The Topological Tverberg Theorem and Winding Numbers

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Abstract

The Topological Tverberg Theorem claims that any continuous map of a $(q-1)(d+1)$-simplex to $\mathbb{R}^d$ identifies points from q disjoint faces. (This has been proved for affine maps, for $d \leq 1$, and if q is a prime power, but not yet in general.)

The Topological Tverberg Theorem can be restricted to maps of the d-skeleton of the simplex. We further show that it is equivalent to a "Winding Number Conjecture" that concerns only maps of the $(d-1)$-skeleton of a $(q-1)(d+1)$-simplex to $\mathbb{R}^d$. "Many Tverberg partitions" arise if and only if there are "many q-winding partitions."

The $d = 2$ case of the Winding Number Conjecture is a problem about drawings of the complete graphs K_{3q-2} in the plane. We investigate graphs that are minimal with respect to the winding number condition.

1 Introduction

Our starting point is the following theorem from affine geometry.

Theorem 1.1 (Tverberg Theorem). *Let d and q be positive integers. Any $(d+1)(q-1)+1$ points in $\mathbb{R}^d$ can be partitioned into q disjoint sets whose convex hulls have a point in common.*

This result is from 1966, due to Helge Tverberg [10]. Today, a number of different proofs are known, including another one by Tverberg [11]. We refer to Matoušek [4, Sect. 6.5] for background, for a state-of-the-art discussion, and for further references.

By Δ_N we denote the N-dimensional simplex, by $\Delta_N^{\leq k}$ its k-skeleton. Usually we will not distinguish between a simplicial complex and its realization. One can express the Tverberg Theorem in terms of a linear map:

Theorem 1.2 (Tverberg Theorem; equivalent version I). *For every linear map*

$$f : \Delta_{(d+1)(q-1)} \to \mathbb{R}^d$$

there are q disjoint faces of $\Delta_{(d+1)(q-1)}$ such that their images have a point in common.

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Definition 1.3 (Tverberg partitions; Tverberg points). For $d \geq 0$ and $k \geq 0$, let

$$f : \Delta^{\leq k}_{(d+1)(q-1)} \longrightarrow \mathbb{R}^d$$

be a map. A set S of q disjoint faces σ of $\Delta^{\leq k}_{(d+1)(q-1)}$ is a *Tverberg partition* for the map f if the images of the faces in S have a point in common, that is, if

$$\bigcap_{\sigma \in S} f(\sigma) \neq \emptyset.$$

Every point in this nonempty intersection is called a *Tverberg point* for the map f.

In terms of this definition, Tverberg's theorem has the following brief statement:

Theorem 1.4 (Tverberg Theorem; equivalent version II). *For every linear map*

$$f : \Delta_{(d+1)(q-1)} \longrightarrow \mathbb{R}^d$$

there is a Tverberg partition.

The "Topological Tverberg Theorem" refers to the validity of this statement for the greater generality of *continuous* maps f. Furthermore, "Sierksma's dutch cheese problem" asks for the minimal number of Tverberg partions, for given $d \geq 1$ and $q \geq 2$.

1.1 The Topological Tverberg Theorem

Conjecture 1.5 ("Topological Tverberg Theorem"). *For every continuous map*

$$f : \Delta_{(d+1)(q-1)} \longrightarrow \mathbb{R}^d$$

there is a Tverberg partition.

For $d = 0$ this is trivial. For $d = 1$ it is equivalent to the mean value theorem for continuous functions $f : \mathbb{R} \to \mathbb{R}$: The median point is a Tverberg point.

For prime q (and arbitrary d) the conjecture was first established by Bárány, Shlosman and Szűcs [1], using deleted products. A proof using deleted joins and the $\mathbb{Z}_q$-index is given in [4]. For prime powers q the conjecture was first proved by Özaydin [6]; different proofs are Volovikov [12] and Sarkaria [8] (see de Longueville [2]). Thus the above conjecture, which has been *proved* only for prime powers q, is known as the "Topological Tverberg Theorem."

Furthermore, it is known (and will be used below) that lower dimensional cases follow from higher dimensional ones:

Proposition 1.6 (de Longueville [2, Prop. 2.5]). *If the Topological Tverberg Theorem holds for q and d, then it also holds for q and $d-1$.*

All cases with $d \geq 2$ and non-primepower q remain open. Thus the smallest unresolved case is $d = 2$, $q = 6$: It deals with maps of the 15-dimensional simplex to $\mathbb{R}^2$. Below, we will reduce this case to a question about the planar drawings of the complete graph K_{16}.

According to Matoušek, "the validity of the Topological Tverberg Theorem for arbitrary (nonprime) q is one of the most challenging problems in this field" (Topological Combinatorics) [4, p.154].

1.2 Reduction to the $(d-1)$-skeleton

The classical version of the Topological Tverberg Theorem deals with maps from the entire $(d+1)(q-1)$-dimensional simplex to $\mathbb{R}^d$. It may seem quite obvious that this can be reduced to the d-skeleton: In a case of "general position" Tverberg partitions can only involve faces of dimension at most d. We establish this in Proposition 2.2.

The main result of our paper is a reduction one step further: We prove that the Topological Tverberg Theorem is equivalent to the "Winding Number Conjecture," Conjecture 1.8, which concerns maps of the $(d-1)$-skeleton of $\Delta_{(d+1)(q-1)}$.

However, the "obvious idea" for a proof does not quite work; consequently, the equivalence is not necessarily valid on a dimension-by-dimension basis. In particular, although the $d=2$ case of the Topological Tverberg Theorem would clearly imply the validity of the Winding Number Conjecture for $d=2$, we do not prove the converse implication.

Definition 1.7 (Winding number of f with respect to a point). Let $f: S^{d-1} \to \mathbb{R}^d$ be a continuous map and let p be a point in $\mathbb{R}^d$. If f does not attain p, then f defines a (singular) cycle $[f]$ in the reduced homology group $\widetilde{H}_{d-1}(\mathbb{R}^d \setminus \{p\}; \mathbb{Z}) \cong \mathbb{Z}$, and thus we define the *winding number of f* with respect to p as

$$W(f,p) \;:=\; [f] \in \mathbb{Z}.$$

The sign of $W(f,p)$ depends on the orientation of S^{d-1} and of $\mathbb{R}^d$, but the expression

$$\text{“}W(f,x) \;=\; 0\text{”}$$

is independent of this choice. In particular, for $d=1$ we get that $W(f,p)$ is zero if the two points $f(S^0)$ lie in the same component of $\mathbb{R} \setminus \{p\}$. Otherwise we say that $W(f,p) \neq 0$.

For any d-simplex Δ_d we have $\partial\Delta_d = \Delta_d^{\leq d-1} \cong S^{d-1}$; the winding number $W(f,x)$ for maps $f: \partial\Delta_d \to \mathbb{R}^d$ and points $x \notin f(\partial\Delta_d)$ is defined the same way. Again it is well-defined up to sign, so the condition "$W(f,x)=0$" is independent of orientations.

Conjecture 1.8 (Winding Number Conjecture). *For any positive integers d and q and every continuous map $f: \Delta_{(d+1)(q-1)}^{\leq d-1} \to \mathbb{R}^d$ there are q disjoint faces $\sigma_1, \dots, \sigma_q$ of $\Delta_{(d+1)(q-1)}^{\leq d}$ and a point $p \in \mathbb{R}^d$ such that for each i, one of the following holds:*

- $\dim(\sigma_i) \leq d-1$ *and* $p \in f(\sigma_i)$,
- $\dim(\sigma_i) = d$, *and either* $p \in f(\partial\sigma_i)$, *or* $p \notin f(\partial\sigma_i)$ *and* $W(f|_{\partial\sigma_i}, p) \neq 0$.

A set $S = \{\sigma_1, \dots, \sigma_q\}$ of faces for which some p satisfies the conditions of the Winding Number Conjecture will be referred to as a *winding partition*; p will be called a *winding point*.

Example 1.9. In the case $d=2$, the continuous map $\Delta_{(d+1)(q-1)}^{\leq d-1} \to \mathbb{R}^d$ is really a drawing of $K_{3(q-1)+1}$, the complete graph with $3(q-1)+1 = 3q-2$ vertices. In general, such a drawing may be quite degenerate; it need not be injective (an embedding), even locally. If the drawing is "in general position" (in a way made precise in the next section), then the Winding Number Conjecture says that in the drawing of K_{3q-2}

- either $q-1$ triangles (that is, drawings of K_3 subgraphs) wind around one vertex,
- or $q-2$ triangles wind around the intersection of two edges,

with the triangles, the edges and the vertex being pairwise disjoint in K_{3q-2}.

