, that is $\mathrm{Tr}(B^s_{pp}(\mathbb{R}^d)) = B^{s-1/p}_{pp}(\mathbb{R}^{d-1})$, see [23]. Let the coorbit space smoothness index r be fixed. Depending on the concrete values of r and p, the direct and the indirect approach can yield the same result. However, in specific cases it turns out that the direct approach is superior as we gain some smoothness: Let $2r - \frac{9}{2} + \frac{3}{p} = 2\kappa + \alpha$ with $\kappa \in \mathbb{Z}$ and $\alpha \in [0,2)$. Then we have for $\alpha \in [0,1)$ by Theorem 10 that $\sigma_1 = \kappa + \alpha$. On the other hand, in case $\alpha + \frac{1}{p} \in [1,2)$ an application of Theorem 11 yields $\mathscr{SCC}_{p,r} \subset B^{\tilde{\sigma}_1}_{pp}$, where $\tilde{\sigma}_1 = \kappa + 1 - \varepsilon$ for arbitrary small $\varepsilon > 0$. Consequently, applying the trace theorem for Besov spaces yields smoothness $\tilde{\sigma}_1 - 1/p = \kappa + 1 - \varepsilon - 1/p < \kappa + \alpha = \sigma_1$.

5.4 Embedding Results

In this subsection, we prove embedding results of certain subspaces of shearlet coorbit spaces into (sums of) homogeneous Besov spaces. But first we provide a result concerning the embedding within shearlet coorbit spaces. In [18, Section 5.7] some embedding theorems for general $L_{p,m}$ coorbit spaces were given. In particular, the authors mentioned that for a fixed weight m, these spaces are monotonically increasing with p. The following corollary is a special results in this direction.

Corollary 3. *For $1 \leq p_1 \leq p_2 \leq \infty$ the embedding $\mathscr{SC}_{p_1,r} \subset \mathscr{SC}_{p_2,r}$ holds true. Introducing the 'smoothness spaces' $\mathscr{G}^r_p := \mathscr{SC}_{p,r+d(\frac{1}{2}-\frac{1}{p})}$, this implies the continuous embedding*

$$\mathscr{G}^{r_1}_{p_1} \subset \mathscr{G}^{r_2}_{p_2}, \quad \text{if} \quad r_1 - \frac{d}{p_1} = r_2 - \frac{d}{p_2}.$$

For convenience we add the simple proof.

Proof. By Theorem 4 we obtain that

$$\|f\|_{\mathscr{SC}_{p_2,r}} \lesssim \|(c_\varepsilon(j,k,l)\|_{\ell_{p_2,r}} \lesssim \Big(\sum_{j\in\mathbb{Z}} \alpha^{jrp_2} \sum_{\substack{k,l\\ \varepsilon\in\{-1,1\}}} |c_\varepsilon(j,k,l)|^{p_2}\Big)^{\frac{1}{p_2}},$$

where $c_\varepsilon(j,k,l)$ is the coefficient belonging in the representation (15) with respect to (14) to the function $\pi(\varepsilon\alpha^{-j},\sigma\alpha^{-j/2}k,S_{\sigma\alpha^{-j/2}k}A_{\alpha^{-j}}\tau l)\psi$. Since $\ell_{p_1}\subset\ell_{p_2}$ for $p_1\le p_2$ we get finally that

$$\begin{aligned}\|f\|_{\mathscr{SC}_{p_2,r}} &\lesssim \Big(\sum_{j\in\mathbb{Z}} \alpha^{jrp_2}\big(\sum_{\substack{k,l\\ \varepsilon\in\{-1,1\}}} |c_\varepsilon(j,k,l)|^{p_1}\big)^{\frac{p_2}{p_1}}\Big)^{\frac{1}{p_2}}\\ &\lesssim \Big(\sum_{j\in\mathbb{Z}} \alpha^{jrp_1} \sum_{\substack{k,l\\ \varepsilon\in\{-1,1\}}} |c_\varepsilon(j,k,l)|^{p_1}\Big)^{\frac{1}{p_1}} \lesssim \|f\|_{\mathscr{SC}_{p_1,r}}.\end{aligned}$$

□

Next we state our final result.

Theorem 11. *The embedding $\mathscr{SCC}_{p,r}\subset B^{\sigma_1}_{p,p}(\mathbb{R}^2)+B^{\sigma_2}_{p,p}(\mathbb{R}^2)$, holds true, where σ_1 is the largest number such that*

$$\sigma_1+\lfloor\sigma_1\rfloor \le 2r-\frac{9}{2}+\frac{4}{p}, \quad \text{and} \quad \sigma_2-\frac{\lfloor\sigma_2\rfloor}{2} = r+\frac{3}{2p}+\frac{1}{4}.$$

Proof. By (18) we know that $f\in\mathscr{SCC}_{p,r}$ can be written as

$$f(x)=\sum_{j\in\mathbb{Z}}\sum_{|k|\le\alpha^{j/2}}\sum_{l\in\mathbb{Z}^2} c(j,k,l)\alpha^{\frac{3}{4}j}\psi(\alpha^j x_1-\alpha^{j/2}kx_2-l_1,\alpha^{j/2}x_2-l_2),$$

where we can choose ψ compactly supported in $[-D,D]\times[-D,D]$ for some $D>1$ and normalized such that the derivatives of order $0\le|\gamma|\le K:=\max\{K_1,K_2\}$, $K_1:=1+\lfloor\sigma_1\rfloor$, $K_2:=1+\lfloor\sigma_2\rfloor$ are not larger than 1.

We split $f\in\mathscr{SCC}_{p,r}$ as in (19) and (20) into f_1 and f_2. Then we obtain with the index transform $l_1=r_1-kl_2$ that

$$\begin{aligned}f_1(x)=&\sum_{j\ge0}\sum_{|k|\le\alpha^{j/2}}\sum_{l_2\in\mathbb{Z}}\sum_{n_1\in\mathbb{Z}}\sum_{r_1\in I(j,n_1)} c(j,k,r_1-kl_2,l_2)\alpha^{\frac{3}{4}j}\\ &\times\psi(\alpha^j x_1-\alpha^{j/2}kx_2-r_1+kl_2,\alpha^{j/2}x_2-l_2)\end{aligned}$$

where $I(j,n_1):=\{r\in\mathbb{Z}:\alpha^{j/2}(n_1-1)<r\le\alpha^{j/2}n_1\}$.

