Trade execution in illiquid markets
Optimal stochastic control and multi-agent equilibria

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Zusammenfassung

In den Modellen der klassischen Finanzmathematik wird angenommen, dass man beliebige Mengen von Gütern zum aktuellen Marktpreis handeln kann, ohne dadurch diesen Preis zu beeinflussen. Dies entspricht für große Transaktionen nicht der Realität: Zum einen muss für den Kauf oder Verkauf großer Positionen eine Preisprämie gezahlt werden, zum anderen haben große Transaktionen einen nachhaltigen Effekt auf zukünftige Preise. Ziel dieser Dissertation ist es, optimale Handelsstrategien in einem solchen „illiquiden Markt" zu finden.

In einem ersten Teil analysieren wir die Situation eines einzelnen Händlers, der ein Portfolio verkaufen möchte. Dieser steht vor einem Dilemma: Auf der einen Seite übt er durch schnelles Handeln einen starken negativen Einfluss auf den Marktpreis aus und reduziert dadurch seinen Handelserlös. Auf der anderen Seite geht er bei langsamem Handeln ein hohes Risiko ein, da der Marktpreis im Laufe der Handelsabwicklung aufgrund von exogenen Ereignissen einbrechen kann. Im ersten Teil dieser Dissertation bestimmen wir den optimalen Mittelweg in diesem Dilemma. Dabei benutzen wir verschiedene Modellierungsansätze mit einem besonderen Fokus auf die Maximierung des erwarteten Nutzens. Die Hamilton-Jacobi-Bellman-Gleichung für dieses Problem ist eine vollständig nicht-lineare, degenerierte partielle Differentialgleichung. Um diese zu lösen, verfolgen wir den ungewöhnlichen Ansatz, zuerst die optimale Kontrolle als Lösung einer partiellen Differentialgleichung herzuleiten und danach mit Hilfe der optimalen Kontrolle eine Lösung der Hamilton-Jacobi-Bellman-Gleichung zu konstruieren. Für den Verkauf eines Portfolios aus mehreren verschiedenen Aktien können wir mittels dieses Ansatzes die hochdimensionale Hamilton-Jacobi-Bellman-Gleichung auf ein zweidimensionales Problem zurückführen, falls der Markt „homogen" in einem bestimmten Sinne ist.

Im zweiten Teil dieser Arbeit betrachten wir mehrere Marktteilnehmer, welche dasselbe Gut in einem illiquiden Markt handeln. Jeder Teilnehmer handelt zum Marktpreis, welcher von den Transaktionen aller Teilnehmer gleichermaßen beeinflusst wird. Dadurch ergibt sich eine Interaktion zwischen den Marktteilnehmern. Wir untersuchen insbesondere die Situation eines Händlers, der innerhalb eines kurzen Zeithorizonts eine Aktienposition liquidieren muss, während andere Marktteilnehmer von seinen Handelsplänen wissen. In einem ersten Marktmodell können wir die optimalen Handelsstrategien aller Agenten in einer komplexen geschlossenen Form herleiten. Wir beleuchten die Interaktionen zwischen den Marktteilnehmern anhand von Beispielfällen und Grenzwerten und finden Erklärungen für die Koexistenz von kooperativen und kompetitiven Verhaltensweisen. Für ein zweites Marktmodell zeigen wir induktiv, dass die Wertfunktion für alle Marktteilnehmer eine spezielle polynomiale Form hat. Dadurch erhalten wir die optimalen Handelsstrategien als lineare Funktionen mit Koeffizienten, welche durch eine explizite Rückwärtsrekursion berechnet werden können. In diesem zweiten Marktmodell ist eine schnelle Abfolge von Käufen und Verkäufen optimal; durch das Betrachten verschiedener Grenzwerte bringen wir dieses Verhalten mit den Kosten von Round-Trip-Transaktionen in Verbindung.

Summary

In the classical models of financial mathematics, it is assumed that arbitrarily large positions of assets can be traded at the current market price without affecting this price. This does not reflect reality for large transactions: First, a price premium must be paid for large positions. Second, large transactions do have a long-lasting effect on future prices. The purpose of this dissertation is to find optimal execution strategies in such an "illiquid market".

In a first part, we analyze the situation of a single trader, who wants to liquidate a portfolio. The trader is facing a dilemma: on the one hand, a quick liquidation results in a strong adverse influence on the market price and thus reduces the liquidation proceeds. On the other hand, a slow execution results in a large risk, since the market price can move significantly during the liquidation time period due to exogenous events. In the first part of the dissertation, we determine the optimal trade-off in this dilemma. We use different modeling approaches with a special focus on utility maximization. The Hamilton-Jacobi-Bellman equation for this problem is a completely non-linear, degenerate partial differential equation. To solve it, we pursue the unusual approach of first obtaining the optimal control as a solution of a partial differential equation and subsequently constructing a solution to the Hamilton-Jacobi-Bellman equation by using the optimal control. For the liquidation of a portfolio consisting of several assets our approach allows us to reduce the high-dimensional Hamilton-Jacobi-Bellman equation to a two-dimensional problem if the market is "homogeneous" in a certain sense.

In the second part of this dissertation, we consider several market participants, who trade the same asset in an illiquid market. Every participant trades at the market price, which is influenced by the transactions of all participants in the same fashion. This leads to an interaction of the market participants. We investigate in particular the situation of a trader who needs to liquidate an asset position in a short time while other market participants are aware of her trading intentions. In a first market model, we can derive the optimal strategies for all agents in a complex closed form. We analyze the interaction of the market participants by reviewing examples and limit cases and find an explanation for the coexistence of cooperative and competitive behavior. For a second market model we show inductively that the value function for all market participants is of a special polynomial form. We thus obtain the optimal trading strategies as linear functions with coefficients which can be calculated by an explicit backward recursion. In this second market model, a quick sequence of buy and sell orders is optimal; by considering different limit cases, we discover that this phenomenon is related to the costs of round trip transactions.

Contents

Chapter 1

Introduction

1.1 Traders' questions on illiquidity

Institutional investors aim to maximize returns by investing in the most promising assets. The total return of an investment however is heavily influenced by the way it is obtained and liquidated (see Perold (1998)). In particular for large funds, the decision to change even a small fraction of their assets under management can result in very large transactions that comprise a major part of the average daily traded volume of the assets. Unlike for small trades, the execution of a large trade is a very complex task: An immediate execution is often not possible or only at a very high cost due to insufficient liquidity. Much value can therefore be added by working the order in a way that minimizes execution costs, which is why institutional investors often rely on the expertise of investment banks for the execution of such large trades.

Triggered by the introduction of electronic trading systems by many exchanges, automatic order execution has become an alternative to manually worked orders. It provides a number of advantages, including both smaller fixed costs of using expensive experienced traders as well as higher execution efficiency due to quicker reactions to incoming orders. Automatic order execution heavily relies on mathematical methods to model illiquid markets and to determine the optimal trading strategies. In particular, the modeling framework introduced by Almgren and Chriss (1999) serves as the basis of many optimal execution algorithms run by practitioners (see e.g., Kissell and Glantz (2003), Schack (2004), Abramowitz (2006), Simmonds (2007) and Leinweber (2007)).

Within the optimal liquidation literature, most research was directed to finding the optimal *deterministic* or *static*[1] liquidation strategy. These strategies do not react to changes in asset price. Instead, it is assumed that at the beginning of the order execution a trading schedule is determined. This schedule is then carried out irrespective of the size and direction of price movements. Such deterministic strategies do not seem to meet real-world needs completely; some investors prefer *adaptive* or *dynamic*[2] strategies, which are provided by many sell side firms (see e.g., Kissell and Malamut (2005) and Kissell and Malamut (2006)). These dynamic trading strategies usually follow one of two opposing philosophies:

- When prices rise, some clients want to sell faster in order to realize these profits; when prices drop, they want to sell slower in order to avoid making losses. These clients follow strategies that are "aggressive in-the-money" (AIM).
- Other clients follow the opposite reasoning: When prices rise, they sell slower since they do not fear making losses in this scenario and thus become more tolerant to risk.

[1]We use the terms "deterministic" and "static" interchangeably.
[2]We use the terms "adaptive" and "dynamic" interchangeably.

However, when prices drop, they speed up their selling in order to avoid large losses. These clients follow "passive in-the-money" (PIM) strategies.

Intuitively, these adaptive strategies are connected to risk aversion. A consistent mathematical treatment however was not available so far, and practitioners had to rely on more or less ad-hoc approaches. In the first part of this thesis, we analyze adaptive liquidation strategies in a sound, fundamental economic framework and identify the precise connection between risk aversion and the dynamics of optimal trading strategies.

In illiquid markets, knowing someone else's trading intention can be very profitable since it allows for exploitation of the expected price impacts. The best known examples are probably the alleged insider trading against LTCM and the forced liquidation of the Amaranth fund. On a smaller scale, trading intentions become known to competitors frequently by several means. For example, when the execution of a trade is commissioned to an investment bank, advance price quotes are usually obtained from several other banks. Banks that are not successful in bidding for the trade will nevertheless be informed about its existence. Another example is provided by market makers and insiders who must report or even preannounce large transactions.

If a market participant receives information, e.g., that someone else is currently selling an asset, then the optimal trading reaction is not clear a priori, as there are in principle two opposing strategies that appear profitable. First, she can also sell the asset when prices are still high and cover her short position later at a lower price; such behavior is called "predatory trading". Second, she can buy the asset at a depressed asset price and sell it later when the asset price recovered, i.e., she can act as a temporary liquidity provider. No mathematical model was available that explained potential drivers of the profits of these two strategies, nor was any criterion available to decide whether of the two strategies should be executed.

The reaction of other market participants on one's own trading intentions has a feedback effect on optimal trade execution. If informed market participants are likely to trade in parallel, then being undetected is important and a "stealth execution" algorithm should be selected. Efforts to conceal the trading intention can be observed when investors who obtain price quotes from several banks only distribute a limited amount of information to the sell side on so called "bid sheets". If informed market participants however can be expected to provide liquidity, then it is profitable to reveal the trading intention, i.e., to perform "sunshine trading". "Indications of interest", which are used in equities trading in the United States are a special tool for this purpose. Again, the choice of stealth or sunshine execution lacked a thorough scientific understanding.

In the second part of this thesis, we propose a mathematical model of the interaction of informed agents in illiquid markets. This model allows us to study the reaction of other agents on one's own trades and thus to quantify the profits of different trading strategies in such an "interactive" market. We determine an objective criterion for the selection of one of the strategies described above and find that all of them can be optimal under certain circumstances.

Before we discuss the mathematical results of this thesis and their connection to the questions raised above in detail, let us first outline the relevant economic literature.

1.2 Economic background

To our knowledge, the first academic study of the price impact of large transactions was carried out by Kraus and Stoll (1972). The market environment has changed significantly in the meantime, most notably by the introduction of electronic trading systems. Several investigations of price impact have been carried out, for example by Holthausen, Leftwich, and Mayers (1987), Holthausen, Leftwich, and Mayers (1990), Barclay and Warner (1993), Chan and Lakonishok (1995), Biais, Hillion, and Spatt (1995), Kempf and Korn (1999), Chordia, Roll, and Subrahmanyam (2001), Chakravarty (2001), Lillo, Farmer, and Mantegna (2003), Mönch (2004), Almgren, Thum, Hauptmann, and Li (2005), Coval and Stafford (2007), Obizhaeva (2007) and Large (2007). Although differing in *quantitative* estimates, all of these studies find the same *qualitative* features of the price impact of large orders. First, the size of an order influences the transaction price itself. Second, the price of transactions subsequent to a large order is affected as well, but not to the same extent. These two price effects were therefore called "temporary" and "permanent" price impact.

Market microstructure research has produced a number of theoretical models to explain these price effects. Most of these models pick up the insight of Bagehot (1971) that information plays a crucial role in trading. Kyle (1985) considers a market with one market maker and a number of informed and uninformed traders. In this model, each order has a permanent impact on market prices, since it might be information-induced. More precisely, the permanent impact is a linear function of the order size. Easley and O'Hara (1987) propose a generalized model in which the market maker does not know whether any of the traders are informed or whether all of them are uninformed. This results in a temporary price impact, since large orders in one direction could be a signal of the arrival of new information. If the order flow decreases, then the market maker's subjective believes about the presence of information revert to a normal level and thus prices recover. A number of other models have been proposed, among them Glosten and Milgrom (1985), Back (1992), Allen and Gorton (1992), Foster and Viswanathan (1996) and Bondarenko (2001). Several other causes of temporary and permanent price impact have been suggested, for example the difficulty to find transaction counterparties in a short time (see Demsetz (1968), Harris and Gurel (1986), Grossman and Miller (1988)). For a list of additional models, see O'Hara (1998) and Biais, Glosten, and Spatt (2005).

While these models endogenously explain the price effects of trading, most of them are either too stylized to sufficiently capture the quantitative real-world price dynamics, or they are too involved to allow for a complete analysis of optimal trading strategies. Therefore a second line of models emerged that exogenously specify the price impact of large orders. These models are designed to be sufficiently complex to realistically capture the price effect of orders, but also to be sufficiently simple to allow for an analysis of optimal trading strategies. Several alternative models have been proposed, including Bertsimas and Lo (1998), Almgren and Chriss (2001), Almgren (2003), Butenko, Golodnikov, and Uryasev (2005), Obizhaeva and Wang (2006), Engle and Ferstenberg (2007) and Alfonsi, Schied, and Schulz (2007b)[3].

[3]Other models have been suggested for the analysis of derivatives in illiquid markets, for example Frey (1997), Frey and Patie (2002), Bank and Baum (2004), Çetin, Jarrow, and Protter (2004) and Jarrow and Protter (2007). Most of these models however are not suitable for the analysis of optimal portfolio liquidation because they assume that sufficiently smooth trading strategies are not affected by illiquidity.

Since this thesis investigates optimal trading in illiquid markets, we follow this second approach and assume exogenous models for the price impact of large orders. While the advantages and disadvantages of the different models are still a topic of ongoing research, we apply the market model introduced by Bertsimas and Lo (1998) and Almgren and Chriss (2001) (respectively Almgren (2003) and several other extensions of it) in most of this thesis for the following reasons. First, it provides a high degree of analytical tractability while still being sufficiently flexible to capture the relevant aspects of both the permanent and temporary price impacts of large trades. It is to our knowledge the only liquidity model that has become the basis of theoretical studies not only on the topic it was designed for (optimal portfolio liquidation), but also on several other topics such as hedging (Rogers and Singh (2007)), investment decision and implementation (Engle and Ferstenberg (2007)) and on the interaction of market participants in illiquid markets (Carlin, Lobo, and Viswanathan (2007)). Second, it demonstrated reasonable properties in real world applications as already pointed out in Section 1.1. See Section 3.2 for a more detailed discussion of this model. To illustrate the robustness of our results, we accompany most of the analyses in this thesis with corresponding investigations of alternative market models.

Within the optimal liquidation literature, initial research was directed to finding strategies that minimize the expected cost incurred by the order execution (see for example Bertsimas and Lo (1998)). More recently, the trade-off between minimizing the cost of fast execution and the risk associated with slow execution moved to the center of interest; see for example Almgren and Chriss (2001), Konishi and Makimoto (2001), Dubil (2002), Almgren (2003), Mönch (2004), Huberman and Stanzl (2005) and Kissell and Malamut (2005). In these investigations, risk aversion is incorporated by assuming that the person liquidating a portfolio is concerned about the mean and variance of execution costs. If the market model is sufficiently simple, then the optimal deterministic portfolio liquidation strategy can be derived explicitly. The proper economic motivation for mean-variance optimization stems from a second-order *approximation* of an expected utility functional as explained, e.g., in Föllmer and Schied (2004). The shortcomings of the mean-variance approach to optimal liquidation were acknowledged early on; e.g., see Bertsimas and Lo (1998):

> Investors are ultimately interested in maximizing the expected utility of their wealth. Therefore, the most natural approach to execution costs is to maximize the investor's expected utility of wealth ...
>
> There is no doubt that such an approach is the 'right' one.

Furthermore, much of the research on optimal liquidation considers only the limited class of deterministic strategies[4]. This might forego optimization potential and a priori excludes aggressive in-the-money and passive in-the-money strategies. Only recently, academic research has started to investigate the optimization potential of dynamic strategies. For example, Almgren and Lorenz (2007) increased the class of admissible strategies by allowing for intertemporal updating and found that aggressive in-the-money strategies can strictly improve mean-variance performance. To overcome the limitations of mean-variance optimization and deterministic strategies, we focus directly on the original problem of *adaptive expected-utility maximization* in the optimal portfolio liquidation part of this thesis. We consider a wide class

[4]Notable exceptions describing optimal adaptive strategies include Submaranian and Jarrow (2001), He and Mamaysky (2005), Almgren and Lorenz (2007) and Çetin and Rogers (2007).

of utility functions, and devote special attention to exponential (CARA) utility functions. This allows us to fully acknowledge the impact of risk aversion on liquidation dynamics in a sound economic framework.

The alleged insider trading against the distressed hedge fund LTCM was covered extensively in public and academic media. The interaction of large, strategic traders in illiquid markets in general however has received much less attention. Most of the classical market microstructure models assume that either there is only one strategic trader (see, e.g., Kyle (1985)), or that each trader can trade only once and therefore does not undertake any strategic considerations (see, e.g., Glosten and Milgrom (1985), Easley and O'Hara (1987)). In both cases, large traders do not need to consider the actions of other large traders. Admati and Pfleiderer (1991) proposed a special model to explain the benefits of sunshine trading and liquidity provision, while Brunnermeier and Pedersen (2005) and Carlin, Lobo, and Viswanathan (2007) suggest models motivating stealth execution and predatory trading. While these models are interesting in their own right, they result in optimal strategies that either always pursue stealth execution or always pursue sunshine trading. To our knowledge, no model existed that explains the coexistence of these strategies; this gap is filled by our analysis in the second part of this thesis. We give a more detailed overview of the existing literature on the interaction of multiple strategic traders in illiquid markets in the introduction to Chapter 8.

1.3 Outline of mathematical and economic results

This thesis consists of two parts. In the first part, we deal with a single large trader facing the task of executing a large trade in an illiquid market. In Part II, we introduce additional informed strategic traders and study their interaction.

1.3.1 Part I: Dynamic portfolio liquidation

In Part I, we investigate the impact of risk aversion on optimal portfolio liquidation. We therefore consider von-Neumann-Morgenstern investors that want to maximize the utility of the liquidation proceeds. Furthermore, we allow for adaptive strategies.

Mathematically, equipping financial models with liquidity features seems similar to the introduction of (fixed or proportional) transaction costs: Both illiquidity and transaction costs make trading expensive and will thus lead to restrictions on trading. The precise effect of these two features of real-world markets however is diametrically opposite: Because of illiquidity, orders are broken up into smaller transactions to reduce the price impact. Transaction costs however lead to a concentration of orders in an effort to reduce fees. This also has an impact on the mathematical tools required to study these effects: While analyses of the effects of transaction costs often employ methods from singular control, we will primarily rely on continuous control techniques.

In Chapter 2, we assume a general additive market model for transaction prices, in which the transaction price is the sum of a "fundamental" market price and the price impact of the trades of the large investor. While the fundamental market price is assumed to be independent of the investor's trades, the price impact can depend on previous trades and the current trade in an (almost) arbitrary way. An important assumption is that the fundamental market price process has independent increments, i.e., that the fundamental market price exhibits no autocorrelation.

Since we consider trading in discrete time only, we can fully leverage the methods of discrete-time stochastic dynamic programming. If a large transaction needs to be broken up into three smaller orders, then we can describe the dynamics of this liquidation in a simple way in spite of the generality of the market model. We find that if the utility function exhibits increasing absolute risk aversion, then an asset position is liquidated faster when the price moved in a favorable direction than when it moved in an unfavorable direction. The optimal strategy is therefore aggressive in-the-money. For a utility function with decreasing absolute risk aversion, we find the opposite optimal behavior, i.e., a passive in-the-money strategy. For exponential utility functions, i.e., constant absolute risk aversion (CARA), we obtain that the optimal strategy is independent of any price moves. Hence for investors with a CARA utility function, the optimal strategy is deterministic. By induction, this result extends to the distribution of a large transaction over an arbitrarily large finite number of orders. In this chapter, the tight connection between the absolute risk aversion profile and the dynamics of optimal portfolio liquidation just described appears for a general market model, but only limited trading time points. It will reemerge in Chapters 3, 4 and 5 for a more restricted class of market models, but for continuous-time trading.

In Chapter 3, we consider the continuous-time optimal portfolio liquidation problem with a finite time horizon for a von Neumann-Morgenstern investor with constant absolute risk aversion (CARA). We turn to the liquidation of baskets, i.e., the simultaneous liquidation of asset positions in several different, but potentially correlated assets. Furthermore, trading in one of the assets can influence the market prices of all other assets (cross-asset price impact). As underlying market impact model, we use a multi-asset extension of the continuous-time liquidity model of Almgren (2003). In this model, the price impact is limited to a purely permanent and a purely temporary component. In this regard, it is a continuous-time special case of the general market model considered in Chapter 2. However, it still includes a wide range of potentially non-linear impact functions and cross-asset impact relationships. We assume that the fundamental asset price is driven by a Brownian motion. In this model, we show that the expected utility of sales revenues, taken over a large class of adapted strategies, is maximized by a deterministic strategy, which is also mean-variance optimal. The classical methods of optimal stochastic control cannot be applied directly to this problem, since the corresponding Hamilton-Jacobi-Bellman (HJB) equation is degenerate and its initial condition is singular. We can overcome these obstacles by first restricting our analysis to deterministic strategies and exploiting the weaker conditions of deterministic optimal control. The value function for optimal deterministic liquidation then solves the degenerate Hamilton-Jacobi-Bellman equation, and by an a priori upper bound on the gains of using adaptive strategies, we can show by a verification argument that these gains are zero, i.e., that the optimal adaptive strategy is deterministic.

After identifying the optimal liquidation strategy for CARA investors with a finite time horizon in Chapter 3, we consider in Chapter 4 the *infinite-horizon* optimal portfolio liquidation problem for a von Neumann-Morgenstern investor *with arbitrary utility function* in a single-asset liquidity model with linear temporary impact. Using a stochastic control approach, we characterize the value function and the optimal strategy as classical solutions of nonlinear parabolic partial differential equations. The optimal strategy is not static (unless the utility function is CARA); intertemporal updating does increase the expected utility in general. We furthermore analyze the sensitivities of the value function and the optimal strategy with respect to the various model parameters. In particular, we find that the optimal strategy is aggressive or passive in-the-money, respectively, if and only if the utility function displays increasing or decreasing risk aversion. Surprisingly, only few further monotonicity relations exist with respect to the other parameters. We point out in particular that the speed by which the remaining asset position is sold can be decreasing in the size of the position but increasing in the liquidity price impact.

By considering the infinite-horizon liquidation problem, we circumvented the singular initial condition of the Hamilton-Jacobi-Bellman equation. The HJB equation however is still fully non-linear, as can be observed by its reduced form:

$$v_X^2 = -2\lambda\sigma^2 X^2 v_R v_{RR}.$$

Here, v is the value function, X is the asset position and R is the state variable. Contrary to our approach in Chapter 3, we cannot resort to deterministic optimal control to find a solution of this partial differential equation. However, we can transform the HJB equation into a well-behaved partial differential equation for the optimal control, for which we can prove the existence and uniqueness of a solution. Using this candidate optimal control, we can construct a solution to the HJB equation and then conclude by a verification argument similar to Chapter 3.

In Chapter 5, we extend the analysis of Chapter 4 to the liquidation of baskets in the multi-asset market model of Chapter 3, including non-linear temporary impact and cross-asset price impact. The Hamilton-Jacobi-Bellman equation for this problem is a degenerate partial differential equation with one "space dimension" (the state variable) and several "time dimensions" (one for each asset). If the market fulfills a scaling property that ensures a certain homogeneity of illiquidity, then we can again exploit the tight connection between deterministic mean-variance optimization and stochastic utility maximization already revealed in Chapter 3, and we can reduce the high-dimensional HJB equation to a two-dimensional partial differential equation. This two-dimensional problem can then be solved by similar methods as in Chapter 4. We find that the set of portfolios that are attained during the liquidation depends on the market volatility and liquidity structure, but is independent of the utility function. The investor's risk aversion only determines how quickly the investor trades, but not which portfolios he holds during the execution.

Instead of focusing on fundamental economic concepts such as adaptive utility maximization, we take a more applied view in Chapter 6 and introduce two new functionals that practitioners are interested in: mean-variance functionals and cost-risk functionals. We find that these two classes of functionals give the same optimal strategies. Furthermore, the optimal strategies with respect to these functionals exhibit desirable properties such as an endogenous liquidation time horizon, slower liquidation of large asset positions and time-consistency.

1.3.2 Part II: Multiple players in illiquid markets

In Part II of this dissertation, we turn to the interaction of strategic traders in illiquid markets. In Chapter 7 we discuss general modeling questions of the multiple player setting: How do the trades of one player affect the transaction price of other players trading at the same point in time? Can traders observe the actions of the other traders and react accordingly, or do they not know what the other traders are doing?

In a number of practical cases, investors need to liquidate large asset positions in a short time. In Chapter 8, we describe the interactions that arise when other market participants are aware of the investor's needs. In particular, we derive the optimal trading strategies. A crucial assumption is that the informed market participants are not limited by the same time constraint the seller is facing.

We solve a competitive trading game in a multi-player extension of the illiquid market model of Chapter 4, incorporating a temporary and a permanent price impact. Each player faces a dynamic programming problem. We decompose the trading time in two stages: a first stage, which encompasses the time during which the seller needs to liquidate, and a second stage, during which the informed competitors continue trading. By the dynamic programming principle, the optimal trading strategy can be found by two consecutive steps. First, the optimal asset position for the informed competitors for the end of the first stage needs to be determined. Then, the optimal trading strategies for the seller and the competitors within the first and second stage can be derived. While the second step was already solved by Carlin, Lobo, and Viswanathan (2007), we provide a solution to the first step. The proofs are significantly complicated by the unwieldy computations.

According to our model, the optimal strategies for these competitors depend on the liquidity characteristics of the market. If the permanent impact affects market prices more heavily than the temporary impact, the competitors will "race" the seller to market, selling in parallel with her and buying back after the seller sold her asset position. If price impact is predominantly temporary, competitors provide liquidity to the seller by buying some of her shares and selling them after the seller has finished her sale. In the first case, the seller should conceal her trading intentions in order not to attract competitors, while in the latter case, pre-announcing a trade can attract liquidity suppliers and thus be beneficial.

As a special case, we investigate behavior in a market with a very large number of competitors. We find that in spite of illiquidity, such a market efficiently determines a new price. Information about the seller's intentions is immediately incorporated into the market price and does not affect it thereafter. The competitors might race the seller to market, but even in markets with high permanent impact, they quickly start buying back shares and sell these after the seller has finished her sale.

In Chapter 9, we determine the optimal trading strategies for two players trading in discrete time in a market with finite price recovery and no spread similar to the model market suggested by Obizhaeva and Wang (2006). By backwards induction, we show that the optimal trading strategies and the expected proceeds are of a certain parametric form, where the parameters can be recursively calculated by backwards induction. By this method we obtain the exact optimal strategies; a straightforward numeric discrete time dynamic programming approach would only return approximately optimal strategies.

We find that if the "transient price impact" in this model is small, if price recovery is slow or if the discretization time step is small, then the optimal trading strategies oscillate: both players perform large round trip trades. From a mathematical point of view, the main issue is that this effect occurs if the discretization time step is small.

For a reasonable multiple player model, one can expect the optimal trading strategies to converge when the discretization time step is reduced further and further. For the model considered in this chapter, we find numerically that such a convergence does not hold, and we support this observation by heuristic explanations and an analytical treatment of limiting cases. The source of the oscillations is that the cost of round trip trading (selling at time t_n and buying back at time t_{n+1}) goes to zero as the discretization time step goes to zero. This leads to an ever decreasing cost of market manipulation and hedging and thus a shift in focus from interacting with the market (liquidating the initial asset position) to interacting with the other strategic player (profiting from her trading intentions respectively hedging against market manipulations by the other player). We argue that this issue can be remedied by introducing a purely temporary impact or by a trading-dependent spread.

Appendices A and B conclude this thesis with supplementary material for Chapters 8 and 9. In particular, long, but explicit formulas are stated, the Mathematica source code used for figure generation is provided and additional numerical examples are presented.

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Part I

Dynamic portfolio liquidation

Chapter 2

Dynamic optimal liquidation in discrete time

2.1 Introduction

We start our investigation of optimal liquidation by considering a discrete-time framework. Due to the simplifying circumstances of discrete-time optimization, we can deal with a very general liquidity model that allows for arbitrary price increments and for price impacts that can depend on time and on the entire previous trade history. Examples of such models include the liquidity model of Almgren and Chriss (1999), which we will extend and use in continuous time in Chapters 3 to 8, and the model of Obizhaeva and Wang (2006), which we will apply in Chapter 9.

In the general liquidity framework of this chapter, we show that if a large trade is split into three transactions, then the optimal strategy is not static, that is, a different number of shares is sold in the second transaction if prices rise during the liquidation than if prices fall. For investors with increasing absolute risk aversion, trading should be accelerated when prices rise, i.e., an aggressive in-the-money strategy should be pursued. On the other hand, investors with decreasing absolute risk aversion should slow down selling when prices rise, i.e., follow a passive in-the-money strategy. A static selling strategy is optimal only for investors with constant absolute risk aversion.

This section serves three main purposes. First, it introduces the main economic idea of the first part of this thesis: the link between absolute risk aversion and the dynamics of optimal liquidation. Second, it illustrates that the results in continuous time presented in the next chapters are of a general nature. They are not limited to the liquidity model of Almgren (2003), although a theoretical analysis becomes much more involved for more general liquidity models in continuous time. Third, it demonstrates that our results also hold in discrete time, which is important for practical applications.

2.2 Market model in discrete time

At the heart of the portfolio liquidation task lies the limited liquidity provided by the marketplace. If there is sufficient liquidity, the entire order can be executed immediately without significant costs. In the case of limited liquidity however, the effect of a (partial) order execution on the market price needs to be taken into account.

The market we consider consists of a risk-free asset and a risky asset[1]. For simplicity of the exposition, we assume that the risk-free asset does not generate interest. Large transactions are usually executed within a few hours or at most a few days; the effect of

[1] An extension of the model and some of the results of this chapter to a multiple asset setting is straightforward. As this chapter serves primarily as a motivating introduction, we limit the discussion to the single asset case.

discounting is therefore marginal, and we will not consider it in this thesis. In the discrete-time setting of this chapter, we assume that trades can be executed at the (not necessarily equidistant[2]) time points t_0, t_1, ..., t_N. At each of these time points, we assume that the seller as well as a number of noise traders execute orders. We denote the orders of the seller at time t_i by x_i, where $x_i > 0$ denotes sell orders and $x_i < 0$ denotes buy orders.

We assume that the transaction price P_i at time t_i can be decomposed into the price impact of the large trader and the "fundamental" asset price $\tilde{P}_i$ that would have occurred in the absence of large trades. We model the fundamental asset price $\tilde{P}_i$ as an arbitrary stochastic process with independent increments ϵ_i:

$$\tilde{P}_{i+1} = \tilde{P}_i \quad + \quad \epsilon_{i+1} \tag{2.1}$$

To avoid technical difficulties, we assume that the underlying probability space Ω is finite[3]. Then we only need to require that the ϵ_i are non-degenerate random variables[4] and independent. We do not make assumptions on the distribution of the ϵ_i. In particular, they can have different distributions. The random price changes ϵ_i reflect the noise traders' actions as well as all external events, e.g., news. The assumption of independence of the ϵ_i implies in particular that the random price changes do not exhibit autocorrelation. The results we derive are sensitive to this assumption; autocorrelation will in principle have an effect on the proceeds of any dynamic trading strategy. Including autocorrelation in the market model however shifts the focus from optimal liquidation to optimal investment: even without any initial asset position, the mathematical model will recommend high-frequency trading to exploit the autocorrelation. But this effect is not related to the original question of optimal execution. Furthermore, many investors do not have an explicit view on autocorrelation and thus choose an execution algorithm that is optimal under the assumption of independence of price increments. Finally, for realistic parameters the effect of autocorrelation on the optimal execution strategy and the resulting execution cost is marginal as was demonstrated by Almgren and Chriss (2001). For these reasons, we will not include autocorrelation in the market models in this thesis.

We allow a general form of the impact of the trades $x_0, x_1, \ldots, x_i$ on the transaction price P_i:

$$P_i = \underset{\substack{\uparrow \\ \text{“Fundamental”} \\ \text{asset price}}}{\tilde{P}_i} \quad - \quad \underset{\substack{\uparrow \\ \text{Price impact} \\ \text{of seller}}}{f_i(x_0, \ldots, x_i)}. \tag{2.2}$$

We assume that the functions $f_i : \mathbb{R}^{i+1} \to \mathbb{R}$ are C^2 and that the "price impact cost of trading" is strictly convex, i.e., that for any two trading trajectories $(x_i) \neq (y_i)$ with

[2]For example, the distance can be taken in volume time to adjust for the U-shaped intraday pattern of market volatility and liquidity.

[3]The results of this chapter also hold for infinite Ω if the price increments ϵ_i satisfy suitable conditions and the set of admissible strategies is chosen appropriately.

[4]More precisely, no ϵ_i may be a.s. constant.

$\sum_i x_i = \sum_i y_i$ and $0 < t < 1$ the following inequality holds:

$$\begin{aligned}\sum_i (tx_i + (1-t)y_i) f_i(tx_0 + (1-t)y_0, \dots, tx_i + (1-t)y_i) \\ < t\left(\sum_i x_i f_i(x_0, \dots)\right) + (1-t)\left(\sum_i y_i f_i(y_0, \dots)\right)\end{aligned} \tag{2.3}$$

Furthermore, we require that the price impact cost of trading grows superlinearly, i.e., that

$$\lim_{|(x_0,\dots,x_N)|\to\infty} \frac{\sum_i x_i f_i(x_0, \dots, x_i)}{|(x_0, \dots, x_N)|} = \infty.$$

This framework generalizes most of the existing market impact models of liquidity. For example, the model suggested by Almgren and Chriss (1999) and Almgren and Chriss (2001) is equivalent to assuming that the ϵ_i are identically normally distributed and describing the market impact as

$$f_i(x_0, \dots, x_i) = \sum_{j=0}^{i-1} \mathrm{PermImp}(x_j) + \mathrm{TempImp}(x_i)$$

Here, $\mathrm{PermImp}, \mathrm{TempImp} : \mathbb{R} \to \mathbb{R}$ are functions describing the permanent and temporary market impact of a trade. If these functions are linear, we have

$$f_i(x_0, \dots, x_i) = \gamma \sum_{j=0}^{i-1} x_j + \lambda x_i$$

for constants $\gamma, \lambda \in \mathbb{R}^+$. Our framework also includes the limit order book model introduced by Obizhaeva and Wang (2006) if we again assume that the ϵ_i are identically normally distributed and that the price impact is given by

$$f_i(x_0, \dots, x_i) = \gamma\left(\sum_{j=0}^{i-1} x_j + x_i/2\right) + \lambda\left(\sum_{j=0}^{i-1} e^{-\rho(t_i - t_j)} x_j + x_i/2\right)$$

for constants $\gamma, \lambda \in \mathbb{R}^+$. Alfonsi, Schied, and Schulz (2007b) suggested an extension of this model. The special case of independent increments of the fundamental price process can also be described in our framework with the following price impact function:

$$f_i(x_0, \dots, x_i) = \begin{cases} \left(\int_0^{x_i} g\left(\sum_{j=0}^{i-1} e^{-\rho(t_i - t_j)} x_j + y\right) dy\right) / x_i & \text{if } x_i \neq 0 \\ g\left(\sum_{j=0}^{i-1} e^{-\rho(t_i - t_j)} x_j\right) & \text{if } x_i = 0 \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a function determined by the shape of the limit order book.

In our model market, we consider an investor who needs to liquidate an asset position. At time t_0, she owns $X_0 \in \mathbb{R}$ shares of the risky asset and $R_0 \in \mathbb{R}$ units of cash. For $X_0 > 0$, this implies selling the asset, whereas $X_0 < 0$ implies buying the asset. In both cases, we will speak of the "liquidation" or "sale" of the respective long or short position. At the end of trading t_N, her cash account is

$$R = R_0 + \sum_{i=0}^{N} x_i P_i = R_0 + \sum_{i=0}^{N} x_i \left(\tilde{P}_i - f_i(x_0, \ldots, x_i) \right).$$

We assume that the investor wants to maximize expected utility of liquidation proceeds by optimally selling off the asset position, i.e., her objective is to achieve

$$\max_{x_0 + \ldots + x_N = X_0} \mathbb{E}\left[u\left(R \right) \right]. \tag{2.4}$$

Here, $u : \mathbb{R} \to \mathbb{R}$ denotes the utility function of the seller. We make the standard assumptions that it is C^2, increasing ($u' > 0$) and concave ($u'' < 0$). We require that the trades x_i at times t_i are adapted, i.e., they can only depend on ϵ_1, ..., ϵ_i. This includes deterministic (also called static) strategies, i.e., strategies that do not depend on *any* ϵ_1, ..., ϵ_N. Note that in Equation 2.4, we require that the investor sells her entire asset position X_0, i.e., $\sum_{i=0}^{N} x_i = X_0$ irrespective of the random price changes ϵ_i. In the following, we call a strategy $(x_0, x_1, ..., x_N)$ *optimal* if it realizes the maximum in Equation (2.4).

We follow Pratt (1964) and refer to the ratio

$$-\frac{u''(R)}{u'(R)} =: A(R)$$

as the *coefficient of absolute risk aversion.* We will distinguish three cases: increasing absolute risk aversion (IARA) for increasing $A(R)$, constant absolute risk aversion (CARA) for constant $A(R)$, and decreasing absolute risk aversion (DARA) for decreasing $A(R)$.

2.3 DYNAMIC TRADING AT THREE POINTS IN TIME

We start our discussion with a simple example. Assume that trading is only allowed at three points in time t_0, t_1 and t_2. The seller now wants to find the optimal adapted liquidation strategy $(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2)$.

At time t_0, the seller is aware of the fundamental price $\tilde{P}_0$ and the distributions of the ϵ_i. Given this information, we know by the dynamic programming principle that the optimal trading strategy has the following form:

- The initial trade $\tilde{x}_0$ is deterministic, i.e., it is a constant.
- The trade $\tilde{x}_1$ depends only on the price $\tilde{P}_1 = \tilde{P}_0 + \epsilon_1$. In the following, we write $\tilde{x}_1(\tilde{P}_1)$ to stress this dependence.
- The last trade $\tilde{x}_2$ is completely determined by the two previous trades: $\tilde{x}_2(\tilde{P}_1) = X_0 - \tilde{x}_0 - \tilde{x}_1(\tilde{P}_1)$.

Let us first assume that $X_0 > 0$, $X_1 := X_0 - \tilde{x}_0 > 0$ and a.s. $X_2 := X_1 - \tilde{x}_1(\tilde{P}_1) > 0$. Then we can describe the dynamic nature of the trading strategy by analyzing $\tilde{x}_1 : \mathbb{R} \to \mathbb{R}$:

- If $\tilde{x}_1(\tilde{P}_1)$ is decreasing in $\tilde{P}_1$, the strategy is called *passive in-the-money* (PIM). If the stock price falls, trading is accelerated; if the stock price rises, trading is slowed down.
- If $\tilde{x}_1(\tilde{P}_1)$ is a constant, we obtain a deterministic trading strategy: The orders executed at time t_0, t_1 and t_2 are predetermined at time t_0. The strategy is therefore *neutral in-the-money* (NIM); the constant order of $\tilde{x}_1$ is executed at time t_1 irrespective of the price $\tilde{P}_1$.
- If $\tilde{x}_1(\tilde{P}_1)$ is increasing in $\tilde{P}_1$, we call the strategy *aggressive in-the-money* (AIM). It exhibits the opposite characteristics of a PIM strategy.

For the general case of arbitrary signs of X_0, X_1 and X_2, the meaning of aggressive (passive) in-the-money refers to strategies in which the absolute asset position $|X_2|$ is smaller (larger) for "favorable" $\tilde{P}_1$ than for "unfavorable" $\tilde{P}_1$. More precisely:

- If $X_1 > 0$ and $|X_2|$ is increasing in $\tilde{P}_1$ or if $X_1 < 0$ and $|X_2|$ is decreasing in $\tilde{P}_1$, then the strategy is *passive in-the-money* (PIM).
- If X_2 is independent of $\tilde{P}_1$, then the strategy is *neutral in-the-money* (NIM).
- If $X_1{>}0$ and $|X_2|$ is decreasing in $\tilde{P}_1$ or if $X_1 < 0$ and $|X_2|$ is increasing in $\tilde{P}_1$, then the strategy is called *aggressive in-the-money* (AIM).

Theorem 2.1. *An optimal trading strategy exists and is a.s. unique. If trading is allowed at three points in time, then the dynamic nature of the optimal liquidation strategy is uniquely determined by the risk aversion characteristics of the utility function. If $X_1 = X_0 - \tilde{x}_0 = 0$ or a.s. $X_2 = X_1 - \tilde{x}_1(\tilde{P}_1) = 0$, then the optimal trade $\tilde{x}_1(\tilde{P}_1)$ is deterministic. If $X_1 \neq 0$ and $\mathbb{P}[X_2 \neq 0] > 0$, then the following equivalence holds:*

Utility function		*Optimal trading strategy*
Decreasing absolute risk aversion (DARA)	$\Leftrightarrow$	*Passive in-the-money (PIM)*
Constant absolute risk aversion (CARA)	$\Leftrightarrow$	*Neutral in-the-money (NIM)*
Increasing absolute risk aversion (IARA)	$\Leftrightarrow$	*Aggressive in-the-money (AIM)*

The previous theorem holds for the general price impact framework introduced in the previous section. In the next three chapters, we will extend it to a continuous-time setting for suitable extensions of the price impact model introduced by Almgren (2003). Exactly the same equivalence as in Theorem 2.1 is established in Theorem 4.5.

Before proceeding with the proof of Theorem 2.1, we need to introduce some notation. The cash position after execution of the optimal first trade $\tilde{x}_0$, a second transaction x_1 and a final trade $X_0 - \tilde{x}_0 - x_1$ can be expressed as

$$\begin{aligned} R(x_1, \tilde{P}_1) &:= R_0 + \tilde{x}_0 P_0 + x_1 P_1 + (X_0 - \tilde{x}_0 - x_1) P_2 && (2.5)\\ &= R_0 + \tilde{x}_0 \cdot (\tilde{P}_0 - f_0(\tilde{x}_0)) + x_1 \cdot (\tilde{P}_1 - f_1(\tilde{x}_0, x_1)) \\ &\quad + (X_0 - \tilde{x}_0 - x_1) \cdot (\tilde{P}_1 + \epsilon_2 - f_2(\tilde{x}_0, x_1, X_0 - \tilde{x}_0 - x_1)). && (2.6)\end{aligned}$$

The term $R(x_1, \tilde{P}_1)$ is in fact a function of ϵ_2, but for notational simplicity, we do not make this dependence explicit. We now define the function

$$\Phi(\tilde{P}_1) := \mathbb{E}\left[u''\left(R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1) \right) \frac{\partial}{\partial x_1} R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1) \;\Big|\; \tilde{P}_1 \right]. \tag{2.7}$$

The proof of Theorem 2.1 uses the following lemma.

Lemma 2.2. *An optimal trading strategy exists and is a.s. unique. Furthermore, it can be chosen such that* $\tilde{x}_1 : \tilde{P}_1 \in \mathbb{R} \to \tilde{x}_1(\tilde{P}_1) \in \mathbb{R}$ *is differentiable everywhere and that the following equivalence holds for all* $\tilde{P}_1 \in \mathbb{R}$*:*

Φ		*Dynamics of optimal strategy*
$\Phi(\tilde{P}_1) < 0$ *and* $X_0 - \tilde{x}_0 \neq 0$	$\Leftrightarrow$	$\frac{d}{d\tilde{P}_1} \frac{\tilde{x}_1}{X_0 - \tilde{x}_0} < 0$
$\Phi(\tilde{P}_1) = 0$ *or* $X_0 - \tilde{x}_0 = 0$	$\Leftrightarrow$	$\frac{d}{d\tilde{P}_1} \tilde{x}_1 = 0$
$\Phi(\tilde{P}_1) > 0$ *and* $X_0 - \tilde{x}_0 \neq 0$	$\Leftrightarrow$	$\frac{d}{d\tilde{P}_1} \frac{\tilde{x}_1}{X_0 - \tilde{x}_0} > 0$

Proof of Lemma 2.2. The existence of an optimal strategy follows from the superlinear growth of the price impact costs of trading, the a.s. uniqueness by the strict convexity of this cost as assumed by Equation (2.3). Let $(\tilde{x}_0, \tilde{x}_1(\tilde{P}_1))$ be such an optimal strategy. Then it a.s. maximizes the expected utility of the total liquidation return conditional on $\tilde{P}_1$:

$$\tilde{x}_1(\tilde{P}_1) = \arg\max_{x_1} \mathbb{E}\left[u\left(R(x_1, \tilde{P}_1) \right) | \tilde{P}_1 \right] \tag{2.8}$$

This implies that almost surely

$$\begin{aligned} 0 &= \left. \frac{d}{dx_1} \right|_{x_1 = \tilde{x}_1(\tilde{P}_1)} \mathbb{E}\left[u\left(R(x_1, \tilde{P}_1) \right) | \tilde{P}_1 \right] \\ &= \mathbb{E}\left[u'\left(R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1) \right) \left(\frac{\partial}{\partial x_1} R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1) \right) \middle| \tilde{P}_1 \right] \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} 0 &\geq \left. \frac{d^2}{dx_1^2} \right|_{x_1 = \tilde{x}_1(\tilde{P}_1)} \mathbb{E}\left[u\left(R(x_1, \tilde{P}_1) \right) | \tilde{P}_1 \right] \qquad (2.10) \\ &= \mathbb{E}\left[u''\left(R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1) \right) \left(\frac{\partial}{\partial x_1} R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1) \right)^2 \middle| \tilde{P}_1 \right] \\ &\quad + \mathbb{E}\left[u'\left(R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1) \right) \left(\frac{\partial^2}{\partial x_1^2} R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1) \right) \middle| \tilde{P}_1 \right]. \qquad (2.11) \end{aligned}$$

Inequality (2.10) is in fact strict, since the first summand of Expression (2.11) is strictly negative[5] due to $u'' < 0$ and the second one is non-positive due to $u' > 0$ and $\partial^2 R / \partial x_1^2 \leq 0$ due to the convexity of the illiquid market (Equation (2.3)). By changing $\tilde{x}_1(\tilde{P}_1)$ almost nowhere, we can obtain an optimal strategy $\tilde{x}_1$ that fulfills Equation (2.9) everywhere (not only almost surely). By the implicit function theorem, $\tilde{x}_1(\tilde{P}_1)$ is differentiable everywhere since it is a solution of Equation (2.9) with non-vanishing differential (2.11). We can now

[5]Here we used that $\tilde{P}_2$ is not a.s. constant.

continue and derive

$$
\begin{aligned}
0 &= \frac{d}{d\tilde{P}_1}\left[\left.\frac{d}{dx_1}\right|_{x_1=\tilde{x}_1(\tilde{P}_1)} \mathbb{E}\left[u\left(R(x_1,\tilde{P}_1)\right)|\tilde{P}_1\right]\right] \\
&= \mathbb{E}\Bigg[u''\left(R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\left(\frac{\partial}{\partial x_1}R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right) \\
&\qquad \left(\left(\frac{\partial}{\partial x_1}R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\frac{d}{d\tilde{P}_1}\tilde{x}_1(\tilde{P}_1)+\left(\frac{\partial}{\partial \tilde{P}_1}R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\right) \\
&\qquad + u'\left(R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\left(\left(\frac{\partial^2}{\partial x_1^2}R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\frac{d}{d\tilde{P}_1}\tilde{x}_1(\tilde{P}_1)\right. \\
&\qquad \left.\left. + \left(\frac{\partial^2}{\partial x_1\partial\tilde{P}_1}R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\right)\right|\tilde{P}_1\Bigg]. \qquad (2.12)
\end{aligned}
$$

Note that by Equation (2.6)

$$
\begin{aligned}
\frac{\partial}{\partial\tilde{P}_1}R(x_1,\tilde{P}_1) &= X_0-\tilde{x}_0 \qquad (2.13)\\
\frac{\partial^2}{\partial x_1\partial\tilde{P}_1}R(x_1,\tilde{P}_1) &= 0.
\end{aligned}
$$

By solving Equation (2.12) for $\frac{d}{d\tilde{P}_1}\tilde{x}_1(\tilde{P}_1)$ and using the two previous identities, we obtain

$$
\begin{aligned}
\frac{d}{d\tilde{P}_1}\tilde{x}_1(\tilde{P}_1) &= -\mathbb{E}\left[u''\left(R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\left(\frac{\partial}{\partial x_1}R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)(X_0-\tilde{x}_0)\middle|\tilde{P}_1\right] \\
&\qquad \Bigg(\mathbb{E}\Bigg[u''\left(R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\left(\frac{\partial}{\partial x_1}R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)^2 \\
&\qquad\qquad + u'\left(R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\left(\frac{\partial^2}{\partial x_1^2}R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\Bigg|\tilde{P}_1\Bigg]\Bigg)^{-1} \\
&= \Phi(\tilde{P}_1)\frac{X_0-\tilde{x}_0}{-\left.\frac{d^2}{dx_1^2}\right|_{x_1=\tilde{x}_1(\tilde{P}_1)}\mathbb{E}\left[u\left(R(x_1,\tilde{P}_1)\right)|\tilde{P}_1\right]}.
\end{aligned}
$$

The statement of the lemma follows because the denominator of the fraction is positive due to Equation (2.10) and the remark thereafter.

□

Proof of Theorem 2.1. Due to Lemma 2.2, we only need to consider the case

$$X_1 = X_0 - \tilde{x}_0 \neq 0.$$

Furthermore, it is clear that the strategy is deterministic if a.s. $X_2 = 0$. We can therefore restrict our discussion to the case where X_2 is not a.s. zero. When we replace the second derivative u'' by the first derivative u' in Φ (Equation (2.7)), then we obtain

$$\mathbb{E}\left[u'\left(R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\left(\frac{\partial}{\partial x_1}R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\middle|\tilde{P}_1\right] = \left.\frac{d}{dx_1}\right|_{x_1=\tilde{x}_1(\tilde{P}_1)}\mathbb{E}\left[u\left(R(x_1,\tilde{P}_1)\right)|\tilde{P}_1\right] = 0.$$

We therefore look into the relationship between the first and second derivative of u which is described by the absolute risk aversion A. We discuss three cases:

1. *Constant absolute risk aversion (CARA)*: This corresponds to the case where
$$-\frac{u''(R)}{u'(R)} = A(R) \equiv A$$
is a constant. We can therefore write $u''(R) = -Au'(R)$ and obtain
$$\Phi(\tilde{P}_1) = -A \left.\frac{d}{dx_1}\right|_{x_1=\tilde{x}_1(\tilde{P}_1)} \mathbb{E}\left[u\left(R(x_1,\tilde{P}_1)\right)\big|\tilde{P}_1\right] = 0.$$
By Lemma 2.2, this implies that $\frac{d}{d\tilde{P}_1}\tilde{x}_1(\tilde{P}_1) = 0$, i.e., the optimal dynamic strategy is a static NIM strategy.