For the "alternating linear" drawing of K_{3q-2} (defined in [7]; see Figure 1) the Winding Number Conjecture is satisfied: The $(2q-1)$st vertex from the left is a winding point. For example, the $q-1$ disjoint triangles $\langle 1, 2, 3q-2\rangle$, $\langle 3, 4, 3q-3\rangle$, ..., $\langle 2q-3, 2q-2, 3q-q\rangle$ wind around it. (This is not surprising, since the alternating linear drawing does have a representation with straight edges, so the Tverberg Theorem applies, and implies the Winding Number Conjecture for this example.)

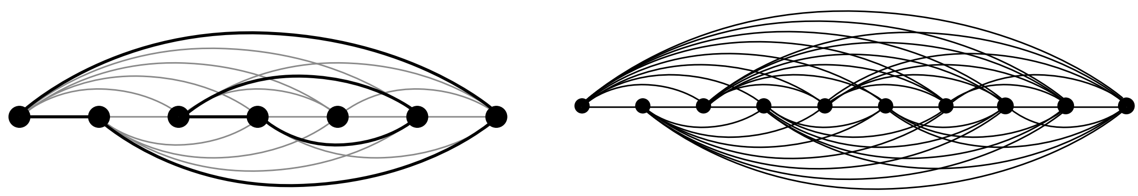

Figure 1: The alternating linear drawings of K_7 and K_{10}. The thick edges in the drawing of K_7 form a winding partition.

Definition 1.10. For any map $f : \partial\Delta_d \to \mathbb{R}^d$, we define

$$W_{\neq 0}(f) \quad := \quad f(\partial\Delta_d) \;\cup\; \{x \in \mathbb{R}^d \setminus f(\partial\Delta_d) \,:\, W(f,x) \neq 0\}.$$

Remark 1. It will be advantageous that $W_{\neq 0}(f)$ is a closed set containing $f(\partial\Delta_d)$, especially in degenerate cases where $\{x \in \mathbb{R}^d \setminus f(\partial\Delta_d) \,:\, W(f,x) \neq 0\}$ might be empty. This is why we add $f(\partial\Delta_d)$ to the definition of $W_{\neq 0}(f)$ and include "$p \in f(\partial\sigma_i)$" in our wording of the Winding Number Conjecture.

Conjecture 1.11 (Winding Number Conjecture, equivalent version). *For every continuous map $f : \Delta^{\leq d-1}_{(d+1)(q-1)} \to \mathbb{R}^d$ there are q disjoint faces $\sigma_1, \ldots, \sigma_q$ of $\Delta^{\leq d}_{(d+1)(q-1)}$ such that*

$$\bigcap_{\dim(\sigma_i)<d} f(\sigma_i) \;\cap\; \bigcap_{\dim(\sigma_i)=d} W_{\neq 0}(f|_{\partial\sigma_i}) \quad \neq \quad \emptyset.$$

This conjecture can be proved easily for $d = 1$ (see Proposition 4.1). Our main result is the following theorem, to be proved in the next two sections.

Theorem 1.12. *For each $q \geq 2$, the Winding Number Conjecture is equivalent to the Topological Tverberg Theorem.*

Remark 2. The basic idea rests on the following two speculations.

- Let $F : \Delta_{(d+1)(q-1)} \to \mathbb{R}^d$ be a continuous map. Every winding partition for $F|_{\Delta^{\leq d-1}_{(d+1)(q-1)}}$ is a Tverberg partition for F.
- Let $f : \Delta^{\leq d-1}_{(d+1)(q-1)} \to \mathbb{R}^d$ be a continuous map. Then f can be extended to a continuous map $F : \Delta_{(d+1)(q-1)} \to \mathbb{R}^d$ such that every Tverberg partition for F is a winding partition for f.

The first statement turns out to be true, but the second one needs adjustments, as we will see in the course of the proof.

1.3 How many Tverberg partitions are there?

Sierksma conjectured that for every linear map $f : \Delta_{(d+1)(q-1)} \to \mathbb{R}^d$ there are at least $((q-1)!)^d$ Tverberg partitions. This number is attained for the configuration of $d+1$ tight clusters, with $q-1$ points each, placed at the vertices of a simplex, and one point in the middle.

For $d = 1$ the mean value theorem implies Sierksma's conjecture. In almost all other cases, Sierksma's conjecture is still unresolved at the time of writing. Nevertheless, for prime powers q, a lower bound is known (for the prime case compare Matoušek [4, Theorem 6.6.1]):

Theorem 1.13 (Vučić and Živaljević [13], Hell [3]). *If $q = p^r$ is a prime power, then for every continuous map $f : \Delta_{(d+1)(q-1)} \to \mathbb{R}^d$ there are at least*

$$\frac{1}{(q-1)!}\Big(\frac{q}{r+1}\Big)^{\lceil (d+1)(q-1)/2 \rceil}$$

Tverberg partitions.

In Section 4 we discuss how such lower bounds translate into lower bounds for the number of winding partitions, with special attention to the case $d = 2$, where a direct translation is not possible. We show that Sierksma's conjecture is nevertheless equivalent to a corresponding lower bound conjecture for the number of winding partions in any map of the $(d-1)$-skeleton of a $(d+1)(q-1)$-simplex.

1.4 Minimal q-winding graphs

The Winding Number Conjecture for $d = 2$ is a problem about the drawings of complete graphs K_{3q-2} in the plane; it asks whether they are "q-winding" — see Definition 5.1. In Section 5 we characterize the 2-winding graphs as the non-outerplanar ones, so K_4 is minimal 2-winding. However, we also show that K_7 is not minimal 3-winding: Its minimal 3-winding subgraph is unique, it is K_7 minus a maximal matching. So the complete graphs K_{3q-2} are not minimal q-winding in general.

2 Reduction to the d-skeleton

The object of this section is to verify that the Topological Tverberg Theorem guarantees the existence of a Tverberg partition in the d-skeleton of $\Delta_{(d+1)(q-1)}$.

Conjecture 2.1 (d-Skeleton Conjecture). *For every continuous map*

$$f : \Delta^{\leq d}_{(d+1)(q-1)} \longrightarrow \mathbb{R}^d$$

there is a Tverberg partition.

Proposition 2.2. *For each $q \geq 2$ and $d \geq 1$, the d-Skeleton Conjecture is equivalent to the Topological Tverberg Theorem.*

It is obvious that the d-Skeleton Conjecture implies the Topological Tverberg Theorem. The converse is harder. For this, we verify that any map in question may be approximated by a piecewise linear map in general position, for which codimension counts yield that only simplices of dimension at most d can be involved in a Tverberg partition. (This needs some care with the definition of "general position," but is rather straightforward otherwise.)

2.1 Maps in general position

For the first lemma, we need the following definition.

Definition 2.3 (Linear maps; general position). Let Δ be a simplicial complex. A map $f : \Delta \to \mathbb{R}^d$ is *linear* if it is linear on every face of Δ. Such a linear map f is *in general position* if for every set of disjoint faces $\{\sigma_1, \sigma_2, \ldots, \sigma_q\}$ of Δ the inequality

$$\operatorname{codim}\Big(\bigcap_{i=1}^{q} f(\sigma_i)\Big) \quad \geq \quad \sum_{i=1}^{q} \operatorname{codim}(f(\sigma_i))$$

holds, where $\operatorname{codim}(\tau) := d - \dim(\tau)$ if $\tau \subset \mathbb{R}^d$. We use the convention that $\dim(\emptyset) = -\infty$ and thus $\operatorname{codim}(\emptyset) = \infty$. Thus in the case $\bigcap_{i=1}^q f(\sigma_i) = \emptyset$ the general position condition holds independently of the right hand side, as then $\operatorname{codim}(\bigcap_{i=1}^q f(\sigma_i)) = \infty$.

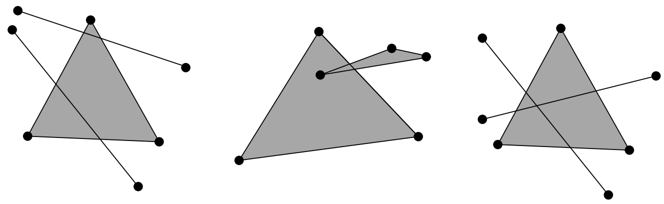

Figure 2: Images $f(\Delta)$ of linear maps $f : \Delta \to \mathbb{R}^2$ in general position.