For $j\ge0$ we set

$$a_{j,n_1,l_2}(x) := \lambda(j,n_1,l_2)^{-1}\alpha^{\frac{3+2K_1}{4}j} \sum_{|k|\le\alpha^{j/2}} \sum_{r_1\in I(j,n_1)} c(j,k,r_1-kl_2,l_2)$$
$$\times\alpha^{-K_1 j/2}\,\psi(\alpha^j x_1-\alpha^{j/2}kx_2-r_1+kl_2,\alpha^{j/2}x_2-l_2),$$

if $\lambda(j,n_1,l_2)\neq 0$ and $a_{j,n_1,l_2}(x)=0$ otherwise, where

$$\lambda(j,n_1,l_2) := \alpha^{\frac{3+2K_1}{4}j} \sum_{|k|\le\alpha^{j/2}} \sum_{r_1\in I(j,n_1)} |c(j,k,r_1-kl_2,l_2)|.$$

By the support assumption on ψ, the functions appearing in the definition of a_{j,n_1,m_2} are only non-zero if the following conditions are satisfied:

$$-D\le\alpha^{j/2}x_2-l_2\le D, \qquad \alpha^{-j/2}(l_2-D)\le x_2\le\alpha^{-j/2}(l_2+D)$$

and

$$-D\le\alpha^j x_1-\alpha^{j/2}kx_2-r_1+kl_2\le D,$$

$$\begin{aligned}
\alpha^{-j}r_1+\alpha^{-j}k(\alpha^{j/2}x_2-l_2)-\alpha^{-j}D\le x_1 \;&\le\; \alpha^{-j}r_1+\alpha^{-j}k(\alpha^{j/2}x_2-l_2)+\alpha^{-j}D,\\
\alpha^{-j}r_1-\alpha^{-j/2}(2D)\le x_1 \;&\le\; \alpha^{-j}r_1+\alpha^{-j/2}(2D),\\
\alpha^{-j/2}n_1-\alpha^{-j/2}(3D)\le x_1 \;&\le\; \alpha^{-j/2}n_1+\alpha^{-j/2}(2D).
\end{aligned}$$

Thus, a_{j,n_1,l_2} is supported in $3DQ_{j,n_1,l_2}$, where the cube is considered with respect to $\sqrt{\alpha}$. The appropriate bounds $|D^\gamma a_{j,n_1,l_2}|\le\alpha^{\frac{j}{2}|\gamma|}$, $|\gamma|\le K_1$ can be derived as in the previous proof. Hence the functions a_{j,n_1,l_2} are K_1-atoms.

Now we obtain for

$$f_1(x)=\sum_{j\ge0}\sum_{l_2\in\mathbb{Z}}\sum_{n_1\in\mathbb{Z}}\lambda(j,n_1,l_2)a_{j,n_1,l_2}(x)$$

that

$$\begin{aligned}
\|f_1\|^p_{B^{\sigma_1}_{p,p}} &\lesssim \sum_{j\in\mathbb{Z}}\alpha^{\frac{j}{2}(\sigma_1-\frac{2}{p})p}\sum_{l_2\in\mathbb{Z}}\sum_{n_1\in\mathbb{Z}}|\lambda(j,n_1,l_2)|^p\\
&= \sum_{j\in\mathbb{Z}}\alpha^{\frac{j}{2}(\sigma_1-\frac{2}{p})p}\alpha^{\frac{j}{2}(\frac{3+2K_1}{2})p}\sum_{l_2\in\mathbb{Z}}\sum_{n_1\in\mathbb{Z}}\left|\sum_{|k|\le\alpha^{j/2}}\sum_{r_1\in I(j,n_1)}|c(j,k,r_1-kl_2,l_2)|\right|^p\\
&\le \sum_{j\in\mathbb{Z}}\alpha^{\frac{j}{2}p(\sigma_1+\frac{7}{2}+K_1-\frac{4}{p})}\sum_{l_2\in\mathbb{Z}}\sum_{n_1\in\mathbb{Z}}\sum_{|k|\le\alpha^{j/2}}\sum_{r_1\in I(j,n_1)}|c(j,k,r_1-kl_2,l_2)|^p\\
&= \sum_{j\in\mathbb{Z}}\alpha^{\frac{j}{2}p(\sigma_1+\frac{9}{2}+\lfloor\sigma_1\rfloor-\frac{4}{p})}\sum_{|k|\le\alpha^{j/2}}\sum_{l_1\in\mathbb{Z}}\sum_{l_2\in\mathbb{Z}}|c(j,k,l_1,l_2)|^p\\
&\lesssim \|f\|^p_{\mathscr{SC}_{p,r}}.
\end{aligned}$$

In the case $j<0$ we obtain with $J(j,n_2):=\{r:\alpha^{-j/2}(n_2-1)<r\le\alpha^{-j/2}n_2\}$ that

$$
\begin{aligned}
f_2(x) &= \sum_{j<0}\sum_{l_1\in\mathbb{Z}}\sum_{l_2\in\mathbb{Z}} c(j,0,l_1,l_2)\alpha^{\frac{3}{4}j}\psi(\alpha^j x_1 - l_1, \alpha^{j/2}x_2 - l_2)\\
&= \sum_{j<0}\sum_{l_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}}\sum_{r_2\in J(j,n_2)} c(j,0,l_1,r_2)\alpha^{\frac{3}{4}j}\psi(\alpha^j x_1 - l_1, \alpha^{j/2}x_2 - r_2)\\
&= \sum_{j<0}\sum_{l_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}} \lambda(j,l_1,n_2) a_{j,l_1,n_2}(x),
\end{aligned}
$$

where

$$
\begin{aligned}
a_{j,l_1,n_2}(x) &:= \lambda(j,l_1,n_2)^{-1}\alpha^{\frac{3-2K_2}{4}j}\sum_{r_2\in J(j,n_2)} c(j,0,l_1,r_2)\alpha^{\frac{jK_2}{2}}\psi(\alpha^j x_1 - l_1, \alpha^{j/2}x_2 - r_2),\\
\lambda(j,l_1,n_2) &:= \alpha^{\frac{3-2K_2}{4}j}\sum_{r_2\in J(j,n_2)} |c(j,0,l_1,r_2)|
\end{aligned}
$$

and $a_{j,l_1,n_2}(x) := 0$ if $\lambda_{j,l_1,n_2} = 0$. By the support assumption on ψ we get

$$
\begin{aligned}
&\alpha^{-j}(l_1 - D) \le x_1 \ \le\ \alpha^{-j}(l_1 + D),\\
&\alpha^{-j/2}(r_2 - D) \le x_2 \ \le\ \alpha^{-j/2}(r_2 + D), \text{ i.e., } \alpha^{-j}(n_2 - 2D) \ \le\ x_2 \ \le\ \alpha^{-j}(n_2 + D).
\end{aligned}
$$