2. *Decreasing absolute risk aversion (DARA)*: In this case, $A(R)$ is a decreasing function of R. Let us first consider the case that $X_2 = X_0 - \tilde{x}_0 - \tilde{x}_1(\tilde{P}_1) = 0$. Then $R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1)$ is independent of ϵ_2, and we have
$$\begin{aligned}\Phi(\tilde{P}_1) &= \mathbb{E}\left[u''\left(R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\left(\frac{\partial}{\partial x_1}R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\Big|\tilde{P}_1\right]\\ &= \frac{u''\left(R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)}{u'\left(R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)}\mathbb{E}\left[u'\left(R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\left(\frac{\partial}{\partial x_1}R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\Big|\tilde{P}_1\right]\\ &= 0.\end{aligned}$$
By Lemma 2.2, we therefore have $\frac{d}{d\tilde{P}_1}\tilde{x}_1(\tilde{P}_1) = 0$. Let us now turn to $X_2 \gtrless 0$. Then $R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1)$ is increasing (decreasing) in ϵ_2. We decompose Φ:
$$\begin{aligned}\Phi(\tilde{P}_1) &= \mathbb{E}\left[u''\left(R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\left(\frac{\partial}{\partial x_1}R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\Big|\tilde{P}_1\right]\\ &= \mathbb{E}\left[u''\left(R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\left(\frac{\partial}{\partial x_1}R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\mathbb{1}_{\epsilon_2<c}\Big|\tilde{P}_1\right]\\ &\quad + \mathbb{E}\left[u''\left(R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\left(\frac{\partial}{\partial x_1}R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\mathbb{1}_{\epsilon_2\geq c}\Big|\tilde{P}_1\right].\end{aligned}$$
Observe that $\frac{\partial}{\partial x_1}R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1)$ is linearly decreasing in ϵ_2 and thus has a unique root. We choose the decomposition point c as this root and find that for $\epsilon_2 < c$
$$\begin{aligned}&u''\left(R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\left(\frac{\partial}{\partial x_1}R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\\ =\;& \underbrace{u'\left(R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)}_{>0}\underbrace{\left(\frac{\partial}{\partial x_1}R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)}_{>0 \text{ since } \epsilon_2<c}\underbrace{\left(-A\left(R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\right)}_{<0}\\ \lessgtr\;& u'\left(R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)\left(\frac{\partial}{\partial x_1}R(\tilde{x}_1(\tilde{P}_1),\tilde{P}_1)\right)(-A(R_c)) \qquad (2.14)\end{aligned}$$

with

$$R_c := R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1)|_{\epsilon_2 = c}.$$

Here, the two cases in Inequality (2.14) refer to the two cases $X_2 \gtrless 0$, and the inequality holds because A is decreasing and thus $A(R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1))$ is decreasing (increasing) in ϵ_2. Similarly, we obtain for $\epsilon_2 \geq c$:

$$u''\left(R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1)\right)\left(\frac{\partial}{\partial x_1} R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1)\right) \\ \lesseqqgtr u'\left(R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1)\right)\left(\frac{\partial}{\partial x_1} R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1)\right)(-A(R_c)).$$

Combining the two inequalities above, we find

$$\begin{aligned} \Phi(\tilde{P}_1) &\lesseqgtr -A(R_c)\,\mathbb{E}\left[u'\left(R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1)\right)\left(\frac{\partial}{\partial x_1} R(\tilde{x}_1(\tilde{P}_1), \tilde{P}_1)\right) \middle| \tilde{P}_1\right] \\ &= -A(R_c)\,\frac{d}{dx_1}\bigg|_{x_1 = \tilde{x}_1(\tilde{P}_1)} \mathbb{E}\left[u\left(R(x_1, \tilde{P}_1)\right) \middle| \tilde{P}_1\right] = 0. \end{aligned}$$

We now differentiate the four cases $X_1 \gtrless 0$ combined with $X_2 \gtrless 0$. If $X_1 > 0$ and $X_2 > 0$, then Lemma 2.2 yields $\frac{d}{d\tilde{P}_1}\tilde{x}_1(\tilde{P}_1) < 0$ and thus $\frac{d}{d\tilde{P}_1}|X_2| > 0$, i.e., the strategy is passive in-the-money. Similarly, we obtain that also in the other three cases the strategy is passive in-the-money. Since X_2 is not a.s. zero, the strategy as a whole is passive in-the-money.

3. *Increasing absolute risk aversion (IARA)*: The last case we consider is the situation where $A(R)$ is increasing in R. By an argument identical to the one above, we find that $\Phi(\tilde{P}_1) \gtrless 0$ and therefore the strategy is AIM by Lemma 2.2.

□

2.4 Trading at multiple points in time with exponential utility

The fact that static strategies are optimal for investors with constant absolute risk aversion can be extended to an arbitrarily large number of trading points $t_0, \ldots, t_n$ by induction.

Theorem 2.3. *Consider an investor with constant absolute risk aversion in our illiquid model market with $n+1$ points in time at which trading is possible. Then the optimal trading strategy for this investor is deterministic (i.e., neutral in-the-money).*

Proof. By induction over the number of time steps $n+1$, we will prove that for CARA investors, the optimal strategy does not depend on $\tilde{P}_0$, ..., $\tilde{P}_n$, but only on the total number X_0 of shares to be sold, on the price impact functions f_i and on the distributions of the ϵ_i. The base case $n = 0$ is obvious. Let us now assume that the statement holds for n time steps and consider the case of $n+1$ time steps $t_0, \dots, t_n$. Then

$$\begin{aligned}\mathbb{E}[u(R)] &= \mathbb{E}\left[u\left(R_0 + \tilde{x}_0 P_0 + \sum_{i=1}^{n} \tilde{x}_i P_i\right)\right] \\ &= e^{-A(R_0 + X_0 \tilde{P}_0)} e^{A\tilde{x}_0 f_0(\tilde{x}_0)} \mathbb{E}\left[u\left(\sum_{i=1}^{n} \tilde{x}_i \left(\sum_{j=1}^{i} \epsilon_j - f_i(\tilde{x}_0, \dots, \tilde{x}_i)\right)\right)\right].\end{aligned}$$

By the inductive hypothesis the optimal $\tilde{x}_i$ only depend on $\tilde{x}_0$, but not on $\tilde{P}_0$, ..., $\tilde{P}_n$. Furthermore, we see that the optimal $\tilde{x}_0$ is independent of $\tilde{P}_0$, which finishes the inductive step. □

Unfortunately, it is not clear how the more general result of Theorem 2.1 can be extended to more than three trading time points by induction. The crucial observation is that Equation (2.13) does not necessarily hold for general utility functions if more than three trading time points are allowed. However, in Chapter 4 we will see that the connection between risk aversion and dynamics of the optimal trading strategy established in Theorem 2.1 holds also in a continuous-time setting. For the continuous-time setting, the inductive methods used in this chapter are not applicable. Furthermore, an analysis of the convergence of optimal strategies when sending the discretization time step to zero is involved (see Gruber (2004)). In the following investigation of continuous time trading we will therefore follow a different route and apply methods of continuous-time optimal stochastic control.

Chapter 3

Optimal basket liquidation with finite time horizon for CARA investors

3.1 Introduction

Investors frequently wish to trade several assets simultaneously. For example, rebalancing an index tracking fund may require trading in several hundred different shares. Optimal execution of such a basket trade depends not only on the (co-)variances of the assets, but also on the (cross-asset) price impact of trading. In this chapter, we introduce a multiple asset market model that accounts for these aspects. It is a multi-asset extension of the continuous-time liquidity model of Almgren (2003)[1].

In this market model, we consider the *continuous-time*, finite-time horizon optimal basket portfolio liquidation problem for a von Neumann-Morgenstern investor with constant absolute risk aversion (CARA). The main result of this chapter states there is no added utility from allowing for intertemporal updating of basket liquidation strategies for CARA investors in continuous time. Thus the expected utility is maximized by a deterministic, mean-variance optimal strategy. This extends the discrete-time Theorem 2.3 to continuous time. Recently, Almgren and Lorenz (2007) suggested a dynamic strategy to maximize mean-variance performance in a special case of the market model that we are applying. Our theorem implies that this strategy will actually *decrease* the expected value of the exact utility.

The proof of our main result, as given in Section 3.4, relies on the observation that the value function of the *deterministic* problem solves the degenerate Hamilton-Jacobi-Bellman equation for the *adaptive* optimization problem and satisfies the singular initial condition. We can thus apply verification arguments along with proper localization to deal with the singularity of the value function.

3.2 Market model in continuous time

We assume that there are $n \geq 1$ risky assets and a risk-free asset traded. In this market, we consider a large investor who needs to liquidate a basket portfolio $\mathrm{X}_0 = (X_0^1, \dots, X_0^n) \in \mathbb{R}^n$ of shares in the n risky assets by time $T > 0$. The investor chooses a liquidation strategy that we describe by the portfolio $\mathrm{X}_t \in \mathbb{R}^n$ held at time t and that satisfies the boundary condition $\mathrm{X}_T = 0$. We assume that $t \mapsto \mathrm{X}_t$ is absolutely continuous with derivative $\dot{\mathrm{X}}_t =: -\xi_t$, i.e.,

$$\mathrm{X}_t = \mathrm{X}_0 - \int_0^t \xi_s \, ds.$$

[1] Due to the increased technical difficulties of stochastic control in continuous time, we cannot maintain the full generality of the liquidity model introduced in Chapter 2.

For questions such as hedging derivatives, the restriction to absolutely continuous strategies is severe, since it excludes for example the Black-Scholes hedging strategy. For an analysis of optimal liquidation strategies, the restriction appears less grave, since reasonable optimal strategies can be expected to have bounded variation. Nevertheless, it would be desirable to allow block trades, i.e., jumps in X_t. Analyses of models that allow for such block trades (e.g., Obizhaeva and Wang (2006) and Alfonsi, Schied, and Schulz (2007b)) reveal that for realistic parameters the optimal trading strategy is absolutely continuous except for very small block trades at the beginning and end of trading. Numerically, the optimal strategy is almost unchanged by the provision of block trades. Unfortunately, allowing for block trades significantly complicates the mathematical analysis. We therefore believe that it is acceptable to limit the discussion to absolutely continuous strategies and will do so in this chapter as well as in Chapters 4 to 8.

Due to insufficient liquidity, the investor's trading rate ξ_t is moving the market prices. We consider an n-dimensional extension of the model introduced by Almgren (2003) (see also Bertsimas and Lo (1998), Almgren and Chriss (1999) and Almgren and Chriss (2001) for discrete-time precursors of this model). This model is one of the standard models for price impact and widely used both for academic studies as well as for optimal execution algorithms used in practice; see Chapter 1 for a discussion. Similar to the single-asset model in Section 2.2, the transaction price vector $\mathrm{P}_t \in \mathbb{R}^n$ in this model is the difference of the fundamental price $\tilde{\mathrm{P}}_t \in \mathbb{R}^n$ and the price impact $f_t((\xi_s)_{0\le s\le t}) \in \mathbb{R}^n$:

$$\mathrm{P}_t = \tilde{\mathrm{P}}_t - f_t((\xi_s)_{0\le s\le t}).$$

The multi-asset price impact f_t is assumed to be of the following special form:

$$f_t((\xi_s)_{0\le s\le t}) := \int_0^t \mathrm{PermImp}(\xi_s)ds + \mathrm{TempImp}(\xi_t) \in \mathbb{R}^n.$$

The incremental order ξ_t therefore induces both a *permanent* and a *temporary impact* on market prices. The permanent impact $\mathrm{PermImp} : \mathbb{R}^n \to \mathbb{R}^n$ accumulates over time and is assumed to be linear:

$$\mathrm{PermImp}(\xi) := \Gamma\xi$$

where $\Gamma = (\Gamma^{ij}) \in \mathbb{R}^{n\times n}$ is a symmetric $n \times n$ matrix. Linearity and symmetry of the permanent impact are necessary to rule out quasi-arbitrage opportunities as was observed by Huberman and Stanzl (2004). The temporary impact $\mathrm{TempImp} : \mathbb{R}^n \to \mathbb{R}^n$ vanishes instantaneously and thus only effects the incremental order ξ_t itself. It is a possibly nonlinear function. The idealization of instantaneous recovery of the temporary impact is derived from the well-known resilience of stock prices after order placement. It approximates reality reasonably well as long as the time intervals between the physical placement of orders are longer than a few minutes. See, e.g., Bouchaud, Gefen, Potters, and Wyart (2004), Potters and Bouchaud (2003), and Weber and Rosenow (2005) for empirical studies on resilience in order books and Obizhaeva and Wang (2006), Alfonsi, Schied, and Schulz (2007a) and Alfonsi, Schied, and Schulz (2007b) for corresponding market impact models.

When the large investor is not active, it is assumed that the fundamental price process $\tilde{\mathrm{P}}$ follows an n-dimensional Bachelier model with linear drift. The resulting vector-valued transaction price dynamics are hence given by

$$\mathrm{P}_t = \tilde{\mathrm{P}}_0 + \sigma \mathrm{B}_t + \mathrm{b}t + \Gamma(\mathrm{X}_t - \mathrm{X}_0) - \mathrm{TempImp}(\xi_t).$$

Equivalently, the transaction price for the i^{th} asset is given by

$$P_t^i = \tilde{P}_0^i + \sum_{j=1}^n \sigma^{ij} B_t^j + b^i t + \sum_{j=1}^n \Gamma^{ij}(X_t^j - X_0^j) - \mathrm{TempImp}^i(\xi_t);$$

for an initial fundamental price vector $\tilde{\mathrm{P}}_0 \in \mathbb{R}^n$, a standard n-dimensional Brownian motion B starting at $\mathrm{B}_0 = 0$, and a (possibly degenerate) $n \times n$ volatility matrix $\sigma = (\sigma^{ij}) \in \mathbb{R}^{n\times n}$. At first sight, it might seem to be a shortcoming of this model that it allows for negative asset prices. But on the scale we are considering, the price process is a random walk on an equidistant lattice and thus perhaps better approximated by an arithmetic rather than, e.g., a geometric Brownian motion.

In the following, we will not be concerned with P itself, but with the proceeds $\mathrm{P}^\top \xi$ of trading. Several different functions $\mathrm{TempImp} : \mathbb{R}^n \to \mathbb{R}^n$ have the same effect on $\mathrm{P}^\top \xi$. For example, in the two asset case the temporary impact functions $\mathrm{TempImp}(\xi)$ and

$$\widetilde{\mathrm{TempImp}}(\xi) := \mathrm{TempImp}(\xi) + \xi^\top \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

give the same proceeds $\mathrm{P}^\top \xi$. We will therefore not specify TempImp, but instead will work directly with the "temporary impact cost of trading"

$$f : \xi \in \mathbb{R}^n \to f(\xi) := \mathrm{TempImp}(\xi)^\top \xi \in \mathbb{R}_0^+.$$

Throughout this chapter, we assume that f is nonnegative, strictly convex, continuously differentiable (C^1) and exhibits superlinear growth, i.e.,

$$\lim_{|\xi|\to\infty} \frac{f(\xi)}{|\xi|} = \infty.$$

We also assume that b and $\Sigma := \sigma\sigma^\top$ are such that

$$\mathrm{b} \perp \ker \Sigma. \tag{3.1}$$

Several market models fit into our framework. The non-linear temporary impact models for a single asset discussed by Almgren (2003) and statistically estimated by Almgren, Thum, Hauptmann, and Li (2005) correspond to $f(\xi) = \lambda \xi^\beta$. For multiple assets, the linear model introduced by Almgren and Chriss (2001) and analyzed by Konishi and Makimoto (2001) can be realized in our framework by setting $f(\xi) = \xi^\top \Lambda \xi$ with a matrix $\Lambda \in \mathbb{R}^{n\times n}$. A non-linear version of this model is given by $f(\xi) = (\xi^\top \Lambda \xi)^\beta$.

We assume that ξ is progressively measurable with respect to a filtration in which B is a Brownian motion. Strategies also need to be *admissible* in the sense that the resulting position in shares, $\mathrm{X}_t(\omega)$, is bounded uniformly in t and ω with a bound that may depend on the choice ξ. By $\mathcal{X}(T, \mathrm{X}_0)$ we denote the class of all admissible strategies ξ that liquidate by T for the initial condition X_0, i.e., that satisfy $\mathrm{X}_0 - \int_0^T \xi_t \, dt = 0$.

3.3 Utility maximizing trading strategies

If the investor holds a portfolio $X_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}$ units of cash at time 0, then her cash position after the execution of the admissible sales strategy $\xi \in \mathcal{X}(T, X_0)$ is given by

$$\begin{aligned}\mathcal{R}_T(\xi) &= r + \int_0^T \xi_t^\top P_t\,dt \\ &= R_0 + \int_0^T X_t^\top \sigma\,dB_t + \int_0^T b^\top X_t\,dt - \int_0^T f(\xi_t)\,dt,\end{aligned} \tag{3.2}$$

where

$$R_0 = r + X_0^\top \tilde{P}_0 - \frac{1}{2} X_0^\top \Gamma X_0. \tag{3.3}$$

The terms in the expressions above have an economic interpretation. $X_0^\top \tilde{P}_0$ is the face value of the portfolio. The term $\frac{1}{2}X_0^\top \Gamma X_0$ corresponds to the liquidation costs resulting from the permanent price impact. Due to the linearity of the permanent impact function, it is independent of the choice of the liquidation strategy. The stochastic integral in (3.2) corresponds to the volatility risk that is accumulated by selling throughout the interval $[0, T]$ rather than liquidating the portfolio instantaneously. The integral $\int_0^T b^\top X_t\,dt$ corresponds to the change of portfolio value incurred by the drift. Finally, the integral $\int_0^T f(\xi)\,dt$ corresponds to the (nonlinear) transaction costs arising from temporary market impact.

We consider the problem of maximizing the expected utility $\mathbb{E}[\,u(\mathcal{R}_T(\xi))\,]$ of the revenues when ξ ranges over $\mathcal{X}(X_0, T)$ where $u : \mathbb{R} \to \mathbb{R}$ is a utility function. When setting up this problem as a stochastic control problem with controlled diffusion process $\mathcal{R}(\xi)$ and control ξ, we face the difficulty that the class $\mathcal{X}(T, X_0)$ of admissible controls depends on both X_0 and T. To explore some of the effects of this dependence, let us denote the value function of the problem by

$$v(T, X_0, R_0) := \sup_{\xi \in \mathcal{X}(T, X_0)} \mathbb{E}[\,u(\mathcal{R}_T(\xi))\,]. \tag{3.4}$$

It depends on the liquidation time, $T \geq 0$, the initial portfolio, $X_0 \in \mathbb{R}^n$, and the initial cash position, R_0, in (3.3). Heuristic arguments suggest that v should satisfy the degenerate Hamilton-Jacobi-Bellman (HJB) equation

$$v_T = -\inf_{c \in \mathbb{R}^n} \left[-\frac{1}{2} X^\top \Sigma X v_{RR} - b^\top X v_R + f(c) v_R + c^\top (\nabla_X v) \right] \tag{3.5}$$

with singular initial condition

$$\lim_{T \downarrow 0} v(T, X, R) = \begin{cases} u(R) & \text{if } X = 0, \\ -\infty & \text{otherwise.} \end{cases} \tag{3.6}$$

The singularity in the initial condition (3.6) reflects the global *fuel constraint* $\int_0^T \xi_t\,dt = X_0$ that is required from strategies in $\mathcal{X}(T, X_0)$, because it penalizes liquidation tasks that have not been completed in time. Solving this singular Cauchy problem for general utility functions u seems to be a very difficult problem at this time. But in this chapter we will show how it—and the corresponding control problem—can be solved in the case of a CARA utility function.

Theorem 3.1. *For a CARA utility function, $u(R) = -e^{-AR}$ with absolute risk aversion $A > 0$, there exists an a.s. unique optimal strategy $\bar{\mathrm{X}}$ respectively $\bar{\xi} \in \mathcal{X}(T, \mathrm{X}_0)$, which is a deterministic function of time. Moreover, the value function $v(T, \mathrm{X}, R)$ is a classical solution of the singular Cauchy problem* (3.5), (3.6).

The proof is presented in Section 3.4. To characterize the optimal strategy $\bar{\mathrm{X}}$, let us now focus on the case in which X ranges only over the subclass $\bar{\mathcal{X}}(T, \mathrm{X}_0) \subset \mathcal{X}(T, \mathrm{X}_0)$ of *deterministic* admissible strategies, i.e., strategies that do not allow for intertemporal updating. In this case, $\mathcal{R}_T(\xi)$ is normally distributed, and we obtain

$$\mathbb{E}[\, u(\mathcal{R}_T(\xi))\,] = -\mathbb{E}[\, e^{-A\mathcal{R}_T(\xi)}\,] = -\exp\Big(-A\mathbb{E}[\,\mathcal{R}_T(\xi)\,] + \frac{A^2}{2}\mathrm{var}\,(\mathcal{R}_T(\xi))\Big). \tag{3.7}$$

Finding the optimal liquidation strategy is thus reduced to the problem of finding the deterministic strategy $\bar{\xi} \in \bar{\mathcal{X}}(T, \mathrm{X}_0)$ that maximizes the mean-variance functional

$$\mathbb{E}[\,\mathcal{R}_T(\xi)\,] - \frac{A}{2}\mathrm{var}\,(\mathcal{R}_T(\xi)). \tag{3.8}$$

This problem of mean-variance optimization has been introduced by Almgren and Chriss (1999) and Almgren and Chriss (2001) for a single asset ($n = 1$) and studied extensively since; see, e.g., Almgren (2003), Almgren and Lorenz (2007), and the references therein. Note that the variance is weighted here by the factor $A/2$, i.e., with *half* of the risk aversion parameter, in contrast to the convention in Almgren and Chriss (2001) of using the full risk aversion.

For simple special cases of the market model, the mean-variance optimal strategies can be derived in closed form. The following corollary is a simple consequence of Theorem 3.1 and the results of Almgren and Chriss (2001).

Corollary 3.2. *Assume that only one risky asset is traded ($n = 1$) and that the temporary impact is linear, i.e.,*

$$f(\xi) := \lambda \xi^2$$

with $\lambda \in \mathbb{R}^+$. Then the trading strategy $\bar{\xi} \in \mathcal{X}(T, X_0)$ maximizing expected utility for the utility function $u(R) = -e^{-AR}$ with $A \in \mathbb{R}^+$ is given in closed form as

$$\bar{\xi}_t = X_0 \sqrt{\frac{A\sigma^2}{2\lambda}} \cdot \frac{\cosh\Big((T-t)\sqrt{\frac{A\sigma^2}{2\lambda}}\Big)}{\sinh\Big(T\sqrt{\frac{A\sigma^2}{2\lambda}}\Big)}.$$

For the setting of the corollary ($n = 1$, linear temporary impact), Figures 3.1 and 3.2 illustrate the optimal trading strategy $\bar{X}$ respectively $\bar{\xi}$ for different levels of risk aversion. The lower the absolute risk aversion, the more uniform the trading is, resulting in a more linear liquidation of the asset position. Figure 3.3 shows $-v(T, X, R)$.

In the case $n = 1$, Almgren and Lorenz (2007) find that allowing for dynamic updating, i.e., replacing $\bar{\mathcal{X}}(T, \mathrm{X}_0)$ by the entire class $\mathcal{X}(T, \mathrm{X}_0)$ of admissible strategies, can improve the *mean-variance* performance compared to deterministic strategies. That is, the maximizer ξ of the functional (3.8) then no longer is a deterministic function of time. For

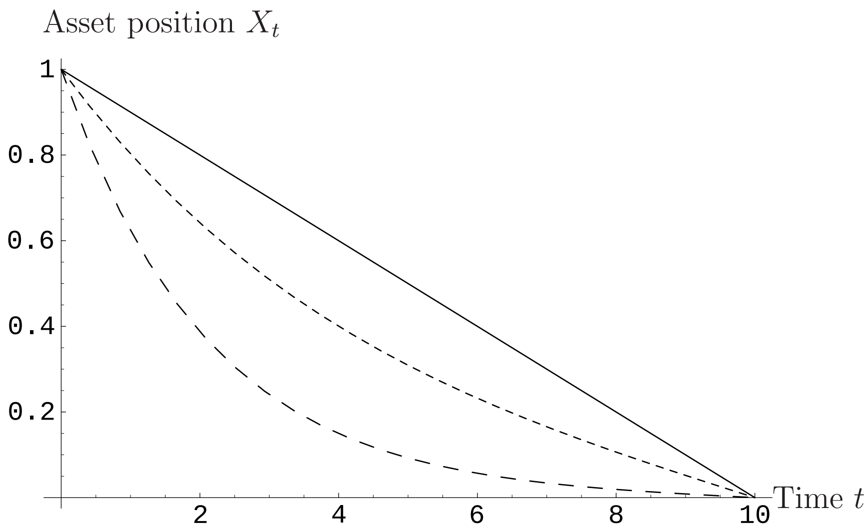


Figure 3.1: Asset position X_t over time for optimal trading strategies with $X_0 = 1$, $T = 10$, $\sigma = 0.03$, $\lambda = 0.01$. The solid line corresponds to $A = 0.0001$, the finely dashed line to $A = 1$, and the coarsely dashed line to $A = 5$.

non-deterministic strategies, however, the identity (3.7) fails, and mean-variance optimization is no longer equivalent to the original problem of maximizing the expected utility of an investor with constant absolute risk aversion. It can only be regarded as a second-order approximation, and Theorem 3.1 shows that in the original problem there is *no* added utility from allowing for intertemporal updating of strategies. In contrast to the special case of CARA utility considered in Theorem 3.1, dynamic strategies *can* improve liquidation performance in general, but only in the case of a utility function with non-constant absolute risk aversion. Such varying risk aversion is clearly insufficiently captured by the mean-variance approximation. We show in Chapters 4 and 5 that for an infinite liquidation horizon, $T = \infty$, deterministic strategies maximize expected utility if and only if the underlying utility function is of CARA type. We strongly believe that this result carries over to the case of a finite liquidation horizon $T < \infty$.

3.4 Proof

For $\xi \in \mathcal{X}(\mathrm{X}_0, T)$, we define the stochastic processes

$$\mathrm{X}_t^\xi := \mathrm{X}_0 - \int_0^t \xi_s \, ds$$

and

$$R_t^\xi := R_0 + \int_0^t \mathrm{X}_s^\xi \cdot \sigma \, d\mathrm{B}_s + \int_0^t \mathrm{b}^\top \mathrm{X}_s^\xi \, ds - \int_0^t f(\xi_s) \, ds, \qquad 0 \le t \le T.$$

If we take $R_0 = r + \tilde{\mathrm{P}}_0^\top \mathrm{X}_0 - \frac{1}{2}\mathrm{X}_0^\top \Gamma \mathrm{X}_0$, then R_T^ξ coincides with the revenues $\mathcal{R}_T(\xi)$ of ξ. But R_0 can also incorporate an initial cash position or revenues that carry over from the 'past' after an application of the Markov property.

Our goal is to identify the value function

$$v(T, \mathrm{X}_0, R_0) = \sup_{\xi \in \mathcal{X}(\mathrm{X}_0, T)} \mathbb{E}[\, u(R_T^\xi) \,] = \sup_{\xi \in \mathcal{X}(\mathrm{X}_0, T)} \mathbb{E}[\, -e^{-AR_T^\xi} \,].$$

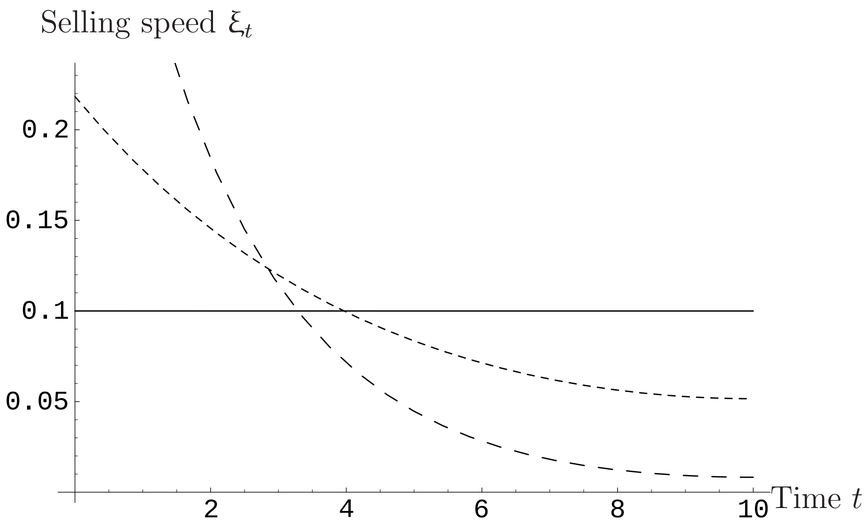


Figure 3.2: Selling speed ξ_t over time for optimal trading strategies with $X_0 = 1$, $T = 10$, $\sigma = 0.03$, $\lambda = 0.01$. The solid line corresponds to $A = 0.0001$, the finely dashed line to $A = 1$, and the coarsely dashed line to $A = 5$.

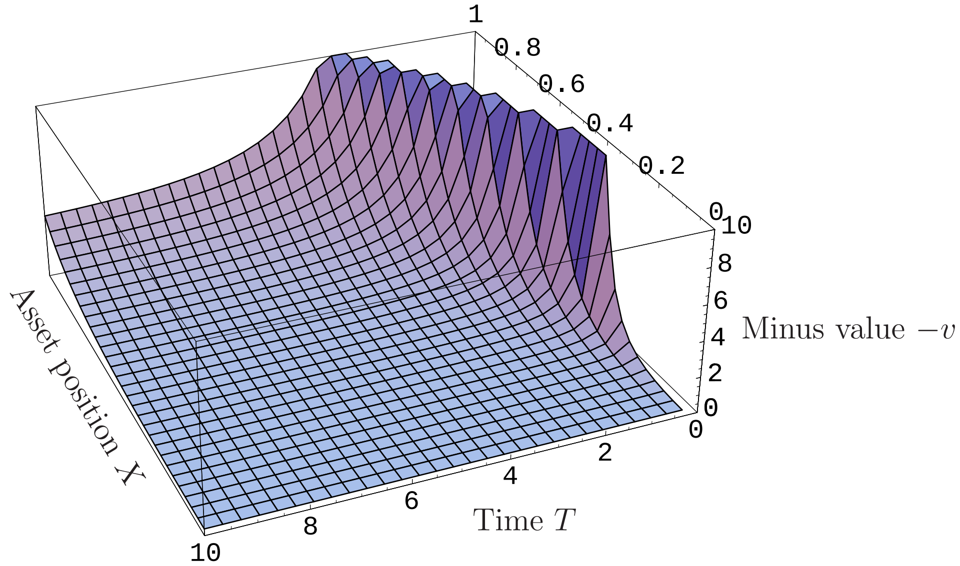


Figure 3.3: Value function $v(T, X, R)$ depending on the remaining time T and asset position X for $A = 1$, $\lambda = 10$, $R = 1$. For better visibility, this figure shows $-v$ instead of v.

with the value function for *deterministic* utility maximization

$$w(T, \mathrm{X}_0, R_0) := \sup_{\xi \in \bar{\mathcal{X}}(T, \mathrm{X}_0)} \mathbb{E}[\, u(R_T^\xi)\,].$$

To this end, we use stochastic control methods. In Subsection 3.4.1, we show that w satisfies the HJB equation (3.5) and the singular initial condition (3.6). Thereafter, we establish the equality of v and w by a verification argument in Subsection 3.4.2.

3.4.1 Deterministic utility maximization

Let us define the mean-variance cost function

$$\bar{v}(T, \mathrm{X}_0) := \inf_{\xi \in \bar{\mathcal{X}}(T,\mathrm{X}_0)} \left[\frac{A}{2} \mathrm{var}\,(R_T^\xi) - \mathbb{E}[\, R_T^\xi - R_0 \,] \right] = \inf_{\xi \in \bar{\mathcal{X}}(T,\mathrm{X}_0)} \int_0^T L(\mathrm{X}_t^\xi, \xi_t)\, dt.$$

where the Lagrangian L is defined as

$$L : (\mathrm{q}, \mathrm{p}) \in \mathbb{R}^{n+n} \to L(\mathrm{q}, \mathrm{p}) = \frac{A}{2} \mathrm{q}^\top \Sigma \mathrm{q} - \mathrm{b}^\top \mathrm{q} + f(\mathrm{p}) \in \mathbb{R}.$$

Minimizing this Lagrangian over $\xi \in \bar{\mathcal{X}}(T, \mathrm{X}_0)$ is a classical problem in the calculus of variations. As discussed in Section 3.3, we have

$$w(T, \mathrm{X}_0, R_0) = \sup_{\xi \in \bar{\mathcal{X}}(T,\mathrm{X}_0)} \mathbb{E}[\, u(R_T^\xi)\,] = -\exp\Big[-AR_0 + A\bar{v}(T, \mathrm{X}_0) \Big]. \tag{3.9}$$

Lemma 3.3. *The function $w(T, \mathrm{X}, R)$ is continuously differentiable in $T > 0$ and $\mathrm{X} \in \mathbb{R}^n$ and satisfies the HJB equation*

$$w_T = -\inf_{\mathrm{c} \in \mathbb{R}^n} \left[-\frac{1}{2} \mathrm{X}^\top \Sigma \mathrm{X} w_{RR} - \mathrm{b}^\top \mathrm{X} w_R + f(\mathrm{c}) w_R + \mathrm{c}^\top (\nabla_\mathrm{X} w) \right] \tag{3.10}$$

with singular initial condition

$$\lim_{T \downarrow 0} w(T, \mathrm{X}, R) = \begin{cases} -e^{-AR} & \text{if } \mathrm{X} = 0, \\ -\infty & \text{if } \mathrm{X} \neq 0. \end{cases} \tag{3.11}$$

For every $\mathrm{X}_0 \in \mathbb{R}^n$ there exists a trading strategy $\bar{\xi}$, respectively $\bar{\mathrm{X}}$ realizing

$$w(T, \mathrm{X}_0, R_0) = \mathbb{E}[\, u(R_T^{\bar{\xi}})\,],$$

i.e., minimizing the action functional $\int_0^T L(\mathrm{X}_t^\xi, \xi_t)\, dt$ over $\xi \in \bar{\mathcal{X}}(T, \mathrm{X}_0)$. Moreover, $\bar{\mathrm{X}}_t$ is Lipschitz continuous in t.

Proof. Our aim is to show that $\bar{v}$ solves the Hamilton-Jacobi equation

$$\bar{v}_T(T, \mathrm{X}) + H(\mathrm{X}, \nabla_\mathrm{X} \bar{v}(T, \mathrm{X})) = 0, \qquad T > 0, \mathrm{X} \in \mathbb{R}^n, \tag{3.12}$$

where

$$H(\mathrm{q}, \mathrm{p}) = -\frac{1}{2} A \mathrm{q}^\top \Sigma \mathrm{q} + \mathrm{b}^\top \mathrm{q} + f^*(\mathrm{p}), \qquad \mathrm{q}, \mathrm{p} \in \mathbb{R}^n,$$

is the Hamiltonian corresponding to the Lagrangian $L(\mathrm{q}, \mathrm{p})$. Here $f^*(z) = \sup_x (x^\top z - f(x))$ denotes the Fenchel-Legendre transform of f.

To see that $\bar{v}(T, \mathrm{X})$ is continuously differentiable and satisfies (3.12), we now apply Theorem 7.1 in Chapter II of Benton (1977). First, note that by taking

$$B := \{(0, 0)\} \subset \mathbb{R} \times \mathbb{R}^n$$

and $g : B \to \mathbb{R}$ as $g(0,0) := 0$, $\bar{v}$ can be written as

$$\bar{v}(T, \mathrm{X}_0) = \inf \Big\{ g(s, \mathrm{y}) + \int_s^T L(\mathrm{Y}_t, \dot{\mathrm{Y}}_t)\, dt \,\big|\, (s, \mathrm{y}) \in B \Big\},$$

where the infimum is taken over all absolutely continuous curves Y such that $\mathrm{Y}_s = \mathrm{y}$ and $\mathrm{Y}_T = \mathrm{X}_0$. Let us assume for a moment that the conditions (H1)–(H6) and (D1)–(D4) in Theorem 7.1 in Chapter II of Benton (1977) are satisfied. Then by Theorem 7.1., Lemma 7.1 and the remark thereafter in Chapter II of Benton (1977), $\bar{v}$ satisfies (3.12), and a Lipschitzian minimizer $\bar{\mathrm{X}}$ exists.

A straightforward computation now shows that

$$w(T, \mathrm{X}, R) = -\exp(-AR + A\bar{v}(T, \mathrm{X})) \tag{3.13}$$

solves

$$w_T = \frac{1}{2}\mathrm{X}^\top \Sigma \mathrm{X} w_{RR} + \mathrm{b}^\top \mathrm{X} w_R + w_R f^*\Big(-\frac{\nabla_{\mathrm{X}} w}{w_R} \Big),$$

and this equation is equivalent to (3.10).

As for the initial condition (3.11), it is clear that $\bar{v}$ satisfies $\bar{v}(T, 0) = 0$ for all T. Moreover, if Y is any curve such that $\mathrm{Y}_0 = 0$ and $\mathrm{Y}_T = \mathrm{X} \neq 0$, then

$$\int_0^T L(\mathrm{Y}_t, \dot{\mathrm{Y}}_t)\, dt = \int_0^T \Big(\frac{1}{2} A \mathrm{Y}_t^\top \Sigma \mathrm{Y}_t + \mathrm{b} \cdot \mathrm{Y}_t \Big)\, dt + \int_0^T f(\dot{\mathrm{Y}}_t)\, dt \geq -TM + Tf\Big(\frac{\mathrm{X}}{T}\Big),$$

by Jensen's inequality and (3.14). Hence, $\bar{v}(T, \mathrm{X}) \geq -MT + Tf(\mathrm{X}/T)$, and this expression blows up as $T \downarrow 0$ by our superlinear growth condition on f.

Let us now check that the conditions (H1)–(H6) and (D1)–(D4) in Theorem 7.1 in Chapter II of Benton (1977) are satisfied. By Theorem 26.6 in Rockafellar (1970), f^* is strictly convex, continuously differentiable, and satisfies the superlinear growth condition. Hence, the same properties are satisfied by $\mathrm{p} \mapsto H(\mathrm{q}, \mathrm{p})$ for any q, and this establishes conditions (H1), (H2), and (H3). Next,

$$\begin{aligned} \mathrm{p}\nabla_{\mathrm{p}} H(\mathrm{q}, \mathrm{p}) - H(\mathrm{q}, \mathrm{p}) &= \mathrm{p}\nabla f^*(\mathrm{p}) - f^*(\mathrm{p}) + \frac{1}{2} A \mathrm{q}^\top \Sigma \mathrm{q} - \mathrm{b}^\top x \\ &= f\big(\nabla f^*(\mathrm{p})\big) + \frac{1}{2} A \mathrm{q}^\top \Sigma \mathrm{q} - \mathrm{b}^\top \mathrm{q}, \end{aligned}$$

due to Theorem 26.5 and 26.6 in Rockafellar (1970). Note that (3.1) implies that there exists a constant M such that

$$|\mathrm{b}^\top \mathrm{q}| \leq \frac{1}{4} A \mathrm{q}^\top \Sigma \mathrm{q} + M. \tag{3.14}$$

Since $f \geq 0$, there is hence a constant C such that for all $\mathrm{q} \in \mathbb{R}^n$

$$|\nabla_{\mathrm{q}} H(\mathrm{q}, \mathrm{p})| \leq C\Big(\mathrm{p}\nabla_{\mathrm{p}} H(\mathrm{q}, \mathrm{p}) - H(\mathrm{q}, \mathrm{p}) + 1 \Big),$$

and this is condition (H4). Moreover,

$$\mathrm{p}\nabla_{\mathrm{p}} H(\mathrm{q}, \mathrm{p}) - H(\mathrm{q}, \mathrm{p}) \geq \frac{1}{2} A \mathrm{q}^\top \Sigma \mathrm{q} - \mathrm{b}^\top \mathrm{q} \geq -M,$$

which is condition (H5). Next, it is clear that both $H(\mathrm{q},\mathrm{p})$ and $|\nabla_{\mathrm{p}} H(\mathrm{q},\mathrm{p})|$ can be bounded from above by an increasing function of $|\mathrm{p}|$, i.e., (H6) holds. Conditions (D1) to (D4) are void in our situation. $\square$

From now on we will refer to the strategy $\bar{\mathrm{X}}$ obtained in the preceding Lemma, and also to its derivative $\bar{\xi}_t = -\dot{\bar{\mathrm{X}}}_t$, as the *optimal strategy in* $\bar{\mathcal{X}}(T,\mathrm{X}_0)$. We then have

$$w(T,\mathrm{X}_0,R_0) = \max_{\xi\in\bar{\mathcal{X}}(T,\mathrm{X})} \mathbb{E}[\,-e^{-AR_T^\xi}\,] = \mathbb{E}[\,-e^{-AR_T^{\bar{\xi}}}\,]. \tag{3.15}$$

3.4.2 Adaptive utility maximization

Let us introduce the sets

$$\mathcal{X}^{(K)}(T,\mathrm{X}_0) := \left\{\xi \in \mathcal{X}(T,\mathrm{X}_0) \,\middle|\, |\mathrm{X}_t^\xi| \le K \text{ for all } t\right\}$$

and the value functions

$$v^{(K)}(T,\mathrm{X},R) := \sup_{\xi\in\mathcal{X}^{(K)}(T,\mathrm{X})} \mathbb{E}[\,-e^{-AR_T^\xi}\,].$$

Note that $\mathcal{X}^{(K)}(T,\mathrm{X})$ is empty for small $K < |\mathrm{X}|$. Moreover, $\bar{\xi} \in \mathcal{X}^{(K)}(T,\mathrm{X})$ for some K. We continue with the following a priori estimate.

Lemma 3.4. *For* $|\mathrm{X}| \le K$*,*

$$v^{(K)}(T,\mathrm{X},R) \le w(T,\mathrm{X},R)\cdot e^{-2A|\mathrm{b}|KT - A^2\mathrm{X}^\top\Sigma\mathrm{X}T/6}.$$

Proof. Suppose that $|\mathrm{X}_0| \le K$ and $\xi \in \mathcal{X}^{(K)}(T,\mathrm{X}_0)$ is such that $\mathbb{E}[\,-e^{-AR_T^\xi}\,]$ is finite. We then have

$$-\infty < \mathbb{E}[\,-e^{-AR_T^\xi}\,] \le -e^{-A\mathbb{E}[\,R_T^\xi\,]}. \tag{3.16}$$

Since X^ξ is bounded, we have

$$\mathbb{E}[\,R_T^\xi\,] = R_0 + \int_0^T \mathbb{E}[\,\mathrm{b}^\top\mathrm{X}_t^\xi - f(\xi_t)\,]\,dt \le R_0 + TK|\mathrm{b}| - \int_0^T \mathbb{E}[\,f(\xi_t)\,]\,dt\,.$$

By (3.16), it follows that $\int_0^T \mathbb{E}[\,f(\xi_t)\,]\,dt$ is finite, and so $\tilde{\xi}_t := \mathbb{E}[\,\xi_t\,]$ is well defined and integrable, due to our assumptions on f. Hence, $\tilde{\xi}$ belongs to $\bar{\mathcal{X}}(T,\mathrm{X}_0)$. Applying Jensen's inequality twice yields

$$\int_0^T \mathbb{E}[\,f(\xi_t)\,]\,dt \ge \int_0^T f(\tilde{\xi}_t)\,dt \ge Tf\Big(\frac{\mathrm{X}_0}{T}\Big).$$

Therefore,

$$\mathbb{E}[\,-e^{-AR_T^\xi}\,] \le -e^{-A\mathbb{E}[\,R_T^\xi\,]} \le -e^{-AR_0 - ATK|\mathrm{b}| + ATf(\mathrm{X}_0/T)} =: \underline{v}(T,\mathrm{X}_0,R_0),$$

and in turn $v^{(K)}(T,\mathrm{X}_0,R_0) \le \underline{v}(T,\mathrm{X}_0,R_0)$.

On the other hand, the constant strategy $\check{\xi}_t = X_0/T$ belongs to $\bar{\mathcal{X}}(T, X_0) \cap \mathcal{X}^{(K)}(T, X_0)$, and so (3.15) yields that

$$\begin{aligned} w(T, X_0, R_0) &\geq \mathbb{E}[\,-e^{-AR_T^{\check{\xi}}}\,] = \underline{v}(T, X_0, R_0) \cdot e^{AT(K|b| - \frac{1}{2}b^\top X_0)} \cdot \mathbb{E}\Big[\,e^{-A\int_0^T \frac{(T-t)}{T} X_0^\top \sigma\, dB_t}\,\Big] \\ &= \underline{v}(T, X_0, R_0) \cdot e^{AT(K|b| - \frac{1}{2}b^\top X_0)} \cdot e^{A^2 X_0^\top \Sigma X_0 T/6} \\ &\geq v^{(K)}(T, X_0, R_0) \cdot e^{2ATK|b| + A^2 X_0^\top \Sigma X_0 T/6}. \end{aligned}$$

This proves the lemma. $\square$

In the next step, we will use a verification argument to identify w with the modified value function

$$v_0(T, X, R) := \inf_{\xi \in \mathcal{X}_0(T,X)} \mathbb{E}[\,-e^{-AR_T^{\xi}}\,] \tag{3.17}$$

that is based on

$$\mathcal{X}_0(T, X) := \Big\{\xi \in \mathcal{X}(X, T) \;\Big|\; \int_0^T f(\xi_t)\, dt \text{ is uniformly bounded in } \omega\Big\}.$$

Lemma 3.5. *We have $v_0(T, X, R) = w(T, X, R)$ and a minimizing strategy in (3.17) is given by the $\bar{\xi}$.*

Proof. The inequality $w \leq v_0$ is obvious from (3.15) and (3.17). To prove the converse inequality, take $K > |X_0|$ and let $\xi \in \mathcal{X}_0(T, X_0) \cap \mathcal{X}^{(K)}(T, X_0)$ be a control process. For $0 < t < T$, Itô's formula yields that

$$\begin{aligned} w(T - t, X_t^\xi, R_t^\xi) - w(T, X_0, R_0) &= \int_0^t w_R(T - s, X_s^\xi, R_s^\xi)(X_s^\xi)^\top \sigma\, dB_s \\ - \int_0^t \Big[f(\xi_s) w_R + \xi_s^\top \nabla_X w + w_T - b^\top X_s^\xi w_R &- \frac{1}{2}(X_s^\xi)^\top \Sigma X_s^\xi w_{RR}\Big](T - s, X_s^\xi, R_s^\xi)\, ds. \end{aligned} \tag{3.18}$$

By (3.10), the latter integral is nonnegative, and by noting that $w_R = -Aw$ we obtain

$$w(T, X_0, R_0) \geq w(T - t, X_t^\xi, R_t^\xi) + A \int_0^t w(T - s, X_s^\xi, R_s^\xi)(X_s^\xi)^\top \sigma\, dB_s. \tag{3.19}$$

We will show next that the stochastic integral in (3.19) is a true martingale. To this end, observe first that, for some constant C_1 depending on the bound for $\int_0^T f(\xi_t)\, dt$ and on the upper bound K for $|X^\xi|$,

$$R_t^\xi := R_0 + \int_0^t (X_s^\xi)^\top \sigma\, dB_s + \int_0^t b^\top X_s^\xi\, ds - \int_0^t f(\xi_s)\, ds \geq -C_1\big(1 + \sup_{s \leq TK^2|\Sigma|} |W_s|\big),$$

where W denotes the DDS-Brownian motion of the continuous martingale $\int_0^t (X_s^\xi)^\top \sigma\, dB_s$ and $|\Sigma|$ is the operator norm of Σ. Hence, by (3.9), for a constant C_2 depending on K and the upper bound of $\int_0^T f(\xi_t)\, dt$,

$$\begin{aligned} w(T - t, X_t^\xi, R_t^\xi) &\geq -\exp\Big(AC_1\big(1 + \sup_{s \leq TK^2|\Sigma|} |W_s|\big) + A \int_t^T L(X_s^\xi, \xi_s)\, ds\Big) \\ &\geq -\exp\Big(AC_1\big(1 + \sup_{s \leq TK^2|\Sigma|} |W_s|\big) + C_2\Big). \end{aligned} \tag{3.20}$$

Since $\sup_{s\leq TK^2|\Sigma|}|W_s|$ has exponential moments of all orders, the martingale property of the stochastic integral in (3.19) follows.

Taking expectations in (3.19) thus yields

$$w(T,\mathrm{X}_0,R_0)\geq\mathbb{E}[\,w(T-t,\mathrm{X}_t^\xi,R_t^\xi)\,]. \tag{3.21}$$

Using the fact that

$$v^{(K)}(T-t,\mathrm{X}_t^\xi,R_t^\xi)\geq\mathbb{E}[-e^{-AR_T^\xi}\,|\,\mathcal{F}_t],$$

Lemma 3.4 then gives that for $C_3:=2A|\mathrm{b}|K+A^2K^2|\Sigma|/6$,

$$w(T,\mathrm{X},R)\geq e^{C_3(T-t)}\mathbb{E}[\,v^{(K)}(T-t,\mathrm{X}_t^\xi,R_t^\xi)\,]\geq e^{C_3(T-t)}\mathbb{E}[\,-e^{-AR_T^\xi}\,].$$

Sending $t\uparrow T$ and taking the infimum over $\xi\in\mathcal{X}_0^{(K)}(T,\mathrm{X}_0)$ and then over $K\geq|\mathrm{X}_0|$ yields $w\geq v_0$.