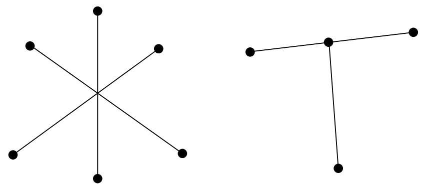

Figure 3: Images $f(\Delta)$ of linear maps $f : \Delta \to \mathbb{R}^2$ *not* in general position. In the last picture, the complex Δ consists of two edges.

Definition 2.4 (Piecewise linear maps; general position). Let Δ be a simplicial complex. A map $f : \Delta \to \mathbb{R}^d$ is *piecewise linear* if there is a subdivision $s : \Delta' \to \Delta$ such that the composition $f \circ s : \Delta' \to \mathbb{R}^d$ is a linear map. Furthermore, we say that f is *in general position* if there is a subdivision s such that the linear map $f \circ s$ is in general position.

Whether $f \circ s$ is in general position depends on the subdivision s. For example, the map f depicted on the very left in Figure 4 combined with the second barycentric subdivision gives a linear map not in general position, although f itself is in general position.

The definition of general position made here may seem overly restrictive for the purpose of this section, but we need it in Proposition 3.5.

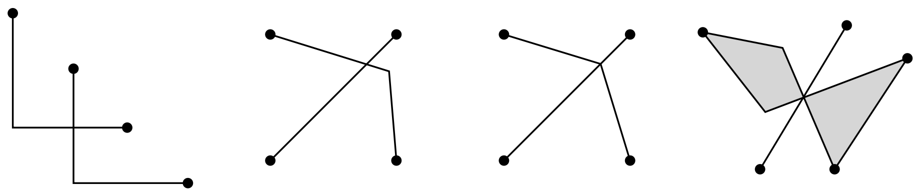

Figure 4: Images $f(\Delta)$ of piecewise linear maps $f : \Delta \to \mathbb{R}^2$. In the first three pictures, Δ consists of two edges, in the last picture Δ consists of a triangle and an edge. The two pictures on the left are in general position, the two on the right are not.

Lemma 2.5. *Let Δ be a simplicial complex and $f : \Delta \to \mathbb{R}^d$ a piecewise linear map in general position. If $\{\sigma_1, \sigma_2, \ldots, \sigma_q\}$ is a set of disjoint faces of Δ, then*

$$\operatorname{codim}\Big(\bigcap_{i=1}^{q} f(\sigma_i)\Big) \;\geq\; \sum_{i=1}^{q} \max\{0, d - \dim \sigma_i\}.$$

Here $\bigcap_{i=1}^q f(\sigma_i)$ might have parts of different dimension. For polyhedral sets A and B we have $\dim(A \cup B) = \max\{\dim A, \dim B\}$ and thus $\operatorname{codim}(A \cup B) = \min\{\operatorname{codim} A, \operatorname{codim} B\}$.

Proof. Let $s : \Delta' \to \Delta$ be a subdivision such that $f \circ s$ is a linear map in general position.

$$\begin{aligned}
\operatorname{codim}(\bigcap_{i=1}^{q} f(\sigma_i)) &= \min_{\substack{\tilde\sigma_i \subset \sigma_i \\ \tilde\sigma_i \text{ simplex in } \Delta'}} \operatorname{codim}(\bigcap_{i=1}^{q} f \circ s(\tilde\sigma_i)) \\
&\geq \min_{\tilde\sigma_i \subset \sigma_i} \sum_{i=1}^{q} \operatorname{codim}(f \circ s(\tilde\sigma_i)) \\
&= \min_{\tilde\sigma_i \subset \sigma_i} \sum_{i=1}^{q} (d - \dim(f \circ s(\tilde\sigma_i))) \\
&= \sum_{i=1}^{q} (d - \max_{\tilde\sigma_i \subset \sigma_i} \dim(f \circ s(\tilde\sigma_i))) \\
&\geq \sum_{i=1}^{q} (d - \min\{d, \dim \sigma_i\}) \quad = \quad \sum_{i=1}^{q} \max\{0, d - \dim \sigma_i\}.
\end{aligned}$$

□

We need an approximation lemma to tackle continuous maps. For our purposes, a version for finite (compact) simplicial complexes is sufficient. See [5, §16] for techniques in this context.

Lemma 2.6 (Piecewise Linear Approximation Lemma). *Let Δ be a finite simplicial complex, and let $f : \Delta \to \mathbb{R}^d$ be a continuous map. Then for each $\varepsilon > 0$ there is a piecewise linear map in general position $\tilde f : \Delta \to \mathbb{R}^d$ with $\|\tilde f - f\|_\infty < \varepsilon$, with*

$$\|\tilde f - f\|_\infty \;:=\; \max\{|\tilde f(x) - f(x)| : x \in \Delta\}.$$

2.2 Tverberg partitions in the d-skeleton

Lemma 2.7. *Any Tverberg partition for a piecewise linear map $f : \Delta_{(d+1)(q-1)} \to \mathbb{R}^d$ in general position contains only faces of dimension at most d.*

Proof. Let f be in general position, with a Tverberg partition $\{\sigma_1, \sigma_2, \ldots, \sigma_q\}$. Then

$$\begin{aligned}
d &\overset{(1)}{\geq} \operatorname{codim}(\bigcap_{i=1}^{q} f(\sigma_i)) \\
&\overset{(2)}{\geq} \sum_{i=1}^{q} \max(0, (d - \dim \sigma_i)) \\
&\overset{(3)}{\geq} \sum_{i=1}^{q} (d - \dim \sigma_i) \\
&= qd - (\sum_{i=1}^{q} ((\text{number of vertices of } \sigma_i) - 1)) \\
&\geq qd - ((\text{number of vertices of } \Delta_{(d+1)(q-1)}) - q) \\
&= qd - ((d+1)(q-1) + 1 - q) \quad = \quad d.
\end{aligned}$$

Here (1) holds because $\{\sigma_1, \sigma_2, \ldots, \sigma_q\}$ is a Tverberg partition and thus $\bigcap_{i=1}^{q} f(\sigma_i) \neq \emptyset$. (2) holds because f is in general position. In (3) we have equality only if $d - \dim(\sigma_i) \geq 0$, or equivalently, if $\dim(\sigma_i) \leq d$ for all i, which is what we had to prove. □

Lemma 2.8. *For every continuous map $f : \Delta^{\le d}_{(d+1)(q-1)} \to \mathbb{R}^d$ there is an $\varepsilon_f > 0$ such that the following holds: If $\tilde{f} : \Delta^{\le d}_{(d+1)(q-1)} \to \mathbb{R}^d$ is a continuous map with $\|\tilde{f} - f\|_\infty < \varepsilon_f$, then every Tverberg partition for $\tilde{f}$ is also a Tverberg partition for f.*

Proof. We have to show that for each S that is *not* a Tverberg partition for f, i.e.

$$\bigcap_{\sigma \in S} f(\sigma) = \emptyset,$$

then there is an $\varepsilon_S > 0$ such that S is not a Tverberg partition for any $\tilde{f}$ with $\|\tilde{f} - f\|_\infty < \varepsilon_S$. Since there are only finitely many choices for S, this implies the lemma (with $\varepsilon_f := \min_S \varepsilon_S$).

If $\tilde{f} : \Delta^{\le d}_{(d+1)(q-1)} \to \mathbb{R}^d$ satisfies $\|\tilde{f} - f\|_\infty < \varepsilon$, then

$$\bigcap_{\sigma \in S} \tilde{f}(\sigma) \quad \subseteq \quad \bigcap_{\sigma \in S} \big\{x \in \mathbb{R}^d \,:\, \mathrm{dist}(x, f(\sigma)) \le \varepsilon\big\} \quad =: \; C_\varepsilon.$$

Taking $\varepsilon = \frac{1}{n}$ we get a chain $C_1 \supset C_{\frac{1}{2}} \supset C_{\frac{1}{3}} \supset \cdots$ of compact sets. If all of them are non-empty, then by compactness also the intersection $\bigcap_n C_{\frac{1}{n}}$ is non-empty, and it would consist of Tverberg points: $C_0 = \bigcap_{\sigma \in S} f(\sigma)$. Thus some $C_{\frac{1}{n(S)}}$ is empty, and we can take $\varepsilon_S := \frac{1}{n(S)}$. ☐

Thus we have established that the Topological Tverberg Theorem implies the d-Skeleton Conjecture (Proposition 2.2). Indeed, for every map $f : \Delta^{\le d}_{(d+1)(q-1)} \to \mathbb{R}^d$ there is an ε_f such that the following holds: If a map $\tilde{f} : \Delta_{(d+1)(q-1)} \to \mathbb{R}^d$ has the property that its restriction to the d-skeleton is ε_f-close to f, then the Tverberg partitions in the d-skeleton of $\tilde{f}$ are also Tverberg partitions for f (Lemma 2.8). Such a map $\tilde{f}$ may be chosen to be general position piecewise linear (Lemma 2.6). So if $\tilde{f}$ has any Tverberg partition, then this lies in the d-skeleton (Lemma 2.7), and thus yields a Tverberg partition in the d-skeleton for f.