Consequently, a_{j,l_1,n_2} is supported in $2DQ_{j,l_1,n_2}$. Since $1 \ge \alpha^{j|\gamma|/2} \ge \alpha^{j|\gamma|} \ge \alpha^{jK_2}$ for $0 \le |\gamma| \le K_2$ and $j < 0$ we obtain further that $|D^\gamma a_{j,n_1,l_2}| \le \alpha^{jK_2/2}\alpha^{j|\gamma|/2} \le \alpha^{j|\gamma|}$ so that a_{j,l_1,n_2} are K_2-atoms. Thus,

$$
\begin{aligned}
\|f_2\|^p_{B^{\sigma_2}_{p,p}} &\lesssim \sum_{j\in\mathbb{Z}} \alpha^{j(\sigma_2 - \frac{2}{p})p} \sum_{l_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}} |\lambda(j,l_1,n_2)|^p\\
&\le \sum_{j<0} \alpha^{j(\sigma_2 - \frac{2}{p} + \frac{3-2K_2}{4})p} \sum_{l_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}} \left|\sum_{r_2\in J(j,n_2)} c(j,0,l_1,r_2)\right|^p\\
&\le \sum_{j<0} \alpha^{j(\sigma_2 - \frac{3}{2p} + \frac{1}{4} - \frac{K_2}{2})p} \sum_{l\in\mathbb{R}^2} |c(j,0,l)|^p\\
&\le \sum_{j\in\mathbb{Z}} \alpha^{jpr} \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{R}^2} |c(j,k,l)|^p\\
&\lesssim \|f\|^p_{\mathscr{SC}_{p,r}},
\end{aligned}
$$

where $r = \sigma_2 - \frac{3}{2p} - \frac{1}{4} - \frac{\lfloor\sigma_2\rfloor}{2}$. □

Remark 6. Embedding results in Besov spaces have also been shown for the curvelet setting by Borup and Nielsen [2]. However, the technique used by these authors is completely different. In contrast to our approach they work in the frequency domain. We prefer to consider the time domain with flexible atomic decompositions for the following reasons. As already outlined above time domain techniques provide a very natural way to derive trace theorems which might be very difficult or even impossible in the Fourier domain. Moreover, since we are working with compactly supported atoms the treatment of shearlet coorbit spaces on bounded domains, including again embedding and trace theorems, seems to be manageable.

It has also turned out that our approach provides some advantages for higher dimensions. Trace theorems for shearlet coorbit spaces on $\mathbb{R}^d$, $d \geq 3$ to higher dimensional hyperplanes are not straightforward since it is not clear that these traces will also be contained in Besov spaces. One natural conjecture would be that the traces of shearlet coorbit spaces on $\mathbb{R}^3$ with respect to two-dimensional hyperplanes are again shearlet coorbit spaces . This conjecture was proved in [8]. As expected flexible atomic and molecular decomposition molecular decomposition techniques for shearlet coorbit spaces can be applied.

6 Analysis of Singularities

In this section, we deal with the decay of the shearlet transform at hyperplane singularities and at special simplex singularities in $\mathbb{R}^d$. For the behaviour of the shearlet transform at singularities in $\mathbb{R}^2$ we refer to [32, 38].

6.1 Hyperplane Singularities

We consider $(d-m)$-dimensional hyperplanes in $\mathbb{R}^d$, $m = 1, \ldots, d-1$ through the origin given by

$$\underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}}_{x_A} + P \underbrace{\begin{pmatrix} x_{m+1} \\ \vdots \\ x_d \end{pmatrix}}_{x_E} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad P := \begin{pmatrix} p_1^{\mathrm{T}} \\ \vdots \\ p_m^{\mathrm{T}} \end{pmatrix} \in \mathbb{R}^{m,d-m}. \tag{21}$$

Note that this setting excludes some special hyperplanes, e.g., for $d = 3$ and $m = 1$ planes containing the x_1-axis and for $d = 3$ and $m = 2$ lines contained within the $x_1 x_2$–plane. To detect such hyperplane singularities one has to perform a simple variable exchange in the shearlet stetting or to define 'cone adapted shearlets' similar to [32].

Let δ denote the Delta distribution. Then we obtain for

$$\nu_m := \delta(x_A + P x_E)$$

that

$$\begin{aligned} \hat{\nu}_m(\omega) &= \int_{\mathbb{R}^d} \delta(x_A + P x_E) e^{-2\pi i(\langle x_A, \omega_A \rangle + \langle x_E, \omega_E \rangle)} \, dx \\ &= \int_{\mathbb{R}^{d-m}} e^{-2\pi i(-\langle P x_E, \omega_A \rangle + \langle x_E, \omega_E \rangle)} \, dx_E \\ &= \delta(\omega_E - P^{\mathrm{T}} \omega_A). \end{aligned} \tag{22}$$

The following theorem describes the decay of the shearlet transform at hyperplane singularities. We use the notation $\mathcal{SH}_\psi f(a,s,t) \sim |a|^r$ as $a \to 0$, if there exist constants $0 < c \le C < \infty$ such that

$$c|a|^r \le |\mathcal{SH}_\psi f(a,s,t)| \le C|a|^r \quad \text{as} \quad a \to 0.$$

Theorem 12. *Let $\psi \in L_2(\mathbb{R}^d)$ be a shearlet satisfying $\hat\psi \in C^\infty(\mathbb{R}^d)$. Assume further that $\hat\psi(\omega) = \hat\psi_1(\omega_1)\hat\psi_2(\tilde\omega/\omega_1)$, where* $\operatorname{supp}\hat\psi_1 \in [-a_1,-a_0]\cup[a_0,a_1]$ *for some $a_1 > a_0 \ge \alpha > 0$ and* $\operatorname{supp}\hat\psi_2 \in Q_b$. *If*

$$(s_m,\dots,s_{d-1}) = (-1,s_1,\dots,s_{m-1})P \quad \text{and} \quad (t_1,\dots,t_m) = -(t_{m+1},\dots,t_d)P^{\mathrm{T}}, \tag{23}$$

then

$$\mathcal{SH}_\psi \nu_m(a,s,t) \sim |a|^{\frac{1-2m}{2d}} \quad \text{as } a \to 0. \tag{24}$$

Otherwise, the shearlet transform $\mathcal{SH}_\psi \nu_m$ decays rapidly, i.e., faster than any polynomial, as $a \to 0$.

The condition (23) requires that the shearlet is aligned with the hyperplane (21) and that t lies within the hyperplane. The condition on $\hat\psi_1$ and $\hat\psi_2$ can be relaxed toward a rapid decay of the functions.