The inequality $w\geq v_0$ and (3.15) show that $\bar{\xi}$ is an optimal strategy in $\mathcal{X}_0(T,\mathrm{X}_0)$. □

Proof of Theorem 3.1: We first show that $w\geq v$. To this end, let $\xi\in\mathcal{X}(T,\mathrm{X}_0)$ be given such that

$$\mathbb{E}[\,-e^{-AR_T^\xi}\,]>-\infty. \tag{3.22}$$

We define for $k=1,2,\dots$

$$\tau_k:=\inf\left\{t\geq 0\,\Big|\,\int_0^t f(\xi_s)\,ds\geq k\text{ or }(T-t)f\left(\frac{\mathrm{X}_t^\xi}{T-t}\right)\geq k\right\}\wedge T.$$

Let K be an upper bound for $|\mathrm{X}^\xi|$ and define $C:=2A|b|K+A^2K^2|\Sigma|/6$. Conditioning on $\mathcal{F}_{\tau_k}$ and applying Lemma 3.4 yields that

$$\begin{aligned}
\mathbb{E}\big[\,-e^{-AR_T^\xi}\,;\,\tau_k<T\,\big] &\leq \mathbb{E}\big[\,v^{(K)}(T-\tau_k,\mathrm{X}_{\tau_k}^\xi,R_{\tau_k}^\xi)\,;\,\tau_k<T\,\big]\\
&\leq \mathbb{E}\big[\,w(T-\tau_k,\mathrm{X}_{\tau_k}^\xi,R_{\tau_k}^\xi)\cdot e^{-C(T-\tau_k)}\,;\,\tau_k<T\,\big]\\
&\leq e^{-CT}\cdot\mathbb{E}\big[\,w(T-\tau_k,\mathrm{X}_{\tau_k}^\xi,R_{\tau_k}^\xi)\,;\,\tau_k<T\,\big]\\
&\leq 0.
\end{aligned}$$

Since $\mathbb{E}[\,-e^{-AR_T^\xi}\,;\,\tau_k<T\,]$ tends to zero as $k\uparrow\infty$ due to (3.22), we conclude that also

$$\mathbb{E}\big[\,w(T-\tau_k,\mathrm{X}_{\tau_k}^\xi,R_{\tau_k}^\xi)\,;\,\tau_k<T\,\big]\longrightarrow 0. \tag{3.23}$$

Moreover,

$$\mathbb{E}\big[\,-e^{-AR_T^\xi}\,;\,\tau_k=T\,\big]=\mathbb{E}\big[\,w(0,0,R_T^\xi)\,;\,\tau_k=T\,\big]=\mathbb{E}\big[\,w(T-\tau_k,\mathrm{X}_{\tau_k}^\xi,R_{\tau_k}^\xi)\,;\,\tau_k=T\,\big].$$

We may thus conclude that

$$\begin{aligned}
&\mathbb{E}[\,-e^{-AR_T^\xi}\,]\\
&\quad= \mathbb{E}\big[\,-e^{-AR_T^\xi}\,;\,\tau_k<T\,\big]+\mathbb{E}\big[\,-e^{-AR_T^\xi}\,;\,\tau_k=T\,\big]\\
&\quad\leq \mathbb{E}\big[\,w(T-\tau_k,\mathrm{X}_{\tau_k}^\xi,R_{\tau_k}^\xi)\,\big]+(e^{-CT}-1)\cdot\mathbb{E}\big[\,w(T-\tau_k,\mathrm{X}_{\tau_k}^\xi,R_{\tau_k}^\xi)\,;\,\tau_k<T\,\big].
\end{aligned}$$

Now define $\xi^{(k)}$ as ξ up to time τ_k, and for $\tau_k < t \leq T$ we let $\xi^{(k)}$ be the $\mathcal{F}_{\tau_k}$-measurable strategy from Lemma 3.3 that optimally liquidates the amount $X^{\xi}_{\tau_k}$ in the remaining time, $T - \tau_k$. Then $\xi^{(k)}$ belongs to $\mathcal{X}_0(T, X_0)$, and Lemma 3.5 yields

$$\mathbb{E}\big[\, w(T - \tau_k, X^{\xi}_{\tau_k}, R^{\xi}_{\tau_k})\,\big] = \mathbb{E}[\, -e^{-AR^{\xi^{(k)}}_T}\,] \leq w(T, X_0, R_0).$$

Therefore

$$\mathbb{E}[\, -e^{-AR^{\xi}_T}\,] \leq w(T, X, R) + (e^{-CT} - 1) \cdot \mathbb{E}\big[\, w(T - \tau_k, X^{\xi}_{\tau_k}, R^{\xi}_{\tau_k})\,;\, \tau_k < T\,\big].$$

Using (3.23) now yields $w \geq v$ and, in view of (3.15), $v = w$ and the optimality of $\bar{\xi}$ in $\mathcal{X}(T, X_0)$.

Let us conclude by arguing that $\bar{\xi}$ is the a.s. unique optimal strategy in $\mathcal{X}(T, X_0)$ since the functional $\mathbb{E}[-e^{-AR^{\xi}_t}]$ is convex in ξ. $\square$

Chapter 4

Optimal liquidation for general utility functions

4.1 Introduction

Our goal in this chapter is to lift the restriction to exponential utility functions of the previous chapter and to determine the adaptive trading strategy that maximizes the expected utility of the proceeds of an asset sale[1] for general utility functions. We address this question in a special case of the continuous-time liquidity model introduced in Section 3.2 with an infinite time horizon. In this chapter, we only discuss the liquidation of a position in a single asset in a market with linear temporary impact, i.e., quadratic temporary impact trading costs; the more general basket liquidation case with a general impact form is analyzed in Chapter 5. We pursue a stochastic control approach and show that the value function and optimal control satisfy certain nonlinear parabolic partial differential equations. These PDEs can be solved numerically, thus providing a computational solution of the problem. But perhaps even more importantly, the PDE characterization facilitates a qualitative sensitivity analysis of the optimal strategy and the value function.

It turns out that the absolute risk aversion of the utility function is the key parameter that determines the optimal strategy by defining the initial condition for the PDE of the optimal strategy. The optimal strategy thus inherits monotonicity properties of the absolute risk aversion. The relation is identical to the one derived in Theorem 2.1 for a discrete-time framework. In particular, we show that investors with increasing absolute risk aversion (IARA) should sell faster when the asset price rises than when it falls. The optimal strategy is hence “aggressive in-the-money” (AIM). On the other hand, investors with decreasing absolute risk aversion (DARA) should sell slower when asset prices rise, i.e., should pursue a strategy that is “passive in-the-money” (PIM). In general, adaptive liquidation strategies can realize higher expected utility than static liquidation strategies which do not react to asset price changes: static strategies are optimal only for investors with constant absolute risk aversion.

The preceding characterization of AIM and PIM strategies is a consequence of the more general fact that the optimal trading strategy is increasing in the absolute risk aversion of the investor. Surprisingly, however, very few monotonicity relations exist with respect to the other model parameters. For example, a larger asset position can lead to a reduced liquidation speed. Moreover, reducing liquidity by increasing the temporary price impact can result in an increased liquidation speed. The occurrence of the preceding anomaly, however, depends on the risk profile of the utility function, and we show that it cannot exist in the IARA case.

Our approach to the PDE characterizations of the value function and the optimal strat-

[1] The focus on sell orders is for convenience of exposition only; our approach and symmetric statements hold for the case of buy orders.

egy deviates from the standard paradigm in control theory. Although our strategies are parameterized by the time rate of liquidation, it is the remaining asset position that plays the role of a "time" variable in the parabolic PDEs. As a consequence, the HJB equation for the value function is nonlinear in the "time" derivative. We therefore do not follow the standard approach of first solving the HJB equation and then identifying the optimal control as the corresponding maximizer or minimizer. Instead we reverse these steps. We first find that a certain transformation $\tilde{c}$ of the optimal strategy can be obtained as the unique bounded classical solution of a fully nonlinear but classical parabolic PDE. Then we show that the solution of a first-order transport equation with coefficient $\tilde{c}$ yields a smooth solution of the HJB equation. A verification theorem finally identifies this function as the value function. Our qualitative results are proved by combining probabilistic and analytic arguments. Some of the results of this chapter are proven in the more general setting of basket liquidation in Chapter 5. The definition and study of aggressive and passive in-the-money strategies however is conceptually clearer in the single asset setting, which is why we first concentrate on this case.

The remainder of this chapter is structured as follows. In Section 4.2, we introduce the problem setup and assumptions. We consider two questions in framework: optimal liquidation (Section 4.3.1) and maximization of asymptotic portfolio value (Section 4.3.2). The solution to these two problems is presented in Section 4.4. All proofs are given in Section 4.5.

4.2 Assumptions

We apply the market model and notation introduced in Section 3.2 in the special case of a single risky asset and linear temporary impact. We will thus assume the following transaction price process:

$$P_t = \tilde{P}_0 + \sigma B_t + \gamma(X_t - X_0) + \lambda \dot{X}_t$$

with a standard Brownian motion B starting at $B_0 = 0$ and positive constants σ (volatility), γ (permanent impact parameter), λ (temporary impact parameter), and $\tilde{P}_0$ (fundamental price at time 0). Note that we assumed that the stock has no drift ($b = 0$ in the notation of Section 3.2). Almgren and Chriss (2001) found that in the linear market model that we consider in this chapter, the effect of a non-zero drift can be separated from the problem of optimal liquidation. More precisely, the optimal strategy in a market with drift is the sum of two strategies. The first of these strategies is the optimal liquidation strategy in the same market but with zero drift. The second strategy is the optimal strategy in the market with drift, but with a zero initial asset position. This second strategy in fact exploits the knowledge about the future drift to make a profit. It is however completely independent of the original liquidation problem; we therefore neglect it in this analysis of optimal liquidation and focus on the first strategy, which can be computed under the assumption of zero drift. Mathematically, the assumption of zero drift is necessary since we will consider liquidations over infinite time horizons, and only in the absence of drift can we expect an investor to actually liquidate a portfolio with a finite time constraint.

By $\mathcal{X}$ we denote the class of all admissible strategies ξ; for notational simplicity, we will not make the dependence on X_0 explicit.

In the following we assume that the investor is a von-Neumann-Morgenstern investor with a utility function $u : \mathbb{R} \to \mathbb{R}$ with absolute risk aversion $A(R)$ that is bounded away from zero and infinity:

$$A(R) := -\frac{u_{RR}(R)}{u_R(R)}$$
$$0 < \inf_{R\in\mathbb{R}} A(R) =: A_{min} \leq \sup_{R\in\mathbb{R}} A(R) =: A_{max} < \infty$$

Furthermore, we assume that the utility function u is sufficiently smooth (C^6). Most of the theorems that we provide are also valid under weaker smoothness conditions, but to keep things simple we only discuss the C^6-case explicitly.

4.3 Liquidation and optimal investment

We now define the problems of optimal liquidation and optimal investment in the illiquid market model.

4.3.1 Optimal liquidation

We consider a large investor who needs to sell a position of $X_0 > 0$ shares of a risky asset and already holds r units of cash. When following an admissible trading strategy ξ, the investor's total cash position is given by

$$\begin{aligned}\mathcal{R}_t(\xi) &= r + \int_0^t \xi_s P_s \, ds \\ &= r + \tilde{P}_0 X_0 - \frac{\gamma}{2} X_0^2 + \underbrace{\sigma \int_0^t X_s^\xi \, dB_s}_{\Phi_t} - \lambda \int_0^t \xi_s^2 \, ds \\ &\quad \underbrace{-\tilde{P}_0 X_t^\xi - \frac{\gamma}{2}\Big((X_t^\xi)^2 - 2X_0 X_t^\xi\Big) - \sigma X_t^\xi B_t}_{\Psi_t} .\end{aligned}$$

Similar to our approach in the finite liquidation time horizon studies of Chapters 2 and 3, we neglect the accumulation of interest. It is not clear a priori that this is acceptable, since over long time horizons a positive interest rate could potentially have a significant impact on wealth dynamics. We will see in Corollary 4.3, that even without interest, the asset position decreases exponentially under the optimal trading strategy. Incorporating a positive interest rate will lead to an even faster decrease of the asset position; however, due to the already fast exponential liquidation, only small changes to the optimal trading strategy are expected for reasonable parameters.

Since the large investor intends to sell the asset position, we expect the liquidation proceeds to converge $\mathbb{P}$-a.s. to a (possibly infinite) limit as $t \to \infty$. Convergence of Φ_t follows if

$$\mathbb{E}\Big[\int_0^\infty (X_s^\xi)^2 \, ds\Big] < \infty \tag{4.1}$$

and a.s. convergence of Ψ_t is guaranteed if a.s.

$$\lim_{t\to\infty}(X_t^\xi)^2 t\ln\ln t = 0.$$

Note that these conditions do not exclude buy orders (negative ξ_t) or short sales (negative X_t^ξ). We will regard strategies admissible for optimal liquidation if they satisfy the preceding two conditions in addition to the assumptions in Section 4.2; we denote the set of such strategies by $\mathcal{X}_1 \subset \mathcal{X}$. For $\xi \in \mathcal{X}_1$, we then have

$$R_\infty^\xi := \lim_{t\to\infty}\mathcal{R}_t(\xi) \tag{4.2}$$

$$= \underbrace{r + \tilde{P}_0 X_0 - \frac{\gamma}{2}X_0^2}_{=:R_0} + \sigma\int_0^\infty X_s^\xi\,dB_s - \lambda\int_0^\infty \xi_s^2\,ds. \tag{4.3}$$

All of the five terms adding up to R_∞^ξ can be interpreted economically. The number r is simply the initial cash endowment of the investor. $\tilde{P}_0 X_0$ is the face value of the initial position. The term $\frac{\gamma}{2}X_0^2$ corresponds to the liquidation costs resulting from the permanent price impact of ξ. Due to the linearity of the permanent impact function, it is independent of the choice of the liquidation strategy. The stochastic integral corresponds to the volatility risk that is accumulated by selling throughout the interval $[0,\infty[$ rather than liquidating the portfolio instantaneously. The integral $\lambda\int_0^\infty \xi_t^2\,dt$ corresponds to the transaction costs arising from temporary market impact.

We assume that the investor wants to maximize the expected utility of her cash position after liquidation:

$$v_1(X_0, R_0) := \sup_{\xi\in\mathcal{X}_1}\mathbb{E}[u(R_\infty^\xi)] \tag{4.4}$$

4.3.2 Maximization of asymptotic portfolio value

Now consider an investor holding x units of the risky asset and r units of cash at time t. In a liquid market, the value of this portfolio is simply $xP_t + r$. If the market is illiquid, there is no canonical portfolio value. The effect of the temporary price impact depends on the liquidation strategy and can be very small for traders with small risk aversion who liquidate the position at a very slow rate. The permanent impact however cannot be avoided, and its impact on a liquidation return is independent of the trading strategy. We therefore suggest to value the portfolio as

$$r + x\left(P_t - \frac{\gamma}{2}x\right) \tag{4.5}$$

where P_t is the market price at time t including permanent but not temporary impact. In practice, P_t can be observed whenever the large investor does not trade. We can think of the portfolio value as the expected liquidation value when the asset position x is sold infinitely slowly. One advantage of this approach is that the portfolio value cannot be permanently manipulated by moving the market; any such market movement is directly accounted for.

When the trading strategy ξ is pursued, the portfolio value[2] in the above sense evolves

[2]Note that R_t denotes the portfolio value (including risky assets) at time t, while $\mathcal{R}_t$ denotes only the cash position at time t.

over time as

$$R_t^\xi = r + \tilde{P}_0 X_0 - \frac{\gamma}{2} X_0^2 + \sigma \int_0^t X_s^\xi \, dB_s - \lambda \int_0^t \xi_s^2 \, ds. \tag{4.6}$$

We assume that the investor trades the risky asset in order to maximize the asymptotic expected utility of portfolio value:

$$v_2(X_0, R_0) := \sup_{\xi \in \mathcal{X}} \lim_{t \to \infty} \mathbb{E}[u(R_t^\xi)]. \tag{4.7}$$

The existence of the limit will be established in Lemma 4.15. Note that our assumptions on strategies admissible for the maximization of asymptotic portfolio value are weaker than those for optimal liquidation. In particular, we do not require that R_t^ξ or X_t^ξ converge.

4.4 Statement of results

Theorem 4.1. *The value functions $v = v_1$ for optimal liquidation and v_2 for maximization of asymptotic portfolio value are equal and are classical solutions of the Hamilton-Jacobi-Bellman equation*

$$\inf_c \left[-\frac{1}{2} \sigma^2 X^2 v_{RR} + \lambda v_R c^2 + v_X c \right] = 0 \tag{4.8}$$

with boundary condition

$$v(0, R) = u(R) \text{ for all } R \in \mathbb{R}. \tag{4.9}$$

The a.s. unique optimal control $\hat{\xi}_t$ is Markovian and given in feedback form by

$$\hat{\xi}_t = c(X_t^{\hat{\xi}}, R_t^{\hat{\xi}}) = -\frac{v_X}{2\lambda v_R}(X_t^{\hat{\xi}}, R_t^{\hat{\xi}}). \tag{4.10}$$

For the value functions, we have convergence:

$$v(X_0, R_0) = \lim_{t \to \infty} \mathbb{E}[u(R_t^{\hat{\xi}})] = \mathbb{E}[u(R_\infty^{\hat{\xi}})] \tag{4.11}$$

Note that the HJB equation in the preceding theorem is fully nonlinear in all partial derivatives of v, even in the "time" derivative, v_X. This can best be observed in the corresponding reduced-form equation:

$$v_X^2 = -2\lambda \sigma^2 X^2 v_R v_{RR}. \tag{4.12}$$

In the following we will use the term "optimal control" to refer to the optimal admissible strategy $\hat{\xi}$ or the optimal feedback function c, depending on the circumstances. At the heart of the above theorem lies the transformed optimal control

$$\tilde{c}(Y, R) := c(\sqrt{Y}, R)/\sqrt{Y}.$$

The existence of a solution to the HJB equation in Theorem 4.1 will be derived from the existence of a smooth solution to the fully nonlinear parabolic PDE given in the following theorem.

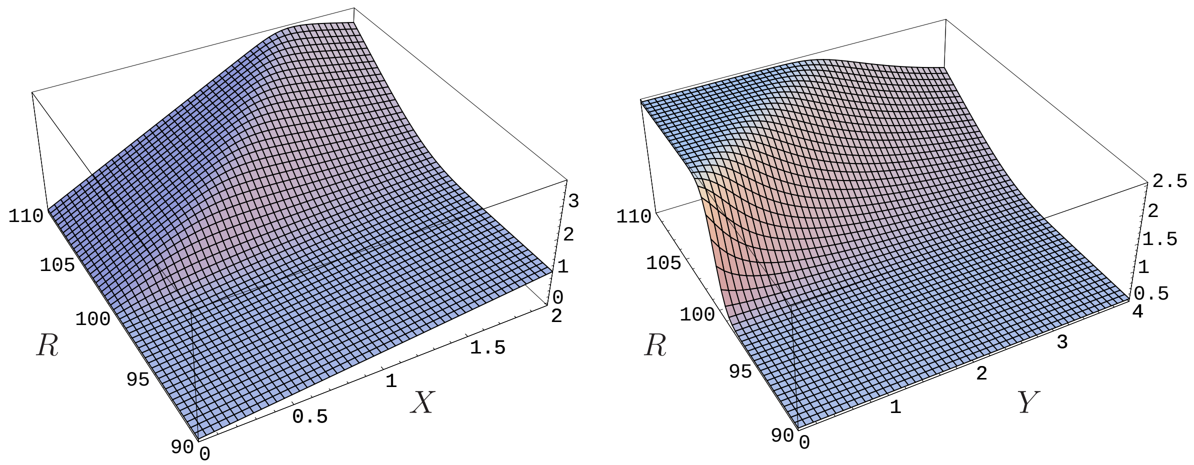


Figure 4.1: Optimal control $c(X, R)$ (left hand figure) and transformed optimal control $\tilde{c}(Y, R)$ (right hand figure) for the utility function with absolute risk aversion $A(R) = 2(1.5 + \tanh(R - 100))^2$ and parameter $\lambda = \sigma = 1$.

Theorem 4.2. *The transformed optimal control $\tilde{c}$ is a classical solution of the fully nonlinear parabolic PDE*

$$\tilde{c}_Y = -\frac{3}{2}\lambda\tilde{c}\tilde{c}_R + \frac{\sigma^2}{4\tilde{c}}\tilde{c}_{RR} \tag{4.13}$$

with initial condition

$$\tilde{c}(0, R) = \sqrt{\frac{\sigma^2 A(R)}{2\lambda}}. \tag{4.14}$$

The bounds of the absolute risk aversion give bounds for the transformed optimal control:

$$\inf_{(Y,R)\in\mathbb{R}_0^+\times\mathbb{R}} \tilde{c}(Y, R) = \inf_{R\in\mathbb{R}} \tilde{c}(0, R) =: \tilde{c}_{min} = \sqrt{\frac{\sigma^2 A_{min}}{2\lambda}}$$

$$\sup_{(Y,R)\in\mathbb{R}_0^+\times\mathbb{R}} \tilde{c}(Y, R) = \sup_{R\in\mathbb{R}} \tilde{c}(0, R) =: \tilde{c}_{max} = \sqrt{\frac{\sigma^2 A_{max}}{2\lambda}}$$

Figure 4.1 shows a numerical example of c and $\tilde{c}$.

Corollary 4.3. *The asset position $X_t^{\hat{\xi}}$ at time t under the optimal control $\hat{\xi}$ is given by*

$$X_t^{\hat{\xi}} = X_0 \exp\left(-\int_0^t \tilde{c}((X_s^{\hat{\xi}})^2, R_s^{\hat{\xi}})\, ds\right) \tag{4.15}$$

and is bounded by

$$X_0 \exp(-t\tilde{c}_{max}) \leq X_t^{\hat{\xi}} \leq X_0 \exp(-t\tilde{c}_{min}).$$

Although we did not a priori exclude intermediate buy orders or short sales, the preceding theorem and corollary reveal that these are never optimal. For investors with constant absolute risk aversion $A = A_{min} = A_{max}$, Corollary 4.3 yields the following explicit formula for the optimal strategy. It is identical to the optimal strategy for mean-variance investors (see Almgren (2003)) and is the limit of optimal execution strategies for finite time horizons (see Chapter 3).

Corollary 4.4. *Assume that the investor has a utility function $u(R) = -e^{-AR}$ with constant risk aversion $A(R) \equiv A$. Then her optimal adaptive liquidation strategy is static and is given by*

$$X_t^{\hat{\xi}} = X_0 \exp\left(-t\sqrt{\frac{\sigma^2 A}{2\lambda}}\right). \tag{4.16}$$

Given the optimal control $c(X,R)$ (or the transformed optimal control $\tilde{c}(X,R)$), we can identify the optimal strategy as aggressive in-the-money (AIM), neutral in-the-money (NIM) and passive in-the-money (PIM) (see also Section 2.3). If prices rise, then R rises. A strategy with an optimal control c that is increasing in R (everything else held constant) sells fast in such a scenario, i.e., is aggressive in-the-money; if c is decreasing in R, it is passive in-the-money, and if c is independent of R, then the strategy is neutral in-the-money. The initial value specification for $\tilde{c}$ given in Theorem 4.2 shows that there is a tight relation between the absolute risk aversion and the optimal adaptive trading strategy: If A is an increasing function, i.e., the utility function u exhibits increasing absolute risk aversion (IARA), then the optimal strategy is aggressive in-the-money at least for small values of X. The next theorem states that such a monotonicity of $\tilde{c}$ propagates to all values of X, not only to small values of X.

Theorem 4.5. *$c(X,R)$ is increasing (decreasing) in R for all values of X if and only if the absolute risk aversion $A(R)$ is increasing (decreasing) in R. In particular, $A(R)$ determines the characteristics of the optimal strategy:*

Utility function		*Optimal trading strategy*
Decreasing absolute risk aversion (DARA)	$\Leftrightarrow$	*Passive in-the-money (PIM)*
Constant absolute risk aversion (CARA)	$\Leftrightarrow$	*Neutral in-the-money (NIM)*
Increasing absolute risk aversion (IARA)	$\Leftrightarrow$	*Aggressive in-the-money (AIM)*

Note that in the numerical example in Figure 4.1, A is increasing. The figure confirms that c and $\tilde{c}$ are also increasing in R. Figure 4.2 shows two sample paths of $X_t^{\hat{\xi}}$. As expected, the asset position is decreased quicker when the asset price is rising than when it is falling.

We now turn to the dependence of the optimal control c on the problem parameters u, X, λ and σ. The following theorem describes the dependence on u. Theorem 4.5 is in fact a corollary to the following general result.

Theorem 4.6. *Suppose u^0 and u^1 are two utility functions such that u^1 has a higher absolute risk aversion than u^0, i.e., $A^1(R) \geq A^0(R)$ for all R. Then an investor with utility function u^1 liquidates the same portfolio X_0 faster than an investor with utility function u^0. More precisely, the corresponding optimal strategies satisfy*

$$c^1 \geq c^0 \qquad \text{and} \qquad \hat{\xi}_t^1 \geq \hat{\xi}_t^0 \qquad \mathbb{P}\text{-a.s.}.$$

An increase of the asset position X has two effects on the optimal liquidation strategy. First, it increases overall risk, leading to a desire to increase the selling speed. Second, it changes the distribution of total proceeds R_∞: it increases its dispersion due to increased risk, and it moves it downwards due to increased temporary impact liquidation cost. This change in return distribution can lead to a reduction in relevant risk aversion and thus a

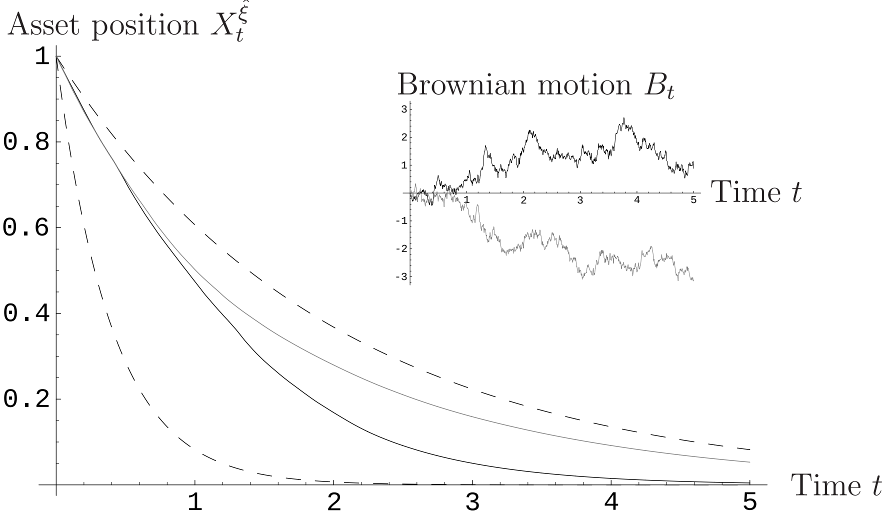


Figure 4.2: Two sample optimal execution paths $X_t^{\hat{\xi}}$ corresponding to the sample paths of the Brownian motion B_t in the inset. The dashed lines represent the upper and lower bounds on $X_t^{\hat{\xi}}$. Parameters are $\lambda = \gamma = \sigma = 1$, $X_0 = 1$, $R_0 = 0$, $\tilde{P}_0 = 100$ and the utility function with absolute risk aversion $A(R) = 2(1.5 + \tanh(R - 100))^2$. 1000 simulation steps were used covering the time span $[0, 5]$.

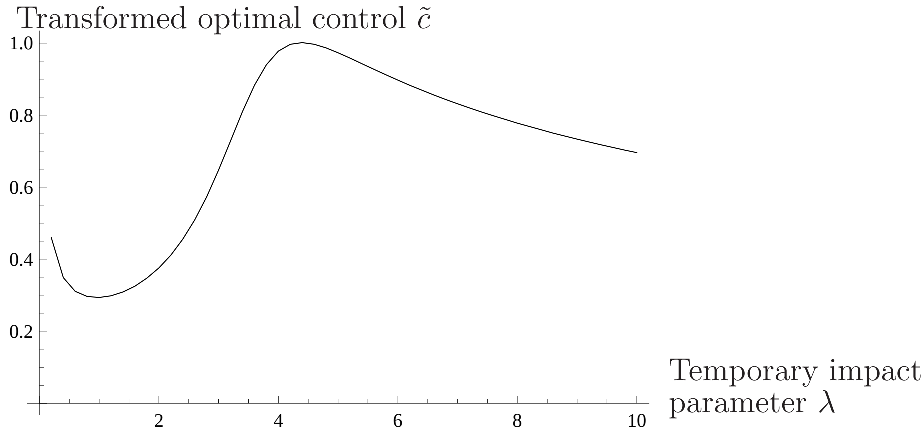


Figure 4.3: Transformed optimal control $\tilde{c}(Y, R, \lambda, \sigma)$ depending on the temporary impact parameter λ. Parameters are $Y = 0.5$, $R = 2$, $\sigma = 1$ and the utility function u with absolute risk aversion $A(R) = 2(1.2 - \tanh(15R))^2$.

desire to reduce the selling speed. In Figure 4.1 one can make the surprising observation that the second effect can outweigh the first, i.e., that the optimal strategy $c(X, R)$ need not be increasing in X. That is, an *increase* of the asset position may lead to a *decrease* of the liquidation rate.

We now turn to the dependence of c on the impact parameters. Perhaps surprisingly, neither the value function v nor the optimal control $\hat{\xi}$ respectively c depend directly on the permanent impact parameter γ. However, γ influences the portfolio value state variable $R = r + X\left(P - \frac{\gamma}{2}X\right)$ and therefore indirectly also the optimal control. For the temporary impact parameter λ, we intuitively expect that the optimal control c decreases when λ increases, since fast trading becomes more expensive. Figure 4.3 shows that this is not necessarily the case: in this example, an increased temporary impact cost leads to faster selling. This counterintuitive behavior cannot occur for IARA utility functions:

Theorem 4.7. *If the utility function u exhibits increasing absolute risk aversion (IARA),* *then the optimal control c is decreasing in the temporary impact parameter λ.*

We conclude our sensitivity analysis with the following theorem that links the dependence on σ to the dependence on λ and X.

Theorem 4.8 (Relation between σ, λ and X)**.** *Let $c(X, R, \lambda, \sigma)$ be the optimal control in a market with temporary impact parameter λ and volatility σ. Then*

$$c(X, R, \lambda, \sigma_1) = \frac{\sigma_2}{\sigma_1} c\left(\frac{\sigma_1}{\sigma_2} X, R, \frac{\sigma_2^2}{\sigma_1^2}\lambda, \sigma_2\right). \tag{4.17}$$

By the boundary condition, we know that $v(0, R) = u(R)$ is a utility function. The next theorem states that for each value of X, $v(X, R)$ can be regarded as a utility function in R.

Theorem 4.9. *The value function $v(X, R)$ is strictly concave, jointly in X and R, increasing in R and decreasing in X. In particular, for every $X > 0$, the value function $v(X, R)$ is again a utility function in R. Moreover, for all X and R, $\tilde{c}(X^2, R)$ is proportional to the square root of the absolute risk aversion $A(X, R) := -v_{RR}(X, R)/v_R(X, R)$ of $v(X, R)$:*

$$\tilde{c}(X^2, R) = \sqrt{\frac{\sigma^2 A(X, R)}{2\lambda}}. \tag{4.18}$$

The value function $v(X, R)$ is only *decreasing* in X when the portfolio value R is kept constant. In this case, increasing X shifts value from the cash account toward the risky asset, which always decreases utility for a risk-averse investor.

In view of non-concave utility functions suggested, e.g., by the prospect theory of Kahneman and Tversky (1979), one might ask to what extend the concavity of u is an essential ingredient of our analysis. Which of our results may carry over to 'utility functions' u that are strictly increasing but not concave? Let us suppose that v is defined as in Equations (4.4) or (4.7). Then it follows immediately that $R \mapsto v(X, R)$ is strictly increasing. If v also satisfies the HJB equation, Equation (4.8), then Equation (4.12) yields

$$v_{RR} = -\frac{v_X^2}{2\sigma^2\lambda v_R} \leq 0.$$

Hence, $R \mapsto v(X, R)$ is concave for every $X > 0$. Therefore v cannot be a solution of the initial value problem in Equations (4.8) and (4.9) unless $v(0, R) = u(R)$ is also concave. This shows that the concavity of u is essential to our approach. Note that the preceding argument can also be used to give an alternative proof of the assertion of concavity in Theorem 4.9.

4.5 Proof of results

This section consists of three parts. First we show that a smooth solution of the HJB equation exists and provide some of its properties. This is achieved by first obtaining a solution of the PDE for the transformed optimal strategy, $\tilde{c}$, and then solving a transport equation with coefficient $\tilde{c}$. In the second part, we apply a verification argument and show that this solution of the HJB equation must be equal to the value function. Theorems 4.1 and 4.2 are direct consequences of the propositions in these two subsections. In the last subsection we prove the qualitative properties of the optimal adaptive strategy and the value function given in Theorems 4.5, 4.6, 4.7, 4.8 and 4.9.

4.5.1 Existence and characterization of a smooth solution of the HJB equation

As a first step, we observe that $\lim_{R\to\infty} u(R) < \infty$ due to the boundedness of the risk aversion, and we can thus assume without loss of generality that

$$\lim_{R\to\infty} u(R) = 0.$$

Proposition 4.10. *There exists a smooth ($C^{2,4}$) solution $\tilde{c} : (Y, R) \in \mathbb{R}_0^+ \times \mathbb{R} \to \tilde{c}(Y, R) \in \mathbb{R}$ of*

$$\tilde{c}_Y = -\frac{3}{2}\lambda\tilde{c}\tilde{c}_R + \frac{\sigma^2}{4\tilde{c}}\tilde{c}_{RR} \tag{4.19}$$

with initial value

$$\tilde{c}(0, R) = \sqrt{\frac{\sigma^2 A(R)}{2\lambda}}. \tag{4.20}$$

The solution satisfies

$$\tilde{c}_{min} := \inf_{R\in\mathbb{R}} \sqrt{\frac{\sigma^2 A(R)}{2\lambda}} \leq \tilde{c}(Y, R) \leq \sup_{R\in\mathbb{R}} \sqrt{\frac{\sigma^2 A(R)}{2\lambda}} =: \tilde{c}_{max}. \tag{4.21}$$

The function $\tilde{c}$ is $C^{2,4}$ in the sense that it has a continuous derivative $\frac{\partial^{i+j}}{\partial Y^i \partial R^j}\tilde{c}(Y, R)$ if $2i + j \leq 4$. In particular, $\tilde{c}_{YRR}$ and $\tilde{c}_{RRR}$ exist and are continuous.

The statement follows from the following auxiliary theorem from the theory of parabolic partial differential equations. We do not establish the uniqueness of $\tilde{c}$ directly in the preceding proposition. However, it follows from Proposition 4.18.

Theorem 4.11 (Auxiliary theorem: Solution of Cauchy problem)**.** *There is a smooth solution ($C^{2,4}$)*

$$f : (t, x) \in \mathbb{R}_0^+ \times \mathbb{R} \to f(t, x) \in \mathbb{R}$$

for the parabolic partial differential equation[3]

$$f_t - \frac{d}{dx}a(x, t, f, f_x) + b(x, t, f, f_x) = 0 \tag{4.22}$$

with initial value condition

$$f(0, x) = \psi_0(x)$$

if all of the following conditions are satisfied:

- *$\psi_0(x)$ is smooth (C^4) and bounded*
- *a and b are smooth (C^3 respectively C^2)*
- *There are constants b_1 and $b_2 \geq 0$ such that for all x and u:*

$$\left(b(x, t, u, 0) - \frac{\partial a}{\partial x}(x, t, u, 0)\right) u \geq -b_1 u^2 - b_2.$$

[3]Here, f_t refers to $\frac{d}{dt}f$ and not $f(t)$.

- *For all $M > 0$, there are constants $\mu_M \geq \nu_M > 0$ such that for all x, t, u and p that are bounded in modulus by M:*

$$\nu_M \leq \frac{\partial a}{\partial p}(x,t,u,p) \leq \mu_M$$

and
$$\left(|a| + \left|\frac{\partial a}{\partial u}\right|\right)(1+|p|) + \left|\frac{\partial a}{\partial x}\right| + |b| \leq \mu_M(1+|p|)^2.$$

Proof. The theorem is a direct consequence of Theorem 8.1 in Chapter V of Ladyzhenskaya, Solonnikov, and Ural'ceva (1968). In the following, we outline the last step of its proof because we will use it for the proof of subsequent propositions.

The conditions of the theorem guarantee the existence of solutions f_N of Equation (4.22) on the strip $\mathbb{R}_0^+ \times [-N, N]$ with boundary conditions

$$f_N(0,x) = \psi_0(x) \text{ for all } x \in [-N, N]$$

and
$$f_N(t, \pm N) = \psi_0(\pm N) \text{ for all } t \in \mathbb{R}_0^+.$$

These solutions converge smoothly as N tends to infinity: $\lim_{N\to\infty} f_N = f$. □

Proof of Proposition 4.10. We want to apply Theorem 4.11 and set

$$\begin{aligned} a(x,t,u,p) &:= h_1(u)p \\ b(x,t,u,p) &:= \frac{3}{2}\lambda h_2(u)p + h_1'(u)p^2 \\ \psi_0(x) &:= \sqrt{\frac{\sigma^2 A(R)}{2\lambda}} \end{aligned}$$

with smooth functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$. With $h_1(u) = \frac{\sigma^2}{4u}$ and $h_2(u) = u$, Equation (4.22) becomes Equation (4.19) by relabeling the coordinates from t to Y and from x to R. All conditions of Auxiliary Theorem 4.11 are fulfilled, except for the last boundedness condition. In order to fulfill these, we take h_1 and h_2 to be smooth nonnegative bounded functions fulfilling $h_1(u) = \frac{\sigma^2}{4u}$ and $h_2(u) = u$ for $\tilde{c}_{min} \leq u \leq \tilde{c}_{max}$. Now all conditions of Theorem 4.11 are fulfilled and there exists a smooth solution to

$$f_t = -\frac{3}{2}\lambda h_2(f) f_x + h_1(f) f_{xx}.$$

We now show that this solution f also fulfills

$$f_t = -\frac{3}{2}\lambda f f_x + \frac{\sigma^2}{4f} f_{xx}$$

by using the maximum principle to show that $\tilde{c}_{min} \leq f \leq \tilde{c}_{max}$. First assume that there is a (t_0, x_0) such that $f(t_0, x_0) > \tilde{c}_{max}$. Then there is an $N > 0$ and $\gamma > 0$ such that also $\tilde{f}_N(t_0, x_0) := f_N(t_0, x_0)e^{-\gamma t_0} > \tilde{c}_{max}$ with f_N as constructed in the proof of Theorem 4.11.

Then $\max_{t\in[0,t_0],x\in[-N,N]} \tilde{f}_N(t,x)$ is attained at an interior point (t_1,x_1), i.e., $0 < t_1 \le t_0$ and $-N < x_1 < N$. We thus have

$$\begin{aligned}\tilde{f}_{N,t}(t_1,x_1) &\ge 0\\ \tilde{f}_{N,x}(t_1,x_1) &= 0\\ \tilde{f}_{N,xx}(t_1,x_1) &\le 0.\end{aligned}$$

We furthermore have that

$$\begin{aligned}\tilde{f}_{N,t} &= e^{-\gamma t} f_{N,t} - \gamma e^{-\gamma t} f_N\\ &= -\frac{3}{2} e^{-\gamma t}\lambda h_2(f_N) f_{N,x} + e^{-\gamma t} h_1(f_N) f_{N,xx} - \gamma e^{-\gamma t} f_N\\ &= -\frac{3}{2}\lambda h_2(f_N)\tilde{f}_{N,x} + h_1(f_N)\tilde{f}_{N,xx} - \gamma \tilde{f}_N\end{aligned}$$

and therefore that

$$\tilde{f}_N(t_1,x_1) \le 0.$$

This however contradicts $\tilde{f}_N(t_1,x_1) \ge \tilde{f}_N(t_0,x_0) \ge \tilde{c}_{max} > 0$.

By a similar argument, we can show that if there is a point (t_0,x_0) with $f(t_0,x_0) < \tilde{c}_{min}$, then the interior minimum (t_1,x_1) of a suitably chosen $\tilde{f}_N := f_N - \tilde{c}_{max} < 0$ satisfies $\tilde{f}_N(t_1,x_1) > 0$ and thus causes a contradiction. □

Proposition 4.12. *There exists a $C^{2,4}$-solution $\tilde{w} : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ of the transport equation*

$$\tilde{w}_Y = -\lambda \tilde{c} \tilde{w}_R \tag{4.23}$$

with initial value

$$\tilde{w}(0,R) = u(R).$$

The solution satisfies

$$0 \ge \tilde{w}(Y,R) \ge u(R - \lambda \tilde{c}_{max} Y)$$

and is increasing in R and decreasing in Y.

Proof. The proof uses the method of characteristics. Consider the function

$$P : (Y,S) \in \mathbb{R}_0^+ \times \mathbb{R} \to P(Y,S) \in \mathbb{R}$$

satisfying the ODE

$$P_Y(Y,S) = \lambda \tilde{c}(Y, P(Y,S))$$

with initial value condition $P(0,S) = S$. Since $\tilde{c}$ is smooth and bounded, a solution of the above ODE exists for each fixed S. For every Y, $P(Y,\cdot)$ is a diffeomorphism mapping $\mathbb{R}$ onto $\mathbb{R}$ and has the same regularity as $\tilde{c}$, i.e., belongs to $C^{2,4}$. We define

$$\tilde{w}(Y,R) = u(S) \quad \text{iff} \quad P(Y,S) = R.$$

Then $\tilde{w}$ is a $C^{2,4}$-function satisfying the initial value condition. By definition, we have

$$\begin{aligned}
0 &= \frac{d}{dY}\tilde{w}(Y, P(Y,S)) \\
&= \tilde{w}_R(Y, P(Y,S))P_Y(Y,S) + \tilde{w}_Y(Y, P(Y,S)) \\
&= \tilde{w}_R(Y, P(Y,S))\lambda\tilde{c}(Y, P(Y,S)) + \tilde{w}_Y(Y, P(Y,S)).
\end{aligned}$$

Therefore $\tilde{w}$ fulfills the desired partial differential equation. Since $\tilde{c} \leq \tilde{c}_{max}$, we know that $P_Y \leq \lambda\tilde{c}_{max}$ and hence $P(Y,S) \leq S + Y\lambda\tilde{c}_{max}$ and thus $\tilde{w}(Y,R) \geq u(R - \lambda\tilde{c}_{max}Y)$.

The monotonicity statements in the proposition follow because the family of solutions of the ODE above do not cross and since $\tilde{c}$ is positive. □

Proposition 4.13. *The function $w(X,R) := \tilde{w}(X^2, R)$ solves the HJB equation*

$$\min_c \left[-\frac{1}{2}\sigma^2 X^2 w_{RR} + \lambda w_R c^2 + w_X c \right] = 0. \tag{4.24}$$

The unique minimum is attained at

$$c(X,R) := \tilde{c}(X^2, R)X. \tag{4.25}$$

Proof. Assume for the moment that

$$\tilde{c}^2 = -\frac{\sigma^2 \tilde{w}_{RR}}{2\lambda\tilde{w}_R}. \tag{4.26}$$

Then with $Y = X^2$:

$$\begin{aligned}
0 &= -\lambda X^2 \tilde{w}_R \left(\frac{\sigma^2\tilde{w}_{RR}}{2\lambda\tilde{w}_R} + \tilde{c}^2 \right) \\
&= -\lambda X^2 \tilde{w}_R \left(\frac{\sigma^2\tilde{w}_{RR}}{2\lambda\tilde{w}_R} + \frac{\tilde{w}_Y^2}{\lambda^2\tilde{w}_R^2} \right) \\
&= -\frac{1}{2}\sigma^2 X^2 w_{RR} - \frac{w_X^2}{4\lambda w_R} \\
&= \inf_c \left[-\frac{1}{2}\sigma^2 X^2 w_{RR} + \lambda w_R c^2 + w_X c \right]
\end{aligned}$$

and Equation (4.25) follows from Equations (4.23) and (4.24).

We now show that Equation (4.26) is fulfilled for all R and $Y = X^2$. First, observe that it holds for $Y = 0$. For general Y, consider the following two equations:

$$\begin{aligned}
\frac{d}{dY}\tilde{c}^2 &= -3\lambda\tilde{c}^2\tilde{c}_R + \frac{\sigma^2}{2}\tilde{c}_{RR} \\
-\frac{d}{dY}\frac{\sigma^2\tilde{w}_{RR}}{2\lambda\tilde{w}_R} &= \sigma^2\tilde{c}\frac{d}{dR}\frac{\tilde{w}_{RR}}{2\tilde{w}_R} + \sigma^2\tilde{c}_R\frac{\tilde{w}_{RR}}{2\tilde{w}_R} + \frac{\sigma^2}{2}\tilde{c}_{RR}.
\end{aligned}$$

The first of these two equations holds because of Equation (4.19) and the second one because of Equation (4.23). Now we have

$$\begin{aligned}\frac{d}{dY}\left(\tilde{c}^2+\frac{\sigma^2\tilde{w}_{RR}}{2\lambda\tilde{w}_R}\right) &= -3\lambda\tilde{c}^2\tilde{c}_R+\frac{\sigma^2}{2}\tilde{c}_{RR}-\sigma^2\tilde{c}\frac{d}{dR}\frac{\tilde{w}_{RR}}{2\tilde{w}_R}-\sigma^2\tilde{c}_R\frac{\tilde{w}_{RR}}{2\tilde{w}_R}-\frac{\sigma^2}{2}\tilde{c}_{RR}\\ &= -\lambda\tilde{c}\frac{d}{dR}\left(\tilde{c}^2+\frac{\sigma^2\tilde{w}_{RR}}{2\lambda\tilde{w}_R}\right)-\lambda\tilde{c}_R\left(\tilde{c}^2+\frac{\sigma^2\tilde{w}_{RR}}{2\lambda\tilde{w}_R}\right).\end{aligned}$$

Hence, the function $f(Y,R):=\tilde{c}^2+\frac{\sigma^2\tilde{w}_{RR}}{2\lambda\tilde{w}_R}$ satisfies the linear PDE

$$f_Y=-\lambda\tilde{c}f_R-\lambda\tilde{c}_R f$$

with initial value condition $f(0,R)=0$. One obvious solution to this PDE is $f(Y,R)\equiv 0$. By the method of characteristics this is the unique solution to the PDE, since $\tilde{c}$ and $\tilde{c}_R$ are smooth and hence locally Lipschitz. □

The next auxiliary lemma will prove useful in the following.

Lemma 4.14 (Auxiliary Lemma)**.** *There are positive constants α, a_1, a_2, a_3 and a_4 such that*

$$\begin{array}{ccccc} u(R) & \geq & w(X,R) & \geq & u(R)\exp(\alpha X^2) \\ 0 & \leq & w_R(X,R) & \leq & a_1+a_2\exp(-a_3R+a_4X^2)\end{array} \tag{4.27}$$

for all $(X,R)\in\mathbb{R}_0^+\times\mathbb{R}$.

Proof of Lemma 4.14. The left hand side of the first inequality follows by the boundary condition for w and the monotonicity of w with respect to X as established in Proposition 4.12. Since the risk aversion of u is bounded from above by $2\lambda\tilde{c}^2_{max}$, we have

$$u(R-\Delta)\geq u(R)e^{2\lambda\tilde{c}^2_{max}\Delta}\text{ for }\Delta\geq 0 \tag{4.28}$$

and thus by Proposition 4.12

$$w(X,R)\geq u(R-\lambda\tilde{c}_{max}X^2)\geq u(R)e^{2\lambda^2\tilde{c}^3_{max}X^2}$$

which establishes the right hand side of the first inequality with $\alpha=2\lambda^2\tilde{c}^3_{max}$.

For the second inequality, we will show the equivalent inequality

$$0\leq\tilde{w}_R(Y,R)\leq a_1+a_2\exp(-a_3R+a_4Y).$$

The left hand side follows since $\tilde{w}$ is increasing in R by Proposition 4.12. For the right hand side, note that $\tilde{w}$ has "bounded absolute risk aversion" due to Equation (4.26) and the bound on $\tilde{c}$ established by Proposition 4.10:

$$-\frac{\tilde{w}_{RR}}{\tilde{w}_R}<\frac{2\lambda\tilde{c}^2_{max}}{\sigma^2}=:\tilde{A}.$$

Then

$$\tilde{w}(Y,R_0)\geq\tilde{w}(Y,R)+\frac{\tilde{w}_R(Y,R)}{\tilde{A}}\left(1-e^{-\tilde{A}(R_0-R)}\right).$$

Since

$$\lim_{R_0\to\infty}\tilde{w}(Y,R_0)=\lim_{R_0\to\infty}u(R_0)=0$$

we have

$$0\geq\tilde{w}(Y,R)+\frac{\tilde{w}_R(Y,R)}{\tilde{A}}$$

and thus

$$\tilde{w}_R(Y,R)\leq-\tilde{w}(Y,R)\tilde{A}\leq-u(R-\lambda\tilde{c}_{max}Y)\tilde{A}.$$

Since u is bounded by an exponential function, we obtain the desired bound on $\tilde{w}_R$. □

4.5.2 Verification argument

We now connect the PDE results from Subsection 4.5.1 with the optimal stochastic control problem introduced in Section 4.3. For any admissible strategy $\xi\in\mathcal{X}$ and $k\in\mathbb{N}$ we define

$$\tau_k^\xi:=\inf\Big\{t\geq 0\,\Big|\int_0^t\xi_s^2\,ds\geq k\Big\}.$$

We proceed by first showing that $u(R_t^\xi)$ and $w(X_t^\xi,R_t^\xi)$ fulfill local supermartingale inequalities. Thereafter we show that $w(X_0,R_0)\geq\lim_{t\to\infty}\mathbb{E}[u(R_t^\xi)]$ with equality for $\xi=\hat{\xi}$. The next lemma in particular justifies our definition of $v_2(X_0,R_0)$ in Equation (4.7).

Lemma 4.15. *For any admissible strategy $\xi\in\mathcal{X}$ the expected utility $\mathbb{E}[\,u(R_t^\xi)\,]$ is decreasing in t. Moreover, we have $\mathbb{E}[\,u(R^\xi_{t\wedge\tau_k^\xi})\,]\geq\mathbb{E}[\,u(R_t^\xi)\,]$.*

Proof. Since $R_t^\xi-R_0$ is the difference of the true martingale $\int_0^t\sigma X_s^\xi\,dB_s$ and the increasing process $\lambda\int_0^t\xi_s^2\,ds$, it satisfies the supermartingale inequality $\mathbb{E}[\,R_t^\xi\,|\,\mathcal{F}_s\,]\leq R_s^\xi$ for $s\leq t$ (even though it may fail to be a supermartingale due to the possible lack of integrability). Hence $\mathbb{E}[\,u(R_t^\xi)\,]$ is decreasing according to Jensen's inequality.