3 Reduction to the $(d-1)$-skeleton

Now we proceed to prove Theorem 1.12, the equivalence of the d-Skeleton Conjecture with the Winding Number Conjecture.

It is quite clear that the Winding Number Conjecture implies the d-Skeleton Conjecture: Every winding partition is indeed a Tverberg partition. This rests on the fact that if $x \in \mathbb{R}^d \setminus f(\partial\Delta_d)$ with $W(f, x) \neq 0$, then every extension of f to Δ_d must hit x. (Any map $F : \Delta_d \to \mathbb{R}^d \setminus \{x\}$ is nullhomotopic, so its restriction to $\partial\Delta_d$ has winding number 0.)

The proof of the converse is harder. For this we want to show that any map

$$f : \Delta^{\le d-1}_{(d+1)(q-1)} \;\longrightarrow\; \mathbb{R}^d$$

can be extended to a map

$$F : \Delta^{\le d}_{(d+1)(q-1)} \;\longrightarrow\; \mathbb{R}^d$$

such that every Tverberg partition for F is a winding partition for f. This would be easy to do if for each d-dimensional face $\sigma \subseteq \Delta_{(d+1)(q-1)}$, we could arrange that $F(\sigma) \subseteq W_{\neq 0}(f|_{\partial\sigma})$. However, this is not always possible because not every continuous map $f : S^{d-1} \to \mathbb{R}^d$ is nullhomotopic within $W_{\neq 0}(f)$. For this, we look at two examples.

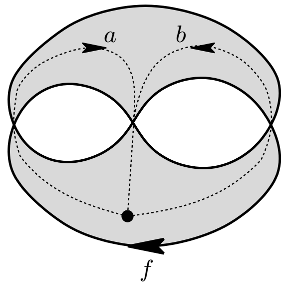

Figure 5: A map $f : S^1 \to \mathbb{R}^2$ that is not nullhomotopic within $W_{\neq 0}(f)$. The shaded area is $W_{\neq 0}(f)$.

Example 3.1. Let $f : S^1 \to \mathbb{R}^2$ be the map illustrated by Figure 5. The topological space $W_{\neq 0}(f)$ is homotopy equivalent to the wedge of two spheres S^1. The fundamental group $\pi_1(W_{\neq 0}(f))$ is $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$, a free product. The element $[f] \in \pi_1(W_{\neq 0}(f))$ can be written as the nonzero term $aba^{-1}b^{-1}$ if we choose generators a, b of $\mathbb{Z} * \mathbb{Z}$ as in the figure.

If we extend f to B^2, then the image covers at least one of the two "holes" in $W_{\neq 0}(f)$ entirely, which are 2-dimensional sets. There is no one-dimensional subset $V \subset \mathbb{R}^2$ such that f is contractible in $W_{\neq 0}(f) \cup V$.

The suspension of this map, $\operatorname{susp} f : S^2 \to \mathbb{R}^3$, does not share this problem: We have

$$W_{\neq 0}(\operatorname{susp} f) = \operatorname{susp} W_{\neq 0}(f) \simeq \operatorname{susp}(S^1 \vee S^1) = S^2 \vee S^2$$

again, but this time the homotopy group $\pi_2(S^2 \vee S^2)$ is not a free product but a direct sum $\mathbb{Z} \oplus \mathbb{Z}$, so $[\operatorname{susp} f] = \hat{a}\hat{b}\hat{a}^{-1}\hat{b}^{-1} = 0$ in $\pi_2(W_{\neq 0}(\operatorname{susp} f))$.

Example 3.2. For $d \geq 4$ the homotopy group $\pi_{d-1}(S^{d-2})$ is nontrivial; for example, the Hopf map $S^3 \to S^2$ is not nullhomotopic. Choose such a map $f : S^{d-1} \to S^{d-2}$ that is not nullhomotopic. Let $i : S^{d-2} \to \mathbb{R}^d$ be an embedding into a $(d-1)$-dimensional linear subspace of $\mathbb{R}^d$. Then $W_{\neq 0}(i \circ f) = i(S^{d-2})$, hence $i \circ f$ can not be contracted in $W_{\neq 0}(i \circ f)$.

An important difference between this example and the previous one is that here, $i \circ f$ can be contracted within the $(d-1)$-dimensional subspace that contains $i(S^{d-2})$; an extension of the range $W_{\neq 0}(i \circ f)$ to a d-dimensional set is not necessary to make the map nullhomotopic.

Because of the problem illustrated by these examples, we take a more technical route. We need an approximation lemma similar to Lemma 2.8; it can be proved along the same lines.

Lemma 3.3. *For every continuous map $f : \Delta^{\leq d-1}_{(d+1)(q-1)} \to \mathbb{R}^d$ there is an $\varepsilon_f > 0$ such that the following holds: If $\tilde{f} : \Delta^{\leq d-1}_{(d+1)(q-1)} \to \mathbb{R}^d$ is a continuous map with $\|\tilde{f} - f\|_\infty < \varepsilon_f$, then every winding partition for $\tilde{f}$ is also a winding partition for f.*

So, if the Winding Number Conjecture holds for piecewise linear maps in general position, then it also holds for all continuous maps.

3.1 The case $d \geq 3$

Definition 3.4 (Triangulations of $\mathbb{R}^d$ in general position). A *triangulation of* $\mathbb{R}^d$ is a simplicial complex Δ with a fixed homeomorphism $\|\Delta\| \cong \mathbb{R}^d$ that is linear on each simplex. We do not distinguish between a face of the triangulation and the corresponding set in $\mathbb{R}^d$.

Triangulations $\Delta_1, \Delta_2, \ldots, \Delta_\ell$ of $\mathbb{R}^d$ are *in general position with respect to each other* if

$$\operatorname{codim}\Big(\bigcap_{i\in S}\sigma_i\Big) \;\geq\; \sum_{i\in S}\operatorname{codim}(\sigma_i)$$

for every subset $S \subset \{1, \ldots, \ell\}$ and faces σ_i of Δ_i.

Proposition 3.5. *For $q \geq 2$ and $d \geq 3$, the d-Skeleton Conjecture implies the corresponding case of the Winding Number Conjecture.*

Proof. By Lemma 3.3 together with the Approximation Lemma 2.6, it suffices to prove the d-dimensional Winding Number Conjecture for the case of piecewise linear maps $f : \Delta^{\leq d-1}_{(d+1)(q-1)} \to \mathbb{R}^d$: It suffices to show the existence of winding partitions for general position piecewise linear maps. Our proof consists of three steps:

1. For every face $\sigma \subset \Delta^{\leq d}_{(d+1)(q-1)}$, choose a triangulation Δ_σ of $\mathbb{R}^d$, such that triangulations for disjoint faces are in general position with respect to each other.
2. Extend f to a continuous map $F : \Delta^{\leq d}_{(d+1)(q-1)} \to \mathbb{R}^d$ that is "compatible" with the Δ_σ.
3. By the d-Skeleton Conjecture, F has a Tverberg partition. Show that every Tverberg partition for F is a winding partition for f.

Step 1: For each face σ of $\Delta^{\leq d}_{(d+1)(q-1)}$ choose a triangulation Δ_σ of $\mathbb{R}^d$ such that the following three conditions are satisfied:

- For each σ of dimension $\dim\sigma \leq d-1$, the set $f(\sigma)$ is a subset of the $\dim(\sigma)$-skeleton of Δ_σ. In the case $\dim\sigma = d$, we need that $f(\partial\sigma)$ is a subset of the $(d-1)$-skeleton of Δ_σ.
- If $\sigma_1, \ldots, \sigma_\ell$ are disjoint faces of $\Delta^{\leq d}_{(d+1)(q-1)}$, then $\Delta_{\sigma_1}, \ldots, \Delta_{\sigma_\ell}$ are in general position with respect to each other. (This is possible because f is in general position. Here we need the restrictive Definition 2.4 of "general position"!)
- For each σ of dimension d, the image $f(\partial\sigma)$ should be contained in a triangulated piecewise linear ball B_σ that is a subcomplex of Δ_σ.