Proof. An application of Plancherel's theorem for tempered distribution together with (22) and (7) yields

$$\begin{aligned}
&\mathcal{SH}_\psi \nu_m(a,s,t) := \langle \nu_m, \psi_{a,s,t}\rangle = \langle \hat\nu_m, \hat\psi_{a,s,t}\rangle \\
&= \int_{\mathbb{R}^d} \delta(\omega_E - P^{\mathrm{T}}\omega_A)|a|^{1-\frac{1}{2d}} e^{2\pi i\langle t,\omega\rangle}\,\bar{\hat\psi}\left(a\omega_1, \operatorname{sgn}(a)|a|^{\frac{1}{d}}(\omega_1 s + \tilde\omega)\right) d\omega \\
&= |a|^{1-\frac{1}{2d}} \int_{\mathbb{R}^m} e^{2\pi i\langle t_A + Pt_E, \omega_A\rangle}\,\bar{\hat\psi}\left(a\omega_1, \operatorname{sgn}(a)|a|^{\frac{1}{d}}(\omega_1 s + \begin{pmatrix}\tilde\omega_A\\ P^{\mathrm{T}}\omega_A\end{pmatrix})\right) d\omega_A
\end{aligned}$$

with $\tilde\omega_A = (\omega_2,\dots,\omega_m)^{\mathrm{T}}$ for $m \ge 2$. In the following, we restrict our attention to the case $m \ge 2$. If $m = 1$, we can simply neglect $\tilde\omega_A$ and the assertion follows in a similar way. By definition of $\hat\psi$ this can be rewritten as

$$|a|^{1-\frac{1}{2d}} \int_{\mathbb{R}^m} e^{2\pi i\langle t_A + Pt_E, \omega_A\rangle}\,\bar{\hat\psi}_1(a\omega_1)\,\bar{\hat\psi}_2\left(|a|^{\frac{1}{d}-1}(s + \frac{1}{\omega_1}\begin{pmatrix}\tilde\omega_A\\ P^{\mathrm{T}}\omega_A\end{pmatrix})\right) d\omega_A.$$

Substituting $\tilde\xi_A = (\xi_2,\dots,\xi_m)^{\mathrm{T}} := \tilde\omega_A/\omega_1$, i.e., $d\tilde\omega_A = |\omega_1|^{m-1}\,d\tilde\xi_A$, we get

$$\begin{aligned}
\mathcal{SH}_\psi \nu_m(a,s,t) = |a|^{1-\frac{1}{2d}} &\int_{\mathbb{R}}\int_{\mathbb{R}^{m-1}} e^{2\pi i\omega_1\langle t_A + Pt_E, (1,\tilde\xi_A^{\mathrm{T}})^{\mathrm{T}}\rangle}\,\bar{\hat\psi}_1(a\omega_1)|\omega_1|^{m-1} \\
&\times \bar{\hat\psi}_2\left(|a|^{\frac{1}{d}-1}(s + \begin{pmatrix}\tilde\xi_A\\ P^{\mathrm{T}}(1,\tilde\xi_A^{\mathrm{T}})^{\mathrm{T}}\end{pmatrix})\right) d\tilde\xi_A\, d\omega_1
\end{aligned}$$

and further by substituting $\xi_1 := a\omega_1$

$$\mathscr{SH}_\psi \nu_m(a,s,t) = |a|^{1-m-\frac{1}{2d}} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}} e^{2\pi i \frac{\xi_1}{a} \langle t_A + Pt_E, (1, \tilde{\xi}_A^{\mathrm{T}})^{\mathrm{T}} \rangle} |\xi_1|^{m-1} \bar{\hat{\psi}}_1(\xi_1)\, d\xi_1 \times \bar{\hat{\psi}}_2 \left(|a|^{\frac{1}{d}-1} (s + \begin{pmatrix} \tilde{\xi}_A \\ P^{\mathrm{T}}(1, \tilde{\xi}_A^{\mathrm{T}})^{\mathrm{T}} \end{pmatrix})) \right) d\tilde{\xi}_A.$$

Finally, by substituting $\tilde{\omega}_A := |a|^{\frac{1}{d}-1}(\tilde{\xi}_A + s_a)$, where $s_a := (s_1, \ldots, s_{m-1})^{\mathrm{T}}$ and $s_e := (s_m, \ldots, s_{d-1})^{\mathrm{T}}$, we obtain

$$\mathscr{SH}_\psi \nu_m(a,s,t) = |a|^{\frac{1-2m}{2d}} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}} e^{2\pi i \frac{\xi_1}{a} \langle t_A + Pt_E, (1, |a|^{1-1/d} \tilde{\omega}_A^{\mathrm{T}} - s_a^{\mathrm{T}}) \rangle} |\xi_1|^{m-1} \bar{\hat{\psi}}_1(\xi_1)\, d\xi_1 \times \bar{\hat{\psi}}_2 \begin{pmatrix} \tilde{\omega}_A \\ |a|^{\frac{1}{d}-1} \left(s_e - P^{\mathrm{T}} \begin{pmatrix} -1 \\ s_a \end{pmatrix}\right) + P^{\mathrm{T}} \begin{pmatrix} 0 \\ \tilde{\omega}_A \end{pmatrix} \end{pmatrix} d\tilde{\omega}_A.$$

If the vector

$$s_e - P^{\mathrm{T}} \begin{pmatrix} -1 \\ s_a \end{pmatrix} \neq 0_{d-m} \tag{25}$$

then at least one component of its product with $|a|^{1/d-1}$ becomes arbitrary large as $a \to 0$. On the other hand, by the support property of $\hat{\psi}_2$, we conclude that $\hat{\psi}_2(\tilde{\omega}_A, \cdot)$ becomes zero if $\tilde{\omega}_A$ is not in $Q_{(b_1,\ldots,b_{m-1})} \subset \mathbb{R}^{m-1}$. But for all $\tilde{\omega}_A \in Q_{(b_1,\ldots,b_{m-1})}$ at least one component of

$$|a|^{\frac{1}{d}-1} \left(s_e - P^{\mathrm{T}} \begin{pmatrix} 1 \\ s_a \end{pmatrix}\right) + P^{\mathrm{T}} \begin{pmatrix} 0 \\ \tilde{\omega}_A \end{pmatrix}$$

is not within the support of $\hat{\psi}_2$ for a sufficiently small so that $\hat{\psi}_2$ becomes zero again. Assume now that we have equality in (25). Then

$$\begin{aligned} \mathscr{SH}_\psi \nu_m(a,s,t) &= |a|^{\frac{1-2m}{2d}} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}} e^{2\pi i \frac{\xi_1}{a} \langle t_A + Pt_E, (1, |a|^{1-1/d} \tilde{\omega}_A^{\mathrm{T}} - s_a^{\mathrm{T}}) \rangle} |\xi_1|^{m-1} \bar{\hat{\psi}}_1(\xi_1)\, d\xi_1 \\ &\quad \times \bar{\hat{\psi}}_2 \begin{pmatrix} \tilde{\omega}_A \\ P^{\mathrm{T}} \begin{pmatrix} 0 \\ \tilde{\omega}_A \end{pmatrix} \end{pmatrix} d\tilde{\omega}_A \\ &= C\, |a|^{\frac{1-2m}{2d}} \int_{\mathbb{R}^{m-1}} \tilde{\psi}_1^{(m-1)} \left(\langle t_A + Pt_E, (1, |a|^{1-1/d} \tilde{\omega}_A^{\mathrm{T}} - s_a^{\mathrm{T}}) \rangle / a \right) \\ &\quad \times \bar{\hat{\psi}}_2 \begin{pmatrix} \tilde{\omega}_A \\ P^{\mathrm{T}} \begin{pmatrix} 0 \\ \tilde{\omega}_A \end{pmatrix} \end{pmatrix} d\tilde{\omega}_A, \end{aligned}$$