For the second assertion, we first take $n=k$ and write for $\tau_m:=\tau_m^\xi$

$$\mathbb{E}[\,u(R^\xi_{t\wedge\tau_k})\,]=\mathbb{E}\Big[\,u\Big(R_0+\sigma\int_0^{t\wedge\tau_n}X_s^\xi\,dB_s-\lambda\int_0^{t\wedge\tau_k}\xi_s^2\,ds\Big)\,\Big].$$

When sending n to infinity, the right-hand side decreases to

$$\mathbb{E}\Big[\,u\Big(R_0+\sigma\int_0^{t}X_s^\xi\,dB_s-\lambda\int_0^{t\wedge\tau_k}\xi_s^2\,ds\Big)\,\Big],\tag{4.29}$$

by dominated convergence because u is bounded from below by an exponential function, the integral of ξ^2 is bounded by k, and the stochastic integrals are uniformly bounded from below by $\inf_{s\leq K^2t}W_s$, where W is the DDS-Brownian motion of $\int X_s^\xi\,dB_s$ and K is an upper bound for $|X^\xi|$. Finally, the term in Equation (4.29) is clearly larger than or equal to $\mathbb{E}[\,u(R_t^\xi)\,]$. □

Lemma 4.16. *For any admissible strategy $\xi\in\mathcal{X}$, $w(X_t^\xi,R_t^\xi)$ is a local supermartingale with localizing sequence (τ_k^ξ).*

Proof. We use a verification argument similar to the one in Chapter 3. For $T > t \geq 0$, Itô's formula yields that

$$w(X_T^\xi, R_T^\xi) - w(X_t^\xi, R_t^\xi) = \int_t^T w_R(X_s^\xi, R_s^\xi)\sigma X_s^\xi \, dB_s - \int_t^T \Big[\lambda w_R \xi_s^2 + w_X \xi_s - \frac{1}{2}(\sigma X_s^\xi)^2 w_{RR}\Big](X_s^\xi, R_s^\xi)\, ds. \quad (4.30)$$

By Proposition 4.13 the latter integral is nonnegative and we obtain

$$w(X_t^\xi, R_t^\xi) \geq w(X_T^\xi, R_T^\xi) - \int_t^T w_R(X_s^\xi, R_s^\xi)\sigma X_s^\xi \, dB_s. \quad (4.31)$$

We will show next that the stochastic integral in Equation (4.31) is a local martingale with localizing sequence $(\tau_k) := (\tau_k^\xi)$. For some constant C_1 depending on t, k, λ, σ, and on the upper bound K of $|X^\xi|$ we have for $s \leq t \wedge \tau_k$

$$R_s^\xi = R_0 + \sigma B_s X_s^\xi + \int_0^s (\sigma \xi_q B_q - \lambda \xi_q^2)\, dq \geq -C_1\big(1 + \sup_{q \leq t} |B_q|\big).$$

Using Lemma 4.14, we see that for $s \leq t \wedge \tau_k$

$$0 \leq w_R(X_s^\xi, R_s^\xi) \leq a_1 + a_2 \exp\left(a_3 C_1\big(1 + \sup_{q \leq t}|B_q|\big) + a_4 K^2\right). \quad (4.32)$$

Since $\sup_{q \leq t} |B_q|$ has exponential moments of all orders, the martingale property of the stochastic integral in Equation (4.31) follows. Taking conditional expectations in Equation (4.31) thus yields the desired supermartingale property

$$w(X_{t\wedge\tau_k}^\xi, R_{t\wedge\tau_k}^\xi) \geq \mathbb{E}[\, w(X_{T\wedge\tau_k}^\xi, R_{T\wedge\tau_k}^\xi) | \mathcal{F}_t\,]. \quad (4.33)$$

The integrability of $w(X_{t\wedge\tau_k}^\xi, R_{t\wedge\tau_k}^\xi)$ follows from Lemma 4.14 and Equation (4.28) in a similar way as in Equation (4.32). □

Lemma 4.17. *There is an adapted strategy $\hat{\xi}$ fulfilling*

$$\hat{\xi}_t := c(X_t^{\hat{\xi}}, R_t^{\hat{\xi}}). \quad (4.34)$$

This $\hat{\xi}$ is admissible for optimal liquidation and maximization of asymptotic portfolio value ($\hat{\xi} \in \mathcal{X}_1 \subset \mathcal{X}$) and satisfies $\int_0^\infty \hat{\xi}_t^2 \, dt < K$ for some constant K. Furthermore, $w(X_t^{\hat{\xi}}, R_t^{\hat{\xi}})$ is a martingale and

$$w(X_0, R_0) = \lim_{t \to \infty} \mathbb{E}[\, u(R_t^{\hat{\xi}})\,] \leq v_2(X_0, R_0). \quad (4.35)$$

Proof. Consider the stochastic differential equation

$$d\begin{pmatrix} \hat{X}_t \\ R_t \end{pmatrix} = \begin{pmatrix} -c(\hat{X}_t, R_t)dt \\ -\lambda c(\hat{X}_t, R_t)dt + \sigma \hat{X}_t dB_t \end{pmatrix}$$

with initial condition $\hat{X}_0 = X_0$. Since c is differentiable, it satisfies local boundedness and Lipschitz conditions, hence a solution to this SDE exists (see for example Durrett (1996)). Note that the solution cannot explode due to the special form of c in Equation (4.25) and the bounds on $\tilde{c}$ established in Proposition 4.10. Furthermore, the resulting stochastic process $\hat{X}$ is absolutely continuous, and by setting $\hat{\xi}_t := -\dot{\hat{X}}_t$ we obtain a solution of Equation (4.34). By Equations (4.21) and (4.25), $X_t^{\hat{\xi}} = \hat{X}_t > 0$ is bounded from above by an exponentially decreasing function of t. Therefore $\hat{\xi}$ is also bounded by such a function and $\int_0^\infty \hat{\xi}_t^2 \, dt < K$ for some constant K, showing that $\hat{\xi}$ is admissible both for optimal liquidation and maximization of asymptotic portfolio value. Next, with the choice $\xi = \hat{\xi}$ the rightmost integral in Equation (4.30) vanishes, and we get equality in Equation (4.33). Since $\tau_K^{\hat{\xi}} = \infty$, this proves the martingale property of $w(X_t^{\hat{\xi}}, R_t^{\hat{\xi}})$. Furthermore, we obtain from Equation (4.27) that

$$u(R_t^{\hat{\xi}}) \geq w(X_t^{\hat{\xi}}, R_t^{\hat{\xi}}) \geq u(R_t^{\hat{\xi}}) \exp(\alpha (X_t^{\hat{\xi}})^2).$$

Since $X_t^{\hat{\xi}}$ is bounded by an exponentially decreasing function, we obtain Equation (4.35). □

Proposition 4.18. *Consider the case of the asymptotic maximization of the portfolio value. We have $v_2 = w$ and the a.s. unique optimal strategy is given by $\hat{\xi}$ respectively c.*

Proof. By Lemma 4.17, we already have $w \leq v_2$. Hence we only need to show that $v_2 \leq w$. Let $\xi \in \mathcal{X}$ be any admissible strategy such that

$$\lim_{t\to\infty} \mathbb{E}[u(R_t^\xi)] > -\infty.$$

By Lemmas 4.16 and 4.14 we have for all k, t and $(\tau_k) := (\tau_k^\xi)$

$$w(X_0, R_0) \geq \mathbb{E}[w(X_{t\wedge\tau_k}^\xi, R_{t\wedge\tau_k}^\xi)] \geq \mathbb{E}\left[u(R_{t\wedge\tau_k}^\xi) \exp(\alpha (X_{t\wedge\tau_k}^\xi)^2)\right].$$

As in the proof of Lemma 4.15 one shows that

$$\begin{aligned} \liminf_{k\to\infty} \mathbb{E}\left[u(R_{t\wedge\tau_k}^\xi) \exp(\alpha (X_{t\wedge\tau_k}^\xi)^2)\right] &\geq \liminf_{k\to\infty} \mathbb{E}\left[u(R_t^\xi) \exp(\alpha (X_{t\wedge\tau_k}^\xi)^2)\right] \\ &= \mathbb{E}\left[u(R_t^\xi) \exp(\alpha (X_t^\xi)^2)\right]. \end{aligned}$$

Hence,

$$w(X_0, R_0) \geq \mathbb{E}[u(R_t^\xi)] + \mathbb{E}\left[u(R_t^\xi)(\exp(\alpha (X_t^\xi)^2) - 1)\right].$$

Let us assume for a moment that the second expectation on the right attains values arbitrarily close to zero. Then

$$w(X_0, R_0) \geq \lim_{t\to\infty} \mathbb{E}[u(R_t^\xi)].$$

Taking the supremum over all admissible strategies $\xi \in \mathcal{X}$ gives $v_2 \leq w$. The optimality of $\hat{\xi}$ follows from Lemma 4.17, its uniqueness from the fact that the functional $\mathbb{E}[u(R_t^\xi)]$ is strictly concave since u is concave and increasing and R_t^ξ is concave.

We now show that $\mathbb{E}\left[u(R_t^\xi)(\exp(\alpha(X_t^\xi)^2)-1)\right]$ attains values arbitrarily close to zero. By Lemma 4.15 and the same line of reasoning as in the proof of Lemma 4.16, we have for all k, t and $(\tau_k) := (\tau_k^\xi)$

$$\begin{aligned}
-\infty < \lim_{s\to\infty} \mathbb{E}[u(R_s^\xi)] &\le \mathbb{E}[u(R_t^\xi)] \le \mathbb{E}[u(R_{t\wedge\tau_k}^\xi)] \\
&= u(R_0) + \mathbb{E}\left[\int_0^{t\wedge\tau_k} u_R(R_s^\xi)\sigma X_s^\xi\, dB_s\right] - \mathbb{E}\left[\int_0^{t\wedge\tau_k} \left[\lambda u_R \xi_s^2 - \frac{1}{2}(\sigma X_s^\xi)^2 u_{RR}\right](R_s^\xi)\, ds\right] \\
&= u(R_0) - \mathbb{E}\left[\int_0^{t\wedge\tau_k} \left[\lambda u_R \xi_s^2 - \frac{1}{2}(\sigma X_s^\xi)^2 u_{RR}\right](R_s^\xi)\, ds\right]. \qquad (4.36)
\end{aligned}$$

Sending k and t to infinity yields

$$\int_0^\infty \mathbb{E}\left[(X_s^\xi)^2 u_{RR}(R_s^\xi)\right]\, ds > -\infty. \qquad (4.37)$$

Next we observe that

$$0 \ge u(R) \ge a_5 u_{RR}(R)$$

for a constant $a_5 > 0$, due to the boundedness of the risk aversion of u, and that

$$\exp(\alpha(X_t^\xi)^2) - 1 \le a_6\alpha(X_t^\xi)^2,$$

due to the bound on X_t^ξ. We now have

$$0 \ge \mathbb{E}\left[u(R_t^\xi)(\exp(\alpha(X_t^\xi)^2)-1)\right] \ge \mathbb{E}[\alpha a_5 a_6 u_{RR}(R_t^\xi)(X_t^\xi)^2].$$

Therefore the right hand side of the above equation attains values arbitrarily close to zero. □

Proposition 4.19. *Consider the case of optimal liquidation. Then $v_1 = w$ and the a.s. unique optimal strategy is given by $\hat{\xi}$ respectively c.*

Proof. For any strategy $\xi \in \mathcal{X}_1$ that is admissible for optimal liquidation, the martingale $\sigma\int_0^t X_s dB_s$ is uniformly integrable due to the requirement in Equation (4.1). Therefore

$$\mathbb{E}[u(R_t^\xi)] \ge \mathbb{E}\left[u(R_\infty^\xi)\right]$$

follows as in the proof of Lemma 4.15. Hence, Proposition 4.18 yields

$$\mathbb{E}[u(R_\infty^\xi)] = \lim_{t\to\infty} \mathbb{E}[u(R_t^\xi)] \le v_2(X_0, R_0) \le w(X_0, R_0).$$

Taking the supremum over all admissible strategies $\xi \in \mathcal{X}_1$ gives $v_1 \le w$. The converse inequality follows from Lemma 4.16, since $\hat{\xi}$ is admissible for optimal liquidation. □

4.5.3 Characterization of the optimal adaptive strategy

Proof of Theorem 4.6. We prove the equivalent inequality $\tilde{c}^1 \geq \tilde{c}^0$. Fix $N > 0$ and let f^i denote the function $\tilde{f}_N$ constructed in the proof of Proposition 4.10 when the parabolic boundary condition is given by $\tilde{f}_N(Y,R) = \sqrt{\sigma^2 A^i(R)/(2\lambda)}$ for $Y = 0$ or $|R| = N$. The result follows if we can show that $g := f^1 - f^0 \geq 0$. A straightforward computation shows that g solves the linear PDE

$$\begin{aligned} g_Y &= -\frac{3}{2}\lambda\big(f^1 g_R + f^0_R g\big) + \frac{\sigma^2}{4} f^1_{RR}\Big(\frac{1}{f^1} - \frac{1}{f^0}\Big) + \frac{\sigma^2}{4f^0} g_{RR} \\ &= \frac{1}{2} a g_{RR} + b g_R + V g, \end{aligned}$$

where the coefficients a and b and the potential V are given by

$$a = \frac{\sigma^2}{2f^0}, \qquad b = -\frac{3}{2}\lambda f^1, \qquad \text{and} \qquad V = -\frac{\sigma^2 f^1_{RR}}{4f^0 f^1} - \frac{3}{2}\lambda f^0_R.$$

The parabolic boundary condition of g is

$$g(Y,R) = \sqrt{\frac{\sigma^2 A^1(R)}{2\lambda}} - \sqrt{\frac{\sigma^2 A^0(R)}{2\lambda}} =: h(R) \qquad \text{for } Y = 0 \text{ or } |R| = N.$$

The functions a, b, V, and h are smooth and (at least locally) bounded on $\mathbb{R}_+ \times [-N, N]$, and a is bounded away from zero. Next, take $T > 0$, $R \in]-N, N[$, and let Z be the solution of the stochastic differential equation

$$dZ_t = \sqrt{a(T-t, Z_t)}\, dB_t + b(T-t, Z_t)dt, \qquad Z_0 = R,$$

which is defined up to time

$$\tau := \inf\big\{t \geq 0 \,\big|\, |Z_t| = N \text{ or } t = T\big\}.$$

By a standard Feynman-Kac argument, g can then be represented as

$$g(T,R) = \mathbb{E}\Big[\, h(Z_\tau) \exp\Big(\int_0^\tau V(T-t, Z_t)\, dt\Big)\Big].$$

Hence $g \geq 0$ as $h \geq 0$ by assumption. □

Proof of Theorem 4.5. In Theorem 4.6 take $u^0(x) := u(x)$ and $u^1(x) := u(x + r)$. If u exhibits IARA, then $A^1 \geq A^0$ if $r > 0$ and hence $c^1 \geq c^0 = c$. But we clearly have $c^1(X,R) = c(X, R + r)$. The result for decreasing A follows by taking $r < 0$. □

The following proof follows the same setup as the proof of Theorem 4.6. The line of argument however is analytic and not probabilistic.

Proof of Theorem 4.7. Let $\lambda^1 > \lambda^0$ be two positive constants. Fix $N > 0$ and let f^i denote the function $\tilde{f}_N$ constructed in the proof of Proposition 4.10 with $\lambda = \lambda^i$. The result follows if we can show that $g := f^0 - f^1 \geq 0$. Let us assume by way of contradiction that (Y_0, R_0)

is a root of g with minimal Y_0. The point (Y_0, R_0) does not lie on the boundary of the strip $\mathbb{R}_0^+ \times [-N, N]$ since $g > 0$ on the boundary due to Equation (4.20). We therefore have that (Y_0, R_0) is a local minimum in $]0, Y_0] \times] - N, N[$ and a root. Hence

$$
\begin{aligned}
g(Y_0, R_0) &= 0 && \Rightarrow f^0 = f^1 \\
g_Y(Y_0, R_0) &\leq 0 && \\
g_R(Y_0, R_0) &= 0 && \Rightarrow f_R^0 = f_R^1 \\
g_{RR}(Y_0, R_0) &\geq 0. &&
\end{aligned}
$$

By Equation (4.19), we now have

$$
\begin{aligned}
0 &\geq g_Y(Y_0, R_0) \\
&= f_Y^0 - f_Y^1 \\
&= \left(-\frac{3}{2}\lambda^0 f^0 f_R^0 + \frac{\sigma^2}{4f^0} f_{RR}^0\right) - \left(-\frac{3}{2}\lambda^1 f^1 f_R^1 + \frac{\sigma^2}{4f^1} f_{RR}^1\right) \\
&= -\frac{3}{2}(\lambda^0 - \lambda^1) f^0 f_R^0 + \frac{\sigma^2}{4f^0} g_{RR} \\
&> 0.
\end{aligned}
$$

The last inequality uses that $f_R^0 > 0$, which holds for IARA utility function u by Theorem 4.5. The established contradiction leads us to conclude that g does not have any roots and thus that $f^0 > f^1$. □

Proof of Theorem 4.8. Equation (4.17) holds since $\tilde{d}(Y, R) = \tilde{c}\left(\frac{\sigma_1^2}{\sigma_2^2} Y, R, \frac{\sigma_2^2}{\sigma_1^2}\lambda, \sigma_2\right)$ is a solution of Equation (4.13) with $\sigma = \sigma_1$. □

Proof of Theorem 4.9. First, it follows immediately from the definition of v in Equation (4.4) that $R \mapsto v(X, R)$ is strictly increasing. Next, take distinct pairs (R_1, X_1), (R_2, X_2) and let $0 < \alpha < 1$ be given. Select the optimal strategies $\hat{\xi}^1, \hat{\xi}^2 \in \mathcal{X}$ such that $v(X_i, R_i) = \mathbb{E}[u(R_\infty^{\hat{\xi}^i})]$ for $i = 1, 2$. Define $\xi := \alpha\hat{\xi}^1 + (1-\alpha)\hat{\xi}^2$. Then

$$
\begin{aligned}
v(\alpha X_1 + (1-\alpha)X_2, \alpha R_1 + (1-\alpha)R_2) &\geq \mathbb{E}[u(R_\infty^\xi)] \\
&> \mathbb{E}[u(\alpha R_\infty^{\hat{\xi}^1} + (1-\alpha)R_\infty^{\hat{\xi}^2})] \\
&> \alpha\mathbb{E}[u(R_\infty^{\hat{\xi}^1})] + (1-\alpha)\mathbb{E}[u(R_\infty^{\hat{\xi}^2})] \\
&= \alpha v(X_1, R_1) + (1-\alpha)v(X_2, R_2).
\end{aligned}
$$

Hence v is strictly concave. By Proposition 4.12, we know that v is decreasing in X. Equation (4.18) follows immediately from Equation (4.26). □

Chapter 5

Adaptive basket liquidation

5.1 Introduction

In this chapter, we determine the utility maximizing trading strategy for *basket liquidations* with respect to a wide range of utility functions and describe it as the solution to a partial differential equation. Surprisingly, the set of portfolios that are held during the liquidation is independent of the investor's utility function but only depends on the market volatility and liquidity structure. The utility function only influences how quickly the investor executes the trades.

For practical applications, we can determine the utility maximizing trading strategy by executing two steps. First, we derive the deterministic mean-variance optimal basket trading strategy. While we show that such a strategy always exists, finding it numerically can be challenging due to the high number of dimensions. Second, we solve a partial differential equation and obtain an optimal "relative trading speed". This PDE depends only on the risk aversion of the utility function, but not on the market parameters such as the covariance structure. Under the utility maximizing trading strategy, the portfolio evolves exactly as in the mean-variance optimal trading strategy, but with a time transformation given by the relative trading speed. This establishes a "separation theorem" for optimal liquidation: Investors with different risk attitudes will choose the same basket liquidation strategy, but execute it at a different speed. Because of this separation, utility maximization becomes a numerically tractable option for implementing adaptive basket liquidation strategies in practice.

We consider the market model described in Section 3.2, i.e., a continuous-time, infinite time-horizon *multiple asset* extension of the model introduced by Almgren and Chriss (2001) and Almgren (2003). In particular, we allow for non-linear cross-asset price impacts. However, we need to assume that price impact scales like a power law, i.e., that trading a times faster results in a price impact multiplied by a^α where $\alpha > 0$ is a constant. In this market model, we first show that a unique mean-variance optimal trading strategy exists and that it satisfies both Bellman's principle of optimality and the Beltrami identity. Furthermore, the mean-variance costs of liquidation fulfill the dynamic programming PDE. Thereafter, we construct a solution to the HJB equation for utility maximization. The key observation is that the expected utility under optimal adaptive liquidation is identical for different portfolios with the same mean-variance cost of deterministic execution. We can therefore construct the utility maximization value function by solving two-dimensional PDEs instead of high-dimensional PDEs. Finally we apply a verification argument to show that the solution to the HJB equation is indeed the value function.

Note that our approach in this chapter differs from the approach in Chapter 4, where we directly derived the optimal adaptive trading strategy without referring to mean-variance optimization. With the results of this Chapter 5 in mind, we however realize that the expression of the optimal control relative to the portfolio size and the exponential bounds of the optimal portfolio evolution derived in Chapter 4 are no coincidence, but rather express the connection between the utility maximizing strategy and the mean-variance optimal exponential strategy. Furthermore, if the price impact is linear in the basket liquidation case, we find that the two-dimensional PDEs of this chapter are special cases of the PDEs in Chapter 4 and that therefore the sensitivity analysis of the single asset case carries over to the basket case. In particular, we find that for linear price impacts the optimal strategy is aggressive in-the-money if the utility function exhibits increasing absolute risk aversion and it is passive in-the-money if the utility function exhibits decreasing absolute risk aversion.

The rest of this chapter is structured as follows. In Section 5.2, we review the multiple asset market model and the investor's trading target. Thereafter, we first show in Section 5.3 that an optimal deterministic strategy exists for mean-variance optimization and subsequently use this strategy to construct the optimal strategy for utility maximization in Section 5.4. All proofs are given in Section 5.5.

5.2 Market model and investor's trading target

We apply the market model and assumptions of Section 3.2. We briefly recall the central definitions. We assume that the transaction prices are given as

$$\mathrm{P}_t = \tilde{\mathrm{P}}_0 + \sigma \mathrm{B}_t - \Gamma(\mathrm{X}_0 - \mathrm{X}_t) - \mathrm{TempImp}(\xi_t).$$

Here, the vector-valued stochastic process B is a standard n-dimensional Brownian motion and σ is the $n \times n$ volatility matrix of the price changes of the n assets. We assume that σ is non-degenerate with covariance matrix $\Sigma := \sigma\sigma^\top \in \mathbb{R}^{n\times n}$. For the same reasons as in Section 4.2, we assume that the drift of the assets and the interest rate are zero. The permanent impact is assumed to be linear with a symmetric matrix $\Gamma \in \mathbb{R}^{n\times n}$ to avoid quasi-arbitrage (see Huberman and Stanzl (2004)). In this chapter, we need to make stronger assumptions on the "temporary impact cost of trading"

$$f : \xi \in \mathbb{R}^n \to f(\xi) := \mathrm{TempImp}(\xi)^\top \xi \in \mathbb{R}_0^+$$

than in Chapter 3. We assume that f is C^1 on $\mathbb{R}^n$, and that it is C^2 and larger than zero on $\mathbb{R}^n \backslash \{0\}$. Furthermore, we require that f has a positive-definite Hessian matrix $D^2 f$ on $\mathbb{R}^n \backslash \{0\}$, or equivalently that is has a nonsingular Hessian matrix and is convex. Finally, we assume that f scales like a power law in the trading speed ξ. More precisely, we assume that there is a constant $\alpha \in \mathbb{R}^+$ such that for all $a \in \mathbb{R}_0^+$:

$$f(a\xi) = a^{\alpha+1} f(\xi). \tag{5.1}$$

Note that this implies $f(0) = 0$. For a discussion of the relevance of the scaling property, see the remark after Theorem 5.3.

By $\mathcal{X}$ we denote the class of all admissible strategies ξ. With respect to the investor's utility function, we apply the assumptions and notation introduced in Section 4.2.

Both optimization frameworks of Section 4.3 can be extended to the multiple asset market model of this chapter. If the investor holds a portfolio $\mathrm{X}_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}$ units of cash at time 0 and follows the admissible trading strategy ξ, then her cash position at time t is given by

$$\begin{aligned}\mathcal{R}_t(\xi) &= r + \int_0^t \mathrm{P}_s \xi_s \, ds \\ &= r + \tilde{\mathrm{P}}_0 \mathrm{X}_0 - \frac{1}{2}(\mathrm{X}_0)^\top \Gamma \mathrm{X}_0 + \underbrace{\int_0^t (\mathrm{X}_s^\xi)^\top \sigma \, d\mathrm{B}_s}_{\Phi_t} - \int_0^t f(\xi_s)\, ds \\ &\quad \underbrace{- \tilde{\mathrm{P}}_0 \mathrm{X}_t^\xi - \frac{1}{2}\left((\mathrm{X}_t^\xi)^\top \Gamma \mathrm{X}_t^\xi - 2(\mathrm{X}_0)^\top \Gamma \mathrm{X}_t^\xi\right) - (\mathrm{X}_t^\xi)^\top \sigma \mathrm{B}_t}_{\Psi_t}.\end{aligned}$$

Convergence of Φ_t follows if

$$\mathbb{E}\Big[\int_0^\infty (\mathrm{X}_s^\xi)^\top \Sigma \mathrm{X}_s^\xi \, ds\Big] < \infty \tag{5.2}$$

and a.s. convergence of Ψ_t is guaranteed if a.s.

$$\lim_{t\to\infty} (\mathrm{X}_t^\xi)^\top \Sigma \mathrm{X}_t^\xi t \ln\ln t = 0. \tag{5.3}$$

For the optimal liquidation setting, we require admissible strategies to fulfill these two conditions and denote the set of such strategies by $\mathcal{X}_1 \subset \mathcal{X}$. We then assume that the investor wants to maximize the expected utility of her cash position after liquidation:

$$v_1(\mathrm{X}_0, R_0) := \sup_{\xi \in \mathcal{X}_1} \mathbb{E}[u(\mathcal{R}_\infty^\xi)] = \sup_{\xi \in \mathcal{X}_1} \mathbb{E}[u(R_\infty^\xi)] \tag{5.4}$$

with

$$R_t^\xi = \underbrace{r + \tilde{\mathrm{P}}_0 \mathrm{X}_0 - \frac{1}{2}(\mathrm{X}_0)^\top \Gamma \mathrm{X}_0}_{R_0} + \int_0^t (\mathrm{X}_s^\xi)^\top \sigma \, d\mathrm{B}_s - \int_0^t f(\xi_s)\, ds. \tag{5.5}$$

The economic interpretation of R_t as the portfolio value at time t given in Section 4.3.2 carries over to the general setting of this chapter. In our second approach, we therefore assume that the investor trades the risky asset in order to maximize the asymptotic expected utility of portfolio value:

$$v_2(\mathrm{X}_0, R_0) := \sup_{\xi \in \mathcal{X}} \lim_{t\to\infty} \mathbb{E}[u(R_t^\xi)]. \tag{5.6}$$

The existence of the limit will be established in Lemma 5.17. Note that our assumptions on strategies admissible for the maximization of asymptotic portfolio value are weaker than those for optimal liquidation. In particular, we do not require that R_t^ξ or X_t^ξ converge.

5.3 Deterministic strategies and mean-variance optimization

Before considering the dynamic maximization of expected utility, we start our analysis with deterministic mean-variance optimization. Let $\bar{\mathcal{X}} \subset \mathcal{X}$ be the set of deterministic admissible strategies. We consider the mean-variance value function[1]:

$$\bar{v}(\mathrm{X}_0) := \inf_{\bar{\xi} \in \bar{\mathcal{X}}} \left[\int_0^\infty f(\bar{\xi}_s) ds + \frac{1}{2} \int_0^\infty (\mathrm{X}_s^{\bar{\xi}})^\top \Sigma \mathrm{X}_s^{\bar{\xi}} ds \right]. \tag{5.7}$$

The following theorem establishes the existence of an optimal trading strategy $\bar{\xi}$ and provides some of its features.

Theorem 5.1. *For each* $\mathrm{X}_0 \in \mathbb{R}^n$, *there is a unique minimizer* $\bar{\xi}^{(\mathrm{X}_0)}$ *of Equation* (5.7). *This minimizer satisfies Bellman's principle of optimality, i.e., there is a continuous vector field*

$$\bar{\mathrm{c}} : \mathrm{X} \in \mathbb{R}^n \to \bar{\mathrm{c}}(\mathrm{X}) \in \mathbb{R}^n$$

such that for all $\mathrm{X}_0 \in \mathbb{R}^n$ *and each* $t \in \mathbb{R}_0$, *we have*

$$\bar{\xi}_t^{(\mathrm{X}_0)} = \bar{\mathrm{c}} \left(\mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}} \right).$$

Furthermore, the vector field $\bar{\mathrm{c}}$ *fulfills the following two equations:*

$$\nabla f(\bar{\mathrm{c}}(\mathrm{X})) = \bar{v}_{\mathrm{X}} \text{ for all } \mathrm{X} \in \mathbb{R}^n \tag{5.8}$$

$$\frac{f(\bar{\mathrm{c}}(\mathrm{X}))}{\mathrm{X}^\top \Sigma \mathrm{X}} = \frac{1}{2\alpha} \text{ for all } \mathrm{X} \in \mathbb{R}^n \backslash \{0\}. \tag{5.9}$$

Equation (5.8) is the dynamic programming PDE (see Cesari (1983)) and Equation (5.9) is the Beltrami identity (see Beltrami (1868)).

For special cases, the vector field $\bar{\mathrm{c}}$ and the mean-variance value function $\bar{v}$ are available in closed form. For the single asset case with non-linear temporary impact $f(\xi) = \lambda \xi^{\alpha+1}$, Almgren (2003) derived

$$\begin{aligned} \bar{c}(X) &= \left(\frac{\sigma^2 X^2}{2\alpha\lambda} \right)^{\frac{1}{\alpha+1}} \\ \bar{v}(X) &= \frac{(\alpha+1)^2}{3\alpha+1} \left(\frac{\lambda \sigma^{2\alpha} X^{3\alpha+1}}{(2\alpha)^\alpha} \right)^{\frac{1}{\alpha+1}}. \end{aligned} \tag{5.10}$$

For the multiple asset case, it is harder to find a closed form expression for $\bar{\mathrm{c}}$. However, if the temporary impact is linear, i.e., $f(\xi) = \xi^\top \Lambda \xi$, Λ is a diagonal matrix and $\Lambda^{-1}\Sigma$ has n different positive eigenvalues, then it is easy to derive from the formulas in Konishi and Makimoto (2001) that

$$\begin{aligned} \bar{\mathrm{c}}(\mathrm{X}) &= \frac{1}{\sqrt{2}} \sqrt{\Lambda^{-1}\Sigma} \mathrm{X} \\ \bar{v}(\mathrm{X}) &= \frac{1}{\sqrt{2}} \mathrm{X}^\top \Sigma \sqrt{\Sigma^{-1}\Lambda} \mathrm{X}. \end{aligned}$$

Figures 5.1 and 5.2 illustrate the trajectories of the optimal deterministic trading strategies in the case of two positively correlated assets with covariance matrix

$$\Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}.$$

[1] More precisely, the function $\bar{v}$ is a simple transformation of the mean-variance value function.

In Figure 5.1, the trajectories for different portfolios X_0 are compared for fixed

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The correlation of the two assets connects the trading in both assets by the investor's desire to reduce portfolio risk by hedging. If the initial asset position in one of the assets is zero, it will not remain zero during the portfolio liquidation; instead, a long or short position is acquired that serves as a hedge for the initial non-zero position in the other asset. For the same reason, a portfolio with long positions in both assets might have a short position in one of the two assets during the optimal liquidation. This short position again serves as a hedge for the long position in the other asset; under certain conditions, it is cheaper to reduce risk by building up the short position as a hedge instead of by selling the long position quicker. For two example portfolios

$$\mathrm{X}_0 = \begin{pmatrix} \pm 1 \\ 1.5 \end{pmatrix},$$

the trajectories for different temporary impact matrices

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & d_2 \end{pmatrix}$$

with $d_2 \in [e^{-3}, e^3]$ are shown in Figure 5.2. The larger the differences in liquidity of the two assets, the larger the incentive to hedge the market risk by trading the more liquid asset quicker than the less liquid asset. For the portfolio

$$\mathrm{X}_0 = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix},$$

this effect is strong, since the initial portfolio market risk is high; for the portfolio

$$\mathrm{X}_0 = \begin{pmatrix} -1 \\ 1.5 \end{pmatrix},$$

the market risk is low already at the beginning of trading and thus the optimal trading trajectories are similar for different temporary impact matrices Λ.

5.4 Dynamic maximization of expected utility

We now turn to the *dynamic* maximization of expected utility.

Theorem 5.2. *The value functions $v = v_1$ for optimal liquidation and v_2 for maximization of asymptotic portfolio value are equal and are classical solutions of the Hamilton-Jacobi-Bellman equation*

$$\inf_{\mathrm{c}} \left[-\frac{1}{2} v_{RR} \mathrm{X}^\top \Sigma \mathrm{X} + v_R f(\mathrm{c}) + v_{\mathrm{X}} \mathrm{c} \right] = 0 \tag{5.11}$$

with boundary condition

$$v(0, R) = u(R) \text{ for all } R \in \mathbb{R}. \tag{5.12}$$

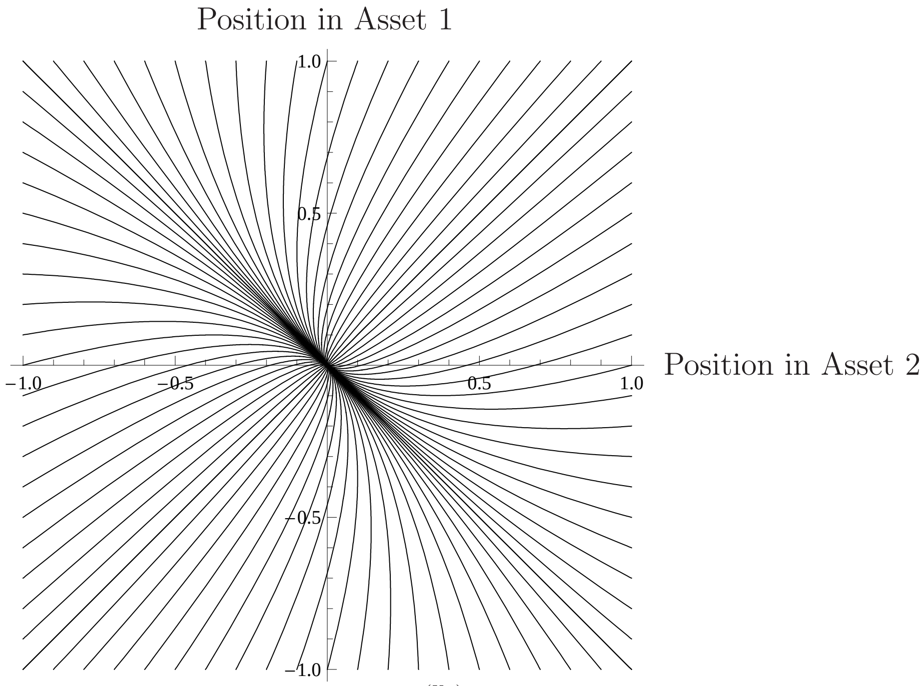


Figure 5.1: Parametric plot of the portfolio trajectories $\mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}}$ under the mean-variance optimal deterministic strategy for different initial portfolios X_0. $\Lambda = ((1,0),(0,1))$, $\Sigma = ((1,0.5),(0.5,1))$.

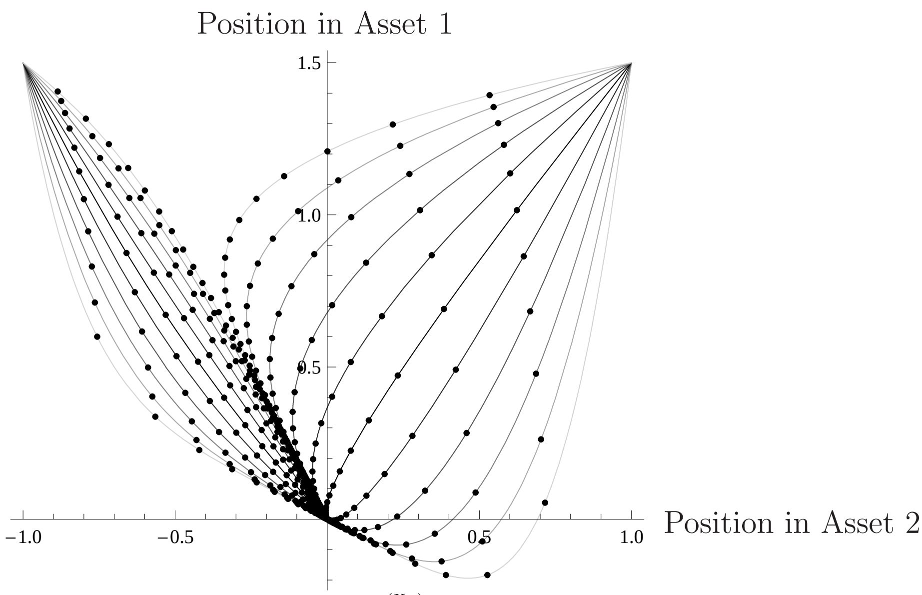


Figure 5.2: Parametric plot of the portfolio trajectories $\mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}}$ under the mean-variance optimal deterministic strategy for two different initial portfolios X_0 and different temporary impact matrices $\Lambda = ((1,0),(0,d_2))$. Darker lines correspond to d_2 closer to 1. $\Sigma = ((1,0.5),(0.5,1))$. The dots show the portfolio X_{t_i} at time points $t_i = i/2$ for $i \in \mathbb{N}$.

The a.s. unique optimal control $\hat{\xi}_t$ is Markovian. We write it in feedback form as

$$\hat{\xi}_t = \mathrm{c}(\mathrm{X}_t^{\hat{\xi}}, R_t^{\hat{\xi}}). \tag{5.13}$$

For the value functions, we have convergence:

$$v(\mathrm{X}_0, R_0) = \lim_{t\to\infty} \mathbb{E}[u(R_t^{\hat{\xi}})] = \mathbb{E}[u(R_\infty^{\hat{\xi}})].$$

Note that in Equation (5.11) and in the rest of this chapter, we use the shorthand notation $v_\mathrm{X} = \nabla_\mathrm{X} v$. In addition to the difficulties with the HJB equation in Theorem 4.1, we now face the complex task of inferring the value function v on $\mathbb{R}^n \times \mathbb{R}$ from initial values only on the line $\{0\} \times \mathbb{R}$. The existence of a solution to the HJB Equation 5.11 is far from obvious; even for the simple integration of vector fields in $\mathbb{R}^n \times \{0\}$, integration conditions need to be fulfilled. Fortunately, the construction of the optimal control c and the value function v can be reduced to a two-dimensional problem involving only the portfolio value R and the mean-variance liquidation cost $Y = \bar{v}(\mathrm{X})$.

Theorem 5.3. *The optimal control* c *is given by*

$$\mathrm{c}(\mathrm{X}, R) = \tilde{c}(\bar{v}(\mathrm{X}), R)\bar{\mathrm{c}}(\mathrm{X})$$

with a "relative liquidation speed" function $\tilde{c} : (Y, R) \in \mathbb{R}_0^+ \times \mathbb{R} \to \tilde{c}(Y, R) \in \mathbb{R}^+$ *that is the unique classical solution of the fully nonlinear parabolic PDE*

$$\tilde{c}_Y = -\frac{2\alpha+1}{\alpha+1}\tilde{c}^\alpha \tilde{c}_R + \frac{\alpha(\alpha-1)}{\alpha+1}\left(\frac{\tilde{c}_R}{\tilde{c}}\right)^2 + \frac{\alpha}{\alpha+1}\frac{\tilde{c}_{RR}}{\tilde{c}} \tag{5.14}$$

with initial condition

$$\tilde{c}(0, R) = A(R)^{\frac{1}{\alpha+1}}. \tag{5.15}$$

The bounds of the absolute risk aversion determine bounds of the relative liquidation speed $\tilde{c}$*:*

$$\inf_{(Y,R)\in\mathbb{R}_0^+\times\mathbb{R}} \tilde{c}(Y, R) = \inf_{R\in\mathbb{R}} \tilde{c}(0, R) =: \tilde{c}_{min} = (A_{min})^{\frac{1}{\alpha+1}}$$

$$\sup_{(Y,R)\in\mathbb{R}_0^+\times\mathbb{R}} \tilde{c}(Y, R) = \sup_{R\in\mathbb{R}} \tilde{c}(0, R) =: \tilde{c}_{max} = (A_{max})^{\frac{1}{\alpha+1}}.$$

Note that here the relative liquidation speed $\tilde{c}$ describes the length of the utility-maximizing control c with respect to the length of the mean-variance optimal control $\bar{\mathrm{c}}$, while the transformed optimal control $\tilde{c}$ in Chapter 4 described the magnitude of c with respect to the portfolio size X. For $\alpha = 1$ as in the linear model of Chapter 4, portfolio size X and mean-variance optimal trading speed $\bar{c}$ are proportional; this is not necessarily the case for $\alpha \neq 1$. For $\alpha = 1$, Equations (5.14) and (5.15) describing the relative liquidation speed $\tilde{c}$ are a special case of Equations (4.13) and (4.14) for the transformed optimal control in Chapter 4 with $\lambda = 1$ and $\sigma^2 = 2$.

The scaling property (Equation (5.1)) is essential for the "Separation Theorem" 5.3. Higher risk aversion leads to faster trading; the relative attractiveness of trading in two directions $\xi^{(1)}$ and $\xi^{(2)}$ with different speeds a is only independent of the trading speed a if $\frac{f(a\xi^{(1)})}{f(a\xi^{(2)})}$ is independent of a, which is equivalent to the scaling property (5.1). If the scaling property does not hold, then we cannot hope for a separation theorem like Theorem 5.3.

Because of Theorem 5.3, utility maximization becomes numerically achievable for practical applications. Bertsimas, Hummel, and Lo (1999) find that even the minimization of expected liquidation costs is numerically challenging for large portfolios. While mean-variance optimal liquidation is by now a standard service of many banks, a utility maximizing dynamic liquidation by brute-force methods of dynamic programming appears out of reach. By Theorem 5.3, such a brute-force approach is fortunately not necessary.

Theorem 5.4. *The value function is given by*

$$v(\mathrm{X}, R) = \tilde{v}(\bar{v}(\mathrm{X}), R)$$

with a function $\tilde{v} : (Y, R) \in \mathbb{R}_0^+ \times \mathbb{R} \to \tilde{v}(Y, R) \in \mathbb{R}$ *that is the unique classical solution of the nonlinear first order PDE*

$$\tilde{v}_Y = -\tilde{v}_R \tilde{c}^{\alpha} \tag{5.16}$$

with initial condition

$$\tilde{v}(0, R) = u(R). \tag{5.17}$$

Theorems 5.3 and 5.4 reveal a tight connection between mean-variance optimization and maximization of expected utility. Both approaches lead to the same liquidation strategy, they only differ by the speed with which this strategy is executed. The expected utility of optimal liquidation then depends only on the current portfolio value R and the mean-variance costs of deterministic liquidation $\bar{v}(\mathrm{X})$.

Corollary 5.5. *The asset position* $\mathrm{X}_t^{\hat{\xi}}$ *at time* t *under the optimal control* $\hat{\xi}$*, is given by*

$$\mathrm{X}_t^{\hat{\xi}} = \mathrm{X}^{\bar{\xi}}_{\int_0^t \tilde{c}(\bar{v}(\mathrm{X}_s^{\hat{\xi}}), R_s^{\hat{\xi}})\, ds}. \tag{5.18}$$

For investors with a utility function $u(R) = -e^{-AR}$ *with constant risk aversion* $A(R) \equiv A$*, the optimal adaptive liquidation strategy is deterministic and is given by*

$$\mathrm{X}_t^{\hat{\xi}} = \mathrm{X}^{\bar{\xi}}_{At}. \tag{5.19}$$

Since for $\alpha = 1$ Equations (5.14) and (5.15) are a special case of Equations (4.13) and (4.14), all the results of Chapter 4 that follow from the properties of Equation (4.13) carry over to the multiple asset setting when $\alpha = 1$. This includes in particular Theorems 4.5, 4.6, 4.7 and 4.8.

The "Separation Theorem" 5.3 does not hold for basket liquidations with a finite time horizon T. Let us consider a simple example of two uncorrelated assets with the same volatility but different liquidity:

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}.$$

By our results of Chapter 3, the optimal strategy for CARA investors is the optimal deterministic strategy for mean-variance investors. For mean-variance investors however there is no interaction between the liquidation of the positions in the two assets due to their independence. Hence the optimal strategy liquidates both asset positions independently with the strategy given in Theorem 3.1. Figure 5.3 shows that the trajectory of the optimal

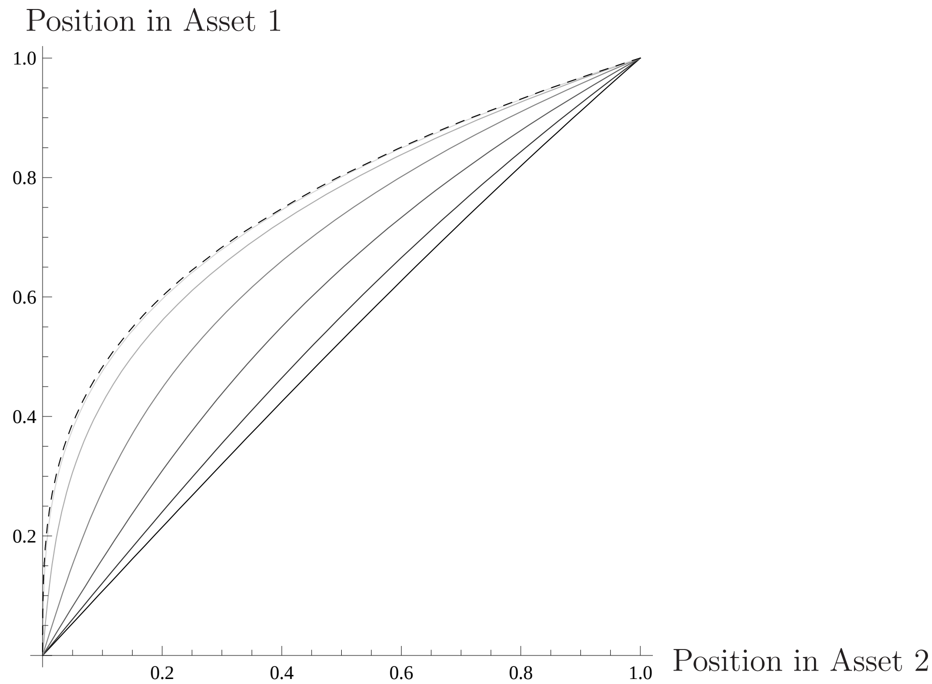


Figure 5.3: Parametric plot of the optimal trading trajectories for a finite liquidation time horizon T for CARA investors with different levels of absolute risk aversion $A \in [1, e^5]$ (darker lines correspond to lower risk aversion). The dashed black line is the trajectory of the optimal liquidation strategy with an infinite time horizon. $\mathrm{X}_0 = (1,1)^\top$, $\Lambda = ((1,0),(0,10))$, $\Sigma = ((1,0),(0,1))$.

liquidation strategy depends on the level of absolute risk aversion A of the utility function. For small values of the risk aversion A, the optimal strategy corresponds roughly to a linear reduction in asset position, since the primary driver of liquidation is the time constraint. For large values of risk aversion, the basket is liquidated quickly irrespective of the time horizon, and the liquidation strategy is primarily driven by market liquidity and volatility; therefore the trajectory of the optimal liquidation strategy is similar to the trajectory of liquidation with an infinite time horizon.

5.5 Proof of results

This section consists of three parts. First we discuss mean-variance optimal strategies and prove Theorem 5.1. By extending methods of calculus of variations to the infinite time setting, we show that optimal strategies exist, that they are unique and that they satisfy Bellman's principle of optimality. In the second subsection, we show that a smooth solution of the HJB equation exists and provide some of its properties. This is achieved by first obtaining a solution of the PDE for $\tilde{c}$ and then defining $\tilde{v}$ by a transport equation with coefficient $\tilde{c}$. In the third subsection, we apply a verification argument and show that this solution of the HJB equation must be equal to the value function. Theorems 5.2, 5.3 and 5.4 and Corollary 5.5 are direct consequences of the propositions in the last two subsections. The proofs in the last two subsections have a similar structure to the proofs in Sections 4.5.1 and 4.5.2. However, they differ in a few subtle points and we therefore provide them in full detail.

5.5.1 Optimal mean-variance strategies

To obtain optimal trading strategies for the infinite horizon setting, we will first show that optimal strategies exist for the setting with finite horizon T (i.e., $\mathrm{X}_t = 0$ for $t \geq T$) and then consider the limit $T \to \infty$.

Lemma 5.6. *If a mean-variance optimal trading strategy exists for* $\mathrm{X}_0 \in \mathbb{R}^n$ *and time horizon* $T \in]0, \infty]$, *then this strategy is unique.*

Proof. This follows directly from the strict convexity of the functional $f(\xi) + \frac{1}{2}\mathrm{X}^\top \Sigma \mathrm{X}$. $\square$

Proposition 5.7. *For* finite *liquidation time horizons* $T \in \mathbb{R}^+$, *a mean-variance optimal liquidation strategy* $\xi^{(\mathrm{X}_0,T)}$ *exists for all initial portfolios* $\mathrm{X}_0 \in \mathbb{R}^n$. *The portfolio evolution* $\mathrm{X}_t^{\xi^{(\mathrm{X}_0,T)}}$ *is* C^1 *in* t *(i.e., the optimal trading vector* $\xi_t^{(\mathrm{X}_0,T)}$ *is continuous). We denote the time at which the portfolio* X_t^ξ *attains zero by*

$$T_0 := \inf\{t > 0 : \mathrm{X}_t^\xi = 0\} \in]0, T].$$

For $t \in [0, T_0[$, *the portfolio evolution* X_t^ξ *is even* C^2 *and fulfills the Euler-Lagrange equation*

$$\Sigma \mathrm{X}_t = D^2 f(-\dot{\mathrm{X}}_t)\ddot{\mathrm{X}}_t.$$

The optimal trading vector $\xi^{(\mathrm{X}_0,T)}$ *satisfies Bellman's principle of optimality, i.e.,*

$$\xi_t^{(\mathrm{X}_0,T)} = \xi_0^{(\mathrm{X}_t,T-t)}.$$

Furthermore, the initial trading speed ξ_0 *is locally uniformly bounded. More precisely, for each portfolio* $\bar{\mathrm{X}}_0 \in \mathbb{R}$ *and each time horizon* $\bar{T}$, *there is a* $\delta > 0$ *and* $C > 0$ *such that* $|\xi_0^{(\mathrm{X}_0,T)}| < C$ *for all* $|\mathrm{X}_0 - \bar{\mathrm{X}}_0| < \delta$ *and* $T \geq \bar{T}$.

Theorem 3.1 establishes the existence of a mean-variance optimal strategy for finite liquidation time horizons, but not the uniform bound on ξ_0, which we need for our proof of Proposition 5.8. We therefore present a self-contained proof of Proposition 5.7 establishing this bound.

Proof. First, we observe that for mean-variance optimal ξ there is an a priori upper bound $K > 0$ independent of T such that

$$\sup\{|\mathrm{X}_t^\xi| : t \in [0, T]\} < K.$$

To see this, select an arbitrary $\tilde{K} > \mathrm{X}_0^\top \Sigma \mathrm{X}_0$ and assume that $\frac{1}{2}\mathrm{X}_t^\top \Sigma \mathrm{X}_t$ attains $\tilde{K}$ at $T_2 := \min\{t > 0 : \frac{1}{2}\mathrm{X}_t^\top \Sigma \mathrm{X}_t \geq \tilde{K}\}$. Then

$$\frac{\tilde{K}}{2} \leq \frac{1}{2}\mathrm{X}_t^\top \Sigma \mathrm{X}_t \leq \tilde{K}$$

for all $t \in [T_1, T_2]$ with $T_1 := \max\{t < T_2 : \frac{1}{2}\mathrm{X}_t^\top \Sigma \mathrm{X}_t \le \frac{\tilde{K}}{2}\}$. Due to the scaling property (Equation (5.1)), we have

$$\int_{T_1}^{T_2} \left(f(\xi_t) + \frac{1}{2}\mathrm{X}_t^\top \Sigma \mathrm{X}_t \right) dt \ge \left(\min_{\substack{\tilde{\mathrm{X}} \in \tilde{\mathcal{X}} \text{ s.t.} \\ \frac{1}{2}\tilde{\mathrm{X}}_{T_1}^\top \Sigma \tilde{\mathrm{X}}_{T_1} = \frac{\tilde{K}}{2}, \frac{1}{2}\tilde{\mathrm{X}}_{T_2}^\top \Sigma \tilde{\mathrm{X}}_{T_2} = \tilde{K}}} \int_{T_1}^{T_2} f(\tilde{\xi}_t) dt \right) + (T_2 - T_1)\frac{\tilde{K}}{2}$$

$$\ge \left(\frac{1}{T_2 - T_1} \right)^{\alpha} \left(\frac{\tilde{K}}{2} \right)^{\alpha+1} \tilde{C} + (T_2 - T_1)\frac{\tilde{K}}{2} \tag{5.20}$$

with the constant

$$\tilde{C} := \min_{\tilde{\mathrm{X}} \in \tilde{\mathcal{X}} \text{ s.t. } \frac{1}{2}\tilde{\mathrm{X}}_0^\top \Sigma \tilde{\mathrm{X}}_0 = 1, \frac{1}{2}\tilde{\mathrm{X}}_1^\top \Sigma \tilde{\mathrm{X}}_1 = 2} \int_0^1 f(\tilde{\xi}_t) dt > 0.$$

Since $\tilde{C}$ is independent of T_1 and T_2, the right-hand side of Equation (5.20) is bounded from below by a function of $\tilde{K}$ that is increasing and unbounded. This establishes that an optimal ξ cannot attain arbitrarily large values of $\mathrm{X}_t^\top \Sigma \mathrm{X}_t$ respectively $\sup_t |\mathrm{X}_t|$.