Step 2: Now we extend f to a d-face $\sigma \subset \Delta^{\leq d}_{(d+1)(q-1)}$. For this let $\tau_1, \ldots, \tau_k$ be the d-faces in $B_\sigma \subseteq \Delta_\sigma$ on which the winding number of $f|_{\partial\sigma}$ is zero, that is, the d-faces in $B_\sigma \cap \overline{(\mathbb{R}^d \setminus W_{\neq 0}(f|_{\partial\sigma}))}$. Then we have

$$W_{\neq 0}(f|_{\partial\sigma}) \;\subseteq\; B_\sigma \setminus (\mathring{\tau}_1 \cup \cdots \cup \mathring{\tau}_k) \;=:\; B_\sigma^\circ.$$

If we choose a point x_i in each τ_i, then the set B_σ° is a retract of $B_\sigma \setminus \{x_1, \ldots, x_k\}$; so this has the homotopy type of a wedge of k $(d-1)$-spheres.

The extension of $f|_{\partial\sigma}$ to σ is possible within B_σ° if and only if $f|_{\partial\sigma}$ is contractible in B_σ°, that is, if the homotopy class $[f|_{\partial\sigma}] \in \pi_{d-1}(B_\sigma^\circ)$ vanishes. However, for $d \geq 3$ we have an isomorphism

$$\pi_{d-1}(B_\sigma^\circ) \;\cong\; \mathbb{Z}^k.$$

Furthermore, the boundary spheres of the simplices τ_i form a homology basis for the wedge, and the evaluation

$$\begin{array}{rcl} \pi_{d-1}(B_\sigma^\circ) & \longrightarrow & \mathbb{Z}^k \\ f|_{\partial\sigma} & \longmapsto & \big(W(f, x_1), \ldots, W(f, x_k)\big) \end{array}$$

gives this isomorphism. Thus any map $f|_{\partial\sigma}$ with trivial winding numbers can be extended to σ. By applying this argument to all d-faces of $\Delta^{\leq d}_{(d+1)(q-1)}$, we obtain a continuous map $F : \Delta^{\leq d}_{(d+1)(q-1)} \to \mathbb{R}^d$.

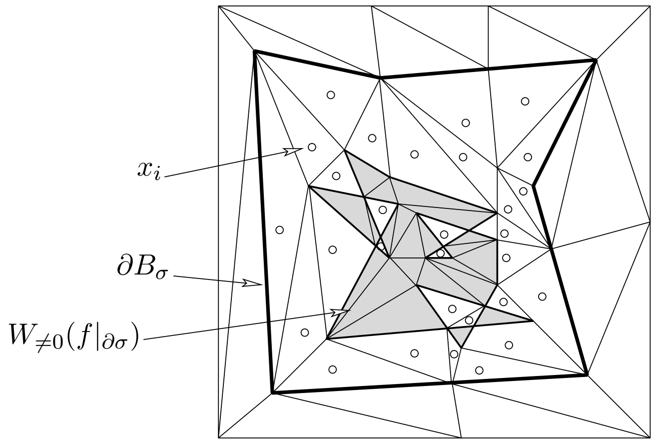

Figure 6: Sketch for the extension problem for $f|_{\partial\sigma}$ within the triangulation Δ_σ. Here B_σ° would consist of the shaded part together with the complete $(d-1)$-skeleton of B_σ.

Step 3: We prove that every Tverberg partition for F is a winding partition for f. Let $p \in \mathbb{R}^d$ be a Tverberg point and $\sigma_1, \ldots, \sigma_q \subset \Delta_{(d+1)(q-1)}^{\leq d}$ a Tverberg partition for F: These exist due to the d-Skeleton Conjecture for continuous maps.[1] We now show that $\sigma_1, \ldots, \sigma_q$ is also a winding partition for f with winding point p:

- $\dim(\sigma_j) \leq d-1$: In that case we immediately have $p \in F(\sigma_j) = f(\sigma_j)$.
- $\dim(\sigma_j) = d$: Suppose $W(f|_{\partial\sigma_j}, p) = 0$. For $1 \leq i \leq q$ let $\tilde{\sigma}_i$ be the face of Δ_{σ_i} that contains p in its relative interior, i.e., the minimal face containing p. We have

$$\begin{aligned} d &\overset{(1)}{\geq} \operatorname{codim}\Big(\bigcap_{i=1}^{q} \tilde{\sigma}_i\Big) \\ &\overset{(2)}{\geq} \sum_{i=1}^{q} \operatorname{codim}(\tilde{\sigma}_i) \quad = \quad \sum_{i=1}^{q} (d - \dim(\tilde{\sigma}_i)) \\ &\overset{(*)}{\geq} \sum_{i=1}^{q} (d - \dim(\sigma_i)) \\ &= \quad qd - ((d+1)(q-1) + 1 - q) \quad = \quad d. \end{aligned}$$

  where (1) holds because $\bigcap_{i=1}^{k} \tilde{\sigma}_i$ contains p and therefore is not empty, and (2) holds because the Δ_{σ_i} are in general position with respect to each other.
  The inequality $(*)$ is an equality if and only if $\dim(\tilde{\sigma}_i) = \dim(\sigma_i)$ for all i and in particular for $i = j$. Hence $\dim(\tilde{\sigma}_j) = \dim(\sigma_j) = d$. Outside of $W_{\neq 0}(f|_{\partial\sigma})$, the image $F(\sigma_i)$ lies entirely in the $(d-1)$-skeleton of Δ_{σ_i}; therefore p must lie in $W_{\neq 0}(f|_{\partial\sigma})$. $\square$

3.2 The case $d = 2$

We do not know whether the cases $d = 2$ of the Winding Number Conjecture and the d-Skeleton Conjecture are equivalent. Thus we take a different route:

Proposition 3.6. *For each $q \geq 2$, if the Winding Number Conjecture holds for $d+1$, then it also holds for d.*

[1] We have to use the version for continuous maps since we can not bring a piecewise linear approximation of F into general position with $F(\sigma) \subset B_\sigma^\circ$.

Proof (suggested by [2, Prop. 2.5]; cf. Prop. 1.6). For any continuous map

$$f : \Delta^{\le d-1}_{(d+1)(q-1)} \longrightarrow \mathbb{R}^d$$

we identify $\mathbb{R}^d$ with $\mathbb{R}^d \times \{0\} = \{x \in \mathbb{R}^{d+1} : x_{d+1} = 0\} \subset \mathbb{R}^{d+1}$, and construct an extension to $F : \Delta^{\le d}_{(d+2)(q-1)} \to \mathbb{R}^{d+1}$, as follows. Choose points $Q_1, \ldots, Q_{q-1}$ (which may coincide) in the upper halfspace $\mathbb{R}^d \times \mathbb{R}^+ = \{x \in \mathbb{R}^{d+1} : x_{d+1} > 0\}$, and an additional cone point Q in the lower halfspace $\mathbb{R}^d \times \mathbb{R}^-$.

We consider $\Delta^{\le d-1}_{(d+1)(q-1)}$ as a subcomplex of $\Delta^{\le d}_{(d+2)(q-1)}$, so the latter has $q-1$ additional vertices $P_1, P_2, \ldots, P_{q-1}$. For F, we map the P_i to Q_i; all faces of $\Delta^{\le d}_{(d+2)(q-1)}$ that involve at least one of the new vertices P_i are mapped accordingly by linear extension. For the d-faces of $\Delta^{\le d}_{(d+1)(q-1)}$ we perform a stellar subdivision, map the new center vertex to Q, and extend canonically.