$$\mathscr{SH}_\psi \nu_m(a,s,t) = C\,|a|^{\frac{1-2m}{2d}} \int_{\mathbb{R}^{m-1}} \tilde{\psi}_1^{(m-1)} \left(\langle t_A + Pt_E, \begin{pmatrix} |a|^{1/d-1} \\ \tilde{\omega}_A^{\mathrm{T}} - |a|^{1/d-1} s_a \end{pmatrix} \rangle \, |a|^{-\frac{1}{d}} \right)$$
$$\times \bar{\hat{\psi}}_2 \begin{pmatrix} \tilde{\omega}_A \\ P^{\mathrm{T}} \begin{pmatrix} 0 \\ \tilde{\omega}_A \end{pmatrix} \end{pmatrix} d\tilde{\omega}_A,$$

where $\tilde{\psi}_1$ has the Fourier transform $\hat{\tilde{\psi}}_1(\xi_1) := \bar{\hat{\psi}}_1(\xi_1)$ for $\xi_1 \geq 0$ and $\hat{\tilde{\psi}}_1(\xi_1) := -\bar{\hat{\psi}}_1(\xi_1)$ for $\xi_1 < 0$. Since by our assumptions the support of $\hat{\psi}_1$ is bounded away from the origin, we see that $\hat{\tilde{\psi}}_1$ is again in $C^\infty(\mathbb{R})$. If $t_A + Pt_E \neq 0_m$, then, since $\hat{\psi}_1 \in C^\infty$ the function $\tilde{\psi}_1^{(m-1)}$ decays rapidly as $a \to 0$ for all $\tilde{\omega}_A$ in the bounded domain, where $\hat{\psi}_2$ doesn't become zero. Consequently, the value of the shearlet transform decays rapidly. If $t_A + Pt_E = 0_m$, then

$$\mathscr{SH}_\psi \nu_m(a,s,t) = C\,|a|^{\frac{1-2m}{2d}}\, \tilde{\psi}_1^{(m-1)}(0) \int_{\mathbb{R}^{m-1}} \bar{\hat{\psi}}_2 \begin{pmatrix} \tilde{\omega}_A \\ P^{\mathrm{T}} \begin{pmatrix} 0 \\ \tilde{\omega}_A \end{pmatrix} \end{pmatrix} d\tilde{\omega}_A \;\sim\; |a|^{\frac{1-2m}{2d}}.$$

This finishes the proof. □

Remark 7. Other choices of the dilation matrix are possible, e.g.,

$$A_a := \begin{pmatrix} a & 0_{d-1}^{\mathrm{T}} \\ 0_{d-1} & \operatorname{sgn}(a)\sqrt{|a|}\, I_{d-1} \end{pmatrix}.$$

Then we have to replace (24) by $|a|^{\frac{d-2m-1}{4}}$ which increases for $d < 2m+1$ as $a \to 0$. Therefore, we prefer our choice.

6.2 Tetrahedron Singularities

In the following, we deal with the cone $\mathscr{C}$ in the first octant of $\mathbb{R}^3$ given by

$$\mathscr{C} := \{x = Ct : t \geq 0 \text{ componentwise}\},$$

where

$$C := (p\; q\; r) = \begin{pmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{pmatrix}, \quad p_j, q_j, r_j > 0,\; j = 1,2$$

and the vectors p, q, r are linearly independent. The vector

$$n_{pq} := \left(1, \frac{p_2 - q_2}{p_1 q_2 - p_2 q_1}, \frac{q_1 - p_1}{p_1 q_2 - p_2 q_1}\right)^{\mathrm{T}} = \left(1, \tilde{n}_{pq}^{\mathrm{T}}\right)^{\mathrm{T}}$$

is a multiple of the normal vector of the plane spanned by p and q. Similarly, we use the notation n_{pr}, n_{qr} for the corresponding vectors perpendicular to the pr-plane and qr-plane. Let $\chi_{\mathscr{C}}$ denote the characteristic function of the cone $\mathscr{C}$. Since the Fourier transform of the Heavyside function H is

$$\hat{H}(\omega) = \frac{1}{2\pi i}\,\mathrm{pv}\left(\frac{1}{\omega}\right) + \sqrt{\frac{\pi}{2}}\,\delta(\omega),$$

see [22, p. 340], we obtain that

$$\begin{aligned}
\hat{\chi}_{\mathscr{C}}(\omega) &= \int_{\mathscr{C}} e^{-2\pi i\langle x,\omega\rangle}\,dx \;=\; |\det \mathrm{C}| \int_{\mathbb{R}^3_+} e^{-2\pi i\langle t, C^{\mathrm{T}}\omega\rangle}\,dt \\
&= c_1\left(\frac{1}{p^{\mathrm{T}}\omega}\frac{1}{q^{\mathrm{T}}\omega}\frac{1}{r^{\mathrm{T}}\omega}\right) \\
&\quad+\; c_2\left(\frac{1}{p^{\mathrm{T}}\omega}\frac{1}{q^{\mathrm{T}}\omega}\,\delta(r^{\mathrm{T}}\omega) + \frac{1}{p^{\mathrm{T}}\omega}\frac{1}{r^{\mathrm{T}}\omega}\,\delta(q^{\mathrm{T}}\omega) + \frac{1}{q^{\mathrm{T}}\omega}\frac{1}{r^{\mathrm{T}}\omega}\,\delta(p^{\mathrm{T}}\omega)\right) \\
&\quad+ c_3\left(\frac{1}{p^{\mathrm{T}}\omega}\,\delta(q^{\mathrm{T}}\omega)\delta(r^{\mathrm{T}}\omega) + \frac{1}{q^{\mathrm{T}}\omega}\,\delta(p^{\mathrm{T}}\omega)\delta(r^{\mathrm{T}}\omega) + \frac{1}{r^{\mathrm{T}}\omega}\,\delta(p^{\mathrm{T}}\omega)\delta(q^{\mathrm{T}}\omega)\right) \\
&\quad+ c_4\left(\delta(p^{\mathrm{T}}\omega)\delta(q^{\mathrm{T}}\omega)\delta(r^{\mathrm{T}}\omega)\right) \qquad (26)
\end{aligned}$$

with nonzero constants $c_j,\ j = 1,2,3,4$. We have omitted the pv to simplify the notation. This can be used to prove the following theorem.