We can therefore reduce the optimization problem with unbounded $\mathrm{X}_t \in \mathbb{R}^n$ to an optimization problem with bounded $\mathrm{X}_t \in [-K, K]^n$. By Tonelli's existence theorem (see, e.g., Cesari (1983), Theorem 2.20), a mean-variance optimal trading strategy exist for the bounded optimization problem; by our previous considerations, this strategy is also optimal for the unbounded optimization problem $\mathrm{X}_t \in \mathbb{R}^n$, and we denote this strategy by $\xi^{(\mathrm{X}_0, T)}$.

In order to apply theorems ensuring continuity of even smoothness of ξ, we need to show that the optimal $\xi = \xi^{(\mathrm{X}_0, T)}$ is essentially bounded. The idea of the following proof is that if ξ trades extremely quickly at some points in time, then the mean-variance costs of ξ can be reduced by "smoothing" the trading speed, i.e., slowing down trading when it is fast and accelerating it when it is slow. To formalize this argument, we first observe that there are bounds $(\mathrm{X}_t^\xi)^\top \Sigma \mathrm{X}_t^\xi < K_0$ and $\int_0^T f(\xi_t) dt = K_1 < \infty$, and we define

$$\mu : t \in \mathbb{R} \to \mu_t := \int_0^t \mathbb{1}_{f(\xi_s) \ge K_2} ds \in \mathbb{R},$$

where $K_2 > 0$ is a large, arbitrary constant. ξ is essentially bounded if there is a $K_2 > 0$ with $\mu \equiv 0$. We assume that $\mu \ne 0$ for all $K_2 \in \mathbb{R}^+$ and establish a contradiction. We define the time transformation

$$\tilde{t}(t, s) := s\mu_t + \frac{T - s\mu_T}{T - \mu_T}(t - \mu_t).$$

For $0 < s < \frac{T}{\mu_T}$, this transformation is a bijection $\tilde{t}(\cdot, s) : [0, T] \to [0, T]$ satisfying $\tilde{t}(0, s) = 0$ and $\tilde{t}(T, s) = T$. When using the variables $\tilde{t}$ and t in the following, we will always assume that they are connected by this bijection, i.e., that $\tilde{t} = \tilde{t}(t, s)$. We can now define a new portfolio evolution Y depending on s:

$$\mathrm{Y}^{(s)} : \tilde{t} \in \mathbb{R}_0^+ \to \mathrm{Y}_{\tilde{t}}^{(s)} := \mathrm{X}_t.$$

The portfolio evolution $\mathrm{Y}^{(s)}$ is absolutely continuous and fulfills

$$\xi_{\tilde{t}}^{(s)} := -\frac{d}{d\tilde{t}}\mathrm{Y}_{\tilde{t}}^{(s)} = \begin{cases} \frac{1}{s}\xi_t & \text{for } f(\xi_t) \geq K_2 \\ \frac{T-\mu_T}{T-s\mu_T}\xi_t & \text{for } f(\xi_t) < K_2. \end{cases}$$

Note that $\xi^{(1)} = \xi$. The mean-variance costs of executing $\mathrm{Y}^{(s)}$ are given by

$$\begin{aligned}
&\int_0^T (f(\xi_{\tilde{t}}^{(s)}) + (\mathrm{Y}_{\tilde{t}}^{(s)})^\top \Sigma \mathrm{Y}_{\tilde{t}}^{(s)}) d\tilde{t} \\
&= \int_{f(\xi_t)\geq K_2} (f(\xi_{\tilde{t}}^{(s)}) + (\mathrm{Y}_{\tilde{t}}^{(s)})^\top \Sigma \mathrm{Y}_{\tilde{t}}^{(s)}) d\tilde{t} \\
&\quad + \int_{f(\xi_t)<K_2} (f(\xi_{\tilde{t}}^{(s)}) + (\mathrm{Y}_{\tilde{t}}^{(s)})^\top \Sigma \mathrm{Y}_{\tilde{t}}^{(s)}) d\tilde{t} \\
&= s\int_{f(\xi_t)\geq K_2} \left(f\left(\frac{1}{s}\xi_t\right) + (\mathrm{X}_t^\xi)^\top \Sigma \mathrm{X}_t^\xi \right) dt \\
&\quad + \frac{T-s\mu_T}{T-\mu_T}\int_{f(\xi_t)<K_2} \left(f\left(\frac{T-\mu_T}{T-s\mu_T}\xi_t\right) + (\mathrm{X}_t^\xi)^\top \Sigma \mathrm{X}_t^\xi \right) dt \\
&= \left(\frac{1}{s}\right)^\alpha \int_{f(\xi_t)\geq K_2} f(\xi_t)dt + s\int_{f(\xi_t)\geq K_2} (\mathrm{X}_t^\xi)^\top \Sigma \mathrm{X}_t^\xi dt \\
&\quad + \left(\frac{T-\mu_T}{T-s\mu_T}\right)^\alpha \int_{f(\xi_t)<K_2} f(\xi_t)dt + \frac{T-s\mu_T}{T-\mu_T}\int_{f(\xi_t)<K_2} (\mathrm{X}_t^\xi)^\top \Sigma \mathrm{X}_t^\xi dt.
\end{aligned}$$

By differentiating with respect to s at $s = 1$, we have

$$\begin{aligned}
&\frac{d}{ds}\bigg|_{s=1} \int_0^T (f(\xi_{\tilde{t}}^{(s)}) + (\mathrm{Y}_{\tilde{t}}^{(s)})^\top \Sigma \mathrm{Y}_{\tilde{t}}^{(s)}) d\tilde{t} \\
&\qquad = -\alpha \underbrace{\int_{f(\xi_t)\geq K_2} f(\xi_t)dt}_{\geq K_2\mu_T} + \underbrace{\int_{f(\xi_t)\geq K_2} (\mathrm{X}_t^\xi)^\top \Sigma \mathrm{X}_t^\xi dt}_{\leq K_0\mu_T} \\
&\qquad\quad + \alpha\frac{\mu_T}{T-\mu_T} \underbrace{\int_{f(\xi_t)<K_2} f(\xi_t)dt}_{\leq K_1 - K_2\mu_T} - \frac{\mu_T}{T-\mu_T} \underbrace{\int_{f(\xi_t)<K_2} (\mathrm{X}_t^\xi)^\top \Sigma \mathrm{X}_t^\xi dt}_{\geq 0}.
\end{aligned}$$

If K_2 is large enough, the right hand side of the above equation is smaller than zero for all possible values $\mu_T \in]0, \frac{K_1}{K_2}]$, which contradicts the optimality of $\xi = \xi^{(1)}$. This completes the proof that ξ is essentially bounded. Note that a suitably large bound K_2 holds for all time horizons longer than T and for all initial portfolios that are close to X_0, establishing the uniform boundedness of ξ_0.

Since ξ is essentially bounded, we can apply the Theorems of Tonelli and Weierstrass (see Cesari (1983), Theorem 2.6) and find that X_t^ξ is C^1 everywhere and C^2 until it attains zero. Furthermore, it fulfills the Euler-Lagrange equation. Bellman's principle of optimality for the optimal trading vector ξ follows by the additivity of mean costs and variance of proceeds, as already noted by Almgren and Chriss (2001). □

Proposition 5.8. *For an* infinite *liquidation time horizon, a mean-variance optimal liquidation strategy* $\bar{\xi}^{(\mathrm{X}_0)}$ *exists for all initial portfolios* $\mathrm{X}_0 \in \mathbb{R}^n$. *The portfolio evolution* $\mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}}$ *is* C^1 *in* t *(i.e., the optimal trading vector* $\bar{\xi}_t^{(\mathrm{X}_0)}$ *is continuous). We denote the time at which the portfolio* X_t *attains zero by*

$$T_0 := \inf\{t > 0 : \mathrm{X}_t^{\bar{\xi}} = 0\} \in]0, \infty].$$

For $t \in [0, T_0[$, *the portfolio evolution* X_t^{ξ} *is* C^2 *and fulfills the Euler-Lagrange equation*

$$\Sigma \mathrm{X}_t = D^2 f(-\dot{\mathrm{X}}_t)\ddot{\mathrm{X}}_t. \tag{5.21}$$

The optimal trading vector $\bar{\xi}^{(\mathrm{X}_0)}$ *satisfies Bellman's principle of optimality, i.e.,*

$$\bar{\xi}_t^{(\mathrm{X}_0)} = \bar{\xi}_0^{(\mathrm{X}_t)} =: \bar{\mathrm{c}}(\mathrm{X}_t).$$

with a continuous vector field $\bar{\mathrm{c}} : \mathrm{X} \in \mathbb{R}^n \to \bar{\mathrm{c}}(\mathrm{X}) \in \mathbb{R}^n$.

Proof. First, we introduce some shorthand notation. For a sequence $(\mathrm{X}_0^{(i)}, T^{(i)}) \in \mathbb{R}^n \times \mathbb{R}$, we define

$$\begin{aligned}
\xi^{(i)} &:= \xi^{(\mathrm{X}_0^{(i)}, T^{(i)})} \\
\mathrm{X}_t^{(i)} &:= \mathrm{X}_t^{\xi^{(i)}} \\
T_0^{(i)} &:= \inf\{t > 0 : \mathrm{X}_t^{(i)} = 0\} \in]0, T^{(i)}].
\end{aligned}$$

In Proposition 5.7, we established the uniform boundedness of ξ_0. For each $\mathrm{X}_0 \in \mathbb{R}^n$, we can therefore select a sequence $(\mathrm{X}_0^{(i)}, T^{(i)})$ with $\lim_{i\to\infty} X_0^{(i)} = X_0$ and $\lim_{i\to\infty} T^{(i)} = \infty$ such that $\xi_0^{(i)}$ converges to $\lim_{i\to\infty} \xi_0^{(i)} =: \xi_0^{(\infty)}$. Then we define $\mathrm{X}_t^{(\infty)}$ as the solution to the Euler-Lagrange equation with initial values $\mathrm{X}_0^{(\infty)} = X_0$ and $\dot{X}_0^{(\infty)} = -\xi_0^{(\infty)}$ until $\mathrm{X}_t^{(\infty)}$ attains zero at time $T_0^{(\infty)} \in]0, \infty]$. On $[T_0^{(\infty)}, \infty[$, we define $\mathrm{X}_t^{(\infty)} \equiv 0$. By the smooth dependence of ODE solutions on the initial parameters, we see that $(\mathrm{X}^{(i)}, \dot{\mathrm{X}}^{(i)})$ converges uniformly to $(\mathrm{X}^{(\infty)}, \dot{\mathrm{X}}^{(\infty)})$ on any compact subset $[0, T] \subset [0, T_0^{(\infty)}[$. Therefore

$$\begin{aligned}
\int_0^\infty (f(\xi_s^{(\infty)}) + (\mathrm{X}_s^{(\infty)})^\top \Sigma \mathrm{X}_s^{(\infty)}) ds &= \lim_{T \to T_0^{(\infty)}} \int_0^T (f(\xi_s^{(\infty)}) + (\mathrm{X}_s^{(\infty)})^\top \Sigma \mathrm{X}_s^{(\infty)}) ds \\
&= \lim_{T \to T_0^{(\infty)}} \lim_{i \to \infty} \int_0^T (f(\xi_s^{(i)}) + (\mathrm{X}_s^{(i)})^\top \Sigma \mathrm{X}_s^{(i)}) ds \\
&\leq \lim_{i \to \infty} \int_0^\infty (f(\xi_s^{(i)}) + (\mathrm{X}_s^{(i)})^\top \Sigma \mathrm{X}_s^{(i)}) ds.
\end{aligned}$$

This establishes that $\mathrm{X}^{(\infty)}$ is "at least as good" as the limit of the finite time strategies $\mathrm{X}^{(i)}$. We now show that no strategy can be any better than this limit. Let $\mathrm{X}^{[\infty]} \in \bar{\mathcal{X}}$ be a deterministic admissible strategy with $\mathrm{X}_0^{[\infty]} = \mathrm{X}_0$ and finite mean-variance cost

$$\int_0^\infty (f(\xi_s^{[\infty]}) + (\mathrm{X}_s^{[\infty]})^\top \Sigma \mathrm{X}_s^{[\infty]}) ds < \infty.$$

Then $X_t^{[\infty]}$ converges to zero as t tends to infinity. We define a sequence of trading strategies $X^{[i]}$ that liquidate the portfolio $X_0^{(i)}$ by time $T^{(i)} > 2$ in the following way:

$$X_t^{[i]} := \begin{cases} X_t^{[\infty]} + (1-t)(X_0^{(i)} - X_0) & \text{for } 0 \le t \le 1 \\ X_t^{[\infty]} & \text{for } 1 < t < T^{(i)} - 1 \\ (T^{(i)} - t)X_{T^{(i)}-1}^{[\infty]} & \text{for } T^{(i)} - 1 \le t \le T^{(i)} \\ 0 & \text{for } t > T^{(i)}. \end{cases}$$

We then have

$$\begin{aligned} \lim_{i\to\infty} \int_0^\infty (f(\xi_s^{(i)}) + (X_s^{(i)})^\top \Sigma X_s^{(i)}) ds &\le \lim_{i\to\infty} \int_0^\infty (f(\xi_s^{[i]}) + (X_s^{[i]})^\top \Sigma X_s^{[i]}) ds \\ &= \lim_{T\to\infty} \lim_{i\to\infty} \int_0^T (f(\xi_s^{[i]}) + (X_s^{[i]})^\top \Sigma X_s^{[i]}) ds \\ &= \int_0^\infty (f(\xi_s^{[\infty]}) + (X_s^{[\infty]})^\top \Sigma X_s^{[\infty]}) ds. \end{aligned}$$

Hence $X^{(\infty)}$ is mean-variance optimal. Because it is unique by Lemma 5.6, we see that $\xi_0^{(i)}$ converges to the same vector ξ_0^∞ for any sequence $(X_0^{(i)}, T^{(i)})$. Therefore $\xi_0^{(\infty)}$ depends continuously on X_0. The validity of the Euler-Lagrange equation carries over by construction; Bellman's principle of optimality follows again by the additivity of mean costs and variance of proceeds. □

The next proposition establishes a special form of the identity established by Beltrami (1868) and rediscovered by Hilbert in 1900; see also Bolza (1909)[pp. 107].

Proposition 5.9. *The vector field* $\bar{c}$ *fulfills*

$$\frac{f(\bar{c}(X))}{X^\top \Sigma X} = \frac{1}{2\alpha} \text{ for all } X \in \mathbb{R}^n \backslash \{0\}.$$

Proof. Let X_t be a mean-variance optimal strategy. Then

$$\begin{aligned} \frac{d}{dt}\left(f(-\dot{X}_t) + \frac{1}{2}X_t^\top \Sigma X_t\right) &= -\nabla f(-\dot{X}_t)\ddot{X}_t + X_t^\top \Sigma \dot{X}_t \\ &= -\nabla f(-\dot{X}_t)\ddot{X}_t + (\ddot{X}_t)^\top D^2 f(-\dot{X}_t)\dot{X}_t \qquad (5.22) \\ &= \frac{d}{dt}(-\nabla f(-\dot{X}_t)\dot{X}_t) \\ &= \frac{d}{dt}((\alpha+1)f(-\dot{X}_t)) \qquad (5.23) \end{aligned}$$

where Equation (5.22) follows by the Euler-Lagrange equation (5.21) and Equation (5.23) by the scaling property (5.1) which implies

$$\nabla f(c)c = \lim_{s\to\infty} \frac{f((1+s)c) - f(c)}{s} = (\alpha+1)f(c). \qquad (5.24)$$

Hence

$$-\alpha f(\bar{c}(X_0)) + \frac{1}{2}X_0^\top \Sigma X_0 = \lim_{t\to\infty}\left(-\alpha f(\bar{c}(X_t)) + \frac{1}{2}X_t^\top \Sigma X_t\right) = 0.$$

The desired equality follows immediately. □

Finally, we show that the mean-variance value function fulfills the dynamic programming PDE.

Proposition 5.10. *The mean-variance value function*

$$\bar{v}(\mathrm{X}_0) := \inf_{\bar{\xi}\in\bar{\mathcal{X}}} \left[\int_0^\infty \left(f(\bar{\xi}_s) + \frac{1}{2}(\mathrm{X}_s^{\bar{\xi}})^\top \Sigma \mathrm{X}_s^{\bar{\xi}} \right) ds \right]$$

is C^1 and fulfills

$$\nabla f(\bar{\mathrm{c}}(\mathrm{X})) = \bar{v}_{\mathrm{X}}. \tag{5.25}$$

Proof. The mean-variance value function is convex because of the convexity of $f(\xi) + \frac{1}{2}\mathrm{X}^\top \Sigma \mathrm{X}$. The function $\bar{v}$ is therefore necessarily differentiable at $\mathrm{X}_0 \in \mathbb{R}^n$, if it is bounded from above by a smooth function $\tilde{v}$ that touches $\bar{v}$ at X_0, i.e., $\tilde{v}(\mathrm{X}_0) = \bar{v}(\mathrm{X}_0)$. Such a function $\tilde{v}$ however can be constructed as

$$\tilde{v}(\mathrm{X}) = \int_0^\infty \left(f(\xi_t^{\mathrm{X}}) + (\mathrm{X}_t^{\xi^{\mathrm{X}}})^\top \Sigma \mathrm{X}_t^{\xi^{\mathrm{X}}} \right) dt$$

with

$$\xi_t^{\mathrm{X}} := \bar{\xi}_t^{(\mathrm{X}_0)} + M_t(\mathrm{X} - \mathrm{X}_0)$$

where

$$M_t := (\bar{\xi}_t^{(\mathrm{X}_0+\mathrm{e}_1)} - \bar{\xi}_t^{(\mathrm{X}_0)}, \bar{\xi}_t^{(\mathrm{X}_0+\mathrm{e}_2)} - \bar{\xi}_t^{(\mathrm{X}_0)}, \ldots, \bar{\xi}_t^{(\mathrm{X}_0+\mathrm{e}_n)} - \bar{\xi}_t^{(\mathrm{X}_0)}) \in \mathbb{R}^{n\times n}.$$

Therefore $\bar{v}$ is differentiable. By the dynamic programming principle, we have that for any absolutely continuous path $\mathrm{X} : \mathbb{R}_0^+ \to \mathbb{R}^n$:

$$\bar{v}(\mathrm{X}_0) \le \bar{v}(\mathrm{X}_t) + \int_0^t \left(f(-\dot{\mathrm{X}}_s) + \frac{1}{2}\mathrm{X}_s^\top \Sigma \mathrm{X}_s \right) ds$$

with equality for the optimal strategy $\mathrm{X}^{\bar{\xi}}$. Since $\bar{v}$ is differentiable, this implies

$$0 \le \bar{v}_{\mathrm{X}}(\mathrm{X}_0)\dot{\mathrm{X}}_0 + f(-\dot{\mathrm{X}}_0) + \frac{1}{2}\mathrm{X}_0^\top \Sigma \mathrm{X}_0.$$

The right hand side therefore attains its minimum at the optimal $\dot{\mathrm{X}}_0 = -\bar{\mathrm{c}}(\mathrm{X}_0)$ and therefore

$$\nabla f(\bar{\mathrm{c}}(\mathrm{X}_0)) = \bar{v}_{\mathrm{X}}(\mathrm{X}_0).$$

This establishes Equation (5.25) and that the mean-variance cost $\bar{v}$ is C^1. □

Proposition 5.11. *For any $\mathrm{X}_0 \in \mathbb{R}^n$, the deterministic mean-variance optimal trading strategy $\bar{\xi} = \bar{\xi}^{(\mathrm{X}_0)}$ satisfies*

$$\lim_{t\to\infty} (\mathrm{X}_t^{\bar{\xi}})^\top \Sigma \mathrm{X}_t^{\bar{\xi}} t \ln\ln t = 0. \tag{5.26}$$

It is admissible both for optimal liquidation and for asymptotic maximization of portfolio value, i.e., $\bar{\xi} \in \bar{\mathcal{X}} \cap \mathcal{X}_1$.

For the proof, we need the following lemma.

Lemma 5.12. *Let* $\mathrm{Y}_0 = a\mathrm{X}_0$ *and let* X *and* Y *be the corresponding mean-variance optimal strategies. Then we have that*

$$\mathrm{Y}_t = a\mathrm{X}_{bt} \text{ with } b := a^{\frac{1-\alpha}{1+\alpha}}.$$

Proof of Lemma 5.12. Let us define

$$\begin{aligned}\hat{\mathrm{X}}_t &:= \frac{1}{a}\mathrm{Y}_{\frac{t}{b}}\\ \hat{\mathrm{Y}}_t &:= a\mathrm{X}_{bt}.\end{aligned}$$

Then $\hat{\mathrm{X}}$ and $\hat{\mathrm{Y}}$ are deterministic strategies with $\hat{\mathrm{X}}_0 = \mathrm{X}_0$ and $\hat{\mathrm{Y}}_0 = \mathrm{Y}_0$, and we obtain

$$\begin{aligned}\bar{v}(\mathrm{X}_0) &\leq \int_0^\infty \left(f(-\dot{\hat{\mathrm{X}}}_s) + \frac{1}{2}\hat{\mathrm{X}}_s^\top \Sigma \hat{\mathrm{X}}_s\right) ds\\ &= \left(\frac{1}{ab}\right)^{\alpha+1} b \int_0^\infty f(-\dot{\mathrm{Y}}_s) ds + \left(\frac{1}{a}\right)^2 b \int_0^\infty \frac{1}{2}\mathrm{Y}_s^\top \Sigma \mathrm{Y}_s ds\\ &= a^{-\frac{3\alpha+1}{\alpha+1}}\bar{v}(\mathrm{Y}_0)\\ &\leq a^{-\frac{3\alpha+1}{\alpha+1}} \int_0^\infty \left(f(-\dot{\hat{\mathrm{Y}}}_s) + \frac{1}{2}\hat{\mathrm{Y}}_s^\top \Sigma \hat{\mathrm{Y}}_s\right) ds\\ &= a^{-\frac{3\alpha+1}{\alpha+1}} \left((ab)^{\alpha+1}\frac{1}{b}\int_0^\infty f(-\dot{\mathrm{X}}_s) ds + a^2\frac{1}{b}\int_0^\infty \frac{1}{2}\mathrm{X}_s^\top \Sigma \mathrm{X}_s ds\right)\\ &= \bar{v}(\mathrm{X}_0).\end{aligned}$$

All the inequalities above are thus equalities, and hence $\hat{\mathrm{X}}$ and $\hat{\mathrm{Y}}$ are optimal. The lemma follows since the optimal strategies are unique. □

Proof of Proposition 5.11. It is clear that $\bar{\xi}^{(\mathrm{X}_0)}$ is admissible for asymptotic maximization of portfolio value, i.e., that $\bar{\xi}^{(\mathrm{X}_0)} \in \mathcal{X}$. To see that it is also admissible for optimal liquidation, i.e., that $\bar{\xi}^{(\mathrm{X}_0)} \in \mathcal{X}_1$, the only thing left to prove is Equation (5.26). First, we observe that by Lemma 5.12, it is sufficient to prove this equation for X_0 with $\mathrm{X}_0^\top \Sigma \mathrm{X}_0 = 1$. Let us first define

$$\tau_0 := \sup_{\mathrm{X}_0 \text{ with } \mathrm{X}_0^\top \Sigma \mathrm{X}_0 = 1} \max\left\{t > 0 : (\mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}})^\top \Sigma \mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}} \geq \frac{1}{2}\right\}.$$

This τ_0 is the time it takes at most until $\mathrm{X}_0^\top \Sigma \mathrm{X}_0$ is reduced from 1 to $\frac{1}{2}$. By Lemma 5.12, we obtain that

$$\tau_1 := \sup_{\mathrm{X}_0 \text{ with } \mathrm{X}_0^\top \Sigma \mathrm{X}_0 = \frac{1}{2}} \max\left\{t > 0 : (\mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}})^\top \Sigma \mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}} \geq \frac{1}{4}\right\} = 2^{\frac{1-\alpha}{1+\alpha}}\tau_0$$

or more generally

$$\tau_k := \sup_{\mathrm{X}_0 \text{ with } \mathrm{X}_0^\top \Sigma \mathrm{X}_0 = \left(\frac{1}{2}\right)^k} \max\left\{t > 0 : (\mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}})^\top \Sigma \mathrm{X}_t^{\bar{\xi}^{(\mathrm{X}_0)}} \geq \left(\frac{1}{2}\right)^{k+1}\right\} = 2^{k\frac{1-\alpha}{1+\alpha}}\tau_0.$$

Let $X_0 \in \mathbb{R}^n$ with $X_0^\top \Sigma X_0 = 1$. Then for all $t \geq \sum_0^k \tau_k$, we have that

$$(X_t^{\bar{\xi}^{(X_0)}})^\top \Sigma X_t^{\bar{\xi}^{(X_0)}} \leq \left(\frac{1}{2}\right)^{k+1}.$$

For $\alpha \geq 1$, we have that $\tau_k \leq \tau_0$; $(X_t^{\bar{\xi}})^\top \Sigma X_t^{\bar{\xi}}$ is therefore bounded from above by an exponential function. For $0 < \alpha < 1$, we see that $(X_t^{\bar{\xi}})^\top \Sigma X_t^{\bar{\xi}}$ is bounded from above by $K(t+1)^{\frac{\alpha+1}{\alpha-1}}$ for a $K > 0$. In both cases we see that Equation (5.26) holds. □

5.5.2 Existence and characterization of a smooth solution of the HJB equation

As a first step, we observe that $\lim_{R\to\infty} u(R) < \infty$ due to the boundedness of the risk aversion, and we can thus assume without loss of generality that

$$\lim_{R\to\infty} u(R) = 0.$$

Proposition 5.13. *There exists a smooth ($C^{2,4}$) solution of*

$$\tilde{c}_Y = -\frac{2\alpha+1}{\alpha+1}\tilde{c}^\alpha \tilde{c}_R + \frac{\alpha(\alpha-1)}{\alpha+1}\left(\frac{\tilde{c}_R}{\tilde{c}}\right)^2 + \frac{\alpha}{\alpha+1}\frac{\tilde{c}_{RR}}{\tilde{c}} \tag{5.27}$$

with initial value

$$\tilde{c}(0,R) = A(R)^{\frac{1}{\alpha+1}}. \tag{5.28}$$

The solution satisfies

$$\tilde{c}_{min} := \inf_{R\in\mathbb{R}} A(R)^{\frac{1}{\alpha+1}} \leq \tilde{c}(Y,R) \leq \sup_{R\in\mathbb{R}} A(R)^{\frac{1}{\alpha+1}} =: \tilde{c}_{max}. \tag{5.29}$$

The function $\tilde{c}$ is $C^{2,4}$ in the sense that it has a continuous derivative $\frac{\partial^{i+j}}{\partial Y^i \partial R^j}\tilde{c}(Y,R)$ if $2i + j \leq 4$. In particular, $\tilde{c}_{YRR}$ and $\tilde{c}_{RRR}$ exist and are continuous. We do not establish the uniqueness of $\tilde{c}$ directly in the preceding proposition. However, it follows from Proposition 5.20.

Proof of Proposition 5.13. We want to apply Theorem 4.11 and set

$$\begin{aligned} a(x,t,u,p) &:= h_1(u)p \\ b(x,t,u,p) &:= h_2(u)p - h_3(u)p^2 + h_1'(u)p^2 \\ \psi_0(x) &:= A(R)^{\frac{1}{\alpha+1}} \end{aligned}$$

with smooth functions $h_1, h_2, h_3 : \mathbb{R} \to \mathbb{R}$. With

$$h_1(u) = \frac{\alpha}{(\alpha+1)u} \qquad h_2(u) = \frac{2\alpha+1}{(\alpha+1)}u^\alpha \qquad h_3(u) = \frac{\alpha(\alpha-1)}{(\alpha+1)u^2}, \tag{5.30}$$

Equation (4.22) becomes Equation (5.27) by relabeling the coordinates from t to Y and from x to R. All conditions of Theorem 4.11 are fulfilled, if we take h_1, h_2 and h_3 to be smooth nonnegative functions bounded away from zero and infinity and fulfilling Equation (5.30) for $\tilde{c}_{min} \leq u \leq \tilde{c}_{max}$. With these functions, there exists a smooth solution to

$$f_t = -h_2(f)f_x + h_3(f)f_x^2 + h_1(f)f_{xx}.$$

We now show that this solution f also fulfills

$$f_t = -\frac{2\alpha+1}{\alpha+1}f^\alpha f_x + \frac{\alpha(\alpha-1)}{\alpha+1}\left(\frac{f_x}{f}\right)^2 + \frac{\alpha}{\alpha+1}\frac{f_{xx}}{f}$$

by using the maximum principle to show that $\tilde{c}_{min} \leq f \leq \tilde{c}_{max}$. First assume that there is a (t_0, x_0) such that $f(t_0, x_0) > \tilde{c}_{max}$. Then there is an $N > 0$ and $\gamma > 0$ such that also $\tilde{f}_N(t_0, x_0) := f_N(t_0, x_0)e^{-\gamma t_0} > \tilde{c}_{max}$ with f_N as constructed in the proof of Theorem 4.11. Then $\max_{t\in[0,t_0],x\in[-N,N]} \tilde{f}_N(t, x)$ is attained at an interior point (t_1, x_1), i.e., $0 < t_1 \leq t_0$ and $-N < x_1 < N$. We thus have

$$\begin{aligned}\tilde{f}_{N,t}(t_1, x_1) &\geq 0\\ \tilde{f}_{N,x}(t_1, x_1) &= 0\\ \tilde{f}_{N,xx}(t_1, x_1) &\leq 0.\end{aligned}$$

We furthermore have that

$$\begin{aligned}\tilde{f}_{N,t} &= e^{-\gamma t}f_{N,t} - \gamma e^{-\gamma t}f_N\\ &= -e^{-\gamma t}h_2(f_N)f_{N,x} + e^{-\gamma t}h_3(f_N)f_{N,x}^2 + e^{-\gamma t}h_1(f_N)f_{N,xx} - \gamma e^{-\gamma t}f_N\\ &= -h_2(f_N)\tilde{f}_{N,x} + h_3(f_N)\tilde{f}_{N,x}f_{N,x} + h_1(f_N)\tilde{f}_{N,xx} - \gamma\tilde{f}_N\end{aligned}$$

and therefore that

$$\tilde{f}_N(t_1, x_1) \leq 0.$$

This however contradicts $\tilde{f}_N(t_1, x_1) \geq \tilde{f}_N(t_0, x_0) \geq \tilde{c}_{max} > 0$.

By a similar argument, we can show that if there is a point (t_0, x_0) with $f(t_0, x_0) < \tilde{c}_{min}$, then the interior minimum (t_1, x_1) of a suitably chosen $\tilde{f}_N := e^{-\gamma t}(f_N - \tilde{c}_{max}) < 0$ satisfies $\tilde{f}_N(t_1, x_1) \geq 0$ and thus causes a contradiction. □

Proposition 5.14. *There exists a $C^{2,4}$-solution $\tilde{w} : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ of the transport equation*

$$\tilde{w}_Y = -\tilde{c}^\alpha \tilde{w}_R \tag{5.31}$$

with initial value

$$\tilde{w}(0, R) = u(R).$$

The solution satisfies

$$0 \geq \tilde{w}(Y, R) \geq u(R - \tilde{c}_{max}^\alpha Y)$$

and is increasing in R and decreasing in Y.

Proof. The proof uses the method of characteristics. Consider the function

$$P : (Y, S) \in \mathbb{R}_0^+ \times \mathbb{R} \to P(Y, S) \in \mathbb{R}$$

satisfying the ODE

$$P_Y(Y, S) = \tilde{c}(Y, P(Y, S))^\alpha \tag{5.32}$$

with initial value condition $P(0, S) = S$. Since $\tilde{c}^\alpha$ is smooth and bounded, a solution of the above ODE exists for each fixed S. For every Y, $P(Y, \cdot)$ is a diffeomorphism mapping $\mathbb{R}$ onto $\mathbb{R}$ that has the same regularity as $\tilde{c}$, i.e., belongs to $C^{2,4}$. We define

$$\tilde{w}(Y, R) = u(S) \quad \text{iff} \quad P(Y, S) = R.$$

Then $\tilde{w}$ is a $C^{2,4}$-function satisfying the initial value condition. By definition, we have

$$\begin{aligned} 0 &= \frac{d}{dY}\tilde{w}(Y, P(Y, S)) \\ &= \tilde{w}_R(Y, P(Y, S))P_Y(Y, S) + \tilde{w}_Y(Y, P(Y, S)) \\ &= \tilde{w}_R(Y, P(Y, S))\tilde{c}(Y, P(Y, S))^\alpha + \tilde{w}_Y(Y, P(Y, S)). \end{aligned}$$

Therefore $\tilde{w}$ fulfills the desired partial differential equation. Since $\tilde{c} \le \tilde{c}_{max}$, we know that $P_Y \le \tilde{c}^\alpha_{max}$ and hence $P(Y, S) \le S + Y\tilde{c}^\alpha_{max}$ and thus $\tilde{w}(Y, R) \ge u(R - \tilde{c}^\alpha_{max}Y)$.

The monotonicity statements in the proposition follow because the family of solutions of the ODE (5.32) do not cross and since $\tilde{c}$ is positive. □

Proposition 5.15. *The function $w(X, R) := \tilde{w}(\bar{v}(\mathrm{X}), R)$ has continuous derivatives up to w_{XRR} and w_{RRRR}, and it solves the HJB equation*

$$\min_{\mathrm{c}} \left[-\frac{1}{2} w_{RR} \mathrm{X}^\top \Sigma \mathrm{X} + w_R f(\mathrm{c}) + w_{\mathrm{X}} \mathrm{c} \right] = 0. \tag{5.33}$$

The unique minimum is attained at

$$\mathrm{c}(X, R) := \tilde{c}(\bar{v}(\mathrm{X}), R)\bar{\mathrm{c}}(\mathrm{X}). \tag{5.34}$$

Note that w is not necessarily everywhere twice differentiable in X; the single asset case with $\alpha < 1$ is a counterexample (see Equation (5.10)).

Proof. Assume for the moment that

$$\tilde{c}^{\alpha+1} = -\frac{\tilde{w}_{RR}}{\tilde{w}_R}. \tag{5.35}$$

Then with $Y = \bar{v}(\mathrm{X})$:

$$\begin{aligned} 0 &= -\frac{1}{2}\mathrm{X}^\top\Sigma\mathrm{X}\tilde{w}_R \left(\frac{\tilde{w}_{RR}}{\tilde{w}_R} + \tilde{c}^{\alpha+1} \right) \\ &= -\frac{1}{2}\mathrm{X}^\top\Sigma\mathrm{X}\tilde{w}_R \left(\frac{\tilde{w}_{RR}}{\tilde{w}_R} + \frac{2\alpha f(\bar{\mathrm{c}})}{\mathrm{X}^\top\Sigma\mathrm{X}}\tilde{c}^{\alpha+1} \right) && (5.36) \\ &= -\frac{1}{2}\tilde{w}_{RR}\mathrm{X}^\top\Sigma\mathrm{X} - \alpha\tilde{w}_R f(\mathrm{c}) && (5.37) \\ &= \inf_{\mathrm{c}} \left[-\frac{1}{2} w_{RR}\mathrm{X}^\top\Sigma\mathrm{X} + w_R f(\mathrm{c}) + w_{\mathrm{X}}\mathrm{c} \right]. && (5.38) \end{aligned}$$

Equation (5.36) holds because of Theorem 5.1, Equation (5.37) because of the scaling property of f (Equation (5.1)), and Equation (5.38) again because of the scaling property of f as in Equation (5.24). Note that the minimizer c as in Equation (5.34) is unique since ∇f is injective due to the convexity of f.

We now show that Equation (5.35) is fulfilled for all R and $Y = \bar{v}(\mathrm{X})$. First, observe that it holds for $Y = 0$. For general Y, consider the following two equations:

$$\begin{aligned} \frac{d}{dY}\tilde{c}^{\alpha+1} &= -(2\alpha+1)\tilde{c}^{2\alpha}\tilde{c}_R + \alpha(\alpha-1)\tilde{c}^{\alpha-2}\tilde{c}_R^2 + \alpha\tilde{c}^{\alpha-1}\tilde{c}_{RR} \\ -\frac{d}{dY}\frac{\tilde{w}_{RR}}{\tilde{w}_R} &= \tilde{c}^{\alpha}\frac{d}{dR}\frac{\tilde{w}_{RR}}{\tilde{w}_R} + \alpha\tilde{c}^{\alpha-1}\tilde{c}_R\frac{\tilde{w}_{RR}}{\tilde{w}_R} + \alpha(\alpha-1)\tilde{c}^{\alpha-2}\tilde{c}_R^2 + \alpha\tilde{c}^{\alpha-1}\tilde{c}_{RR}. \end{aligned}$$

The first of these two equations holds because of Equation (5.27) and the second one because of Equation (5.31). Now we have

$$\frac{d}{dY}\left(\tilde{c}^{\alpha+1} + \frac{\tilde{w}_{RR}}{\tilde{w}_R}\right) = -\tilde{c}^{\alpha}\frac{d}{dR}\left(\tilde{c}^{\alpha+1} + \frac{\tilde{w}_{RR}}{\tilde{w}_R}\right) - \alpha\tilde{c}^{\alpha-1}\tilde{c}_R\left(\tilde{c}^{\alpha+1} + \frac{\tilde{w}_{RR}}{\tilde{w}_R}\right).$$

Hence, the function $f(Y,R) := \tilde{c}^{\alpha+1} + \frac{\tilde{w}_{RR}}{\tilde{w}_R}$ satisfies the linear PDE

$$f_Y = -\tilde{c}^{\alpha} f_R - \alpha\tilde{c}^{\alpha-1}\tilde{c}_R f$$

with initial value condition $f(0,R) = 0$. One obvious solution to this PDE is $f(Y,R) \equiv 0$. By the method of characteristics this is the unique solution to the PDE, since $\tilde{c}$ and $\tilde{c}_R$ are smooth and hence locally Lipschitz. $\square$

The next auxiliary lemma will prove useful in the following.

Lemma 5.16 (Auxiliary Lemma). *There are positive constants a_1, a_2, a_3, a_4 and b such that*

$$\begin{array}{rcccl} u(R) & \geq & w(\mathrm{X},R) & \geq & u(R)\exp(b\bar{v}(\mathrm{X})) \\ 0 & \leq & w_R(\mathrm{X},R) & \leq & a_1 + a_2\exp(-a_3R + a_4\bar{v}(\mathrm{X})) \end{array} \tag{5.39}$$

for all $(\mathrm{X},R) \in \mathbb{R}^n \times \mathbb{R}$.

Proof of Lemma 5.16. The left hand side of the first inequality follows by the boundary condition for w and the monotonicity of w with respect to X as established in Proposition 5.14. Since the risk aversion of u is bounded from above by $\tilde{c}_{max}^{\alpha+1}$, we have

$$u(R-\Delta) \geq u(R)e^{\tilde{c}_{max}^{\alpha+1}\Delta} \text{ for } \Delta \geq 0 \tag{5.40}$$

and thus by Proposition 5.14

$$w(\mathrm{X},R) \geq u(R - \tilde{c}_{max}^{\alpha}\bar{v}(\mathrm{X})) \geq u(R)e^{\tilde{c}_{max}^{2\alpha+1}\bar{v}(\mathrm{X})}$$

which establishes the right hand side of the first inequality with $b = \tilde{c}_{max}^{2\alpha+1}$.

For the second inequality, we will show the equivalent inequality

$$0 \leq \tilde{w}_R(Y,R) \leq a_1 + a_2\exp(-a_3R + a_4Y).$$

The left hand side follows since $\tilde{w}$ is increasing in R by Proposition 5.14. For the right hand side, note that also the "risk aversion" of $\tilde{w}$ is bounded by $\tilde{c}^{\alpha+1}_{max}$ due to Equation (5.35). Hence

$$\tilde{w}(Y,R_0) \geq \tilde{w}(Y,R) + \frac{\tilde{w}_R(Y,R)}{\tilde{c}^{\alpha+1}_{max}}\left(1-e^{-\tilde{c}^{\alpha+1}_{max}(R_0-R)}\right).$$

Since

$$\lim_{R_0\to\infty}\tilde{w}(Y,R_0) \leq \lim_{R_0\to\infty} u(R_0) = 0$$

we have

$$0 \geq \tilde{w}(Y,R) + \frac{\tilde{w}_R(Y,R)}{\tilde{c}^{\alpha+1}_{max}}$$

and thus

$$\tilde{w}_R(Y,R) \leq -\tilde{w}(Y,R)\tilde{c}^{\alpha+1}_{max} \leq -u(R-\tilde{c}^{\alpha}_{max}Y)\tilde{c}^{\alpha+1}_{max}.$$

Since u is bounded by an exponential function, we obtain the desired bound on $\tilde{w}_R$. □

5.5.3 Verification argument

We now connect the PDE results from Subsection 5.5.2 with the optimal stochastic control problem introduced in Section 5.2. For any admissible strategy $\xi \in \mathcal{X}$ and $k \in \mathbb{N}$ we define

$$\tau^{\xi}_k := \inf\Big\{t \geq 0 \,\Big|\, \int_0^t f(\xi_s)\,ds \geq k\Big\}.$$

We proceed by first showing that $u(R^{\xi}_t)$ and $w(\mathrm{X}^{\xi}_t, R^{\xi}_t)$ fulfill local supermartingale inequalities. Thereafter we show that $w(\mathrm{X}_0,R_0) \geq \lim_{t\to\infty}\mathbb{E}[u(R^{\xi}_t)]$ with equality for $\xi = \hat{\xi}$. The next lemma in particular justifies our definition of $v_2(\mathrm{X}_0,R_0)$ in Equation (5.6).

Lemma 5.17. *For any admissible strategy ξ the expected utility $\mathbb{E}[\,u(R^{\xi}_t)\,]$ is decreasing in t. Moreover, we have $\mathbb{E}[\,u(R^{\xi}_{t\wedge\tau^{\xi}_k})\,] \geq \mathbb{E}[\,u(R^{\xi}_t)\,]$.*

Proof. Since $R^{\xi}_t - R_0$ is the difference of the true martingale $\int_0^t (\mathrm{X}^{\xi}_s)^\top\sigma\,d\mathrm{B}_s$ and the increasing process $\int_0^t f(\xi_s)\,ds$, it satisfies the supermartingale inequality $\mathbb{E}[\,R^{\xi}_t\,|\,\mathcal{F}_s\,] \leq R^{\xi}_s$ for $s \leq t$ (even though it may fail to be a supermartingale due to the possible lack of integrability). Hence $\mathbb{E}[\,u(R^{\xi}_t)\,]$ is decreasing according to Jensen's inequality.

For the second assertion, we first take $n = k$ and write for $\tau_m := \tau^{\xi}_m$

$$\mathbb{E}[\,u(R^{\xi}_{t\wedge\tau_k})\,] = \mathbb{E}\Big[u\Big(R_0 + \int_0^{t\wedge\tau_n}(\mathrm{X}^{\xi}_s)^\top\sigma\,d\mathrm{B}_s - \int_0^{t\wedge\tau_k} f(\xi_s)\,ds\Big)\Big].$$

When sending n to infinity, the right-hand side decreases to

$$\mathbb{E}\Big[u\Big(R_0 + \int_0^{t}(\mathrm{X}^{\xi}_s)^\top\sigma\,d\mathrm{B}_s - \int_0^{t\wedge\tau_k} f(\xi_s)\,ds\Big)\Big], \tag{5.41}$$

by dominated convergence because u is bounded from below by an exponential function, the integral of $f(\xi)$ is bounded by k, and the stochastic integrals are uniformly bounded from below by $\inf_{s\leq Kt} W_s$, where W is the DDS-Brownian motion of $\int(\mathrm{X}^{\xi}_s)^\top\sigma\,d\mathrm{B}_s$ and K is an upper bound for $(\mathrm{X}^{\xi})^\top\Sigma\mathrm{X}^{\xi}$. Finally, the term in Equation (5.41) is clearly larger than or equal to $\mathbb{E}[\,u(R^{\xi}_t)\,]$. □

Lemma 5.18. *For any admissible strategy* ξ*,* $w(\mathrm{X}_t^\xi, R_t^\xi)$ *is a local supermartingale with localizing sequence* (τ_k^ξ)*.*

Proof. We use a verification argument similar to the ones in Chapters 3 and 4. For $T > t \geq 0$, Itô's formula yields that

$$\begin{aligned} w(\mathrm{X}_T^\xi, R_T^\xi) - w(\mathrm{X}_t^\xi, R_t^\xi) &= \int_t^T w_R(\mathrm{X}_s^\xi, R_s^\xi)(\mathrm{X}_s^\xi)^\top \sigma \, d\mathrm{B}_s \\ &\quad - \int_t^T \Big[w_R f(\xi_s) + w_\mathrm{X} \xi_s - \frac{1}{2}(\mathrm{X}_s^\xi)^\top \Sigma \mathrm{X}_s^\xi w_{RR} \Big] (\mathrm{X}_s^\xi, R_s^\xi) \, ds. \end{aligned} \tag{5.42}$$

By Proposition 5.15 the latter integral is nonnegative and we obtain

$$w(\mathrm{X}_t^\xi, R_t^\xi) \geq w(\mathrm{X}_T^\xi, R_T^\xi) - \int_t^T w_R(\mathrm{X}_s^\xi, R_s^\xi)(\mathrm{X}_s^\xi)^\top \sigma \, d\mathrm{B}_s. \tag{5.43}$$

We will show next that the stochastic integral in Equation (5.43) is a local martingale with localizing sequence $(\tau_k) := (\tau_k^\xi)$. For some constant C_1 depending on t, k, $|\sigma|$, R_0, and on the upper bound of $|\mathrm{X}^\xi|$ we have for $s \leq t \wedge \tau_k$

$$R_s^\xi = R_0 + (\mathrm{X}_s^\xi)^\top \sigma \mathrm{B}_s + \int_0^s (\xi_q^\top \sigma \mathrm{B}_q - f(\xi_q)) \, dq \geq -C_1 \big(1 + \sup_{q \leq t} |\mathrm{B}_q|\big).$$

Using Lemma 5.16, we see that for $s \leq t \wedge \tau_k$

$$0 \leq w_R(\mathrm{X}_s^\xi, R_s^\xi) \leq a_1 + a_2 \exp\left(a_3 C_1 \big(1 + \sup_{q \leq t} |\mathrm{B}_q|\big) + a_4 K^2 \right) \tag{5.44}$$

where K is the upper bound of $\bar{v}(\mathrm{X}^\xi)$. Since $\sup_{q \leq t} |\mathrm{B}_q|$ has exponential moments of all orders, the martingale property of the stochastic integral in Equation (5.43) follows. Taking conditional expectations in Equation (5.43) thus yields the desired supermartingale property

$$w(\mathrm{X}_{t\wedge\tau_k}^\xi, R_{t\wedge\tau_k}^\xi) \geq \mathbb{E}[\, w(\mathrm{X}_{T\wedge\tau_k}^\xi, R_{T\wedge\tau_k}^\xi) | \mathcal{F}_t \,]. \tag{5.45}$$

The integrability of $w(\mathrm{X}_{t\wedge\tau_k}^\xi, R_{t\wedge\tau_k}^\xi)$ follows from Lemma 5.16 and Equation (5.40) in a similar way as in Equation (5.44). $\square$

Lemma 5.19. *There is an adapted strategy* $\hat{\xi}$ *fulfilling*

$$\hat{\xi}_t := \mathrm{c}(\mathrm{X}_t^{\hat{\xi}}, R_t^{\hat{\xi}}). \tag{5.46}$$

This $\hat{\xi}$ *is admissible for optimal liquidation and maximization of asymptotic portfolio value and satisfies* $\int_0^\infty f(\hat{\xi}_t)\, dt < K$ *for some constant* K*. Furthermore,* $w(\mathrm{X}_t^{\hat{\xi}}, R_t^{\hat{\xi}})$ *is a martingale and*

$$w(\mathrm{X}_0, R_0) = \lim_{t \to \infty} \mathbb{E}[\, u(R_t^{\hat{\xi}}) \,] \leq v_2(\mathrm{X}_0, R_0). \tag{5.47}$$

Proof. Consider the stochastic differential equation

$$d\begin{pmatrix} s_t \\ R_t \end{pmatrix} = \begin{pmatrix} \tilde{c}(\bar{v}(\mathrm{X}^{\bar{\xi}}_{s_t}), R_t)dt \\ -\tilde{c}(\bar{v}(\mathrm{X}^{\bar{\xi}}_{s_t}), R_t)^{\alpha+1} f(\bar{\mathrm{c}}(\mathrm{X}^{\bar{\xi}}_{s_t}))dt + (\mathrm{X}^{\bar{\xi}}_{s_t})^\top \sigma dB_t \end{pmatrix} \tag{5.48}$$

with initial condition $s_0 = 0$. The functions $\tilde{c}$ and $\bar{v}$ are differentiable, $\mathrm{X}^{\bar{\xi}}_s$ is differentiable in s, and by the Beltrami identity (5.9) we have

$$f(\bar{\mathrm{c}}(\mathrm{X}^{\bar{\xi}}_s)) = \frac{(\mathrm{X}^{\bar{\xi}}_s)^\top \Sigma \mathrm{X}^{\bar{\xi}}_s}{2\alpha}$$

which establishes that $f(\bar{\mathrm{c}}(\mathrm{X}^{\bar{\xi}}_s))$ is differentiable in s. Hence, Equation (5.48) satisfies local boundedness and Lipschitz conditions and hence has a solution; see for example Durrett (1996). We can now set $\hat{\mathrm{X}}_t := \mathrm{X}^{\bar{\xi}}_{s_t}$; the resulting stochastic process $\hat{\mathrm{X}}$ is absolutely continuous, and by setting $\hat{\xi}_t := -\dot{\hat{\mathrm{X}}}_t$ we obtain a solution of Equation (5.46). We observe that $\hat{\xi}$ is admissible both for optimal liquidation and maximization of asymptotic portfolio value if $\int_0^\infty f(\hat{\xi}_t)\,dt < K$ for some constant K; conditions (5.2) and (5.3) are clear by Proposition 5.11 and the lower bound on $\tilde{c}$ (Proposition 5.13).. The upper bound for $\int_0^\infty f(\hat{\xi}_t)\,dt$ can be derived as follows:

$$\begin{aligned}\int_0^\infty f(\hat{\xi}_t)dt &= \int_0^\infty f(\tilde{c}(\mathrm{X}^{\hat{\xi}}_t, R^{\hat{\xi}}_t)\bar{\mathrm{c}}(\mathrm{X}^{\hat{\xi}}_t))dt = \int_0^\infty \tilde{c}^{\alpha+1}(\mathrm{X}^{\hat{\xi}}_t, R^{\hat{\xi}}_t) f(\bar{\mathrm{c}}(\mathrm{X}^{\hat{\xi}}_t))dt \\ &\le \tilde{c}^{\alpha+1}_{max} \int_0^\infty f(\bar{\mathrm{c}}(\mathrm{X}^{\hat{\xi}}_t))dt \le \frac{\tilde{c}^{\alpha+1}_{max}}{\tilde{c}_{min}} \bar{v}(\mathrm{X}_0).\end{aligned}$$

Next, with the choice $\xi = \hat{\xi}$ the rightmost integral in Equation (5.42) vanishes, and we get equality in Equation (5.45). Since $\tau^{\hat{\xi}}_K = \infty$, this proves the martingale property of $w(\mathrm{X}^{\hat{\xi}}_t, R^{\hat{\xi}}_t)$. Furthermore, we obtain from Equation (5.39) that

$$u(R^{\hat{\xi}}_t) \ge w(\mathrm{X}^{\hat{\xi}}_t, R^{\hat{\xi}}_t) \ge u(R^{\hat{\xi}}_t)\exp(b\bar{v}(\mathrm{X}^{\hat{\xi}}_t)).$$

Since $\bar{v}(\mathrm{X}^{\hat{\xi}}_t)$ uniformly converges to zero as t tends to infinity, we obtain Equation (5.47). □

Proposition 5.20. *Consider the case of the asymptotic maximization of the portfolio value. We have $v_2 = w$ and the a.s. unique optimal strategy is given by $\hat{\xi}$ respectively* c.