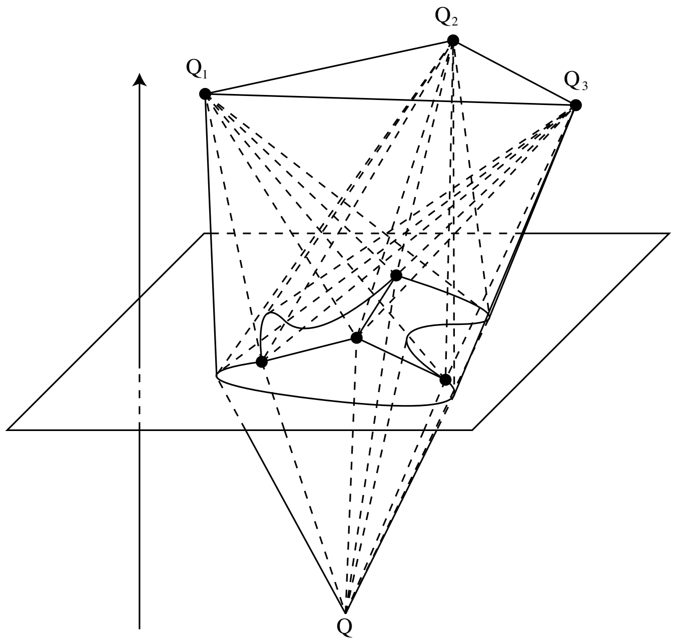

Figure 7: The map F. The plane $\mathbb{R}^2$ contains the image of $\Delta^{\le 1}_4 = K_4$, the three points above the plane are Q_1, Q_2, Q_3 and the point below is Q.

The Winding Number Conjecture for $d+1$ applied to F yields a winding point p in $\mathbb{R}^{d+1}$ with a winding partition consisting of q disjoint faces $\sigma_1, \ldots, \sigma_q$ of $\Delta^{\le d+1}_{(d+2)(q-1)}$.

The winding point cannot be in the upper halfspace, since then all the $F(\sigma_i)$ would need to intersect the upper halfspace, so the disjoint faces σ_i would need to contain distinct vertices P_j, and there are only $q-1$ of these. If p were in the lower halfspace, then all the σ_i would need to be d-faces of $\Delta^{\le d}_{(d+1)(q-1)}$. For these disjoint d-faces we would need $q(d+1)$ vertices in $\Delta_{(d+1)(q-1)}$, which has only $(d+1)(q-1)+1 = q(d+1)-d$ vertices.

Thus p has to be in $\mathbb{R}^d$. Define $\tilde{\sigma}_i := \sigma_i \cap \Delta^{\le d}_{(d+1)(q-1)}$. We claim that $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_q$ are q disjoint faces that form a winding partition for f. For this, we have three cases.

1. If $\dim(\sigma_i) \le d-1$, then $p \in F(\sigma_i) \cap \mathbb{R}^d = F(\sigma_i \cap \Delta^{\le d-1}_{(d+1)(q-1)}) = F(\tilde{\sigma}_i) = f(\tilde{\sigma}_i)$.
2. If $\dim(\sigma_i) = d$, then $p \in F(\sigma_i) \cap \mathbb{R}^d = F(\sigma_i \cap \Delta^{\le d-1}_{(d+1)(q-1)})$. Now either $\sigma_i \subseteq \Delta^{\le d}_{(d+1)(q-1)}$, then we have $F(\sigma_i \cap \Delta^{\le d-1}_{(d+1)(q-1)}) = F(\partial\tilde{\sigma}_i) = f(\partial\tilde{\sigma}_i)$; or $\sigma_i \not\subseteq \Delta^{\le d}_{(d+1)(q-1)}$, so we have $F(\sigma_i \cap \Delta^{\le d-1}_{(d+1)(q-1)}) = F(\tilde{\sigma}_i) = f(\tilde{\sigma}_i)$.

3. For $\dim(\sigma_i) = d+1$ we may assume that p is not in $F(\partial\sigma_i)$. We know that p lies in $W_{\neq 0}(F|_{\partial\sigma_i}) \cap \mathbb{R}^d$, therefore $F(\partial\sigma_i)$ must contain points in both halfspaces. Thus σ_i contains exactly one of the P_j, $\tilde{\sigma}_i$ is d-dimensional, and

$$\begin{aligned} p \in W_{\neq 0}(F|_{\partial\sigma_i}) \cap \mathbb{R}^d &= \{x \in \mathbb{R}^{d+1} : W(F|_{\partial\sigma_i}, x) \neq 0\} \cap \mathbb{R}^d \\ &= \{x \in \mathbb{R}^d : W(f|_{\partial\tilde{\sigma}_i}, x) \neq 0\} = W_{\neq 0}(f|_{\partial\tilde{\sigma}_i}). \end{aligned}$$ □

4 The number of winding partitions and Tverberg partitions

The Winding Number Conjecture, and the analogue of the Sierksma conjecture for winding partitions, are trivial in the case $d = 1$:

Proposition 4.1 (The case d=1).
For every continuous mapping $f : \Delta_{2(q-1)}^{\leq 0} \to \mathbb{R}$, there are at least $(q-1)!$ winding partitions.

Proof. $\Delta_{2(q-1)}^{\leq 0}$ is a set of $2(q-1)+1 = 2q-1$ vertices. $f(\Delta_{2(q-1)}^{\leq 0})$ is a set of $2(q-1)+1$ real numbers (counted with multiplicity). Denote the vertices of $\Delta_{2(q-1)}^{\leq 0}$, ordered by their function value, by $P_1, \dots, P_{q-1}, M, Q_1, \dots, Q_{q-1}$. A partition of these points into q sets is a winding partition for f if one of the sets is $\{M\}$ and all the other sets contain exactly one of the P_i and one of the Q_j. There are $(q-1)!$ such partitions. □

Corollary 4.2 (The case d≥3). *For each $q \geq 2$ and $d \geq 3$, the following three numbers are equal:*
1. *the minimal number of Tverberg partitions for continuous maps $f : \Delta_{(d+1)(q-1)} \to \mathbb{R}^d$,*
2. *the minimal number of Tverberg partitions for continuous maps $f : \Delta_{(d+1)(q-1)}^{\leq d} \to \mathbb{R}^d$,*
3. *the minimal number of winding partitions for continuous maps $f : \Delta_{(d+1)(q-1)}^{\leq d-1} \to \mathbb{R}^d$.*

If $d = 2$, then the first two of these numbers are equal.

Proof. By Lemmas 2.8 and 3.3, the minimal numbers will be achieved for general position maps f. For these, all Tverberg partitions lie in the d-skeleton by Lemma 2.7. The proof of Proposition 3.5 shows that for $d \geq 3$, each $f : \Delta_{(d+1)(q-1)}^{\leq d-1} \to \mathbb{R}^d$ can be extended to an $F : \Delta_{(d+1)(q-1)}^{\leq d} \to \mathbb{R}^d$ such that all Tverberg partitions for F are winding partitions for f. □

If Sierksma's conjecture on the minimal number of Tverberg partitions is correct, then the equivalence established in the previous proposition carries over to the case $d = 2$:

Theorem 4.3. *For each $q \geq 2$, the following three statements are equivalent:*
1. Sierksma's conjecture: *For all positive integers d and q and every continuous map $f : \Delta_{(d+1)(q-1)} \to \mathbb{R}^d$ there are at least $((q-1)!)^d$ Tverberg partitions.*
2. *For every continuous map $f : \Delta_{(d+1)(q-1)}^{\leq d} \to \mathbb{R}^d$ there are at least $((q-1)!)^d$ Tverberg partitions.*
3. *For every continuous map $f : \Delta_{(d+1)(q-1)}^{\leq d-1} \to \mathbb{R}^d$ there are at least $((q-1)!)^d$ winding partitions.*

Proof. By our proof for Theorem 1.12, we know that Statements 1 and 2 are equivalent and that Statement 3 implies Statement 2, which in turn guarantees Statement 3 if $d \neq 2$.

We now prove that the case $d = 3$ of Statement 3 implies the case $d = 2$. By Lemma 3.3, it is sufficient to examine piecewise linear maps $f : \Delta_{3(q-1)}^{\leq 1} \to \mathbb{R}^2$ in general position. Regard f as a map $\Delta_{3(q-1)}^{\leq 1} \to \mathbb{R}^3$ in the way we did in the proof of Proposition 3.6. For each pair e_1, e_2 of 1-dimensional faces of $\Delta_{3(q-1)}^{\leq 1}$, define one of them to be the "upper" and the other one to be the

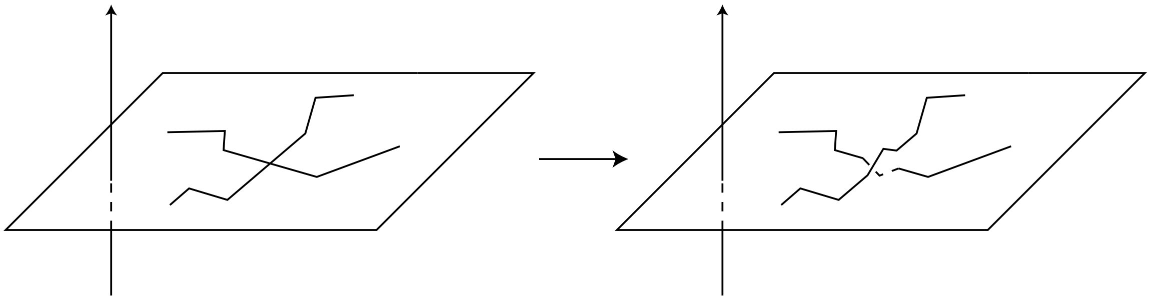

Figure 8: How $\tilde{f}$ is obtained from f

"lower" one of the pair. Now alter f in the following way: For each intersection $P \in f(e_1) \cap f(e_2)$ of the images of two edges, change f slightly so that the image of the "upper" line runs above the image of the "lower" line at P, i.e., has a bigger last coordinate (see Figure 8). We call this new map $\tilde{f} : \Delta^{\leq 1}_{3(q-1)} \to \mathbb{R}^3$.