Theorem 13. *Let $\psi \in L_2(\mathbb{R}^3)$ be a shearlet satisfying $\hat{\psi} \in C^\infty(\mathbb{R}^3)$. Assume further that $\hat{\psi}(\omega) = \hat{\psi}_1(\omega_1)\hat{\psi}_2(\tilde{\omega}/\omega_1)$, where $\operatorname{supp}\hat{\psi}_1 \in [-a_1,-a_0]\cup[a_0,a_1]$ for some $a_1 > a_0 \geq \alpha > 0$ and $\operatorname{supp}\hat{\psi}_2 \in Q_b$. Let $a > 0$. If*

$$1 - p_1s_1 - p_2s_2 \neq 0,\ 1 - q_1s_1 - q_2s_2 \neq 0,\ 1 - r_1s_1 - r_2s_2 \neq 0 \quad \text{and} \quad t = (0,0,0)^T,$$

then

$$\mathscr{SH}_\psi \chi_{\mathscr{C}}(a,s,t) \sim a^{13/9}.$$

If

$$\begin{aligned}
&1 - p_1s_1 - p_2s_2 = 0,\ 1 - q_1s_1 - q_2s_2 \neq 0,\ 1 - r_1s_1 - r_2s_2 \neq 0 \ \text{ or}\\
&1 - q_1s_1 - q_2s_2 = 0,\ 1 - p_1s_1 - p_2s_2 \neq 0,\ 1 - r_1s_1 - r_2s_2 \neq 0 \ \text{ or}\\
&1 - r_1s_1 - r_2s_2 = 0,\ 1 - p_1s_1 - p_2s_2 \neq 0,\ 1 - q_1s_1 - q_2s_2 \neq 0
\end{aligned}$$

and $t_1 - t_2s_1 - t_3s_2 = 0$ which is in particular the case if $t = cp$, $t = cq$ or $t = cr$, resp., then

$$\mathscr{SH}_\psi \chi_{\mathscr{C}}(a,s,t) \lesssim a^{3/2}.$$

If

$$s = -\tilde{n}_{pq},\ n_{pq}^T t = 0 \quad \text{or} \quad s = -\tilde{n}_{pr},\ n_{pr}^T t = 0 \quad \text{or} \quad s = -\tilde{n}_{qr},\ n_{qr}^T t = 0$$

then

$$\mathcal{SH}_\psi \chi_{\mathcal{C}}(a,s,t) \lesssim a^{5/6}.$$

Otherwise, the shearlet transform $\mathcal{SH}_\psi \chi_{\mathcal{C}}(a,s,t)$ *decays rapidly as* $a \to 0$.

Fig. 2 illustrates the decay of the shearlet transform.

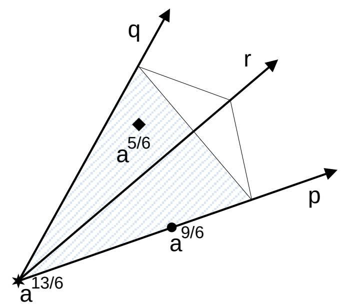

Fig. 2 Decay of the shearlet transform of the characteristic function of the cone $\mathcal{C}$: we have $\star$ if $\hat{s} \not\perp p,q,r$, $\bullet$ if $\hat{s} \perp p$ but $\hat{s} \not\perp q,r$ and $\diamond$ if $\hat{s} \perp p,q$, where $\hat{s} := (1,-s_1,-s_2)^{\mathrm{T}}$ is orthogonal to the plane containing the largest shearlet value.

Proof. To determine the decay of $\mathcal{SH}_\psi \chi_{\mathcal{C}}(a,s,t) = \langle \hat{\chi}_{\mathcal{C}}, \hat{\psi}_{a,s,t} \rangle$ as $a \to 0$, we consider the four parts of (26) separately.

1. Since p,q,r are linearly independent, we have by the support of $\hat{\psi}$ that

$$\langle \delta(p^{\mathrm{T}}\cdot)\delta(q^{\mathrm{T}}\cdot)\delta(r^{\mathrm{T}}\cdot), \hat{\psi}_{a,s,t} \rangle = \hat{\psi}_{a,s,t}(0) = 0.$$

2. Next we obtain

$$\begin{aligned} &\langle \delta(p^{\mathrm{T}}\cdot)\delta(q^{\mathrm{T}}\cdot) \frac{1}{r^{\mathrm{T}}\cdot}, \hat{\psi}_{a,s,t} \rangle \\ &= a^{5/6} \frac{1}{r^{\mathrm{T}} n_{pq}} \int_{\mathbb{R}} e^{2\pi i \omega_1 \langle t, n_{pq} \rangle} \frac{\bar{\hat{\psi}}_1(a\omega_1)}{\omega_1} \bar{\hat{\psi}}_2 \left(a^{-2/3}(s + \tilde{n}_{pq}) \right) d\omega_1 \\ &\sim a^{5/6} \bar{\hat{\psi}}_2 \left(a^{-2/3}(s + \tilde{n}_{pq}) \right) \int_{\mathbb{R}} e^{2\pi i \xi_1 \langle t, n_{pq} \rangle / a} \frac{\bar{\hat{\psi}}_1(\xi_1)}{\xi_1} d\xi_1. \end{aligned} \tag{27}$$

If $s \neq -\tilde{n}_{pq}$, then (27) becomes zero for sufficiently small a since $\bar{\hat{\psi}}_2$ is compactly supported. If $s = -\tilde{n}_{pq}$, then

$$\langle \delta(p^{\mathrm{T}}\cdot)\delta(q^{\mathrm{T}}\cdot) \frac{1}{r^{\mathrm{T}}\cdot}, \hat{\psi}_{a,s,t} \rangle \sim a^{5/6} \phi_1(\langle t, n_{pq} \rangle / a),$$

where ϕ_1 defined by $\hat{\phi}_1(\xi) := \bar{\hat{\psi}}_1(\xi)/\xi \in \mathcal{S}$ is rapidly decaying. Thus, the above expression decays rapidly as $a \to 0$ except for $n_{pq}^{\mathrm{T}} t = 0$, i.e., t is in the pq-plane, where the decay is $a^{5/6}$.

3. For $I_3 := \langle \delta(p^{\mathrm{T}}\cdot)\frac{1}{q^{\mathrm{T}}\cdot}\frac{1}{r^{\mathrm{T}}\cdot}, \hat{\psi}_{a,s,t}\rangle$ we get with $\omega_3 = -(\omega_1 + p_1\omega_2)/p_2$ that

$$I_3 = a^{5/6}\int_{\mathbb{R}^2} e^{2\pi i\langle t,\omega\rangle}\,\bar{\hat{\psi}}_1(a\omega_1)\bar{\hat{\psi}}_2\left(a^{-2/3}\left(s+\frac{1}{\omega_1}\begin{pmatrix}\omega_2\\ \omega_3\end{pmatrix}\right)\right)\frac{1}{q^{\mathrm{T}}\omega}\,\frac{1}{r^{\mathrm{T}}\omega}\,d\omega_1 d\omega_2.$$