Proof. By Lemma 5.19, we already have $w \le v_2$. We now show that $v_2 \le w$. Let ξ be any admissible strategy such that

$$\lim_{t\to\infty} \mathbb{E}[u(R^{\xi}_t)] > -\infty. \tag{5.49}$$

By Lemmas 5.18 and 5.16 we have for all k, t and $(\tau_k) := (\tau^{\xi}_k)$

$$w(\mathrm{X}_0, R_0) \ge \mathbb{E}[w(\mathrm{X}^{\xi}_{t\wedge\tau_k}, R^{\xi}_{t\wedge\tau_k})] \ge \mathbb{E}\left[u(R^{\xi}_{t\wedge\tau_k})\exp(b\bar{v}(\mathrm{X}^{\xi}_{t\wedge\tau_k}))\right].$$

As in the proof of Lemma 5.17 one shows that

$$\begin{aligned}\liminf_{k\to\infty} \mathbb{E}\left[u(R^{\xi}_{t\wedge\tau_k})\exp(b\bar{v}(\mathrm{X}^{\xi}_{t\wedge\tau_k}))\right] &\ge \liminf_{k\to\infty} \mathbb{E}\left[u(R^{\xi}_t)\exp(b\bar{v}(\mathrm{X}^{\xi}_{t\wedge\tau_k}))\right] \\ &= \mathbb{E}\left[u(R^{\xi}_t)\exp(b\bar{v}(\mathrm{X}^{\xi}_t))\right].\end{aligned}$$

Hence,

$$w(\mathrm{X}_0, R_0) \geq \mathbb{E}[u(R_t^\xi)] + \mathbb{E}\left[u(R_t^\xi)(\exp(b\bar{v}(\mathrm{X}_t^\xi)) - 1)\right].$$

Let us assume for a moment that the second expectation on the right attains values arbitrarily close to zero. Then

$$w(\mathrm{X}_0, R_0) \geq \lim_{t\to\infty} \mathbb{E}[u(R_t^\xi)].$$

Taking the supremum over all admissible strategies ξ, gives $w \geq v_2$. The optimality of $\hat{\xi}$, follows from Lemma 5.19, its uniqueness from the fact that the functional $\mathbb{E}[u(R_t^\xi)]$ is strictly concave since u is concave and increasing and R_t^ξ is concave.

We now show that $\mathbb{E}\left[u(R_t^\xi)(\exp(b\bar{v}(\mathrm{X}_t^\xi)) - 1)\right]$ attains values arbitrarily close to zero. First we observe that

$$0 \geq u(R) \geq a_5 u_{RR}(R)$$

for a constant $a_5 > 0$, due to the boundedness of the risk aversion of u, and that

$$\exp(b\bar{v}(\mathrm{X}_t^\xi)) - 1 \leq a_6 b\bar{v}(\mathrm{X}_t^\xi),$$

due to the bound on X_t^ξ. Since X_t^ξ is uniformly bounded, we see that for every $\epsilon_1 > 0$ there is a $\epsilon_2 > 0$ such that the following bound holds uniformly:

$$\bar{v}(\mathrm{X}_t^\xi) < \epsilon_1 + \epsilon_2 (\mathrm{X}_t^\xi)^\top \Sigma \mathrm{X}_t^\xi.$$

Combining the last three inequalities, we obtain

$$\begin{aligned} 0 &\geq \mathbb{E}\left[u(R_t^\xi)(\exp(b\bar{v}(\mathrm{X}_t^\xi)) - 1)\right] \\ &\geq b a_6 \epsilon_1 \mathbb{E}[u(R_t^\xi)] + b a_5 a_6 \epsilon_2 \mathbb{E}[(\mathrm{X}_t^\xi)^\top \Sigma \mathrm{X}_t^\xi u_{RR}(R_t^\xi)]. \end{aligned} \tag{5.50}$$

Let us now assume that the second expectation of Equation (5.50) attains values arbitrarily close to zero. Then for each $\epsilon_1 > 0$ there is a $\tilde{t} \in \mathbb{R}^+$ such that

$$0 \geq \mathbb{E}\left[u(R_{\tilde{t}}^\xi)(\exp(b\bar{v}(\mathrm{X}_{\tilde{t}}^\xi)) - 1)\right] \geq b a_6 \epsilon_1 \lim_{t\to\infty} \mathbb{E}[u(R_t^\xi)].$$

Sending ϵ_1 to zero yields that $\mathbb{E}\left[u(R_t^\xi)(\exp(b\bar{v}(\mathrm{X}_t^\xi)) - 1)\right]$ attains values arbitrarily close to zero, since $\lim_{t\to\infty} \mathbb{E}[u(R_t^\xi)]$ is bounded by assumption (see Equation (5.49)).

We finish the proof by showing that the second expectation of Equation (5.50) attains values arbitrarily close to zero. By Lemma 5.17 and the same line of reasoning as in the proof of Lemma 5.18, we have for all k, t and $(\tau_k) := (\tau_k^\xi)$

$$\begin{aligned} -\infty < \lim_{s\to\infty} \mathbb{E}[u(R_s^\xi)] &\leq \mathbb{E}[u(R_t^\xi)] \leq \mathbb{E}[u(R_{t\wedge\tau_k}^\xi)] \\ &= u(R_0) + \mathbb{E}\left[\int_0^{t\wedge\tau_k} u_R(R_s^\xi)(\mathrm{X}_s^\xi)^\top \sigma\, d\mathrm{B}_s\right] \\ &\quad - \mathbb{E}\left[\int_0^{t\wedge\tau_k} \left[u_R f(\xi_s) - \frac{1}{2}(\mathrm{X}_s^\xi)^\top \Sigma \mathrm{X}_s^\xi u_{RR}\right](R_s^\xi)\, ds\right] \\ &= u(R_0) - \mathbb{E}\left[\int_0^{t\wedge\tau_k} \left[u_R f(\xi_s) - \frac{1}{2}(\mathrm{X}_s^\xi)^\top \Sigma \mathrm{X}_s^\xi u_{RR}\right](R_s^\xi)\, ds\right]. \end{aligned} \tag{5.51}$$

Sending k and t to infinity yields

$$\int_0^\infty \mathbb{E}\left[(\mathrm{X}_s^\xi)^\top \Sigma \mathrm{X}_s^\xi u_{RR}(R_s^\xi)\right]\, ds > -\infty \tag{5.52}$$

which concludes the proof. □

Proposition 5.21. *Consider the case of optimal liquidation. Then $v_1 = w$ and the a.s. unique optimal strategy is given by $\hat{\xi}$, respectively* c.

Proof. For any strategy ξ that is admissible for optimal liquidation, the martingale

$$\int_0^t (\mathrm{X}_s)^\top \sigma d\mathrm{B}_s$$

is uniformly integrable due to the requirement in Equation (5.2). Therefore

$$\mathbb{E}[u(R_t^\xi)] \geq \mathbb{E}\left[u(R_\infty^\xi)\right]$$

follows as in the proof of Lemma 5.17. Hence, Proposition 5.20 yields

$$\mathbb{E}[u(R_\infty^\xi)] = \lim_{t\to\infty} \mathbb{E}[u(R_t^\xi)] \leq v_2(\mathrm{X}_0, R_0) \leq w(\mathrm{X}_0, R_0).$$

Taking the supremum over all admissible strategies ξ gives $v_1 \leq w$. The converse inequality follows from Lemma 5.18, since $\hat{\xi}$ is admissible for optimal liquidation. □

Chapter 6

Equivalence of local mean-variance and global risk measure optimization

6.1 Introduction

In the previous chapters, we determined optimal trading strategies for von-Neumann-Morgenstern investors. Professional traders however often do not think in terms of utility functions but rather in terms of expected proceeds and variance of proceeds. Risk managers on the other hand often think about principal trades in terms of their value-at-risk and calculate the corresponding risk cost as the cost of economic capital. We introduce two general approaches to measuring the subjective benefit of liquidations that either rely on the trader's terms of mean and variance of proceeds or on the risk manager's terms of risk (costs) of holding an asset position over time. Depending on the exact specifications, the resulting strategies posses many desirable properties. For example, endogenous liquidation time horizons are obtained, large trades are executed slower than small trades, and optimal liquidation strategies are time-consistent. Our main result is that the trader's and risk manager's approach introduced in this chapter are in fact identical, i.e., with suitable parameters, they result in the same optimal liquidation strategies.

The first approach that we consider is the trader's optimization of mean-variance functionals

$$\text{Benefit}_1 = \mathbb{E}[\text{Liquidation proceeds}] - f(\text{var}\,[\text{Liquidation proceeds}])$$

where $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous, increasing function. This includes mean-variance optimization for linear f and mean-standard-deviation optimization for $f(x) = a\sqrt{x}$. A trader maximizing this type of benefit will first determine her trading strategy at the current point in time and then start executing it. While in theory this strategy could be executed until the asset position is liquidated completely, in practice the liquidation specifications are changed continuously, e.g., by the client who reduces the asset position to be sold, or by the trader who manually deviates from the liquidation time plan to exploit an unexpected internal crossing opportunity. Any such unanticipated change in the liquidation specifications requires a reoptimization of the liquidation strategy for the remaining portfolio. This however can lead to a significant change in the liquidation strategy, since the optimal strategies for nonlinear f are not time-consistent. In practice, such reoptimization is necessary frequently; in this chapter, we consider the limit of *continuous* reoptimization. This limit is interesting from a practical point of view since it ensures that a small change of liquidation specifications by the client during the execution does not result in a significant change in strategy due to unexpected reoptimization. Without reoptimization, the function f only influences which mean-variance efficient strategy is chosen. With continuous reoptimization, f determines the liquidation strategy to a greater extent.

Our second approach is to consider the risk or risk cost $g(X)$ of holding an asset position X and to maximize the difference of expected proceeds and total risk (cost):

$$\text{Benefit}_2 = \mathbb{E}[\text{Liquidation proceeds}] - \int_0^\infty g(\text{Asset position at time } t)dt$$

where $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous, increasing function. This approach again includes mean-variance optimization if g is a quadratic function. The interpretation of g as risk cost however can also imply different shapes of g. For example, risk cost is often measured as the cost of economic capital, which can be calculated as

$$\text{cost of capital} \times \text{standard deviation } \sigma \text{ of asset price} \times \text{multiplier} \times \text{asset position } X.$$

This implies a function g that is linear in the asset position X. Depending on the risk measurement framework and the other asset positions of the bank, several other functions g can be appropriate.

Our main result is the equivalence of the two approaches above. More precisely, there is a bijection between functions[1] f and g such that the strategy maximizing the mean-variance functional given by f with continuous reoptimization also maximizes the benefit under the risk framework with a risk function g. For example, if f is linear (mean-variance optimization), g is a quadratic function. If f is a power law, then g is also a power law; in particular, if f is proportional to the square root function, then g is proportional to $x^{2/3}$, and if f is proportional to $x^{3/5}$, then g is linear.

The equivalence of the two approaches is important for practical applications. The "local mean-variance optimization" of the first approach is intuitive for traders, however it is economically not clear why the "shortsighted" approach of continuous reoptimization is sensible. Such an economic justification however is provided by the corresponding risk function, which results in the same strategy but does not require reoptimization since it is a priori time-consistent.

Much of the existing literature focuses on mean-variance optimization (see, e.g., Almgren and Chriss (2001), Almgren (2003), Huberman and Stanzl (2005), Almgren and Lorenz (2007)). In spite of the theoretical and numerical benefits of this approach, it has several practical shortcomings. Most notably, the liquidation of large portfolios in practice requires a longer time and smaller relative selling speed than the liquidation of small portfolios. Mean-variance optimal liquidation however results in the same[2] relative selling speed and time horizon for small and large positions. As a remedy, Konishi and Makimoto (2001) suggest mean-standard-deviation optimization (f proportional to the square root function), although the resulting strategies are not time-consistent. This approach was picked up both by academics and practitioners (see, e.g., Dubil (2002), Mönch (2004) and Kissell and Malamut (2005)); however, no mathematical approach to achieve time-consistency was suggested so far.

The impact of liquidity on risk measures has been acknowledged early on (see, e.g., Bangia, Diebold, Schuermann, and Stroughair (1998), Berkowitz (2000) and Hisata and Yamai

[1]The functions f and g need to fulfill certain conditions; see Theorem 6.13 for details.

[2]Here, we assumed linear temporary impact.

(2000)). The inverse influence of risk management on optimal trading strategies has received much less attention[3]. Roux (2007) analyzes a certain interpretation of mean-variance optimization with relative risk aversion and finds that the resulting optimal strategies are also optimal for the linear risk measure suggested by Konishi and Makimoto (2001).

In the next section, we discuss some assumptions and notation. In Section 6.3, we introduce the trader's approach of maximizing mean-variance functionals. Subsequently, we analyze the effect of continuous reoptimization in Section 6.4. Several properties of the resulting optimal strategies are derived, including an explicit solution for power law functions f. In Section 6.5, we introduce the risk manager's approach of minimizing execution costs including risk (costs), and show in Section 6.6 that it is equivalent to the optimization of mean-variance functionals. This equivalence holds in particular within the class of power law functions, and we obtain closed form expressions of the power law risk function g corresponding to a power law f. All proofs are given in Section 6.7.

6.2 Assumptions and notation

We apply the single-asset market model introduced in Section 4.2. Some of the results in this Chapter hold in the more general case of nonlinear price impacts; for notational simplicity, we limit the discussion to the linear impact case. We parameterize strategies with $\xi(t) := -\dot{X}(t)$ such that $X_t = X_0 - \int_0^t \xi_s\, ds$. In this chapter, we limit the discussion to deterministic functions $\xi : \mathbb{R}_0^+ \to \mathbb{R}$. We only require the trading strategies X_t to be absolutely continuous such that ξ_t exists; in particular, we do not require that X_t is C^1.

Instead of specifying our model in terms of the expected cash proceeds of the liquidation, we will equivalently use the cost of liquidation which is the difference between the book value $X_0\tilde{P}_0$ and the actual proceeds. The expected cost of liquidation up to time t with strategy X is then given by

$$C_X(t) := \mathbb{E}\left[\int_0^t (\tilde{P}_0 - P_s)\xi_s ds\right] = \frac{1}{2}\gamma(X_0 - X_t)^2 + \lambda \int_0^t \xi_s^2 ds$$

and the variance is given by

$$V_X(t) := \mathrm{var}\left[\int_0^t (\tilde{P}_0 - P_s)\xi_s ds\right] = \sigma^2 \int_0^t X_s^2 ds.$$

Analogously to Chapters 4 and 5, we do not impose any exogenous time horizon for the liquidation. Instead, we only require that $\lim_{t\to\infty} X_t = 0$. The infinite time horizon mean and variance of the liquidation cost are then given by

$$C_X := \lim_{t\to\infty} C_X(t) = \frac{1}{2}\gamma X_0^2 + \lambda \int_0^\infty \xi_s^2 ds \in \mathbb{R}_0^+ \cup \{\infty\}$$
$$V_X := \lim_{t\to\infty} V_X(t) = \sigma^2 \int_0^\infty X_s^2 ds \in \mathbb{R}_0^+ \cup \{\infty\}.$$

[3] A notable exception is Duffie and Ziegler (2003). Konishi and Makimoto (2001) also briefly discuss the use of linear g, however do not give the interpretation as a measure of risk (cost).

Since the first term of the expected liquidation cost does not depend on the liquidation strategy, we can set $\gamma = 0$ without loss of generality and have:

$$C_X = \lambda \int_0^\infty \xi_s^2 ds.$$

6.3 Optimization of mean-variance functionals

Let us first consider the minimization of mean-variance functionals of the form

$$C_X + f(V_X) = \lambda \int_0^\infty \xi_s^2 ds + f\left(\sigma^2 \int_0^\infty X_s^2 ds\right) \tag{6.1}$$

for an arbitrary continuous, increasing and unbounded function $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ that is smooth (C^3) on $\mathbb{R}^+$ and satisfies $f(0) = 0$ and $f(V) > 0$ for all $V > 0$. Furthermore, we require that f is bounded from above by a power law, i.e., that there are $a, b, c > 0$ such that $f(V) < aV^b + c$ for all $V \in \mathbb{R}^+$. The purpose of this restriction is that for large asset positions, the aversion to variance does not impose an upper bound on the accepted variance (see Lemma 6.3). Our approach includes mean-variance optimization ($f(V) = aV$) as well as mean-standard-deviation optimization ($f(V) = a\sqrt{V}$).

We call a trading strategy optimal if it minimizes the mean-variance functional (6.1). Since only the mean and variance of the liquidation costs are considered in our objective functional, the optimal strategy must be mean-variance efficient, i.e., there is no other strategy that realizes the same expected cost but a lower variance or the same variance but a lower expected cost. The efficient frontier consisting of the optimal points in the expected-cost-variance-plane was introduced by Almgren and Chriss (1999) and was computed in continuous time for the infinite time horizon case in Almgren (2003).

Proposition 6.1 (Almgren (2003)). *The set of mean-variance efficient strategies consists of the strategies*

$$X_t^{(\delta)} = X_0 \exp\left(-\sqrt{\frac{\delta}{\lambda}}\sigma t\right)$$
$$\xi_t^{(\delta)} = \sqrt{\frac{\delta}{\lambda}}\sigma X_t$$

with parameter $\delta > 0$. They trace out the efficient frontier in the following way:

$$C_\delta := C_{X^{(\delta)}} = \frac{1}{2}\sqrt{\delta\lambda}\sigma X_0^2 \tag{6.2}$$
$$V_\delta := V_{X^{(\delta)}} = \frac{1}{2}\sqrt{\frac{\lambda}{\delta}}\sigma X_0^2. \tag{6.3}$$

Given a mean-variance functional $C_X + f(V_X)$, we can now determine the optimal value of δ.

Proposition 6.2. *There exists a parameter $\delta \in \mathbb{R}^+$ such that the corresponding mean-variance efficient strategy $X^{(\delta)}$ maximizes the mean-variance functional $C_X + f(V_X)$. This parameter δ fulfills*

$$\delta = f'\left(\frac{1}{2}\sqrt{\frac{\lambda}{\delta}}\sigma X_0^2\right). \tag{6.4}$$

If f is a power law, i.e., $f(V) = aV^b$, then

$$\delta = \left(ab\left(\frac{\sqrt{\lambda}\sigma X_0^2}{2}\right)^{b-1}\right)^{\frac{2}{b+1}} \tag{6.5}$$

$$\xi_0 = \left(\frac{ab}{\lambda}\frac{\sigma^{2b}}{2^{b-1}}\right)^{\frac{1}{b+1}} X_0^{\frac{3b-1}{b+1}}. \tag{6.6}$$

So far, the only difference between mean-variance optimization and optimization with respect to mean-variance functionals $C_X + f(V_X)$ is that the latter implies a non-constant Lagrangian multiplier. This changes when the strategy is continuously reoptimized, as we will see in Section 6.4.

For each $X_0 \in \mathbb{R}^+$, we select an optimal δ_{X_0} and denote the corresponding mean and variance of costs by $C_{X_0} = C_{X^{(\delta_{X_0})}}$ respectively $V_{X_0} = V_{X^{(\delta_{X_0})}}$. The following Lemmas 6.3, 6.4 and 6.5 provide simple properties of the optimal strategies for mean-variance functionals and provide conditions under which these optimal strategies are "well-behaved".

Lemma 6.3. *Let $Y_0 > X_0 > 0$. Then we have*

$$V_{Y_0} \geq V_{X_0}.$$

The variance V_{X_0} attains arbitrarily large and small values. More precisely,

$$\lim_{X_0 \to 0} V_{X_0} = 0 \qquad\qquad \lim_{X_0 \to \infty} V_{X_0} = \infty.$$

Furthermore, there are $a, c > 0$ and $0 < b < 4$ such that $\delta_{X_0} < aX_0^b + c$ for all $X_0 \geq 1$.

The next lemma clarifies under which conditions δ_{X_0} depends smoothly on X_0.

Lemma 6.4. *The optimal δ_{X_0} given by Equation* (6.4) *is unique and a twice continuously differentiable (C^2) function of X_0 on $\mathbb{R}^+$ if and only if $f''(V)V > -2f'(V)$ for all $V > 0$. In this case, we furthermore have that $\frac{d}{dX_0}V_{X_0} > 0$ for all $X_0 > 0$.*

Note that the condition $f''(V)V > -2f'(V)$ of the proposition is fulfilled for example by all power laws $f(V) = aV^b$ for $a, b > 0$.

In the following, we will always assume that δ is unique and a C^2 function[4] of X_0. Then also V_{X_0} and C_{X_0} are C^2 by Equations (6.2) and (6.3).

For "sensible" f, larger asset positions X_0 should lead to more intensive trading, i.e., to larger ξ_0. The next lemma describes for which f this is the case.

[4] The proof of Lemma 6.4 establishes that it suffices to assume that δ is unique and differentiable, since then it is necessarily C^2.

Lemma 6.5. *The initial selling speed*

$$\xi_0^{(X_0)} := -\frac{d}{dt}\bigg|_{t=0} X_t^{(\delta_{X_0})} = \sqrt{\frac{\delta_{X_0}}{\lambda}}\sigma X_0$$

is increasing in X_0 if and only if $f''(V)V \geq -\frac{2}{3}f'(V)$ for all $V > 0$.

The condition $f''(V)V \geq -\frac{2}{3}f'(V)$ is satisfied, e.g., by all power laws $f(V) = aV^b$ with exponent[5] $b > \frac{1}{3}$.

In practice, large asset positions are liquidated at a larger *absolute* rate, but a smaller *relative* rate than small asset positions, i.e., X_t/X_0 is large for large X_0 and small for small X_0. The following proposition shows that this property only holds when f is strictly concave; in particular, it does not hold for mean-variance optimization (linear f). This indicates that traders' attitudes towards variance are better captured by concave functions f, such as the square root function $f(V) = a\sqrt{V}$ in the mean-standard-deviation optimization.

Proposition 6.6. *Under optimal liquidation with respect to the mean-variance functional $C + f(V)$, the fraction X_t/X_0 (strictly) increases in X_0 (i.e., large asset positions are liquidated at a smaller relative rate) if and only if f is (strictly) concave.*

Power laws $f(V) = aV^b$ with $\frac{1}{3} < b < 1$ thus result in the desired scaling behavior: large positions are liquidated at a higher absolute but at a lower relative rate than small positions.

Let us assume that at time 0, we computed the optimal liquidation strategy for the mean-variance functional. At time t, we might have to recompute the optimal strategy for selling the remaining asset position X_t, for example due to a change in liquidation specifications. As explained in Section 6.1, such recomputations are frequently necessary in practice, and it is important that this newly computed strategy is similar to the originally planned strategy. In particular, when the liquidation specifications are not changed at all, the reoptimization should not result in a change of strategy, i.e., the strategy should be time-consistent. The following proposition states that the only mean-variance functional having this property is mean-variance itself.

Proposition 6.7. *If the optimal strategies for the mean-variance functional $C_X + f(V_X)$ are time-consistent, then f is a linear function, i.e., the mean-variance functional reduces to mean-variance optimization.*

6.4 Mean-variance functionals with continuous reoptimization

One way of imposing time-consistency on the trading trajectory is to compute the optimal liquidation strategy at time t given the asset position X_t, execute it for a small period of time Δt and then to recompute the optimal strategy given the asset position $X_{t+\Delta t}$. Sending the time interval Δt to zero results in *continuous reoptimization of mean-variance functionals*

[5] The case $b = \frac{1}{3}$ satisfies the condition, but corresponds to the degenerate case of a constant $\xi_0^{(X_0)}$.

and an execution strategy given by

$$\xi_t = \sqrt{\frac{\delta_t}{\lambda}}\sigma X_t \tag{6.7}$$

$$\delta_t = f'\left(\frac{1}{2}\sqrt{\frac{\lambda}{\delta_t}}\sigma X_t^2\right). \tag{6.8}$$

These equations constitute a first-order ODE for the strategy X. Note that δ_t depends only on X_t; we can therefore consider δ as a function δ_t of time t or as a function δ_X of the remaining asset position X. By our assumption that δ_{X_0} depends smoothly on X_0, we see that this ODE has a unique solution for each start portfolio X_0. The resulting strategy is by definition time-consistent, while maintaining the scaling properties established in Lemma 6.5 and Proposition 6.6. In particular, $\frac{\xi_t}{X_t}$ is decreasing in X_t if and only if f is concave. For power laws $f(V) = aV^b$, we can explicitly solve the above Equations (6.7) and (6.8).

Proposition 6.8. *If f is a power law, i.e., $f(V) = aV^b$, then the strategy given by continuous reoptimization of the mean-variance functional $C_X + f(V_X)$ fulfills the equation*

$$\xi_t = \left(\frac{ab}{\lambda}\frac{\sigma^{2b}}{2^{b-1}}\right)^{\frac{1}{b+1}} X_t^{\frac{3b-1}{b+1}} \text{ for all } t \text{ with } X_t > 0 \tag{6.9}$$

which is solved by

$$\begin{aligned} &\text{For } b<1: \quad X(t) = A(B-t)^C \text{ for } 0 \le t \le B, \qquad X(t) = 0 \text{ for } t > B \qquad (6.10)\\ &\text{For } b=1: \quad X(t) = e^{-\sqrt{\frac{a}{\lambda}}\sigma t} X_0 \text{ for } t \ge 0\\ &\text{For } b>1: \quad X(t) = A(B+t)^C \text{ for } t \ge 0 \end{aligned}$$

with parameters

$$\begin{aligned} A &:= \left(\left|\frac{2b-2}{b+1}\right|\left(\frac{ab}{\lambda}\frac{\sigma^{2b}}{2^{b-1}}\right)^{\frac{1}{b+1}}\right)^{-\frac{b+1}{2b-2}} > 0\\ B &:= \left(\frac{X_0}{A}\right)^{\frac{1}{C}} > 0\\ C &:= -\frac{b+1}{2b-2}. \end{aligned} \tag{6.11}$$

Figures 6.1 and 6.2 illustrate the continuously reoptimized strategy for power laws $f(V) = aV^b$ with different values of the exponent b. The faster relative selling of smaller asset positions for $b < 1$ is illustrated in Figure 6.3.

The strategies for power laws f with exponent $\frac{1}{3} < b < 1$ have many desirable properties, e.g., a realistic scaling behavior, time-consistency, an endogenous liquidation time horizon and an intuitive explanation in terms of mean and variance. All of these properties are important for practical applications. On the other hand, it is hard to find an economically sound justification for the approach of continuous reoptimization. If reoptimization is foreseen now already, it should be taken into account in the current optimization. A priori,

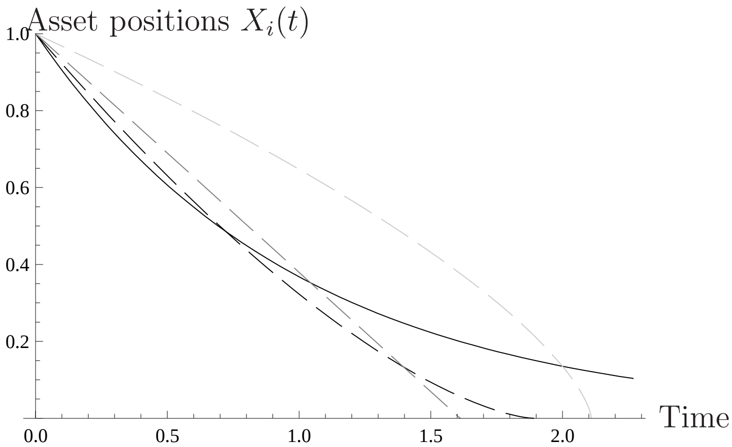


Figure 6.1: Asset position under continuously reoptimized selling strategy. The continuous line corresponds to $b = 1$, the dashed lines to $b = \frac{1}{2}$, $b = \frac{1}{3}$ and $b = 0.15$ in descending order of darkness. $X_0 = \lambda = \sigma = a = 1$.

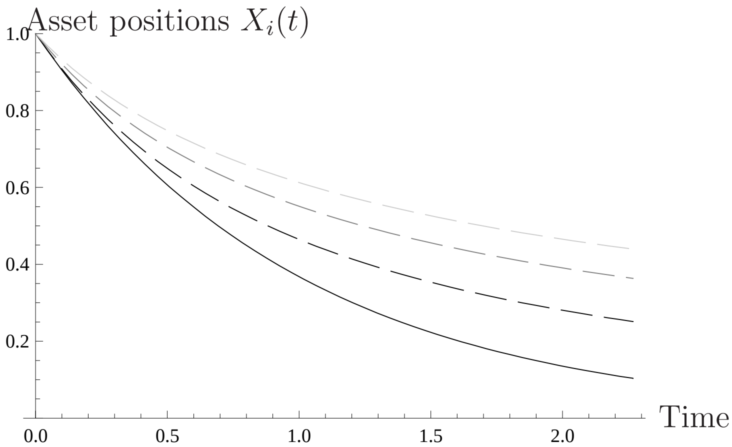


Figure 6.2: Asset position under continuously reoptimized selling strategy. The continuous line corresponds to $b = 1$, the dashed lines to $b = 2$, $b = 4$ and $b = 8$ in descending order of darkness. $X_0 = \lambda = \sigma = a = 1$.

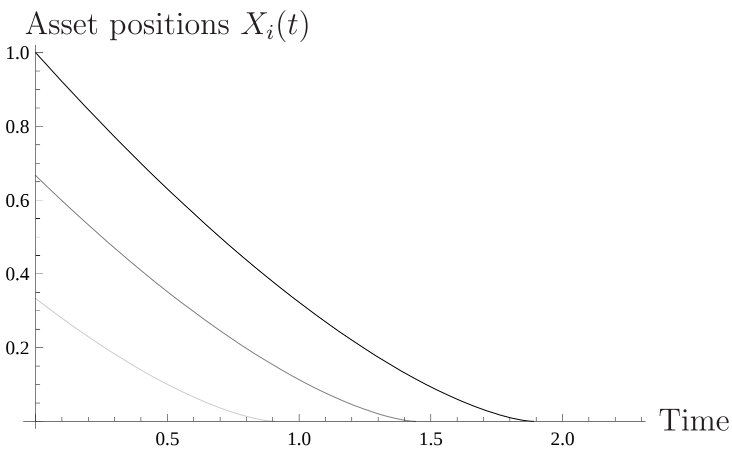


Figure 6.3: Asset position under continuously reoptimized selling strategy. The black line corresponds to $X_0 = 1$, the dark grey line to $X_0 = \frac{2}{3}$ and the light grey line to $X_0 = \frac{1}{3}$. $b = \frac{1}{2}$, $\lambda = \sigma = a = 1$.

it is not clear whether the strategies obtained by continuous reoptimization only exhibit desirable properties, but are otherwise "shooting at a moving target and constantly missing it". We will resolve this issue in Theorem 6.13, where we show that the strategies obtained by continuous reoptimization minimize an economically fundamental cost-risk functional.

In Figures 6.1 and 6.2 we observe that a zero asset position is either never attained ($b \geq 1$) or it is attained with a zero selling speed ($\frac{1}{3} < b < 1$), with finite selling speed larger than zero ($b = \frac{1}{3}$) or with an infinite selling speed ($0 < b < \frac{1}{3}$). The next proposition shows that the critical role of the power exponent $\frac{1}{3}$ extends to general functions f.

Proposition 6.9. *If there are $a', \epsilon > 0$ and $b' > -\frac{2}{3}$ such that $f'(V) < a'V^{b'}$ for all $0 < V < \epsilon$, then either X never attains 0 or it attains 0 at time*

$$T_0 := \inf\{t > 0 : X_t = 0\}$$

with selling speed $\xi_{T_0} = 0$. If there are $a', \epsilon > 0$ and $b' < -\frac{2}{3}$ such that $f'(V) > a'V^{b'}$ for all $0 < V < \epsilon$, then X attains 0 after a finite time T_0, but with infinite selling speed $\lim_{t \to T_0} \xi_t = \infty$.

The proposition is a direct consequence of the following auxiliary lemma and Equation (6.7).

Lemma 6.10. *Assume that there are $a', b', \epsilon > 0$ such that $f'(V) \gtrless a'V^{b'}$ for all $0 < V < \epsilon$. Then there is a $\tilde{\epsilon} > 0$ such that*

$$\delta_{X_0} \gtrless \left(a' \left(\frac{1}{2} \sqrt{\lambda} \sigma X_0^2 \right)^{b'} \right)^{\frac{2}{b'+2}}$$

for all $0 < X_0 < \tilde{\epsilon}$.

6.5 Optimization of risk measures

An alternative approach to using mean-variance functionals is using general measures of risk. In this chapter, we consider a risk function to be a continuous and increasing function[6] $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ that is smooth (C^2) on $\mathbb{R}^+$ and fulfills $g(0) = 0$. We think of $g(x)$ as the subjective risk or risk cost associated with holding the asset position x. We consider strategies minimizing the expected liquidation costs and the total risk (costs) of liquidating an asset position:

$$C_X + \underbrace{\int_0^\infty g(X_s) ds}_{:=R_X}.$$

This includes mean-variance optimization ($g(X) = aX^2$) as well as the risk measure suggested by Konishi and Makimoto (2001) ($g(X) = aX$). As discussed in Section 6.1, several different shapes of g can be appropriate in practice. We first establish the existence of an optimal strategy.

[6] In principle, we could consider risk functions $g : \mathbb{R} \to \mathbb{R}$. It is clear that the optimal strategy never includes shortselling; we therefore restrict our modeling framework to nonnegative asset positions. For the dual buying problem, functions $g : \mathbb{R}_0^- \to \mathbb{R}_0^+$ can be considered.

Lemma 6.11. *For each risk function g, there exists at least one optimal strategy X_t, i.e., a minimizer of $C_X + R_X$. All such optimal strategies are classical C^2-solutions of the Euler-Lagrange equation*

$$\ddot{X}_t = \frac{g'(X_t)}{2\lambda} \tag{6.12}$$

until they attain 0[7].

We now turn to the uniqueness of the minimizer of $C_X + R_X$.

Proposition 6.12. *Let g be locally bounded from above by a power law at 0, i.e., there are $c, d, \epsilon > 0$ such that $g(x) \leq cx^d$ for all $0 \leq x \leq \epsilon$. Then there exists a unique minimizer X of $C_X + R_X$. This minimizer is the only strategy fulfilling either of the following two conditions:*

1. *X is a classical C^2-solution of the Euler-Lagrange Equation (6.12) on all of $\mathbb{R}_0^+$, and X approaches 0, but never attains the value 0, i.e., $\lim_{t\to\infty} X_t = 0$ and for all $t \geq 0$ we have $X_t > 0$.*

2. *X attains zero at the finite time $T_0 := \inf\{t \geq 0 : X_t = 0\} \in \mathbb{R}_0^+$, is a classical C^2-solution of the Euler-Lagrange Equation (6.12) on $t \in [0, T_0[$, is zero on $[T_0, \infty[$ and is differentiable at T_0 with $\dot{X}_{T_0} = -\xi_{T_0} = 0$.*

In the second case of the proposition, X is not necessarily twice differentiable at time T_0.

6.6 Equivalence of mean-variance functionals and risk measures

Let the set $\mathcal{F}$ consist of all functions $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ that satisfy all of the following conditions:

- f is continuous and increasing without bound on $\mathbb{R}_0^+$ and is C^3 on $\mathbb{R}^+$
- $f(0) = 0$ and $f(V) > 0$ for all $V > 0$
- $f''(V)V \geq -\frac{2}{3}f'(V)$ for all $V > 0$
- Global bound: there are $a, b, c > 0$ such that $f(V) < aV^b + c$ for all $V > 0$
- Local bound: there are $a', \epsilon > 0$ and $b' > -\frac{2}{3}$ such that $f'(V) < a'V^{b'}$ for all $0 < V < \epsilon$

Similarly, let the set $\mathcal{G}$ consist of all functions $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ that satisfy all of the following conditions:

- g is continuous and increasing on $\mathbb{R}_0^+$ and is C^2 on $\mathbb{R}^+$
- $g(0) = 0$
- $g'(X)X < 6g(X)$ for all $X > 0$

[7]They may well never attain the value 0, but only converge to it.

- Global bound: there are $a, c > 0$ and $0 < b < 6$ such that $g(X) < aX^b + c$ for all $X > 0$
- Local bound: there are $a', \epsilon > 0$ and $0 < b' < 6$ such that $g(X) < a'X^{b'}$ for all $0 < X < \epsilon$

The sets $\mathcal{F}$ and $\mathcal{G}$ contain a wide range of functions. Among many others, the set $\mathcal{F}$ contains all power laws with exponent larger than $\frac{1}{3}$, while $\mathcal{G}$ contains all power laws with exponent between 0 and 6.

The following theorem is the main result of this chapter and establishes a bijection between mean-variance functionals and risk measures.

Theorem 6.13. *There is a unique bijection between functions $f \in \mathcal{F}$ and functions $g \in \mathcal{G}$ such that the mean-variance functional $C_X + f(V_X)$ for f with continuous reoptimization and the cost-risk functional $C_X + R_X$ for the corresponding g give the same optimal selling strategy.*

We have thus established the critical link between the two approaches introduced in this chapter. This link reveals a sound economic meaning of continuous reoptimization of mean-variance functionals, and it provides an intuitive interpretation of the minimization of cost-risk functionals in terms of mean and variance.

In Theorem 6.13, the condition $f \in \mathcal{F}$ respectively $g \in \mathcal{G}$ is important. The following proposition shows that if f strictly violates the local boundedness condition, then there is no cost-functional $C_X + R_X$ with the same optimal trading strategies as the mean-variance functional $C_X + f(V_X)$ with continuous reoptimization. Similar propositions hold for the other conditions in the definition of $\mathcal{F}$ and $\mathcal{G}$.

Proposition 6.14. *Consider the strategy X that minimizes the mean-variance functional $C_X + f(V_X)$ with continuous reoptimization. If there are $a', \epsilon > 0$ and $b' < -\frac{2}{3}$ such that $f'(V) > a'V^{b'}$ for all $0 < V < \epsilon$, then the strategy X is not optimal with respect to any risk function g.*

For the special case of power laws f, the next theorem establishes that the corresponding risk function g is again a power law.

Theorem 6.15. *The bijection of Theorem 6.13 maps power laws to power laws. More precisely, let $f(V) = aV^b$ be a power law. The strategy minimizing the mean-variance functional $C_X + f(V_X)$ with continuous reoptimization also minimizes $C_X + R_X$ for a risk function g if and only if $b > \frac{1}{3}$. The risk function g is then given by*

$$g(X) = \left(\frac{a^2 b^2 \sigma^{4b} \lambda^{b-1}}{2^{2b-2}} \right)^{\frac{1}{b+1}} X^{\frac{6b-2}{b+1}}. \tag{6.13}$$

In particular, we obtain the following equivalences:

Optimization of mean-variance functional $C_X + f(V_X)$ with continuous reoptimization	*Optimization of $C_X + R_X$ with respect to risk function g*
$f(V) \sim V$ *(mean-variance)*	$g(X) \sim X^2$
$f(V) \sim V^{1/2}$ *(mean-standard-deviation)*	$g(X) \sim X^{2/3}$
$f(V) \sim V^{3/5}$	$g(X) \sim X$ *(Konishi and Makimoto (2001))*

Note that the power laws that are in $\mathcal{F}$ are exactly those with an exponent larger than $\frac{1}{3}$. Furthermore, all power laws in $\mathcal{G}$ can be written in the form of Equation (6.13).

6.7 Proofs

Proof of Proposition 6.2: Since $C_\delta + f(V_\delta)$ depends smoothly on δ and

$$\lim_{\delta\to 0}(C_\delta + f(V_\delta)) = \infty$$
$$\lim_{\delta\to\infty}(C_\delta + f(V_\delta)) = \infty,$$

there must be a global minimizer δ with $0 < \delta < \infty$. It therefore must satisfy

$$\begin{aligned} 0 &= \frac{d}{d\delta}\left(C_\delta + f(V_\delta)\right) \\ &= \frac{1}{4}\sqrt{\frac{\lambda}{\delta}}\sigma X_0^2 - \frac{1}{4}\sqrt{\frac{\lambda}{\delta^3}}\sigma X_0^2 f'\left(\frac{1}{2}\sqrt{\frac{\lambda}{\delta}}\sigma X_0^2\right), \end{aligned}$$

and Equation (6.4) follows. If f is a power law, then δ as given in Equation (6.5) is the only solution to Equation (6.4) and thus optimal. □

Proof of Lemma 6.3: We first show that $V_{Y_0} \geq V_{X_0}$. By Equations (6.2) and (6.3), we have that for any $\delta > 0$

$$C_{X^\delta} = \frac{K_{X_0}}{V_{X^\delta}} \text{ with } K_{X_0} := \frac{1}{4}\lambda\sigma^2 X_0^4.$$

By the optimality of δ_{X_0} and δ_{Y_0}, we have

$$\frac{K_{X_0}}{V_{X_0}} + f(V_{X_0}) \leq \frac{K_{X_0}}{V_{Y_0}} + f(V_{Y_0}) \tag{6.14}$$

$$\frac{K_{Y_0}}{V_{X_0}} + f(V_{X_0}) \geq \frac{K_{Y_0}}{V_{Y_0}} + f(V_{Y_0}) \tag{6.15}$$

and hence

$$K_{X_0}\left(\frac{1}{V_{X_0}} - \frac{1}{V_{Y_0}}\right) \leq f(V_{Y_0}) - f(V_{X_0}) \leq K_{Y_0}\left(\frac{1}{V_{X_0}} - \frac{1}{V_{Y_0}}\right).$$

This establishes that for $Y_0 > X_0 > 0$ we must have $V_{Y_0} \geq V_{X_0}$, since $K_{Y_0} > K_{X_0}$.

It is clear that when X_0 tends to zero, then the mean-variance functional $C_{X_0} + f(V_{X_0})$ tends to zero; hence V_{X_0} also tends to zero.

Since f is bounded from above by a power law by assumption, we can find $\tilde{a}, \tilde{b}, \tilde{c} > 0$ such that $f(V) \leq \tilde{f}(V) := \tilde{a}V^{\tilde{b}} + \tilde{c}$ for all $V \in \mathbb{R}_0^+$. Then

$$C_{X_0} + f(V_{X_0}) \leq \inf_{\tilde{X} \text{ s.t. } \tilde{X}_0 = X_0}\left(C_{\tilde{X}} + \tilde{f}(V_{\tilde{X}})\right) \leq a' X_0^{b'} + c'$$

with $a', c' > 0$, and $0 < b' < 4$ because of Equations (6.4), (6.2) and (6.3). By Equation (6.2), $\sqrt{\delta_{X_0}}$ therefore is bounded from above for $X_0 \geq 1$ by a function $\hat{a}X_0^{\hat{b}} + \hat{c}$ with $\hat{a}, \hat{c} > 0$ and $0 < \hat{b} < 2$. This establishes the desired upper bound on δ_{X_0} and thus also the divergence of $\lim_{X_0\to\infty} V_{X_0}$ by Equation (6.3). □

Proof of Lemma 6.4: Let us first assume that $f''(V)V > -2f'(V)$. Consider $C_\delta + f(V_\delta)$ as a function of $\sqrt{\delta}$. The solution to Equation (6.4) is also the solution to the equation

$$\begin{aligned} 0 &= \frac{d}{d\sqrt{\delta}}\left(C_\delta + f(V_\delta)\right) \\ &= \frac{1}{2}\sqrt{\lambda}\sigma X_0^2 - \frac{1}{2\delta}\sqrt{\lambda}\sigma X_0^2 f'(V_\delta). \end{aligned}$$

The solution δ to this equation is unique and differentiable in X_0 if

$$\begin{aligned} 0 &< \frac{d^2}{d\sqrt{\delta}^2}\left(C_\delta + f(V_\delta)\right) \\ &= \frac{\sqrt{\lambda}}{\sqrt{\delta^3}}\sigma X_0^2 f'(V_\delta) + \frac{1}{4\delta^2}\lambda\sigma^2 X_0^4 f''(V_\delta). \end{aligned}$$

This is equivalent to our assumption. Furthermore, the smoothness of $f' \in C^2(\mathbb{R}^+)$ carries over to δ_{X_0}.

Now assume that δ is unique and differentiable in X_0. Then V_{X_0} is differentiable with

$$\begin{aligned} \frac{d}{dX_0}V_{X_0} &= -\left(\frac{d}{dX_0}\delta_{X_0}\right)\frac{1}{4}\sqrt{\frac{\lambda}{\delta_{X_0}^3}}\sigma X_0^2 + \sqrt{\frac{\lambda}{\delta_{X_0}}}\sigma X_0 \\ &= -\left(f''(V_{X_0})\frac{d}{dX_0}V_{X_0}\right)\frac{1}{4}\sqrt{\frac{\lambda}{\delta_{X_0}^3}}\sigma X_0^2 + \sqrt{\frac{\lambda}{\delta_{X_0}}}\sigma X_0 \end{aligned}$$

and thus

$$\frac{d}{dX_0}V_{X_0}\left(1 + f''(V_{X_0})\frac{1}{4}\sqrt{\frac{\lambda}{\delta_{X_0}^3}}\sigma X_0^2\right) = \sqrt{\frac{\lambda}{\delta_{X_0}}}\sigma X_0 > 0. \tag{6.16}$$

By Lemma 6.3 we know $\frac{d}{dX_0}V_{X_0} \geq 0$ and therefore

$$1 + f''(V_{X_0})\frac{1}{4}\sqrt{\frac{\lambda}{\delta_{X_0}^3}}\sigma X_0^2 > 0.$$

This however is equivalent to the first inequality given in the statement of the proposition for the value $V = V_{X_0}$. Since V_{X_0} attains all values in $\mathbb{R}^+$ by Lemma 6.3 and the assumed continuity of δ_{X_0}, we obtain $f''(V)V + 2f'(V) > 0$ for all $V > 0$.