We continue similar to the proof of Proposition 3.6 and choose points $Q_1, \dots, Q_{q-1}$ high above $\mathbb{R}^2$ and a point Q far below $\mathbb{R}^2$ and extend $\tilde{f}$ to a map $F : \Delta^{\leq 2}_{4(q-1)} \to \mathbb{R}^3$ by taking cones using the Q_i and Q. Let $\{\sigma_1, \dots, \sigma_q\}$ be a winding partition for F and denote $\tilde{\sigma}_i := \sigma_i \cap \Delta^{\leq 1}_{3(q-1)}$. By the argument given in that proof, $\{\tilde{\sigma}_1, \dots, \tilde{\sigma}_q\}$ is a winding partition for f. Since f is in general position, there are two possibilities for the $\tilde{\sigma}_i$.

- The $\tilde{\sigma}_i$ are 2-dimensional, except for one, say $\tilde{\sigma}_1$, that is 0-dimensional. Since $\{\sigma_1, \dots, \sigma_q\}$ is a winding partition for our constructed F, the faces $\sigma_2, \dots, \sigma_q$ have to be 3-dimensional and the face σ_1 has to be 0-dimensional. Therefore each of the faces $\sigma_2, \dots \sigma_q$ contains exactly one of the vertices P_i. Hence the winding partition $\{\tilde{\sigma}_1, \dots, \tilde{\sigma}_q\}$ for f corresponds to $(q-1)!$ winding partitions of F.
- All but two of the $\tilde{\sigma}_i$ are 2-dimensional, and the other two, say $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$, are 1-dimensional. W.l.o.g. let $\tilde{\sigma}_1$ be the "upper" one of the two. Since $\{\sigma_1, \dots, \sigma_q\}$ is a winding partition for F, the faces $\sigma_3, \dots, \sigma_q$ have to be 3-dimensional, the face σ_2 has to be 2-dimensional and the face σ_1 has to be 1-dimensional. Hence the winding partition $\{\tilde{\sigma}_1, \dots, \tilde{\sigma}_q\}$ for f corresponds to $(q-1)!$ winding partitions of F.

In both cases there is a 1-to-$(q-1)!$ map between winding partitions of f and winding partitions of F. Since there are at least $((q-1)!)^3$ winding partitions of F, there have to be at least $((q-1)!)^2$ winding partitions of f. $\square$

Example 4.4. For the alternating linear model of K_n described in Example 1.9, there are $((q-1)!)^2$ winding partitions, exactly the bound conjectured in the previous Theorem.

We now know that for $d \geq 3$ the proved and conjectured lower bounds for the number of Tverberg partitions also apply to the number of winding partitions; in the following we derive a nontrivial lower bound on the number of winding partitions also for the case $d = 2$.

Proposition 4.5 (The case d=2). *Let $q = p^r$ be a prime power. For every map $f\colon \Delta^{\leq 1}_{3(q-1)} \to \mathbb{R}^2$ there are at least*

$$\frac{1}{((q-1)!)^2} \Big(\frac{q}{r+1} \Big)^{2(q-1)}$$

winding partitions.

Proof. In the case $d = 3$, there are at least $b := \frac{1}{(q-1)!} \cdot (\frac{q}{r+1})^{2(q-1)}$ Tverberg partitions (Theorems 1.13 and 4.2) and thus the same number of winding partitions. By the proof of the previous theorem, $\frac{b}{(q-1)!}$ is a bound for the number of winding partitions for $d = 2$. $\square$

5 q-Winding Graphs

For $d = 2$ the Winding Number Conjecture claims that complete graphs $K_{3q-2} = \Delta^{\leq 1}_{3(q-1)}$ have a certain property. We now consider all graphs that have this property. For this, we interpret graphs as (1-dimensional) topological spaces if needed. Thus, a *drawing* of G is just a continuous map $G \to \mathbb{R}^2$. (Nevertheless, the *paths* and *cycles* in the following definition are required to be subgraphs, so the paths start and end at vertices. A single vertex is a path of length 0.)

Definition 5.1 (q-winding). A graph is G *q-winding* if for every drawing $f : G \to \mathbb{R}^2$ there are q disjoint paths or cycles $P_1, \ldots, P_q$ in G with

$$\Big(\bigcap_{P_i \text{ is a path}} f(P_i) \Big) \cap \Big(\bigcap_{P_i \text{ is a cycle}} W_{\neq 0}(f|_{P_i}) \Big) \neq \emptyset.$$

In this situation $P_1, \ldots, P_q$ is a *q-winding partition* for f.

The case $d = 2$ of the Winding Number Conjecture claims that K_{3q-2} is q-winding. This is proved in the case when q is a prime power. So the first "undecided case" is $q = 6$: Does every drawing of K_{16} have a 6-winding partition, into either a vertex and five triangles, or into two edges and four triangles?

We now take a closer look at 2- and 3-winding graphs. (Every non-empty graph is 1-winding.)

5.1 2-Winding graphs and Δ-to-Y operations

Proposition 5.2. *K_4 and $K_{2,3}$ are 2-winding.*

Our proof will be phrased in terms of Δ-to-Y operations (compare [14, Sect. 4.1]). We discuss their effect on q-winding graphs in general before we return to the proof of the proposition.

Definition 5.3 (Δ-to-Y operations). A *Δ-to-Y operation* transforms a graph G into a graph G' by deletion of the three edges of a triangle, and addition of a new 3-valent vertex that is joined to the three vertices of the triangle. A *Y-to-Δ operation* is the reverse of a Δ-to-Y operation.

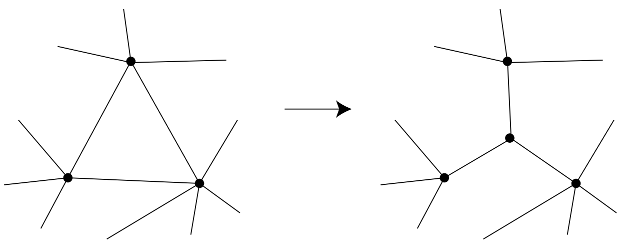

Figure 9: A Δ-to-Y operation.

Lemma 5.4. *If there is a continuous map $f : G \to G'$ that maps disjoint paths and cycles to disjoint paths resp. cycles, and if G is q-winding, then G' is also q-winding.*

Proof. Let $g : G' \to \mathbb{R}^2$ be any drawing of G'. Then $g \circ f : G \to \mathbb{R}^2$ is a drawing of G. Since G is q-winding, there are q disjoint paths or cycles in G that form a q-winding partition for $g \circ f$. These paths/cycles are mapped under f to q disjoint paths/cycles in G', which form a q-winding partition for g. ☐

Any inclusion $G \subset G'$ satisfies the assumptions of this lemma, as does any series reduction (inverse subdivision of an edge).

Lemma 5.5. *A graph G' obtained from a Δ-to-Y operation on a q-winding graph G' is again q-winding.*

Proof. Assume that G' is obtained from G by a Δ-to-Y operation, more precisely by deleting the edges v_1v_2, v_2v_3 and v_1v_3 and adding the vertex v together with the edges vv_1, vv_2 and vv_3. Define $f : G \to G'$ as the identity on all vertices of G and all edges of G except the three deleted ones. For these, define $f(v_iv_j) := v_ivv_j$. The function f maps disjoint paths and cycles to disjoint paths/cycles. ☐

We note that Y-to-Δ operations do not preserve the property to be q-winding: See Figure 10.