Substituting first $\xi_2 := a^{-2/3}(s_1+\omega_2/\omega_1)$ and then $\xi_1 := a\omega_1$ this becomes

$$\begin{aligned} I_3 = a^{3/2}\int_{\mathbb{R}^2} & e^{2\pi i\xi_1\left(t_1-\frac{t_3}{p_2}-s_1(t_2-\frac{p_1t_3}{p_2})\right)/a}\, e^{2\pi i\xi_1\xi_2(t_2-\frac{p_1t_3}{p_2})/a^{1/3}}\;\frac{\bar{\hat{\psi}}_1(\xi_1)}{\xi_1} \\ & \times\,\bar{\hat{\psi}}_2\begin{pmatrix}\xi_2\\ a^{-2/3}(-\frac{1}{p_2}+\frac{p_1}{p_2}s_1+s_2)-\frac{p_1}{p_2}\xi_2\end{pmatrix}\frac{1}{g_{pq}(\xi_2)}\,\frac{1}{g_{pr}(\xi_2)}\,d\xi_1 d\xi_2\end{aligned}$$

where $g_{pq}(\xi_2) := 1-\frac{q_2}{p_2}-s_1(q_1-\frac{p_1q_2}{p_2})+a^{2/3}\xi_2(q_1-\frac{p_1q_2}{p_2})$. If $1-p_1s_1-p_2s_2\neq 0$, then $\bar{\hat{\psi}}_2\big((\xi_2, a^{-2/3}(-\frac{1}{p_2}+\frac{p_1}{p_2}s_1+s_2)-\frac{p_1}{p_2}\xi_2)^{\mathrm{T}}\big)$ becomes zero for sufficiently small a by the support property of $\hat{\psi}_2$.
Let $1-p_1s_1-p_2s_2 = 0$.
3.1. If $1-\frac{q_2}{p_2}-s_1(q_1-\frac{p_1q_2}{p_2})\neq 0$, i.e., $s_1\neq -\frac{p_2-q_2}{p_1q_2-p_2q_1}$ and $1-\frac{r_2}{p_2}-s_1(r_1-\frac{p_1r_2}{p_2})\neq 0$, i.e., $s_1\neq -\frac{p_2-r_2}{p_1r_2-p_2r_1}$, then the function ϕ_2 defined by $\hat{\phi}_2 := \frac{\bar{\hat{\psi}}_2\left(\xi_2(1,-\frac{p_1}{p_2})^{\mathrm{T}}\right)}{g_{pq}(\xi_2)\,g_{pr}(\xi_2)}\in\mathscr{S}$ is rapidly decaying and we obtain

$$I_3 = a^{3/2}\int_{\mathbb{R}^1} e^{2\pi i\xi_1\left(t_1-\frac{t_3}{p_2}-s_1(t_2-\frac{p_1t_3}{p_2})\right)/a}\;\frac{\bar{\hat{\psi}}_1(\xi_1)}{\xi_1}\;\phi_2\left(\frac{\xi_1(t_2p_2-p_1t_3)}{p_2\,a^{1/3}}\right)d\xi_1$$

If $t_2p_2-p_1t_3\neq 0$, then

$$\phi_2\left(\frac{\xi_1(t_2p_2-p_1t_3)}{p_2\,a^{1/3}}\right)\le C\,\frac{a^{2r/3}}{a^{2r/3}+\|\xi_1(t_2-p_1t_3/p_2)\|^{2r}}\qquad\forall r\in\mathbb{N}$$

and since $\bar{\hat{\psi}}_1(\xi_1)=0$ for $\xi_1\in[-a_0,a_0]$, we see that I_3 is rapidly decaying as $a\to 0$.
If $t_2p_2-p_1t_3 = 0$, then

$$I_3\sim a^{3/2}\phi_1\left(\frac{t_1-\frac{t_3}{p_2}}{a}\right)$$

which decays rapidly as $a\to 0$ except for $t_1p_2 = t_3$. Now $t_2p_2-p_1t_3 = 0$ and $t_1p_2 = t_3$ imply that $t = c\,p$, $c\in\mathbb{R}$. In this case we have that $I_3\sim a^{3/2}$.
3.2. If $s_1 = -\frac{p_2-q_2}{p_1q_2-p_2q_1}$ and consequently $s = -\tilde{n}_{pq}$, then

$$\begin{aligned}
I_3 &\sim a^{5/6} \int_{\mathbb{R}^2} e^{2\pi i \xi_1 \left(t_1 - \frac{t_3}{p_2} - s_1 (t_2 - \frac{p_1 t_3}{p_2})\right)/a} e^{2\pi i \xi_1 \xi_2 (t_2 - \frac{p_1 t_3}{p_2})/a^{1/3}} \frac{\bar{\hat{\psi}}_1(\xi_1)}{\xi_1} \\
&\quad \times \frac{\bar{\hat{\psi}}_2\left(\xi_2 (1, -p_1/p_2)^{\mathrm{T}}\right)}{g_{pr}(\xi_2)} \frac{1}{\xi_2} \, d\xi_2 d\xi_1 \\
&\sim a^{5/6} \int_{\mathbb{R}} e^{2\pi i \xi_1 \left(t_1 - \frac{t_3}{p_2} - s_1 (t_2 - \frac{p_1 t_3}{p_2})\right)/a} \frac{\bar{\hat{\psi}}_1(\xi_1)}{\xi_1} (\phi_2 * \mathrm{sgn}) \left(\frac{\xi_1 (p_2 t_2 - p_1 t_3)}{p_2 a^{1/3}} \right) d\xi_1 \\
&\lesssim a^{5/6} \phi_1 \left(\frac{t_1 - \frac{t_3}{p_2} - s_1 (t_2 - \frac{p_1 t_3}{p_2})}{a} \right),
\end{aligned}$$

where ϕ_2 and ϕ_1 are defined by $\hat{\phi}_2(\xi_2) := \frac{\bar{\hat{\psi}}_2\left(\xi_2(1,-p_1/p_2)^{\mathrm{T}}\right)}{g_{pr}(\xi_2)} \in \mathscr{S}$ and $\hat{\phi}_1(\xi_1) := \frac{\bar{\hat{\psi}}_1(\xi_1)}{\xi_1} \in \mathscr{S}$. The last expression decays rapidly as $a \to 0$ except for $t_1 - \frac{t_3}{p_2} - s_1(t_2 - \frac{p_1 t_3}{p_2}) = 0$, where $I_3 \lesssim a^{5/6}$. Together with the conditions on s the later is the case if $n_{pq}^{\mathrm{T}} t = 0$.