The inequality $\frac{d}{dX_0}V_{X_0} > 0$ follows since Lemma 6.3 implies $\frac{d}{dX_0}V_{X_0} \geq 0$ and Equation (6.16) contradicts $\frac{d}{dX_0}V_{X_0} = 0$. $\square$

In the following we will use the shorthand notation

$$\delta'_{X_0} := \frac{d}{dX_0}\delta_{X_0} \qquad V'_{X_0} := \frac{d}{dX_0}V_{X_0} \qquad C'_{X_0} := \frac{d}{dX_0}C_{X_0}.$$

Proof of Lemma 6.5: By Equation (6.4) we have

$$\delta'_{X_0} = f''(V_{X_0})V'_{X_0} = f''(V_{X_0})\left(V_{X_0}\frac{2}{X_0} - \frac{1}{2\delta_{X_0}}\delta'_{X_0}V_{X_0}\right)$$

and thus

$$\delta'_{X_0} = \frac{f''(V_{X_0})V_{X_0}\frac{2}{X_0}}{1 + f''(V_{X_0})\frac{V_{X_0}}{2\delta_{X_0}}} = \frac{2}{X_0}\frac{2f'(V_{X_0})f''(V_{X_0})V_{X_0}}{2f'(V_{X_0}) + f''(V_{X_0})V_{X_0}}.$$

For $\xi_0^{(X_0)}$, we obtain

$$\frac{d}{dX_0}\xi_0^{(X_0)} = \sqrt{\frac{\delta_{X_0}}{\lambda}}\sigma + \frac{1}{2}\sqrt{\frac{1}{\lambda\delta_{X_0}}}\sigma X_0\delta'_{X_0}.$$

Therefore $\frac{d}{dX_0}\xi_0^{(X_0)} \geq 0$ if and only if

$$0 \leq \delta_{X_0} + \frac{1}{2}\delta'_{X_0}X_0 = f'(V_{X_0}) + \frac{2f'(V_{X_0})f''(V_{X_0})V_{X_0}}{2f'(V_{X_0}) + f''(V_{X_0})V_{X_0}}.$$

This inequality however is equivalent to the inequality stated in the proposition, since $2f'(V) + f''(V)V > 0$ by Lemma 6.4 and our assumption that δ is unique and C^2. □

Proof of Proposition 6.6: We have by Proposition 6.1 that

$$\frac{d}{dX_0}\frac{X_t^{(\delta_{X_0})}}{X_0} = -\left(\frac{d}{dX_0}\delta_{X_0}\right)\frac{1}{2}\sqrt{\frac{1}{\delta_{X_0}\lambda}}\sigma t e^{\left(-\sqrt{\frac{\delta_{X_0}}{\lambda}}\sigma t\right)}.$$

Hence we need to show that $\frac{d}{dX_0}\delta_{X_0} \leq 0$ if and only if f is concave. By Equation (6.4), we have

$$\delta'_{X_0} = V'_{X_0}f''(V_{X_0}).$$

By Lemma 6.4, we know that $V'_{X_0} > 0$, which completes our proof. □

Proof of Proposition 6.7: The optimal strategy computed at time 0 implies selling

$$\xi_t^{(\delta_0)} = \sqrt{\frac{\delta_0}{\lambda}}\sigma X_t$$

shares at time t with

$$\delta_0 = f'\left(\frac{1}{2}\sqrt{\frac{\lambda}{\delta_0}}\sigma X_0^2\right).$$

The optimal strategy computed at time t however implies selling

$$\xi_t^{(\delta_t)} = \sqrt{\frac{\delta_t}{\lambda}}\sigma X_t$$

shares at time t with

$$\delta_t = f'\left(\frac{1}{2}\sqrt{\frac{\lambda}{\delta_t}}\sigma X_t^2\right).$$

For time-consistency, we need $\xi_t^{(\delta_0)} = \xi_t^{(\delta_t)}$ and thus $\delta_0 = \delta_t$, which implies $f' \equiv \text{const}$. Hence f is a linear function. □

Proof of Proposition 6.8: Equation (6.9) follows directly from Equation (6.6). Since $\xi_t = -\frac{d}{dt}X_t$, Equation (6.9) is in fact an ordinary differential equation with a unique solution determined by the initial condition X_0. It is easy to check that the strategies given in closed form in Equations (6.10) - (6.11) fulfill the ODE and the initial condition. $\square$

Proof of Lemma 6.10: First note that under optimal liquidation,

$$\lim_{X_0 \to 0}(C_{X_0} + f(V_{X_0})) = 0$$

and thus $\lim_{X_0 \to 0} V_{X_0} = 0$. Hence for small enough X_0 we can combine Equation (6.8) with the bound on f' assumed in the proposition and we obtain

$$\delta_{X_0} \gtrless a' \left(\frac{1}{2} \sqrt{\frac{\lambda}{\delta_{X_0}}} \sigma X_0^2 \right)^{b'} .$$

The desired equation follows directly. $\square$

Proof of Lemma 6.11: By standard methods of calculus of variations, all optimal solutions are C^2 and solve the Euler-Lagrange equation until they attain the boundary value zero; see for example Cesari (1983). The existence of an optimal strategy is not clear however, since we consider an infinite time horizon; we will therefore construct it as the limit of finite time horizon strategies. For each finite liquidation time horizon T, there is at least one solution to the liquidation problem (i.e., minimizing $C_X + R_X$ with $X_T = 0$), and this solution is C^2 and also fulfills the Euler-Lagrange equation from time 0 until X attains zero (see again Cesari (1983)). Let us denote the initial selling speed of this optimal solution $X^{(T)}$ for the time horizon T by $\xi_0^{(T)}$. Then we can select an increasing sequence of time horizons $T^{(1)}, T^{(2)}, \ldots$ with $\lim_{n\to\infty} T^{(n)} = \infty$ such that $\xi_0^{(T^{(n)})}$ converges, and we define $\xi_0^{(\infty)} := \lim_{n\to\infty} \xi_0^{(T^{(n)})}$. The ODE (6.12) with boundary conditions X_0 and $\dot{X}_0 = -\xi_0^{(\infty)}$ gives a trajectory $X_t^{(\infty)}$ until X_t attains zero at time $T_0 \in]0, \infty]$; we define $X_t^{(\infty)} \equiv 0$ for all $t > T_0$. We then have

$$C_{X^{(\infty)}} + R_{X^{(\infty)}} \leq \lim_{n\to\infty} \left(C_{X^{(T^{(n)})}} + R_{X^{(T^{(n)})}} \right).$$

On the other hand, let $X^{[\infty]}$ be any liquidation strategy for the infinite time horizon that realizes a finite cost-risk measure, i.e.,

$$C_{X^{[\infty]}} + R_{X^{[\infty]}} < \infty.$$

We define the finite time liquidation strategies $X^{[T^{(n)}]}$ as executing $X^{[\infty]}$ until time $T^{(n)} - 1$ and linearly liquidating $X^{[\infty]}_{T^{(n)}-1}$ from $T^{(n)} - 1$ until $T^{(n)}$. Since $\lim_{t\to\infty} X_t^{[\infty]} = 0$, we have

$$C_{X^{[\infty]}} + R_{X^{[\infty]}} = \lim_{n\to\infty} \left(C_{X^{[T^{(n)}]}} + R_{X^{[T^{(n)}]}} \right).$$

Because $C_{X^{[T^{(n)}]}} + R_{X^{[T^{(n)}]}} \geq C_{X^{(T^{(n)})}} + R_{X^{(T^{(n)})}}$, we have $C_{X^{[\infty]}} + R_{X^{[\infty]}} \geq C_{X^{(\infty)}} + R_{X^{(\infty)}}$, which establishes the optimality of $X^{(\infty)}$. $\square$

Proof of Proposition 6.12: Let X be a minimizer of $C_X + R_X$. We first show that if X attains zero at a finite time T_0, then $\dot{X}_{T_0} = 0$. First, we see by the optimality of X that $\xi_t = -\dot{X}_t$ is decreasing in t. Now let us assume that $T_0 < \infty$ and that the left limit of the selling speed $\underline{\xi} := \lim_{t \nearrow T_0} \xi_t > 0$. Again by optimality of X, we have $\xi_t \geq \underline{\xi}$ for all $0 \leq t < T_0$. For each $\epsilon > 0$, we define the time T_ϵ as the time at which $X_{T_\epsilon} = \epsilon$. Then the contribution of the sale of ϵ to $C + R$ is at least:

$$\int_{T_\epsilon}^{\infty} (\lambda \xi_s^2 + g(X_s))ds \geq \int_{T_\epsilon}^{\infty} \lambda \xi_s^2 ds \geq \int_{T_\epsilon}^{T_\epsilon + \epsilon/\underline{\xi}} \lambda \underline{\xi}^2 ds = \lambda \epsilon \underline{\xi}. \tag{6.17}$$

Because $g(x) \leq cx^d$ for $0 \leq x \leq \epsilon$ by our assumption in the proposition, selling the asset position ϵ with constant speed $\tilde{\xi} > 0$ from time T_ϵ until $T_\epsilon + \epsilon/\tilde{\xi}$ results in a contribution to $C + R$ of

$$\int_{T_\epsilon}^{T_\epsilon + \epsilon/\tilde{\xi}} (\lambda \tilde{\xi}^2 + g(\epsilon - \tilde{\xi}(s - T_\epsilon)))ds \leq \lambda \epsilon \tilde{\xi} + \int_0^{\epsilon/\tilde{\xi}} c(\tilde{\xi} s)^d ds \leq \lambda \epsilon \tilde{\xi} + c\frac{\epsilon^{1+d}}{(1+d)\tilde{\xi}}. \tag{6.18}$$

With $\tilde{\xi} = \underline{\xi}/2$ and sufficiently small ϵ, the right hand side of Equation (6.18) is smaller than the right hand side of Equation (6.17), which establishes a contradiction to the optimality of X. Since it is obvious that any optimal X satisfies $\lim_{t\to\infty} X_t = 0$, we have established the characterization of optimal strategies given in the proposition.

For the uniqueness of the optimal strategy, let X and Y be two different minimizers of $C + R$ with times $T_0^{(X)}, T_0^{(Y)} \in]0, \infty]$ at which they attain zero. We will now construct a contradiction. By Lemma 6.11, both X and Y solve the Euler-Lagrange equation. Since a solution to the Euler-Lagrange equation is uniquely determined by its initial values, we have $\xi_0^{(X)} \neq \xi_0^{(Y)}$. Without loss of generality, let $\xi_0^{(X)} > \xi_0^{(Y)}$. Let us compare the selling speeds of X and Y when an asset position z remains to be liquidated. We easily see that $\xi_{X^{-1}(z)}^{(X)} - \xi_{Y^{-1}(z)}^{(Y)} > \xi_0^{(X)} - \xi_0^{(Y)}$ for all $0 \leq z < X_0$. This implies

$$\xi_{X^{-1}(z)}^{(X)} > \xi_{Y^{-1}(z)}^{(Y)} + \xi_0^{(X)} - \xi_0^{(Y)} > \xi_0^{(X)} - \xi_0^{(Y)} > 0. \tag{6.19}$$

The strategy X hence reaches 0 in a finite time $T_0^{(X)}$ and with a positive selling speed $\xi_{T_0}^{(X)} > 0$, which contradicts the optimality of X by the first part of this proof.

By a similar argument, it follows that there is at most one solution to the Euler-Lagrange equation that fulfills either of the two conditions stated in the proposition. $\square$

Proof of Theorem 6.13: Let us assume that the trading strategy X is optimal for the mean-variance functional given by $f \in \mathcal{F}$ with continuous reoptimization. By differentiating Equation (6.7) by t, we obtain

$$\begin{aligned} \ddot{X}_t &= -\sqrt{\frac{\delta_t}{\lambda}}\sigma \dot{X}_t - \dot{\delta}_t \frac{1}{2}\sqrt{\frac{1}{\lambda \delta_t}}\sigma X_t \\ &= \frac{\delta_{X_t}}{\lambda}\sigma^2 X_t + \frac{1}{2\lambda}\delta'_{X_t}\sigma^2 X_t^2. \end{aligned}$$

We used the interpretation of δ as a function of time t and as a function of the asset position X_t (see the remark after Equation (6.8)) and the conventions $\delta'_X = \frac{d}{dX}\delta_X$ and

$\dot{\delta}_t = \frac{d}{dt}\delta_t = \delta'_{X_t}\dot{X}_t$. If X also minimizes $C_X + R_X$ for a risk function g, then by Lemma 6.11 this risk function must satisfy

$$g'(X) = 2\delta_X \sigma^2 X + \delta'_X \sigma^2 X^2$$

with initial condition $g(0) = 0$. This ODE has a solution if and only if

$$\begin{aligned} -\infty &< \lim_{\epsilon \to 0} \int_\epsilon^1 (2\delta_X \sigma^2 X + \delta'_X \sigma^2 X^2) dX \\ &= \lim_{\epsilon \to 0} [\sigma^2 \delta_X X^2]_\epsilon^1 \\ &= \sigma^2 \delta_1 - \lim_{\epsilon \to 0} \sigma^2 \delta_\epsilon \epsilon^2. \end{aligned} \tag{6.20}$$

Since f' is bounded from above by $a'V^{b'}$ with $b' > -\frac{2}{3}$ for small V, we see by Lemma 6.10 that the limit in Equation (6.20) converges to zero, and we obtain

$$g(X) = \sigma^2 \delta_X X^2. \tag{6.21}$$

Hence X satisfies the Euler-Lagrange equation for a risk function g. X is indeed optimal for the risk function g by Proposition 6.12. To see this, observe that g is locally bounded from above by a power law with exponent between 0 and 6 by Lemma 6.10. Furthermore, we know by Proposition 6.9 that X either never attains 0 or attains it with zero selling speed.

We now show that $g \in \mathcal{G}$. The risk function g is C^2 by Lemma 6.4, it is increasing by Lemma 6.5, and it is globally bounded by a power law with exponent between 0 and 6 by Lemma 6.3. The only property left to show is that $6g(X) > g'(X)X$. A simple calculation and Equation (6.21) show that

$$V_{X_0} = \frac{\sqrt{\lambda}}{2\sqrt{g(X_0)}} \sigma^2 X_0^3. \tag{6.22}$$

By Lemma 6.4, we have

$$0 < V'_{X_0} = \frac{3\sqrt{\lambda}}{2\sqrt{g(X_0)}} \sigma^2 X_0^2 - g'(X_0) \frac{\sqrt{\lambda}}{4\sqrt{g^3(X_0)}} \sigma^2 X_0^3 \tag{6.23}$$

which is equivalent to $6g(X) > g'(X)X$ for all $X > 0$.

Let us now consider a risk function $g \in \mathcal{G}$ with optimal strategy X. We define

$$\tilde{\delta}_{X_0} := \frac{g(X_0)}{\sigma^2 X_0^2}. \tag{6.24}$$

By Equation (6.21), all that we have to do is find a $f \in \mathcal{F}$ such that $\delta_{X_0} = \tilde{\delta}_{X_0}$ for all $X_0 > 0$, where δ_{X_0} is the optimal δ for the liquidation of X_0 with respect to the mean-variance functional $C_X + f(V_X)$. To construct this f, we first set

$$\tilde{V}_{X_0} := \frac{1}{2} \sqrt{\frac{\lambda}{\tilde{\delta}_{X_0}}} \sigma X_0^2$$

and define f as the solution of the ODE

$$f'(\tilde{V}_{X_0}) = \tilde{\delta}_{X_0}. \tag{6.25}$$

Since $6g(X) > g'(X)X$, we know by the transformation (6.22)–(6.23) that

$$\tilde{V}'_{X_0} > 0. \tag{6.26}$$

Furthermore $\tilde{V}_{X_0}$ is not bounded from above because of the global bound on g. Therefore the ODE (6.25) is well-defined. Since $g(X) < a'X^{b'}$ with $a' > 0$ and $0 < b' < 6$ for small X, we obtain $\tilde{\delta}_{X_0} < \frac{a'}{\sigma^2} X_0^{b'-2}$ and thus $\tilde{V}_{X_0} > \frac{1}{2}\sqrt{\frac{\lambda}{a'}}\sigma^2 X_0^{3-\frac{b'}{2}}$.We therefore have

$$f'(\tilde{V}_{X_0}) = \tilde{\delta}_{X_0} < \frac{a'}{\sigma^2} X_0^{b'-2} < \frac{a'}{\sigma^2}\left(\tilde{V}_{X_0}\frac{2}{\sigma^2}\sqrt{\frac{a'}{\lambda}}\right)^{\frac{b'-2}{3-\frac{b'}{2}}} = \tilde{a}\tilde{V}_{X_0}^{\tilde{b}}$$

with $\tilde{a} > 0$ and $\tilde{b} > -\frac{2}{3}$. Therefore there exists a solution f of the ODE (6.25) that satisfies $f(0) = 0$. As we have just shown, its derivative f' is locally bounded at zero by a power law with exponent larger than $-\frac{2}{3}$. By the same line of argument, f' is globally bounded by a power law, thus giving a global power law bound for f. Furthermore, f is positive and C^3 on $\mathbb{R}^+$, and it is increasing and continuous on $\mathbb{R}_0^+$. To see that f is unbounded, we first observe that

$$\tilde{\delta}_{X_0} = \frac{g(X_0)}{\sigma^2 X_0^2} = \frac{g(X_0)}{\sigma 2\tilde{V}_{X_0}\sqrt{\frac{\tilde{\delta}_{X_0}}{\lambda}}}$$

and thus for $X_0 > 1$

$$\tilde{\delta}_{X_0} = \left(\frac{g(X_0)\sqrt{\lambda}}{\sigma 2\tilde{V}_{X_0}}\right)^{\frac{2}{3}} > \left(\frac{g(1)\sqrt{\lambda}}{\sigma 2\tilde{V}_{X_0}}\right)^{\frac{2}{3}} = \bar{a}\tilde{V}_{X_0}^{-\frac{2}{3}}$$

with $\bar{a} > 0$. This establishes that f is unbounded by Equation (6.25). All that is left to show is that f satisfies $f''(V)V \geq -\frac{2}{3}f'(V)$ for all $V > 0$. To see this, first observe that by (6.25)

$$\tilde{V}'_{X_0} = \frac{2}{X_0}\tilde{V}_{X_0} - \frac{\tilde{\delta}'_{X_0}}{2\tilde{\delta}_{X_0}}\tilde{V}_{X_0} = \frac{2}{X_0}\tilde{V}_{X_0} - \frac{f''(\tilde{V}_{X_0})\tilde{V}'_{X_0}}{2f'(\tilde{V}_{X_0})}\tilde{V}_{X_0}$$

and thus

$$\tilde{V}'_{X_0}\left(1 + \frac{f''(\tilde{V}_{X_0})\tilde{V}_{X_0}}{2f'(\tilde{V}_{X_0})}\right) = \frac{2}{X_0}\tilde{V}_{X_0}.$$

The second factor on the right hand side is positive since the left hand side $\frac{2}{X_0}\tilde{V}_{X_0}$ and the first factor on the right hand side $\tilde{V}'_{X_0}$ are positive by Equation (6.26). We can therefore divide by it and obtain

$$\tilde{V}'_{X_0} = \frac{\frac{2}{X_0}\tilde{V}_{X_0}}{1 + \frac{f''(\tilde{V}_{X_0})\tilde{V}_{X_0}}{2f'(\tilde{V}_{X_0})}}.$$

Now we have by Equation (6.24) and (6.25)

$$
\begin{aligned}
0 \le g'(X_0) =& \frac{d}{dX_0}\left(f'(\tilde{V}_{X_0})\sigma^2 X_0^2\right)\\
=& f''(\tilde{V}_{X_0})\sigma^2 X_0^2 \tilde{V}'_{X_0} + 2f'(\tilde{V}_{X_0})\sigma^2 X_0\\
=& f''(\tilde{V}_{X_0})\sigma^2 X_0^2 \left(\frac{\frac{2}{X_0}\tilde{V}_{X_0}}{1+\frac{f''(\tilde{V}_{X_0})\tilde{V}_{X_0}}{2f'(\tilde{V}_{X_0})}}\right) + 2f'(\tilde{V}_{X_0})\sigma^2 X_0\\
=& \sigma^2 X_0 \frac{3f''(\tilde{V}_{X_0})\tilde{V}_{X_0} + 2f'(\tilde{V}_{X_0})}{1+\frac{f''(\tilde{V}_{X_0})\tilde{V}_{X_0}}{2f'(\tilde{V}_{X_0})}}.
\end{aligned}
$$

Since the denominator is positive, the numerator is positive. This is equivalent to

$$
f''(V)V \ge -\frac{2}{3}f'(V).
$$

□

Proof of Proposition 6.14: By Equation (6.20) in the proof of Theorem 6.13, we know that the limit $\lim_{\epsilon\to 0}\sigma^2\delta_\epsilon\epsilon^2$ needs to exist in order to allow for the existence of a corresponding risk function g. Lemma 6.10 however establishes the divergence of this limit for the conditions assumed in the proposition (f' bounded from below by $a'V^{b'}$ with $b' < -\frac{2}{3}$ for small V). □

Proof of Theorem 6.15: Let $f(V) = aV^b$ with $a, b > 0$. We will only consider the cases $0 < b < 1$ and $1 < b < \infty$; the case $b = 1$ follows analogously. By Proposition 6.8, we know that for $b \ne 1$, the strategy minimizing the mean-variance functional with $f(V) = aV^b$ fulfills

$$
\ddot{X}_t = AC(C-1)(B \pm t)^{C-2} = A^{\frac{2}{C}}C(C-1)X_t^{\frac{5b-3}{b+1}}
$$

with A and C as defined in Proposition 6.8. If this strategy minimizes $C+R$ for a risk function g, then the right hand side of the above equation must be equal to $\frac{g'(X)}{2\lambda}$ by Lemma 6.11. This determines a first order ODE for g with initial value $g(0) = 0$. This ODE has the following solution for $\frac{6b-2}{b+1} > 0$

$$
g(X) = 2\lambda A^{\frac{2}{C}}C(C-1)\frac{b+1}{6b-2}X^{\frac{6b-2}{b+1}}
$$

and no solution for $\frac{6b-2}{b+1} \le 0$. By Theorem 6.13, X is also optimal for this g since $f \in \mathcal{F}$ if $b > \frac{1}{3}$. □

Part II

Multiple players in illiquid markets

Chapter 7

Single vs. multiple players: economic and mathematical differences

7.1 Introduction

In Part I of this thesis, we investigated optimal behavior of a single large trader in an illiquid market. All other market participants were modeled exogenously in the market model, but we did not solve for their endogenous optimal behavior. This approach seems acceptable as long as no strategic agent is aware of the large trader's intentions. In such a situation, all market participants only react to order flow, and this reaction can be captured by market models such as those considered in the first part of this thesis.

In this second part, we will include additional informed strategic agents in our model, i.e., large traders that are aware of each other's intentions. This leads to an interaction: each agent predicts the actions of the other agents and selects the most profitable trading strategy for herself, which in turn influences the actions of the other agents. Each agent faces a dual challenge: on the one hand, she needs to optimize her trading such that her own adverse impact on trading proceeds remains small. On the other hand, she needs to trade such that the price impact of the other agents does not affect her liquidation proceeds too negatively. We will see that the relative importance of these considerations depends on the liquidity of the market.

A situation we are particularly interested in is a single large seller who is facing a number of competitors who are aware of her selling intentions. This situation has occurred frequently in the financial markets, with the forced liquidations of the LTCM and Amaranth hedge funds being the most prominent examples. It appears on a much smaller scale in everyday trade execution when investors ask several banks for principal bid quotes and then select only the bank offering the most attractive quote. All other banks are then aware of the impending trade. Two trading strategies for these banks come to mind: First, they can quickly trade in the same direction as the bank that won the principal bid. After the investor's portfolio liquidation has moved the price, they can close their position at a profit. Alternatively, informed banks can initially trade in the opposite direction, i.e., provide liquidity, and thus exploit the temporary move in asset prices induced by the liquidation of the investor's portfolio. After the liquidation, the banks can close their asset position at the re-established "fundamental" price. It is not clear a priori which of these two strategies offers a higher profit, and we will see that the answer again depends on the liquidity of the market.

Before proceeding to the actual analysis of the interaction of multiple players, we discuss conceptually how transactions prices are set and which dynamic trading opportunities the agents can pursue.

7.2 Price impact and transaction prices

In transparent markets, trades by different agents might result in different price impacts. For example, a passive index-tracking fund is likely to incur a smaller price impact than an active fund that is known to be well informed. For our analysis, we assume that the price impact of a trade does not depend on the strategic agent that initiated it, either because the strategic agents are sufficiently similar or because the trading venue is anonymous and therefore the agent's identity is not revealed.

For each transaction, we need to determine a transaction price. In reality, discrete trades arrive at the trading venue one after another, and no two trades arrive at the same time. Therefore the transaction price can be determined for each transaction individually. Since we want to allow for continuous trading in continuous time, we need to specify the price for transactions that simultaneously arrive at the trading venue. We make the fundamental assumption that at each point in time, all transactions are executed at the same price. By this assumption, we can accommodate a set of equal agents and do not have to worry about multiple market prices. When several agents submit orders simultaneously, then the net order of the agents needs to be matched by supply and demand of the general market. Since we assume that the market does not (or cannot) differentiate between agents, the transaction price of the net order depends only on its size. Hence, all transactions at this point in time are executed at the price at which the net order is matched by the supply and demand of the general market. We can think of trade execution as a two step process: first, the net order of the agents is executed at the trading venue and a transaction price is established[1]. At this price, the remaining orders of the agents (which add up to zero) are then crossed. This procedure is similar to the trading regime at the upstairs market or at electronic crossing networks, where incoming orders are crossed at (or close to) the price established by the primary exchange (see Butler (2007)).

7.3 Risk aversion and dynamic reactions to changes in fundamental market price

In the first part of this thesis, we focused on the effects of risk aversion and on trading strategies that dynamically react to changes in market prices. In this second part, we want to shift the focus of attention from the single trader to the interaction of several traders. In principle, each trader in a multi-agent market needs to strike a balance between three effects: the adverse impact of her own trading, the expected price impact of the other agents' trades, and the risk of movements in the fundamental market price. In order to keep the mathematical analysis tractable, we need to disregard the third effect and limit our discussion to risk-neutral agents that only pursue strategies that are independent of changes of the fundamental market price. The mathematical analysis even with these simplifying assumptions is already quite involved, as we will see in Chapters 8 and 9. We believe that in spite of these restrictions, our analysis is still meaningful, as the core effects of agent interaction appear irrespective of the agents' attitudes toward risk.

[1] If the net order is zero, the previous transaction price is used.

7.4 Market transparency and dynamic reactions to trades of other agents

One important attribute of a financial market is its transparency. While many financial markets used to be rather transparent, they have become more and more opaque by the introduction of electronic trading systems in recent years. By now, many exchanges feature both pre- and post-trade anonymity, and electronic crossing networks provide an arena for completely anonymous "dark liquidity". Market transparency is important for the interaction of multiple agents since it allows agents to dynamically react to other agents' trades. There is no such feedback opportunity in anonymous markets; even if all agents can observe the size and price of all transactions[2], a single agent does not know which (if any) informed agents participated in them.

Many assets are being traded at several exchanges. Even if trading is anonymous at only one of them, then feedback strategies are no longer possible since each agent can direct part of her trading to the transparent and another part to the anonymous trading venue. It is therefore impossible to estimate the real size or direction of an agents trades just by observing the transparent order flow; each visible trade might be offset by an invisible trade at the anonymous venue.

In game theoretic terms, agents in anonymous markets can only adopt open-loop strategies, i.e., strategies that depend only on time. Agents in transparent markets however can follow closed-loop strategies, i.e., feedback strategies that depend on time but also on the previous trades of the other agents. In this thesis, we are mainly concerned with trading in highly developed markets such as the equities markets. Since at least one anonymous trading venue exists for most of these markets, we believe that the open-loop setting is the appropriate modeling framework. Although being significantly different in economic meaning and mathematical approach, open-loop and closed-loop equilibria are often similar (see, e.g., Fudenberg and Levine (1988)). In particular, Carlin, Lobo, and Viswanathan (2007) found that the optimal open-loop and closed-loop strategies in the multiple player illiquid market model used in Chapter 8 exhibit the same qualitative properties. Furthermore, we determine both open-loop and closed-loop equilibrium strategies in a multiple player extension of the liquidity model introduced by Obizhaeva and Wang (2006) and find that they are almost identical.

Part II of this thesis is structured as follows. In Chapter 8, we introduce a multiple player extension of the single-asset market model that we already considered in Section 4.2. We then solve for the optimal trading strategies for a seller and several informed competitors in a two stage model, where the seller needs to quickly liquidate an asset position in the first stage, while the informed competitors are more patient and can trade in the first and also in a subsequent second stage. In Chapter 9, we determine the optimal trading strategies in a multiple player extension of the limit order book model developed by Obizhaeva and Wang (2006) and contrast them with the results of Chapter 8. We find that the cost of round trip trades is an important feature of multiple player illiquid market models. Appendices A and B contain supplementary material, including explicit statements of complex equations, additional numerical illustrations and the Mathematica source code used to generate the

[2] Even this minimum level of transparency is often not provided by electronic crossing networks.

figures in Chapters 8 and 9.

Chapter 8

Stealth vs. sunshine trading, predatory trading vs. liquidity provision

8.1 Introduction

A variety of circumstances such as a margin call or a stop-loss strategy in combination with a large price drop can force a market participant (the "seller") to liquidate a large asset position urgently. Such a swift liquidation may result in a significant impact on the asset price. Hence, intuitively it seems to be crucial to prevent information leakage while executing the trade, for informed market participants (the "competitors") could otherwise try to earn a profit by predatory trading: They can sell in parallel with the seller and cover their short positions later at a lower price. Probably the most widely known example of such a situation is the alleged predation on the hedge fund LTCM[1]. Surprisingly, however, some sellers do not follow a secretive "stealth trading" strategy but rather practice "sunshine trading", which consists in pre-announcing the trade to competitors so as to attract liquidity[2].

Our goal in this chapter is to propose a new model of a competitive trading environment that explains the tradeoff that leads the seller to choose between stealth and sunshine execution and the competitors to choose between predation and liquidity provision. We argue that these choices are driven by the relations between the different liquidity parameters of the market, the number of competitors of the seller and the trading time horizons. In particular, different behavioral patterns may coexist within the same set of agents when they are trading in markets of different liquidity types. Since our model market is semi-strong efficient and allows for anonymous trading possibilities, our results are applicable to a wide variety of real-world markets including most equity exchange markets.

To fully acknowledge the roles of the different liquidity parameters of the market and of the number of competitors of the seller, we need to relax all exogenous trading constraints in our model. In particular, we do not require that predators face the same time constraint as the seller. This assumption is reasonable as sellers typically must achieve a trading target in a fixed and relatively short time horizon—e.g., a margin call has to be covered by the end of the day—while predators often may afford to maintain a long or a short position for a number of days. In order to capture the structure of this situation, we consider a two stage model of an illiquid market. In the first stage, the seller as well as the competitors trade; in the second stage, only the competitors trade and unwind the asset positions they acquired during the first stage. Liquidity effects are incorporated into our market model by

[1] See, e.g., Lowenstein (2001), Jorion (2000) and Cai (2003)).

[2] See, e.g., Harris (1997) and Dia and Pouget (2006). A similar phenomenon occurs in the sometimes widespread distribution of so-called "indications of interest" in which brokers announce tentative conditions for certain liquidity trades.

applying a permanent as well as a temporary impact as in the market model introduced in Section 4.2; this model was proposed by Almgren and Chriss (2001) and used by Carlin, Lobo, and Viswanathan (2007). For the sake of simplicity, throughout this chapter we focus on the liquidation of a long position of assets; equivalent statements hold for the liquidation of a short position.

In our analysis of the optimal agent behavior in this model, we first assume that all agents know the seller's liquidation intentions. We derive a Nash equilibrium of optimal trading strategies for the seller and the competitors, and we show that, in equilibrium, the competitors' optimal strategy depends heavily on the liquidity type of the market. We identify two distinct types of illiquid markets: First, if the temporary price impact dominates the permanent impact then prices show a high resilience after a large transaction. The price in such "elastic" markets behaves similar to a rubber band: trading pressure can stretch it, but after the trading pressure reduces, the price recovers. Such market conditions can occur when it is difficult to find counterparties for a specific deal within a short time. In such a market, the optimal strategy for the competitors is to cooperate with the seller: they should buy some of the seller's assets and sell them at a later point in time. On the other hand, if the permanent price impact of a trade outweighs the temporary impact, then large transactions have a long-lasting influence. In such "plastic" markets, the trading pressure exerts a "plastic deformation" on the market price. Such a situation can arise when a large supply or demand of the asset is interpreted as the revelation of new information on the fundamentals of the asset. Under these conditions the optimal behavior of the competitors is the opposite: they should sell in parallel to the seller and buy back at a later point in time (predatory trading). In this case, the price is pushed far down during the first stage and recovers during the second stage, resulting in price overshooting. The latter effect disappears as the number of competitors increases; for a large number of competitors, the market price incorporates the seller's intentions almost instantly and exhibits little drift thereafter. This effect indicates that our model market fulfills the semi-strong form of the efficient markets hypothesis.

Through sunshine trading, the seller can increase the number of competitors. We find that in elastic markets, the seller always achieves a higher return when competitors are participating than when she is selling by herself. Therefore, sunshine trading appears to be sensible in such a market. In a plastic market, the seller's return can be significantly reduced by competitors; however, as the number of competitors increases, the optimal strategy for the competitors changes from predation to cooperation and the return for the seller increases back, sometimes even above the level of return obtained in the absence of competitors. Hence, if the seller has reason to believe that there is *some* leakage of information[3], it may be sensible to take the initiative of publicly announcing the impending trade so as to turn around the adverse situation of predation by few competitors into the beneficial

[3]In practice, information leakage can occur due to a variety of circumstances. For instance, as in the case of the LTCM crisis, the position may simply be too large to keep its liquidation secret. In a much more common situation, the execution of the trade will be commissioned to an investment bank, but advance price quotes are obtained from several banks. Banks that are not successful in bidding for the trade will nevertheless be informed about its existence and hence constitute potential competitors. When obtaining price quotes, it is therefore common practice for the client to distribute only a limited amount of information on their "bid sheets" so as to to reduce the potentially adverse effects of predatory trading. Another example is provided by market makers who must report large transactions.

situation of liquidity provision.

Although our approach is normative rather than descriptive, our model provides a number of empirically testable hypotheses for both seller and competitor behavior. In our model, sunshine trading is rational in elastic markets or when the trading horizon of the seller is comparatively short. We therefore suspect that sunshine trades and indications of interest are usually short-term and occur in markets with high temporary impact, while we conjecture that efforts to conceal trading intentions are particularly strong in plastic markets.

We predict that competitors in plastic markets pursue predatory trading if they know about selling intentions of other agents, while we expect them to provide liquidity in elastic markets. Unfortunately, we are not aware of any systematic study of informed competitors reactions to trading under varying market liquidity[4]. However, the analysis of distressed hedge funds lends anecdotal support to our hypothesis. During the LTCM crisis in 1998, several competitors allegedly engaged in front-running and predatory trading, while no individual investor was willing to acquire LTCM's positions and thus provide liquidity. According to our results, such a behavior is rational in plastic markets. The price evolution after the LTCM crisis indicates that its liquidation had a predominantly permanent effect[5], i.e., that the market was indeed plastic.

More recently, the hedge fund Amaranth experienced severe losses resulting in the need for urgent liquidation[6]. Contrary to LTCM, Amaranth quickly found a buyer for its portfolio[7]. In the Amaranth case, liquidity provision apparently appeared as the more profitable option for competitors compared to predatory trading. How can the differences between competitors' behavior in the LTCM and Amaranth cases be explained? In both cases very large market participants were in distress, promising large profit opportunities for competitors. However, Amaranth operated in the natural gas market, which behaved elastic during a previous hedge fund liquidation[8]. According to our model liquidity provision is the most profitable behavior in such an elastic market.

The profitability of liquidity provision in elastic markets is confirmed by Coval and Stafford (2007), who find that providing liquidity to open-ended mutual funds that suffer severe cash outflows promises average annual abnormal excess returns well over 10%. This supports our hypothesis since these profits are made on the temporary nature of the price impact. Interestingly, the impact of stock sales in markets that do not suffer from extreme cash outflows appears to be predominantly permanent, resulting in profitable predatory

[4]This could be carried out, e.g., by analyzing the order flow after pre-announcement of a sale. In plastic markets, we expect to see an initial increase of additional seller initiated trades. In elastic markets, we expect to see an increase in buy orders.

[5]Lowenstein (2001) notes that (Epilogue, page 229): "(...) a year after the bail-out [of LTCM], swap spreads remained (...) far higher than when Long-Term had entered the (...) trade."

[6]For a description of the Amaranth case, see Till (2006) and Chincarini (2007). Finger (2006) finds that "The events of September [2006] led to the greatest losses ever by a single hedge fund, close to twice the money lost by Long Term Capital Management."

[7]Till (2006) notes that "Amaranth sold its entire energy-trading portfolio to J.P. Morgan Chase and Citadel Investment Group on Wednesday, September 20th [2006]."

[8]Till (2006) observes that "There was a preview of the intense liquidation pressure on the Natural Gas curve on 8/2/06, the day before the [natural-gas-oriented] energy hedge fund, MotherRock, announced that they were shutting down. (...) A near-month calendar spread in Natural Gas experienced a 4.5 standard-deviation move intraday before the spread market normalized by the close of trading on 8/2/06."

trading opportunities for insiders.

This chapter builds on previous work in three research areas. The first area to which our work is connected is research on predatory trading. In previous studies, the size of the liquidation completely determines the optimal action of the competitors. In these models, predatory trading is always optimal for large liquidations. For small liquidations, predatory trading is always or never optimal, depending of the model at hand.

Brunnermeier and Pedersen (2005) suggest a model in which the total rate of trading as well as the asset positions of all traders face exogenous constraints. They show that in equilibrium in their model predation and price overshooting occur necessarily, irrespective of the market environment[9]. As a side effect of the exogenous trading constraint, their model market is weakly inefficient: even if the number of informed competitors is large, the market price changes continuously in reaction to the trading of the seller and the competitors.

Carlin, Lobo, and Viswanathan (2007) propose a model in which competitors can engage in and refrain from predatory trading, however there is no room for optimal liquidity provision. To explain abstinence from predatory trading, they assume that all market participants repeatedly execute large transactions in a fully transparent market[10]; in such a repeated game, predation can be punished by applying a tit-for-tat strategy. In their model, competitors always refrain from predatory trading while others are liquidating small positions, but cooperation always breaks down if an unusually large distressed sale is occurring. Although their analysis of a one stage game is also at the foundation of our model, the two models diverge in their qualitative predictions of trading decisions: their model predicts that predatory trading is most widespread in elastic markets, while our model predicts the opposite.

Attari, Mello, and Ruckes (2005) discuss trading strategies against a financially constraint arbitrageur. Price impact in their model market is completely temporary, resulting in an elastic market with profitable liquidity provision. By clever exploitation of the arbitrageur's capital constraint, the competitors can profitably engage in predatory trading, but only for arbitrageurs with very large asset positions.

In a second line of research, the effects of sunshine trading are investigated. In a theoretical investigation, Admati and Pfleiderer (1991) propose a model in which sunshine trading is always increasing the seller's return as long as speculators do not face market entry costs. The underlying motives for sunshine trading in this model and in our model are very different[11]. Empirical evidence on the benefit of trade pre-announcements appears to be mixed (see, e.g., Harris (1997), Dia and Pouget (2006)), which is in line with our observation

[9]The only situation in which predatory trading does not occur in the model of Brunnermeier and Pedersen (2005) is when there is significant capacity on the sideline. In their model, this implies that the asset is heavily undervalued. They show that this cannot be the case in equilibrium.

[10]Our model explains cooperation in a different way; in particular, our model is also applicable to anonymous markets.

[11]In the model of Admati and Pfleiderer (1991), sunshine traders can expect to obtain better trade conditions in the market since it is assumed that their actions are not based on private information. In our model, we do not assume that sunshine trades have a special motivation; instead, we show that sunshine trading under certain market conditions can raise the attention of competitors and attract them to provide liquidity. A different market perception of sunshine trades can easily be incorporated in our framework by applying different liquidity parameters for sunshine trades and for unannounced trades.

that the potential benefit of sunshine trading depends on the liquidity characteristics of the market.

The third line of research consists of empirical investigations and theoretical modeling of the market impact of large transactions. See Section 1.2 and Part I for an overview.

The remainder of this chapter is structured as follows. In Section 8.2, we introduce the market model and specify the game theoretic optimization problem. As a preparation for the general two stage model, we review predation in a one stage model in Section 8.3. In this model, the seller and the competitors face the same time constraint, i.e., the competitors do not have the opportunity to trade after the seller finished selling. In the main Section 8.4, we turn to the more general two stage framework and derive our main results. After identifying the Nash equilibrium of optimal trading strategies in Section 8.5, we investigate the qualitative properties of our model in three example markets in Section 8.6. Thereafter, we summarize the general properties in Section 8.7. Appendix 8.A contains additional propositions on the one stage model. All proofs of propositions are given in Appendix 8.B. Appendix A of this dissertation contains supplementary material. An explicit statement of some of the coefficients in this section is presented in Appendix A.1. Appendix A.2 contains the Mathematica source code used to generate the figures in this chapter.

8.2 Market model for multiple players

We start by describing the market dynamics and trade motives of market participants. The market consists of a risk-free asset and a risky asset. Trading takes place in continuous time. We assume that the risk-free asset does not generate interest. In this market we consider $n+1$ strategic players and a number of noise traders. The strategic players are aware of liquidity needs in the market and optimize their trading to profit from these needs, whereas noise traders have less information and trade based on exogenous liquidity and investment needs. We assume that the number of strategic players $(n+1)$ is given a priori. During our analysis, we will perform comparative statics and discuss the incentives for each player to change the number of strategic players in the market.

We denote the time-dependent risky asset positions of the strategic players by $X_0(t)$, $X_1(t)$, ..., $X_n(t)$ and assume that they are differentiable in t. Their trading $\dot{X}_i(t)$ affects the market price in the form of a permanent impact and a temporary impact. Trades at time t are thus executed at the price

$$P(t) = \tilde{P}(t) + \gamma \sum_{i=0}^{n} (X_i(t) - X_i(0)) + \lambda \sum_{i=0}^{n} \dot{X}_i(t).$$

Here, $\tilde{P}(t)$ is an arbitrary martingale, starting at $\tilde{P}(0) = \tilde{P}_0$ and defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This term reflects the price changes due to the random trades of noise traders. The second term on the right hand side represents the permanent price impact resulting from the change in total asset position of all strategic players. Its magnitude is determined by the parameter $\gamma > 0$. The third term reflects the temporary impact caused by the net trading speed of all strategic investors. Its magnitude is controlled by the parameter $\lambda > 0$. This price dynamics model is a multi-player extension of the market model introduced in Section 4.2.

In this market, the strategic players are facing the following optimization problem. Each player i knows all other players' initial asset positions $X_j(0)$ and their target asset positions $X_j(T)$ for some fixed point $T > 0$ in the future[12]. We assume that these trading targets are binding; players are not allowed to violate their targets. We assume that all players are risk-neutral[13]; therefore, players want to maximize their own *expected* return by choosing an optimal trading strategy $X_i(t)$ given their boundary constraints on $X_i(0)$ and $X_i(T)$. In mathematical terms, each player is looking for a strategy that realizes the maximum

$$\begin{aligned} r_i &:= \max_{X_i} \mathbb{E}(\text{Return for player } i) = \max_{X_i} \mathbb{E}\left(\int_0^T (-\dot{X}_i(t))P(t)dt\right) \\ &= \max_{X_i} \mathbb{E}\left(-\int_0^T \dot{X}_i(t)\left(\tilde{P}(t) + \gamma \sum_{j=0}^{n}(X_j(t) - X_j(0)) + \lambda \sum_{j=0}^{n} \dot{X}_j(t)\right) dt\right). \end{aligned}$$

Although in principle the strategies X_i might be adapted, we limit our discussion to deterministic strategies, where the function X_i does not depend on the stochastic price component $\tilde{P}(t)$ or on the previous trades of the other agents. In such open-loop strategies, all players determine their trade schedules ex ante[14]. Hence,

$$r_i = \max_{X_i}\left(-\int_0^T \dot{X}_i(t)\left(\tilde{P}_0 + \gamma \sum_{j=0}^{n}(X_j(t) - X_j(0)) + \lambda \sum_{j=0}^{n} \dot{X}_j(t)\right) dt\right). \tag{8.1}$$

A set of strategies $(X_0, X_1, ..., X_n)$ satisfying Equation (8.1) for all $i = 0, 1, ..., n$ constitutes a Nash equilibrium; we call such a set of strategies *optimal*[15] and denote the corresponding optimal returns in equilibrium by $R_i := r_i$. These are determined by the *expected* price

$$\bar{P}(t) := \mathbb{E}(P(t)) = \tilde{P}_0 + \gamma \sum_{i=0}^{n}(X_i(t) - X_i(0)) + \lambda \sum_{i=0}^{n} \dot{X}_i(t).$$

Whenever we refer to *price* or *return* in the following, we will always refer to the *expected price* $\bar{P}(t)$ and the *expected return* $-\int \dot{X}_i(t)\bar{P}(t)dt$.

[12]For the purposes of this chapter, we assume that all strategic players have perfect information. For imperfect information, we expect to obtain slightly changed dynamics (potentially including a "waiting game" as in Foster and Viswanathan (1996)), but expect the qualitative results on predatory trading and liquidity provision to remain unchanged.

[13]See also Footnote 19.

[14]See Section 7.4 for a discussion of open-loop and closed-loop strategies. The analysis of closed-loop strategies in which players can dynamically react to other players' actions is mathematically more difficult. It is often not possible to derive closed form solutions, on which we rely in the proof of Theorem 8.2. Carlin, Lobo, and Viswanathan (2007) show numerically that closed-loop solutions of the one stage model (see Section 8.3) are similar to the open-loop solutions and do not exhibit any new qualitative features. Therefore, no major differences are expected in the two stage model introduced in Section 8.4.

[15]These strategies remain optimal for the entire trading time. At a future point in time $t \in [0, T]$, there is no reason to deviate from the trade schedule chosen at time 0 as long as no other player deviated from her trade schedule.

8.3 A one stage model: agents with uniform time constraint

In this section, we investigate the optimal strategies in a one stage framework: *all* players trade over the same time interval $[0, T_1]$. The results in this section will be used in the analysis of a two stage model in the following sections.

The optimal strategies in the one stage framework were derived by Carlin, Lobo, and Viswanathan (2007). We repeat their result:

Theorem 8.1 (Carlin, Lobo, and Viswanathan (2007))**.** *Assume that $n+1$ players are trading simultaneously in a time period $t \in [0, T_1]$. They start with asset positions $X_i(0)$ and need to achieve a target asset position $X_i(T_1)$. Furthermore, these players are risk-neutral and are aware of all other players' asset positions and trading targets. Then the unique optimal strategies for these $n+1$ players (in the sense of a Nash equilibrium) are given by:*

$$\dot{X}_i(t) = a e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t} + b_i e^{\frac{\gamma}{\lambda}t}$$

with

$$\begin{aligned} a &= \frac{n}{n+2}\frac{\gamma}{\lambda}\left(1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}\right)^{-1} \frac{\sum_{i=0}^{n}(X_i(T_1) - X_i(0))}{n+1} \\ b_i &= \frac{\gamma}{\lambda}\left(e^{\frac{\gamma}{\lambda}T_1} - 1\right)^{-1}\left(X_i(T_1) - X_i(0) - \frac{\sum_{j=0}^{n}(X_j(T_1) - X_j(0))}{n+1}\right). \end{aligned}$$

Proof. See Carlin, Lobo, and Viswanathan (2007). □

For the rest of this section, we consider the following more specific situation: One player (say player 0) wants to sell an asset position $X_0(0) = X_0$ in the time interval $[0, T_1)$, i.e. the target is given by $X_0(T_1) = 0$. All other players (i.e., players 1, 2, ..., n) do not want to change their initial and terminal asset positions (for simplicity, we assume that $X_i(0) = X_i(T_1) = 0$ for $i \neq 0$), but they want to exploit their knowledge of player 0's sales.

The result is *preying* of the n players on the first player (see Figure 8.1 and 8.2; see Table 8.1 for the parameter values used for the figures): while the first player is starting to sell off her asset position, the other players sell short the asset and realize a comparatively high price per share. At the end of the trading period, the price has been pushed down by the combined sales of both seller and competitors. While the seller liquidates the remaining part of her long position at a fairly low price, the other players can now close their short positions at a favorable price. Since the price has dropped, the preying players need to spend less on average for buying back than they received for initially selling short. In the following, we refer to player 0 as the "seller" and to the players 1, 2, ..., n as the "competitors".

In the one stage model considered so far, there is no room for cooperation; preying *always* occurs. The seller's return is further deteriorating as the number of competitors increases; preying becomes more competitive with more players being involved (see Figure 8.3). We will see in the next section that relaxing the exogenous time constraint on the positions of competitors can lead to a more differentiated behavior. It includes in particular the possibility of liquidity provision to the seller.

Parameter	Value
Asset position X_0	1
Initial price $\widetilde{P}_0$	10
Duration T_1	1
Permanent impact sensitivity γ	3
Temporary impact sensitivity λ	1

Table 8.1: Parameter values used for numerical computation of the figures in Section 8.3.

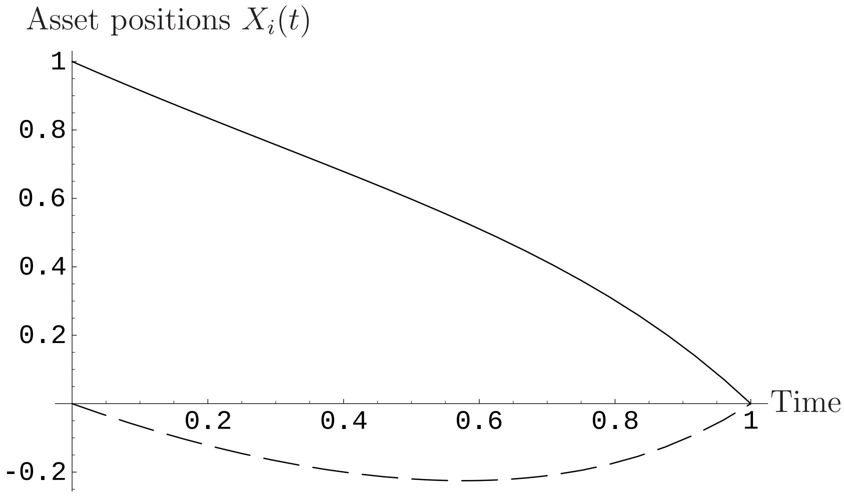


Figure 8.1: Asset positions $X_i(t)$ over time. The solid line represents the seller, the dashed line the competitor ($n = 1$).

8.4 A two stage model: agents with different time constraints

In the previous section, we have assumed that the seller and the competitors are limited to trade during the same time interval. As we have mentioned earlier, in reality the seller is often facing a stricter time constraint than the competitors do. While the seller usually needs to liquidate her asset position within a few hours, the competitors can often afford to close their positions at a later point in time. In the following, we therefore extend the one stage model considered so far to a two stage framework[16] and assume that:

- In stage 1, *all* players (the seller and the competitors) are trading.
- In stage 2, *only* the competitors are trading; the seller is not active.

[16]The framework can be extended further to a three stage model including a stage 0 in which only the competitors are allowed to trade. Such a setup can capture the effects of front-running, which results in different results in particular for price overshooting. We limit our analysis to the two stage model since in most practical cases, there is little room for front-running due to legal constraints or insufficient time (i.e., stage 0 is very short); see the introduction for examples.
As another alternative, the model can account for a different trading horizon for each competitor. This increases the mathematical complexity, but does not lead to qualitatively new phenomena within stage 1.

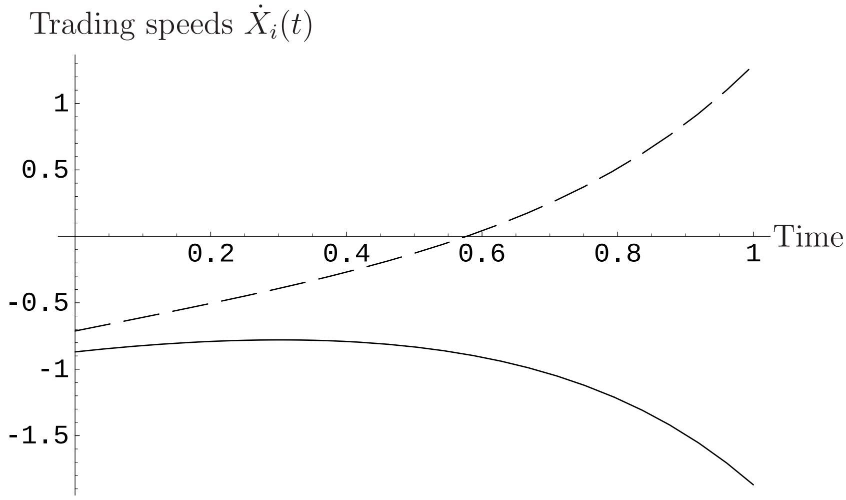


Figure 8.2: Trading speeds $\dot{X}_i(t)$ over time. The solid line represents the seller, the dashed line the competitor ($n = 1$).

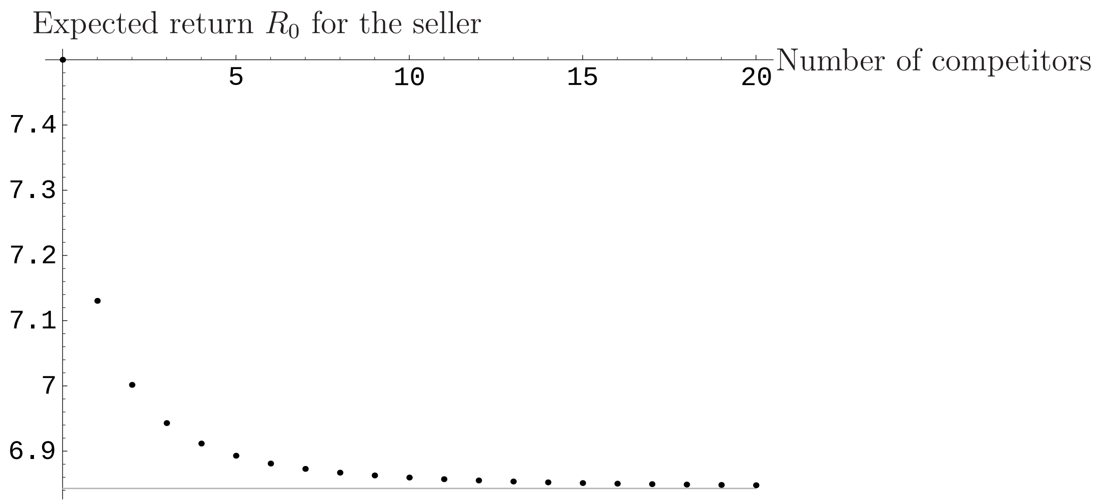


Figure 8.3: Expected cash return for the seller (player 0) from selling X_0 shares, depending on the number n of competitors. The expected return in absence of competitors is 7.5 (intersection point of x- and y-axes). The grey line at the bottom corresponds to the limit $n \to \infty$.