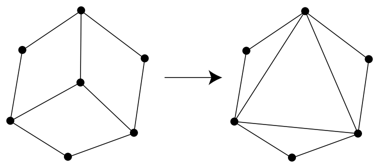

Figure 10: A Y-to-Δ operation that transforms this 2-winding graph into a graph that is not 2-winding.

Lemma 5.6. *If G has a q-winding minor, then G is q-winding.*

Proof. The "minor of" relation is generated by addition of edges, and splitting of vertices. Both operations satisfy the condition of Lemma 5.4. ☐

We return to the discussion of 2-winding graphs.

Proof of Proposition 5.2. The Winding Number Conjecture holds for $q = 2$, so K_4 is 2-winding. The graph $K_{2,3}$ is obtained from K_4 by a Δ-to-Y operation and hence is 2-winding as well. ☐

Theorem 5.7. *A graph is 2-winding if and only if it contains K_4 or $K_{2,3}$ as a minor.*

Proof. Every graph that has a q-winding minor is itself q-winding. Therefore every graph containing K_4 or $K_{2,3}$ as a minor is 2-winding.

On the other hand, if a graph does not contain one of these two minors, then it is *outerplanar*, that is, it has a planar drawing with all vertices lying on the exterior region. In such a drawing no two edges intersect, and no cycle winds around a vertex. Hence the graph is not 2-winding. ☐

5.2 3-Winding graphs and q-winding subgraphs of complete graphs

We prove two general results about q-winding subgraphs of K_{3q-2}, and obtain the minimal 3-winding subgraph of K_7. For the Topological Combinatorics notation and basics employed in the following, we refer to Matoušek [4, Chap. 6].

Theorem 5.8. *Let $p \geq 3$ be a prime and M a maximal matching in K_{3p-2}. Then $K_{3p-2} - M$ is p-winding.*

Proof (suggested by Vučić and Živaljević [13], in the presentation of Matoušek [4, Sect. 6.6]). Let $N := 4(p-1)$ and let $f : K_{3p-2} \to \mathbb{R}^2$ be a drawing of K_{3p-2}, which we may assume to be piecewise linear in general position.

We divide the proof in three steps.

1. We describe a $\mathbb{Z}_p$-invariant subcomplex L of $(\Delta_N)^{*p}_{\Delta(2)}$.
2. We show that $\mathrm{ind}_{\mathbb{Z}_p}(L) \geq N > N-1 = \mathrm{ind}_{\mathbb{Z}_p}((\mathbb{R}^3)^{*p}_{\Delta})$. Thus by the defining property of the index (see Matoušek [4, Sects. 6.2, 6.3]), L cannot be mapped to $(\mathbb{R}^3)^{*p}_{\Delta}$ $\mathbb{Z}_p$-equivariantly.
3. We extend the drawing f to a map $F : \Delta_N \to \mathbb{R}^3$ and examine the Tverberg partitions of F and winding partitions of f that are obtained from the equivariant map $F^*|_L : L \to (\mathbb{R}^3)^{*p}$.

Step 1: The vertex set of the deleted join complex $(\Delta_N)^{*p}_{\Delta(2)}$ can be arranged in an array of $(N+1) \times p$ points, as in Figure 11. The maximal simplices then have exactly one vertex in each of the $N+1$ levels.

We extend the matching M of K_{3p-2} to a maximal matching on the vertices of Δ_N and group the rows into pairs accordingly. One row remains single. For each pair of rows we choose a $\mathbb{Z}_p$-invariant cycle in the complete bipartite graph generated by these two shores, such that the cycles contain no vertical edges. (This requires $p \geq 3$.) The maximal simplices of L shall be the maximal simplices of $(\Delta_N)^{*p}_{\Delta(2)}$ which contain an edge from each cycle (compare Figure 11). This defines L as a $\mathbb{Z}_p$-invariant N-dimensional subcomplex of $(\Delta_N)^{*p}_{\Delta(2)}$.

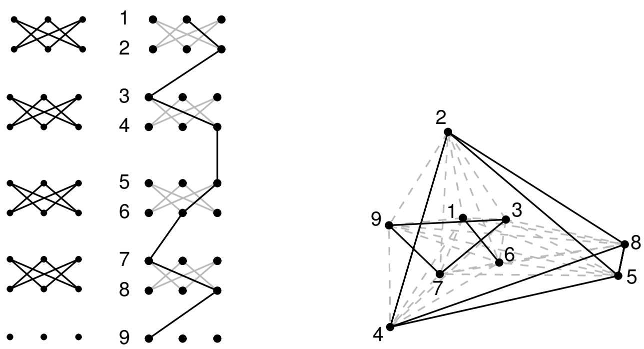

Figure 11: The left figure indicates the complex L in the case $p = 3$ and $N = 8$: For each pair of rows a cycle is drawn; the bold chain indicates a maximal face of L. The figure on the right illustrates the partition of the vertices of $(\Delta_8)^{*3}_{\Delta(2)}$ represented by this face.

Step 2: L can be interpreted as the join of its $N/2$ cycles and the remaining row of p points,

$$L \;\cong\; (S^1)^{*N/2} * D_p \;\cong\; S^{N-1} * D_p.$$

This space is N-dimensional and $(N-1)$-connected, so $\mathrm{ind}_{\mathbb{Z}_p}(L) = N$.

The identity $\mathrm{ind}_{\mathbb{Z}_p}((\mathbb{R}^3)^{*p}_{\Delta}) = N-1$ is elementary as well; see [4, Sect. 6.3].

Step 3: Now we can extend f to a continuous map $F : \Delta_{4(p-1)} \to \mathbb{R}^3$, such that for every Tverberg partition $\{\sigma_1, \ldots, \sigma_p\}$ for F, the set $\{\sigma_1 \cap \Delta_{3(p-1)}, \ldots, \sigma_p \cap \Delta_{3(p-1)}\}$ is a winding partition for f.

According to the pattern of [4, Sect. 6.3], this yields a $\mathbb{Z}_p$-equivariant map $F^{*p} : \Delta^{*p}_{4(p-1)} \to (\mathbb{R}^3)^{*p}$. In view of the index computation of Step 2, the restriction $F^{*p}|_L$ hits the diagonal, which yields a p-fold coincidence point in L, and thus a Tverberg partition for $F : \Delta_{4(p-1)} \to \mathbb{R}^3$ which does not use a matching edge. $\square$

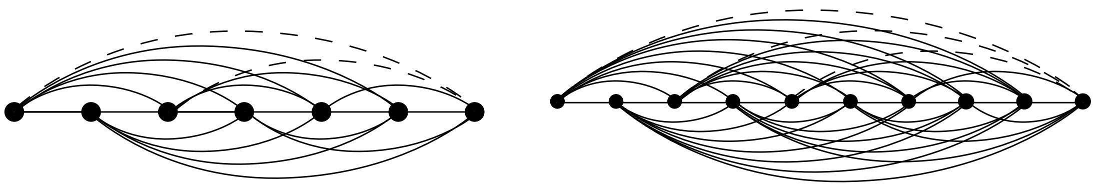

Figure 12: Drawing of K_7 and K_{10}. The edges that form X are dashed.

Proposition 5.9. *Let X be $q-1$ edges of K_{3q-2} meeting in one vertex. Then $K_{3q-2} - X$ is not q-winding.*

Proof. All we need is a drawing of $K_{3q-2} - X$ without a q-winding partition. We can use the alternating linear model of K_n described in Example 1.9. Order the vertices such that the meeting vertex is at the right end of the drawing and the other vertices of X have the numbers $1, 3, 5, \dots, 2q-5, 2q-3$. The edges of X are then in the upper half. (Compare Figure 12.) It is a nice, elementary exercise to verify that in this situation there is no winding partition that doesn't use an edge of X. □

Corollary 5.10. *The unique minimal 3-winding minor of K_7 is $K_7 - M$, where M is a maximal matching.*

Proof. $K_7 - M$ is a 3-winding minor of K_7 (Theorem 5.8, for $p = 3$). It is minimal, because all edges not in M are adjacent to an edge in M and thus must not be deleted (Proposition 5.9).

If on the other hand K is a 3-winding minor of K_7, then only a matching can be deleted (again by Proposition 5.9). For K to be minimal, this matching must be maximal. □

Proposition 5.11. *Not every 3-winding graph has K_7 minus a maximal matching as a minor.*

Proof. Let M be a maximal matching in K_7. Execute a Δ-to-Y operation on $K_7 - M$; the resulting graph is 3-winding, but does not have $K_7 - M$ as a minor. □

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