4. Finally, we examine $I_4 := \langle \frac{1}{p^{\mathrm{T}}\cdot} \frac{1}{q^{\mathrm{T}}\cdot} \frac{1}{r^{\mathrm{T}}\cdot}, \hat{\psi}_{a,s,t} \rangle$. We obtain

$$I_4 = a^{5/6} \int_{\mathbb{R}^3} e^{2\pi i \langle t, \omega \rangle} \bar{\hat{\psi}}_1(a\omega_1) \bar{\hat{\psi}}_2 \left(a^{-2/3} \left(s + \frac{1}{\omega_1} \begin{pmatrix} \omega_2 \\ \omega_3 \end{pmatrix} \right) \right) \frac{1}{p^{\mathrm{T}}\omega} \frac{1}{q^{\mathrm{T}}\omega} \frac{1}{r^{\mathrm{T}}\omega} \, d\omega$$

and further by substituting $\xi_j := a^{-2/3}(s_{j-1} + \omega_j/\omega_1)$, $j = 2,3$ and $\xi_1 := a\omega_1$

$$\begin{aligned}
I_4 = a^{13/6} \int_{\mathbb{R}^3} & e^{2\pi i \xi_1 \left(t_1 + t_2 (a^{2/3}\xi_2 - s_1) + t_3 (a^{2/3}\xi_3 - s_2)\right)/a} \frac{\bar{\hat{\psi}}_1(\xi_1)}{\xi_1} \\
& \times \frac{\bar{\hat{\psi}}_2\left((\xi_2, \xi_3)^{\mathrm{T}}\right)}{g_p(\xi_2,\xi_3)\, g_q(\xi_2,\xi_3)\, g_r(\xi_2,\xi_3)} \, d\xi,
\end{aligned}$$

where $g_p(\xi_2, \xi_3) := 1 - p_1 s_1 - p_2 s_2 + a^{2/3}(\xi_2 p_1 + \xi_3 p_2)$.

4.1. If $1 - p_1 s_1 - p_2 s_2 \neq 0$, $1 - q_1 s_1 - q_2 s_2 \neq 0$ and $1 - r_1 s_1 - r_2 s_2 \neq 0$, then ϕ_2 defined by $\hat{\phi}_2(\xi_2, \xi_3) := \frac{\bar{\hat{\psi}}_1\left((\xi_2,\xi_3)^{\mathrm{T}}\right)}{g_p(\xi_2,\xi_3)\, g_q(\xi_2,\xi_3)\, g_r(\xi_2,\xi_3)} \in \mathscr{S}$ is rapidly decaying and

$$I_4 = a^{13/6} \int_{\mathbb{R}} e^{2\pi i \xi_1 (t_1 - t_2 s_1 - t_3 s_2)/a} \frac{\bar{\hat{\psi}}_1(\xi_1)}{\xi_1} \phi_2(\xi_1 (t_2, t_3)/a^{1/3}) \, d\xi_1,$$

Similarly as before, we see that I_4 decays rapidly as $a \to 0$ if $(t_2, t_3) \neq (0,0)$. For $t_2 = t_3 = 0$ we conclude that $I_4 \sim a^{13/6} \phi_1\left((t_1 - t_2 s_1 - t_3 s_2)/a\right)$. The right-hand side is rapidly decaying as $a \to 0$ except for $t_1 - t_2 s_1 - t_3 s_2 = 0$, i.e., for $t = (0,0,0)^{\mathrm{T}}$, where $I_4 \sim a^{13/6}$.

4.2. If $1 - p_1 s_1 - p_2 s_2 = 0$ and $1 - q_1 s_1 - q_2 s_2 \neq 0$, $1 - r_1 s_1 - r_2 s_2 \neq 0$, we obtain with $\hat{\phi}_2(\xi_2, \xi_3) := \frac{\bar{\hat{\psi}}_1\left((\xi_2,\xi_3)^{\mathrm{T}}\right)}{g_q(\xi_2,\xi_3)\, g_r(\xi_2,\xi_3)} \in \mathscr{S}$ that

$$
\begin{aligned}
I_4 &= a^{3/2} \int_{\mathbb{R}^3} e^{2\pi i \xi_1 (t_1 - t_2 s_1 - t_3 s_2)/a} \, e^{2\pi i \xi_1 (t_2 \xi_2 + t_3 \xi_3)/a^{1/3}} \, \frac{\bar{\hat{\psi}}_1(\xi_1)}{\xi_1} \\
&\quad \times \hat{\phi}_2(\xi_2, \xi_3) \frac{1}{p_1 \xi_2 + p_2 \xi_3} \, d\xi \\
&\sim a^{3/2} \int_{\mathbb{R}} e^{2\pi i \xi_1 (t_1 - t_2 s_1 - t_3 s_2)/a} \, \frac{\bar{\hat{\psi}}_1(\xi_1)}{\xi_1} \, (\phi_2 * h)(\xi_1 (t_2, t_3)/a^{1/3}) \, d\xi_1 \\
&\lesssim a^{3/2} \phi_1 (t_1 - t_2 s_1 - t_3 s_2)/a),
\end{aligned}
$$

where $h(u,v) := \mathrm{sgn}(-v/p_2)\,\delta(t_2 - p_1 t_3/p_2)$. Thus I_4 decays rapidly as $a \to 0$ except for $t_1 - t_2 s_1 - t_3 s_2 = 0$.

4.3. Let $1 - p_1 s_1 - p_2 s_2 = 0$ and $1 - q_1 s_1 - q_2 s_2 = 0$, i.e., $s = -\tilde{n}_{pq}$. Then we obtain with $\hat{\phi}_2(\xi_2, \xi_3) := \frac{\bar{\hat{\psi}}_1((\xi_2,\xi_3)^{\mathrm{T}})}{g_r(\xi_2,\xi_3)} \in \mathscr{S}$ that

$$
\begin{aligned}
I_4 &= a^{5/6} \int_{\mathbb{R}^3} e^{2\pi i \xi_1 (t_1 - t_2 s_1 - t_3 s_2)/a} \, e^{2\pi i \xi_1 (t_2 \xi_2 + t_3 \xi_3)/a^{1/3}} \, \frac{\bar{\hat{\psi}}_1(\xi_1)}{\xi_1} \\
&\quad \times \hat{\phi}_2(\xi_2, \xi_3) \frac{1}{p_1 \xi_2 + p_2 \xi_3} \frac{1}{q_1 \xi_2 + q_2 \xi_3} \, d\xi \\
&= a^{5/6} \int_{\mathbb{R}} e^{2\pi i \xi_1 (t_1 - t_2 s_1 - t_3 s_2)/a} \, \frac{\bar{\hat{\psi}}_1(\xi_1)}{\xi_1} (\phi_2 * h)(\xi_1 (t_2, t_3)/a^{1/3}) \, d\xi_1 \\
&\lesssim a^{5/6} \phi_1 (t_1 - t_2 s_1 - t_3 s_2)/a),
\end{aligned}
$$

where $h(u,v) := \mathrm{sgn} \frac{p_2 u - p_1 v}{p_1 q_2 - q_1 p_2} \mathrm{sgn} \frac{q_2 u - q_1 v}{p_1 q_2 - q_1 p_2}$. If $t_1 - t_2 s_1 - t_3 s_2 = 0$, i.e., $n_{pq}^{\mathrm{T}} t = 0$, then $I_4 \lesssim a^{5/6}$, otherwise we have a rapid decay as $a \to 0$. This finishes the proof. □

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