The first stage runs from $t = 0$ to T_1, the second stage[17] from T_1 to T_2. The asset position of player i is denoted by $X_i(t)$ with $t \in [0, T_2]$. We require the strategies $X_i(t)$ to be differentiable within each stage, but they need not be differentiable at $t = T_1$.

[17] In reality, the seller usually has to liquidate an asset position by the end of the trading day. In this case, the second stage begins at the open of the next trading day. Our framework can easily be extended to accommodate for this setting by having the second stage run from $\tilde{T}_1 > T_1$ to T_2. Since we assumed that the seller and the competitors are risk-neutral, this does not change any of the statements in this exposition; for notational simplicity, we therefore restrict ourselves to the case where the second stage starts immediately after the first stage.

The market prices are governed by

$$P(t) = \tilde{P}(t) + \gamma \sum_{i=0}^{n} (X_i(t) - X_i(0)) + \lambda \sum_{i=0}^{n} \dot{X}_i(t)$$

for $t \in [0, T_2] \backslash T_1$. Again, $\tilde{P}(t)$ is a martingale, starting at $\tilde{P}(0) = \tilde{P}_0$. Since the $X_i(t)$ might be non-differentiable at $t = T_1$, the above formula might not be well-defined; we therefore set

$$P(T_1) = \lim_{t \searrow T_1} P(t), \qquad P(T_1-) := \lim_{t \nearrow T_1} P(t),$$

forcing the price to be right-continuous.

The seller (player 0) is assumed to liquidate an asset position $X_0 = X_0(0)$ during stage 1: $X_0(t) = 0$ for all $t \in [T_1, T_2]$. We assume that the n competitors want to exploit their knowledge of the seller's intentions, but do not want to change their asset position permanently. We therefore require that the competitors have the same asset positions at the beginning of stage 1 and at the end of stage 2: $X_i(0) = X_i(T_2)$. For notational simplicity, we assume[18] $X_i(0) = 0$. All assumptions and notation introduced in Section 8.2 apply in our two stage model; in particular, we restrict our analysis to risk-neutral players[19] following deterministic strategies.

There are no a-priori restrictions on competitors' asset positions $X_i(T_1)$ at the end of stage 1. They can be positive, i.e., the competitors buy some of the seller's shares in stage 1 and thereby provide liquidity to the seller. Alternatively, they can be negative, i.e., the competitors sell parallel to the seller, driving the market price further down and preying on the seller. In the next section, we show that the occurrence of liquidity provision or predation depends on the market characteristics, in particular on the balance between temporary and permanent impact.

8.5 Optimal strategies in the two stage model

We can now describe the optimal behavior of all $n+1$ strategic players in the two stage model introduced in the previous section. If the optimal asset positions $X_i(T_1)$ of the competitors at the end of stage 1 are known, the entire optimal strategies are determined by Theorem 8.1: In stage 1, $n + 1$ players are trading and the initial and final asset positions are known; in stage 2, n players are trading and again the initial and final asset positions are known[20].

[18]The optimal trading speed $\dot{X}_i(t)$ of the competitors is independent of their initial asset position $X_i(0)$. In particular, our results also hold in the case where competitors have different initial asset positions.

[19]Risk aversion can be incorporated in two different ways. The first is to regard the different execution time frame of the seller and the competitors as proxies of their risk aversion. This provides a simple model of a highly risk averse seller in a market environment with relatively risk-neutral competitors. Alternatively, risk aversion can explicitly be modeled by introducing utility functions for the seller and the competitors. This leads to the coexistence of liquidity provision and preying already in the one stage model introduced in Section 8.3. The dynamics for a risk averse seller facing relatively risk-neutral competitors is qualitatively very similar to the two stage model presented here. A detailed discussion of the effects of risk aversion lie beyond the scope of this chapter and are subject of ongoing research.

[20]In the case $n = 1$, it follows from the results in Almgren and Chriss (2001) and Almgren (2003) that the optimal trading strategy in stage 2 is a linear increase / decrease of the competitor's asset position.

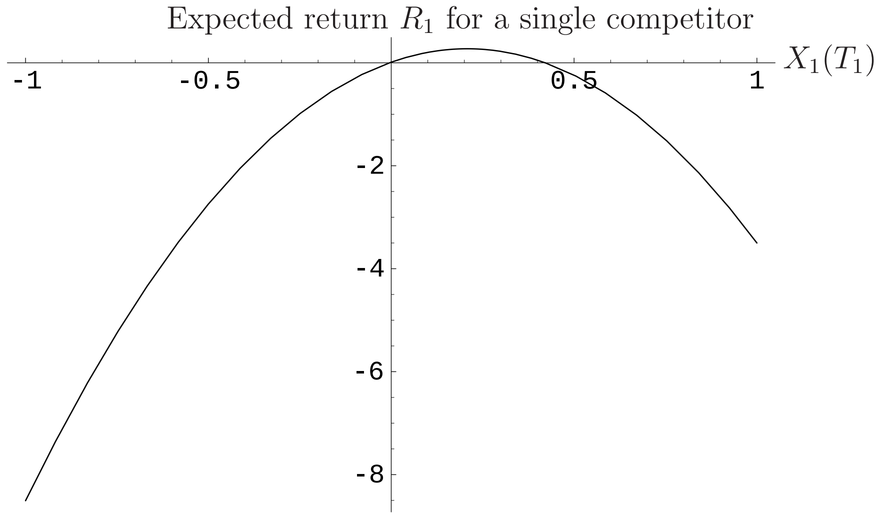


Figure 8.4: Expected return R_1 for a single competitor depending on her asset position $X_1(T_1)$. Optimal trading within stage 1 and stage 2 is assumed. Parameters are chosen as in the elastic market in Table 8.2.

Therefore, we only need to derive the optimal asset positions[21] $X_i(T_1)$ for all competitors $i = 1, 2, ..., n$ (see Figure 8.4 for an illustration).

Theorem 8.2. *In the unique Nash equilibrium, all competitors acquire the same asset position during stage 1:*

$$X_i(T_1) = F\left(\frac{\gamma T_1}{\lambda}, \frac{T_2}{T_1}, n\right) X_0. \tag{8.2}$$

The function F is given in closed form in the proof in Appendix 8.B. For the special case $n = 1$, we obtain

$$X_1(T_1) = -\frac{\left(-2 - e^{\frac{\gamma T_1}{3\lambda}} - e^{\frac{2\gamma T_1}{3\lambda}} + e^{\frac{\gamma T_1}{\lambda}}\right)\frac{\gamma}{\lambda}(T_2 - T_1)}{6\left(-1 + e^{\frac{\gamma T_1}{\lambda}}\right) + \left(2 + e^{\frac{\gamma T_1}{3\lambda}} + e^{\frac{2\gamma T_1}{3\lambda}} + 2e^{\frac{\gamma T_1}{\lambda}}\right)\frac{\gamma}{\lambda}(T_2 - T_1)} X_0. \tag{8.3}$$

Formulas (8.2) and (8.3) do not depend on γ and λ separately, but only on the fraction[22] $\frac{\gamma T_1}{\lambda} = \frac{\gamma}{\lambda/T_1}$, which can be interpreted as a normalized ratio of liquidity parameters. The permanent impact parameter γ has unit “dollars per share” and is independent of the time unit. The temporary impact parameter λ has unit “dollars per share per time unit” and thus depends on the time unit. The fraction λ/T_1 can be interpreted as the temporary impact parameter normalized to the length of the first stage.

In the next section, we will analyze the qualitative influence of the ratio $\frac{\gamma T_1}{\lambda}$ by reviewing some specific example markets. For notational simplicity, we will implicitly assume that

[21] Carlin, Lobo, and Viswanathan (2007) noted this for the single competitor case. They also conjectured that in a two stage model there will be price overshooting. As we will see in Section 8.6 and Proposition 8.9, the source of this price overshooting is not necessarily the presence of strategic players. In fact, price overshooting is reduced by competitors in elastic markets.

[22] Since the dependence of F on n is non-reciprocal, the joint strategy of the competitors changes as the number of competitors increases (see also the dependence on n in Theorem 8.1), resulting in a reduced joint profit of the competitors. Hence, the competitors have an incentive to collude.

Parameter	Elastic market	Plastic market	Intermediate market
Asset position X_0		1	
Initial price $\tilde{P}_0$		10	
Duration T_1 of stage 1		1	
Duration $T_2 - T_1$ of stage 2		1	
Permanent impact sensitivity γ	1	3	1.8
Temporary impact sensitivity λ	3	1	1

Table 8.2: Parameter values used for numerical computation in Section 8.6.

$T_1 = 1$ and thus restrict our discussion to γ and λ. We will return to the general situation again in Section 8.7.

8.6 Example markets

8.6.1 Definition of the example markets

In an illiquid market, each market order causes a price impact. Some part of this initial price impact is temporary and therefore vanishes after the execution of the market order. In the following, we will analyze two polar market extremes in more detail:

- *Elastic markets*, in which the major part of the initial total market impact vanishes after the execution of a market order (i.e., temporary impact $\lambda >>$ permanent impact γ). The market price in such markets behaves similar to an elastic rubber band: trading pressure can stretch it, but after the trading pressure reduces, the price recovers.

- *Plastic markets*, in which the price impact of market orders is predominantly permanent (i.e., permanent impact $\gamma >>$ temporary impact λ). In such markets, the trading pressure exerts a "plastic deformation" on the market price.

Empirical studies report that markets are indeed sometimes plastic and sometimes elastic[23]. In many practical cases however, the market will fall into neither of these two categories, but instead temporary and permanent impact will be balanced; we therefore conclude our case analysis by reviewing an *intermediate market*, that is, a market where temporary and permanent impact are balanced: $\lambda \approx \gamma$. For the numerical computations, we used the parameter values given in Table 8.2.

[23]Holthausen, Leftwich, and Mayers (1987) find that for their data sample, 75% of the total price impact of large transactions was temporary, while the follow-up study Holthausen, Leftwich, and Mayers (1990) finds that for a different sample, 85% of the total price impact was permanent. Coval and Stafford (2007) show that in markets where investors withdraw their money from open-ended mutual funds, the total price impact of transactions is predominantly temporary, while in other markets the price impact is predominantly permanent. The anecdotal evidence presented in the introduction indicates that the market for derivatives traded by LTCM was plastic, whereas the energy market was elastic during the Amaranth crisis.

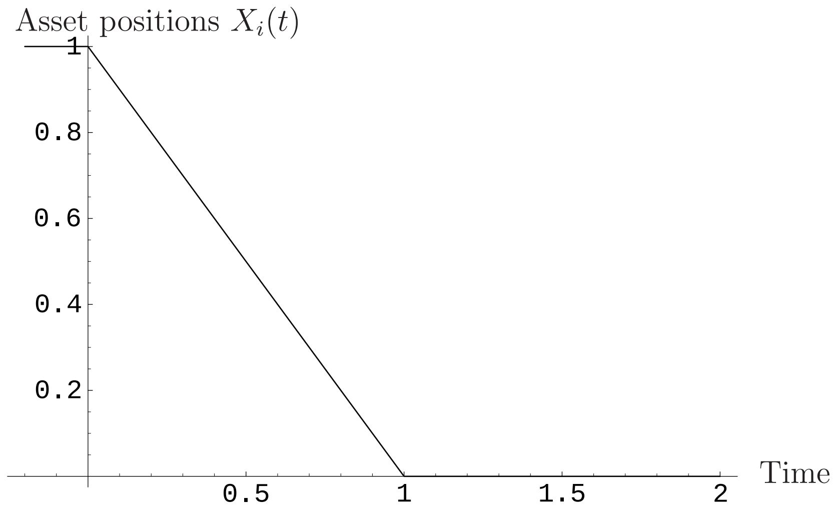


Figure 8.5: Asset position $X_0(t)$ of the seller when no competitors are active.

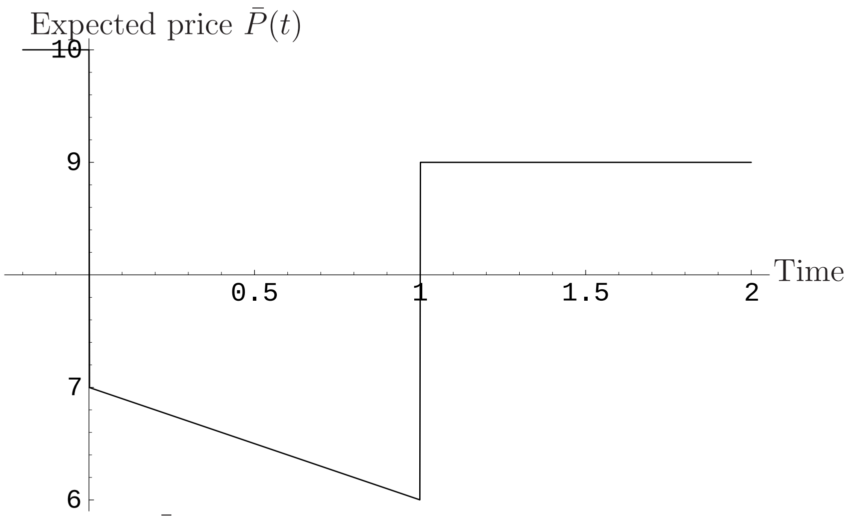


Figure 8.6: Expected price $\bar{P}(t)$ in an elastic market over time when no competitors are active; at time $t = 1$, stage 1 ends and stage 2 begins.

8.6.2 Example market 1: Elastic market

To begin with, let us assume that no competitors are active in the market. In such a situation, it is optimal for the seller to sell her asset position linearly (Figure 8.5). We therefore expect that the market price in stage 1 drops dramatically (Figure 8.6), since in order to satisfy the seller's trading needs, liquidity is required fast — which is expensive in an elastic market. In stage 2, no selling pressure from the seller exists any more; hence, the market price will bounce back. Furthermore, since the permanent impact is comparatively small, it will bounce back almost completely.

A competitor knowing of the seller's intentions would expect this price pattern. Her natural reaction would therefore be to buy some of the seller's shares in stage 1 at the very low price and to sell them in stage 2 at the much higher price. Figure 8.7 shows that this is indeed what happens when the seller and the competitors follow their optimal strategies.

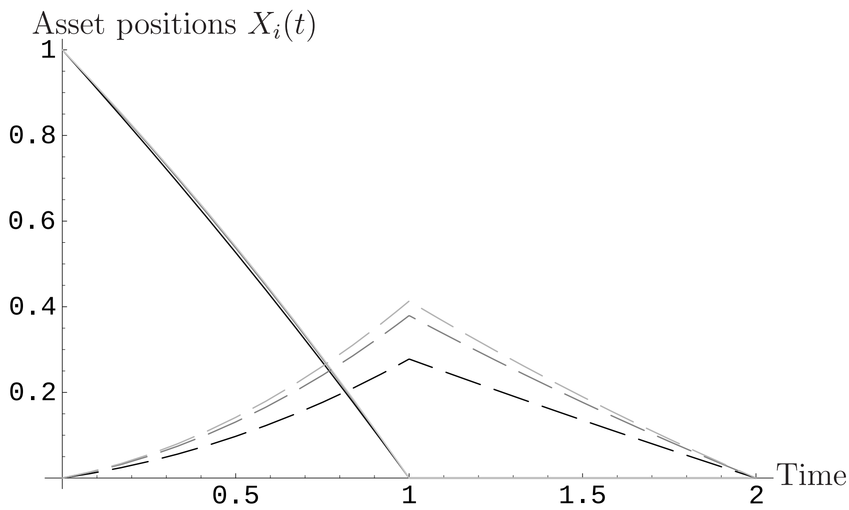


Figure 8.7: Asset positions $X_i(t)$ over time in an elastic market; at time $t = 1$, stage 1 ends and stage 2 begins. The solid lines represents the seller, the dashed lines the combined asset position of all n competitors. The black lines correspond to $n = 2$, the dark grey lines to $n = 10$ and the light grey lines to $n = 100$.

As can be seen in these figures, the total asset position $\sum_{i=1}^{n} X_i(T_1)$ acquired by the competitors at the end of stage 1 increases as the number of competitors increases (see also Figure 8.8). To gain some intuition for this phenomenon, let us assume that n_1 competitors optimally acquire a joint asset position of $n_1 Y_1$ shares. Imagine one of the competitors increases her target asset position by 1. This will decrease the profit per share that she makes, but adds another share to her profitable portfolio. If the original target position Y_1 is optimal, then this increase will leave her total profit roughly unchanged:

$$\text{Profit per share} \times 1 - \text{Decrease in profit per share} \times Y_1 \approx 0.$$

Let us now assume that $n_2 > n_1$ competitors are active and that they jointly acquire $n_1 Y_1$ shares. Now, increasing the target position $\frac{n_1 Y_1}{n_2}$ of an individual competitor by one share changes the competitor's total profit by

$$\text{Profit per share} \times 1 - \text{Decrease in profit per share} \times \frac{n_1 Y_1}{n_2} > 0.$$

Therefore each competitor has an incentive to increase the trading target for the end of stage 1, resulting in an increased joint trading target.

The effect of the competitors' trading (buying in stage 1, selling in stage 2) is that prices between stage 1 and stage 2 will even out; the large price jumps expected in the absence of competitors will disappear if the number of competitors is large enough (see Figure 8.9). The price overshooting created by the selling pressure of the seller is therefore reduced by the competitors.

From the seller's perspective, the competitors' trading is beneficial; by buying some of her shares, the competitors reduce the seller's market impact and thus increase her return. As we have just discussed, a larger number of competitors implies a larger combined purchase by the competitors. Hence, the seller can expect to profit from each additional competitor, i.e., the larger the number of competitors, the larger her profit. This is illustrated by

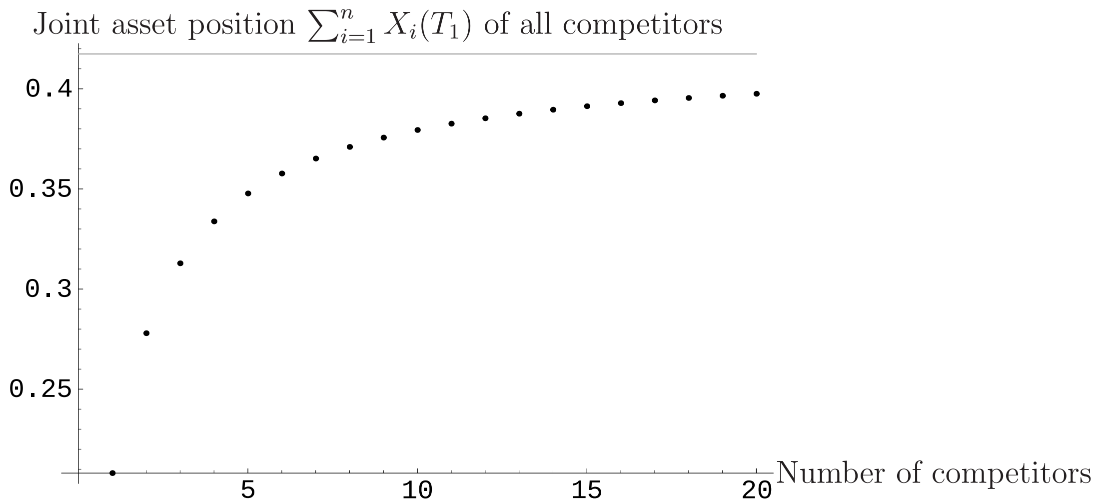


Figure 8.8: Joint asset position $\sum_{i=1}^{n} X_i(T_1)$ of all competitors in an elastic market at time T_1 depending on the total number n of all competitors. The grey line represents the limit $\lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1)$.

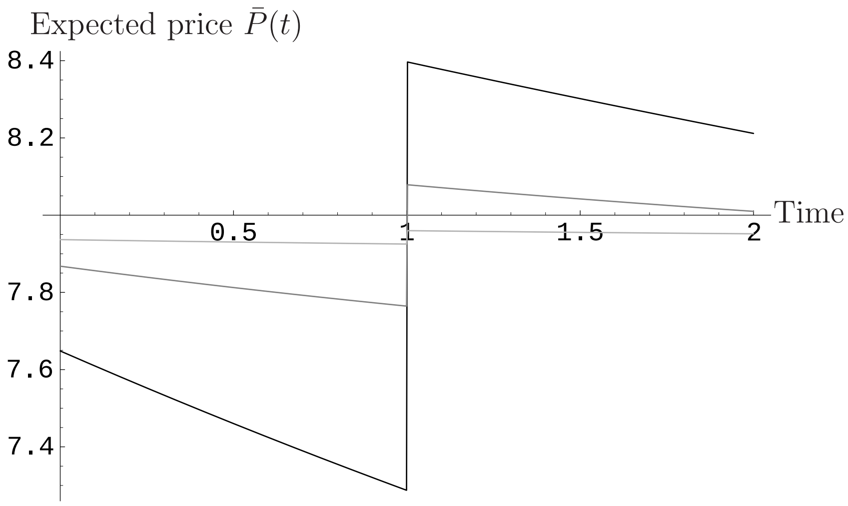


Figure 8.9: Expected price $\bar{P}(t)$ in an elastic market over time depending on the number of competitors n; at time $t = 1$, stage 1 ends and stage 2 begins. The black line corresponds to $n = 2$, the dark grey line to $n = 10$ and the light grey line to $n = 100$. A significant reduction in price drift can be observed; furthermore, $\bar{P}(0)$ is smaller than $\tilde{P}_0 = 10$.

Figure 8.10; the seller's return is higher when competitors are active than it is when there are no competitors.

The practical implications are evident: in an elastic market, it is sensible to announce any large, time-constrained asset transaction directly at the beginning of trading in order to attract liquidity.

8.6.3 Example market 2: Plastic market

We will now turn to plastic markets, i.e., markets with a permanent impact that considerably exceeds the temporary impact. In such a setting, we expect the price dynamics to be very

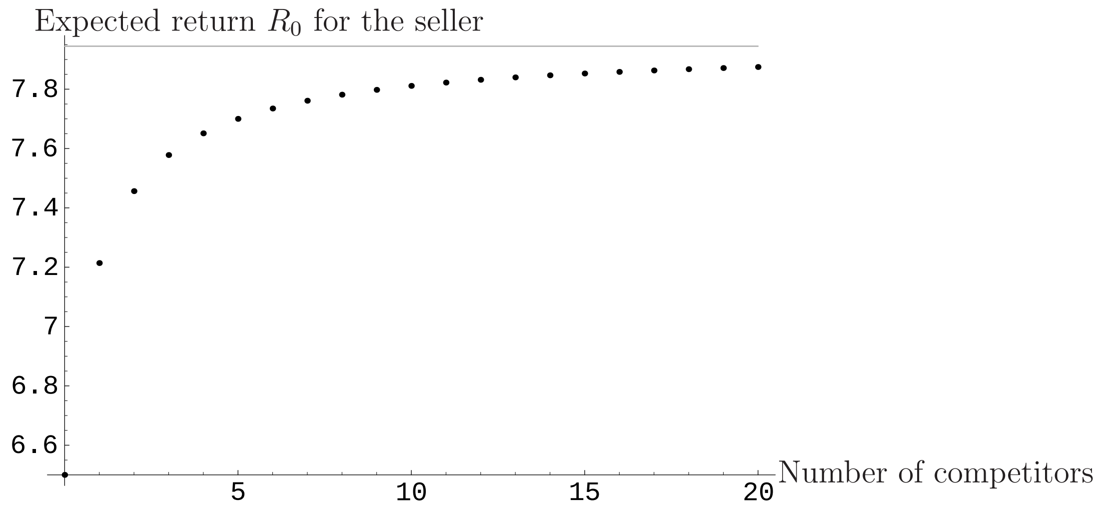


Figure 8.10: Expected return R_0 for the seller in an elastic market, depending on the number of competitors. The grey line represents the limit $n \to \infty$. The return for the seller without competitors is at the intersection of x- and y-axis.

different from the dynamics described for elastic markets in the previous section.

Let us again assume that no competitors are active. Then, the optimal trading strategy for the seller is again a linear decrease of the asset position (see Figure 8.5). In stage 1, the seller is constantly pushing the market price further and further down; we therefore expect the price to be high at the beginning of stage 1 and low at the end of stage 1 (see Figure 8.11). In stage 2, the price will bounce back, since the temporary impact of the seller's trading has vanishes. However, this jump will be comparatively small because the temporary price impact is small.

For a competitor, this implies that buying some of the seller's shares in stage 1 does not promise any large profit; the price reversion in stage 2 is too small. Instead, it appears more profitable to exploit the price changes *within* stage 1 instead of the price changes *between* stage 1 and stage 2. By selling short the asset at the beginning of stage 1 and buying it back at the end of stage 1, she can likely make a large profit. Thus, we expect to see preying behavior similar to the behavior in the one stage framework discussed in Section 8.3. Our hypothesis is verified by the numerical results shown in Figure 8.12.

It might be surprising that the asset position $X_i(T_1)$ of the competitors at the end of the first stage changes from a short position to a long position as the number of competitors increases. This can be explained in the following way. For a small number of competitors the price evolution will be sufficiently close to the one shown in Figure 8.11, therefore preying is attractive and the competitors will enter stage 2 with a short position. As the number of competitors increases, the price curve flattens within the first stage due to the increased competition for profit from predatory trading[24] (Figure 8.13). In comparison, the recovery of prices between stage 1 and stage 2 now becomes attractive, even though it is relatively small. Similar to the line of argument in elastic markets, it now pays off for the competitors to acquire a small asset position during stage 1 in order to sell it during stage 2. This is illustrated in Figure 8.14. If the number of competitors is small, it is beneficial to enter

[24]See also Proposition 8.A.1 in the appendix.

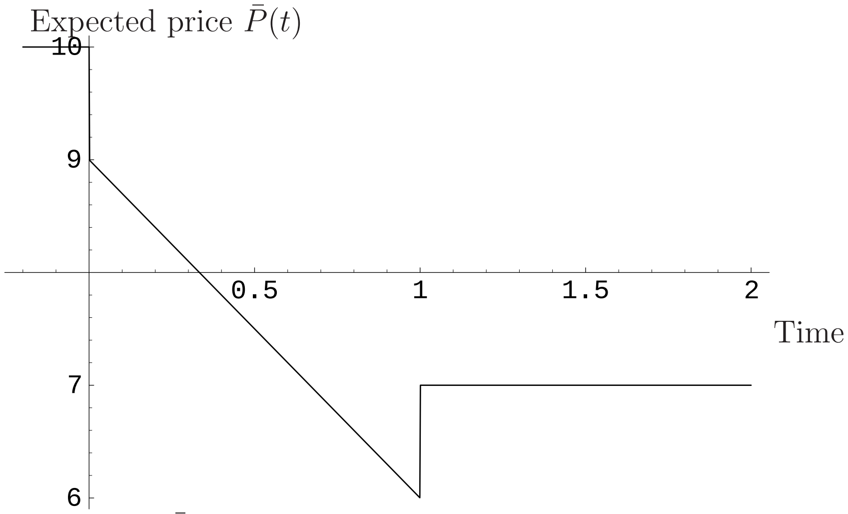


Figure 8.11: Expected price $\bar{P}(t)$ in a plastic market over time when no competitors are active; at time $t = 1$, stage 1 ends and stage 2 begins.

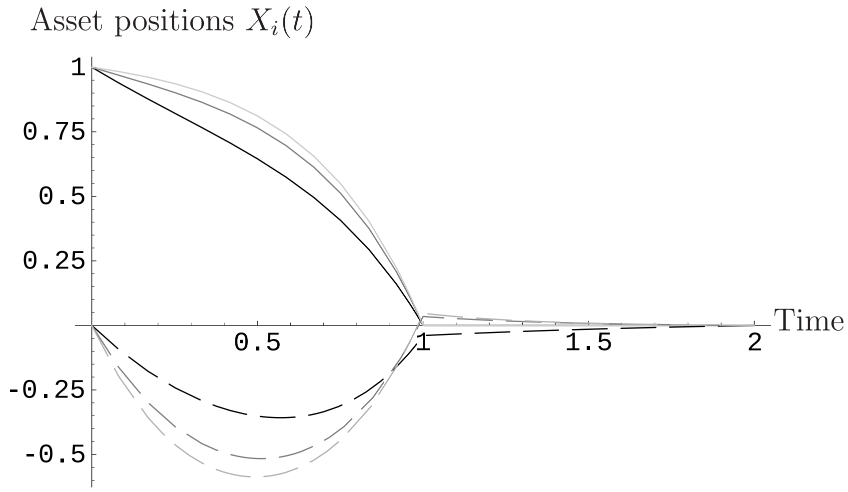


Figure 8.12: Asset positions $X_i(t)$ over time in a plastic market; at time $t = 1$, stage 1 ends and stage 2 begins. The solid lines represents the seller, the dashed lines the combined asset position of all n competitors. The black lines correspond to $n = 2$, the dark grey lines to $n = 10$ and the light grey lines to $n = 100$.

stage 2 with a short position; if the number of competitors is large, it is more attractive to enter stage 2 with a long position.

Based on this line of argument, we expect the price overshooting to disappear if the number of competitors is large. A single competitor however can decrease or increase price overshooting, depending on how plastic the market is. In the plastic market considered in this section, even a single competitor reduces price overshooting; if the permanent impact is increased to 7.0 and all other parameters are unchanged, a single competitor increases price overshooting.

Similar to the results of Section 8.3, we might be tempted to expect that the return for the seller is decreasing as the number of competitors increases and predation becomes more fierce. Figure 8.15 shows that this is not the case. The return for the seller is significantly

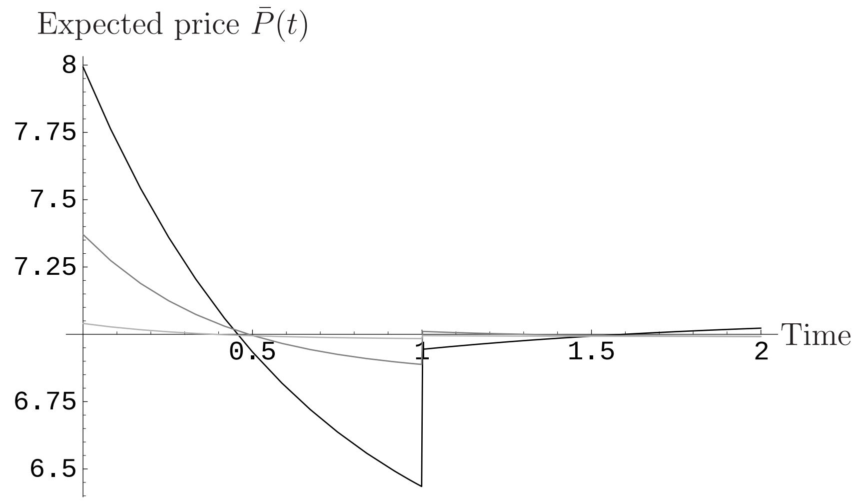


Figure 8.13: Expected price $\bar{P}(t)$ in a plastic market over time depending on the number of competitors n; at time $t = 1$, stage 1 ends and stage 2 begins. The black line corresponds to $n = 2$, the dark grey line to $n = 10$ and the light grey line to $n = 100$. A significant reduction in price drift can be observed.

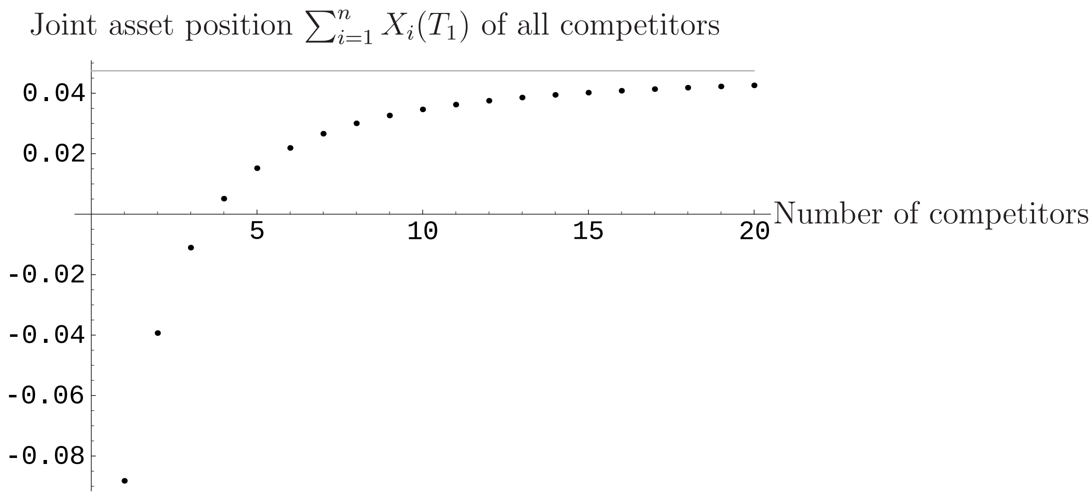


Figure 8.14: Joint asset position $\sum_{i=1}^{n} X_i(T_1)$ of all competitors in a plastic market at time T_1 depending on the total number n of all competitors. The grey line represents the limit $\lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1)$.

decreased by competitors; furthermore, two competitors decrease it more than a single competitor. However, the return for the seller is higher when three competitors are active than when only two competitors are active; as soon as at least two competitors are active, each additional competitor is beneficial for the seller.

The connection between the return for the seller and the number of competitors is a combination of effects from the one stage model and the two stage model in an elastic market. The first effect (already observed in the one stage model) is that a larger number of competitors leads to more aggressive preying and hence to a reduced return for the seller. This effect is very strong for a small number of competitors, but not for a large number of competitors. The second effect is that a larger number of competitors also results in an increased total asset position $\sum_{i=1}^{n} X_i(T_1)$ of all competitors at the end of stage 1. This

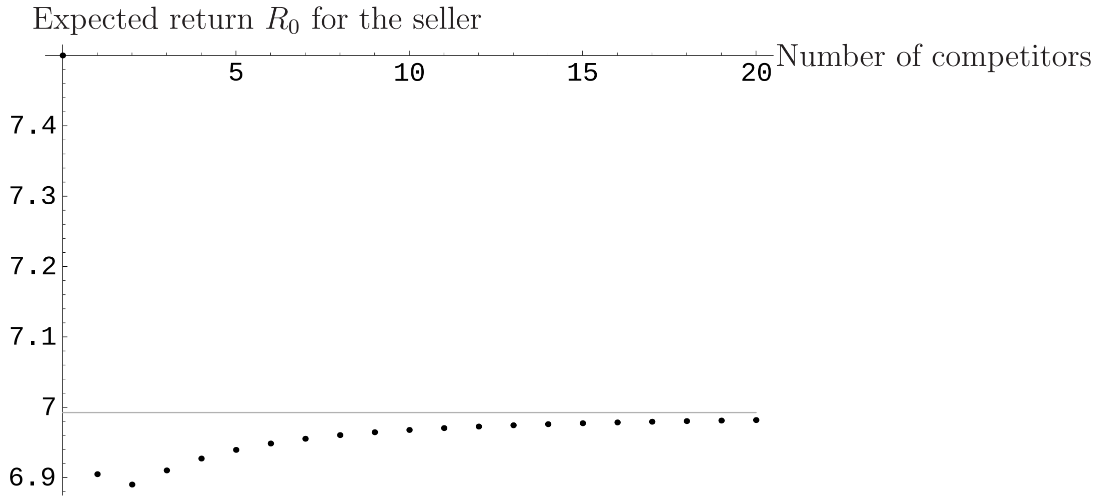


Figure 8.15: Expected return R_0 for the seller in a plastic market, depending on the number of competitors. The grey line represents the limit $n \to \infty$. The return for the seller without competitors is at the intersection of x- and y-axis.

reduces the trading pressure in stage 1 and therefore increases the return for the seller. The latter effect dominates the first if the number of competitors is large.

8.6.4 Example market 3: Intermediate market

In most cases, the differences between the temporary and permanent impact factors γ and λ will not be as extreme as depicted above. If the two parameters are closer together, we can expect to observe characteristics of both elastic as well as plastic markets:

- At the beginning of the first stage, the competitors "race the seller to market", that is, they sell in parallel to her. We say that *intra-stage predation* occurs.
- For a small number of competitors, the competitors end the first stage with either a long or a short position depending on whether the market is more elastic or more plastic (see Figure 8.16).
- For a large number of competitors, the competitors buy back more shares than they sold at the beginning of stage 1; we say that *inter-stage cooperation* takes place subsequently to the intra-stage predation.
- If the number of competitors is large, then prices do not overshoot. Instead, market prices are almost flat and almost the same in stage 1 and stage 2.
- If a certain minimum number of competitors is active, then additional competitors increase the return for the seller since the increase in inter-stage cooperation outweighs the increase in intra-stage predation.

All of these characteristics hold; we prove them in general in the next section. However, one interesting question remains open so far. We have already seen that in elastic markets

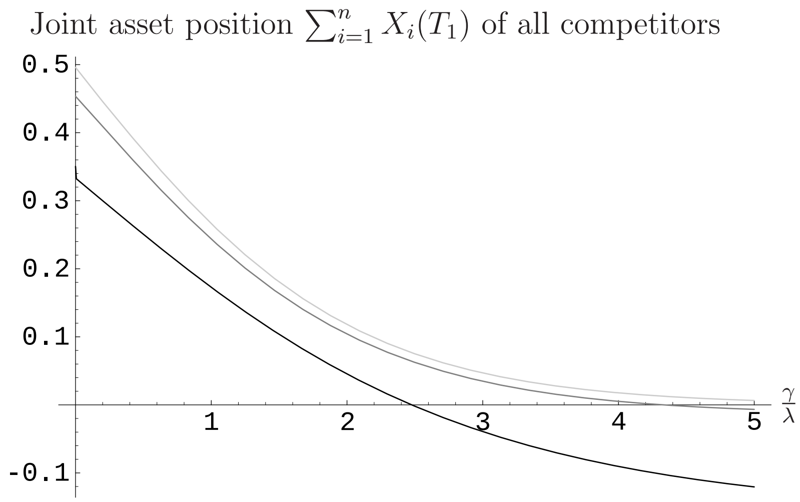


Figure 8.16: Asset position $X_1(T_1)$ of the competitors, depending on $\frac{\gamma}{\lambda}$. The black line corresponds to $n = 2$, the dark grey line to $n = 10$ and the light grey line to $n = 100$. The other parameters are chosen as in Table 8.2.

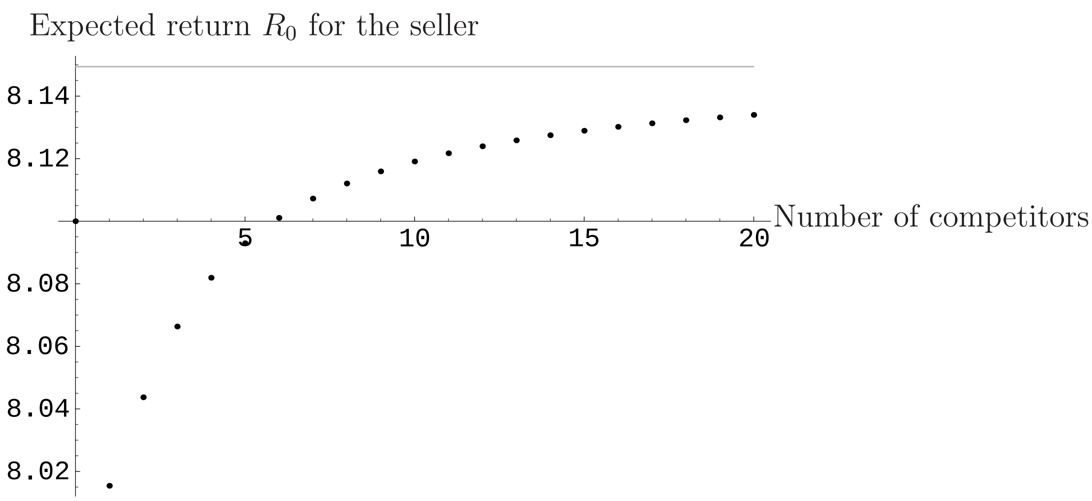


Figure 8.17: Expected return R_0 for the seller in an intermediate market, depending on the number of competitors. The grey line represents the limit $n \to \infty$. The return for the seller without competitors is at the intersection of x- and y-axis.

the seller benefits from competitors, whereas in plastic markets the seller prefers to have no competitors at all. What is the situation in an intermediate market? Of course, both effects may apply depending on whether the market is more plastic or more elastic in nature. However, a new phenomenon can also arise: It might be the case that a small number of competitors is harmful to the seller's profits, but a large number increases the profits even beyond the case of no predation (see Figure 8.17 for an example).

The practical implications are evident: If there are already some informed traders or if the seller expects to be able to attract a sufficient number of competitors, announcing her trading intentions can be attractive; if there is only a limited number of potential competitors she is best advised to conceal her intentions.

8.7 General properties of the two stage model

After having reviewed three explicit market examples, we summarize their common equilibrium properties.

8.7.1 Competitor behavior: Predatory trading versus liquidity provision

Proposition 8.3. *As the number of competitors n tends to infinity, the combined asset position of all competitors at the end of stage 1 converges to*

$$\lim_{n\to\infty}\sum_{i=1}^{n} X_i(T_1) = \lim_{n\to\infty} nX_1(T_1) = \frac{e^{\frac{\gamma(T_2-T_1)}{\lambda}}-1}{e^{\frac{\gamma(T_2)}{\lambda}}-1}X_0.$$

In economic terms, this implies that for large n, intra-stage cooperation between the seller and the competitors occurs regardless of the market parameters: in stage 1, the competitors buy a portion of the seller's asset position and sell this portion in stage 2. Thereby the market impact in stage 1 is reduced.

We can draw an intuitive consequence for elastic markets: If the number of competitors is high, then the net sale of seller and competitors in each stage is proportional to the time available for selling. The following corollary expresses this in mathematical terms when sending λ to ∞.

Corollary 8.4. *As the number of competitors n and the temporary price impact coefficient λ tend to infinity, the combined asset position of all competitors at time T_1 converges:*

$$\lim_{\lambda\to\infty}\lim_{n\to\infty}\sum_{i=1}^{n} X_i(T_1) = \frac{T_2-T_1}{T_2}X_0$$

We summarize the drivers of inter-stage cooperation.

Corollary 8.5. *For a large enough number n of competitors, the total net amount of liquidity $\sum_{i=1}^{n} X_i(T_1)$ provided by strategic players in stage 1 is*

- *decreasing in $\gamma T_1/\lambda$,*
- *increasing in T_2/T_1, and*
- *increasing in n.*

The first driver highlights the importance of the market environment; inter-stage cooperation is reduced in plastic markets[25]. The second driver relates to the influence of risk management. If the competitors have enough capital, they will be willing to hold inventory for a long period of time, i.e., $T_2 > T_1$. On the other hand, if they are in a financially weak condition, risk management is likely to limit the maximum holding period T_2 in order to reduce the associated risk. The third driver reflects the effect of limited competition among

[25] In the repeated game model of Carlin, Lobo, and Viswanathan (2007), the opposite result is obtained and cooperation is increased in plastic markets.

strategic players. By a combination of the latter two drivers, liquidity can disappear in a self-exciting vicious circle: Financial distress of some market participants can result in a general tightening of risk management practices and a smaller number of players engaging in strategic trading, leading to increased predatory trading and more distressed players.

8.7.2 Seller behavior: Stealth versus sunshine trading

We now turn to the return that the seller can expect to receive in a market with a certain number n of strategic competitors.

Theorem 8.6. *By selling an asset position X_0 in stage 1, the seller receives an average total cash position of*

$$R_0 = X_0 \left(\tilde{P}_0 - \gamma X_0 G \left(\frac{\gamma T_1}{\lambda}, \frac{T_2}{T_1}, n \right) \right).$$

The function G is given in closed form in the proof in Appendix 8.B. For large n, the seller's return is

- *decreasing in $\gamma T_1/\lambda$,*
- *increasing in T_2/T_1, and*
- *increasing in n.*

It converges to:

$$\lim_{n\to\infty} R_0 = X_0 \left(\tilde{P}_0 - \gamma X_0 \frac{1}{1 - e^{-\frac{\gamma}{\lambda} T_2}} \right)$$

The cash received in the limit case $n \to \infty$ is exactly the initial asset position multiplied by the limit of the expected market price derived in Proposition 8.8.

Given the result above, the benefits of sunshine trading can easily be quantified[26]. If the seller's intentions remain secret[27], she can expect a return of[28]

$$X_0 \left(\tilde{P}_0 - \gamma X_0/2 - \lambda X_0/T_1 \right). \tag{8.4}$$

Alternatively, she can pre-announce her intentions, attract a large number of competitors and thus expect a return of

$$X_0 \left(\tilde{P}_0 - \gamma X_0 \frac{1}{1 - e^{-\frac{\gamma}{\lambda} T_2}} \right). \tag{8.5}$$

[26] We assume that pre-announcing a trade does not change market-wide liquidity. In case sunshine traders are structurally special, this can be modeled by changing λ and γ for sunshine trades. For example, Admati and Pfleiderer (1991) assume that sunshine traders are uninformed; their trades should therefore result in a smaller (or possibly even no) permanent price change. This can be incorporated by assuming a smaller value for γ for sunshine trades.

[27] Even without pre-announcement, the market will try to infer the complete trading intentions from the trading pattern observable in the market. Barclay and Warner (1993) and Chakravarty (2001) find that the market reacts strongest to orders of medium size because such orders are most likely to be part of the execution of a large, informed transaction. Such observations should be taken into account when performing "stealth execution".

[28] See Almgren and Chriss (2001) and Almgren (2003) for a discussion of the case without competitors.

Corollary 8.7. *Assuming that pre-announcement attracts a large number of competitors ($n \approx \infty$), sunshine trading is superior to stealth trading if*

$$\frac{1}{2} + \frac{\lambda}{\gamma T_1} > \frac{1}{1 - e^{-\frac{\gamma}{\lambda} T_2}}. \tag{8.6}$$

If the competitors do not face any material time constraint ($T_2 \to \infty$), sunshine trading is beneficial if

$$\frac{\lambda}{\gamma} > \frac{T_1}{2}. \tag{8.7}$$

In our model, the ratio γ/λ of the market liquidity parameters γ and λ and the length of the two stages T_1 and $T_2 - T_1$ determine whether sunshine trading is beneficial. These drivers are not relevant in existing models. Most notably, sunshine trading is always beneficial in the model used by Admati and Pfleiderer (1991), while it is never beneficial in equilibrium in the model of Brunnermeier and Pedersen (2005).

8.7.3 Price evolution

We now analyze the market prices resulting from the combined trading activities of the seller and the competitors in more detail. In Figures 8.9 and 8.13, we observe that when trading commences in $t = 0$, the expected price jumps downward from its level $\bar{P}(0-) = \tilde{P}_0$ to $\bar{P}(0)$ due to the temporary impact of the selling. After the initial price jump, the expected price $\bar{P}(t)$ is exhibiting a downward trend. This indicates that our model market does not fulfill the strong form of the efficient markets hypothesis as introduced by Fama (1970): if relevant information is shared by only a *small* number of market participants, then this information is only slowly reflected in market prices. On the other hand, empirical evidence suggests that capital markets are efficient in the semi-strong sense. We would therefore expect that if the seller's intentions are known by a *sufficiently large* number of market participants, this information is instantaneously fully reflected in market prices. Public information can thus not be used to predict price changes. The following proposition states that this is indeed the case in our market model.

Proposition 8.8. *The absolute value of the drift $|\dot{\bar{P}}(t)|$ is a decreasing function of n. In the limit, the expected market price instantaneously jumps to*

$$\tilde{P}_0 - \frac{\gamma}{1 - e^{-\frac{\gamma(T_2)}{\lambda}}} X_0$$

and is constant from thereon throughout stage 1 and stage 2 until the end of stage 2.

In plastic markets, the initial price jump $|\bar{P}(0) - \tilde{P}_0|$ is an increasing function of the number n of competitors, while it is a decreasing function of n in elastic markets. It is interesting to note that the new equilibrium price $\tilde{P}_0 - \frac{\gamma}{1-e^{-\frac{\gamma(T_2)}{\lambda}}} X_0$ does not depend on whether the seller can trade in stage 2 (see Proposition 8.A.1).

To formally discuss price overshooting, we include the time after T_2 in our analysis, i.e., the time after the seller and the competitors have stopped trading. The temporary impact of the trades during $[0, T_2]$ vanishes immediately at T_2; therefore, only the permanent impact remains. The seller sold X_0 while the competitors did not change their asset positions. Therefore we obtain an expected market price of $\bar{P}(T_2+) = \tilde{P}_0 - \gamma X_0$ for the time after T_2. If during the trading phase $[0, T_2]$ the price drops below $\bar{P}(T_2+)$, i.e.,

$$\min_{t\in[0,T_2]} \bar{P}(t) - \bar{P}(T_2+) < 0$$

we say that the price *overshoots*. We can now describe the relationship between price overshooting and predatory activity.

Proposition 8.9. *The price $\bar{P}(t)$ attains its minimum in the interval $[0, T_2]$ at the end of the first stage:*

$$\min_{t\in[0,T_2]} \bar{P}(t) = \bar{P}(T_1-).$$

Price overshooting occurs irrespective of the presence of competitors:

$$\bar{P}(T_1-) < \bar{P}(T_2+).$$

The level of price overshooting $\bar{P}(T_2+) - \bar{P}(T_1-)$ is increased by competitors only in very plastic markets, i.e., only if the permanent impact is much larger than the temporary impact. In all other circumstances, price overshooting is reduced by competitors. If competitors are already active in the market ($n \geq 1$), then additional competitors reduce price overshooting irrespective of the market character.

It is interesting to compare our results to the models introduced by Brunnermeier and Pedersen (2005) and by Carlin, Lobo, and Viswanathan (2007). Preying introduces price overshooting in the first framework, but it reduces price overshooting in the latter (see Proposition 8.A.2); in our model, the effect of preying on price overshooting depends on the market. In all three models, price overshooting is reduced by additional competitors (assuming that at least one competitor is active).

8.A Propositions on the one stage model

We first state two propositions concerning the one stage model introduced in Section 8.3. These are used for comparison of the one stage model and the two stage model as well as in the proofs presented in Appendix 8.B.

Proposition 8.A.1. In the one stage model, the absolute value of the drift $|\dot{\bar{P}}(t)|$ is a decreasing function of n. In the limit case $n \to \infty$, the expected market price instantaneously jumps to

$$\tilde{P}_0 - \frac{\gamma}{1 - e^{-\frac{\gamma T_1}{\lambda}}} X_0$$

and is constant from thereon until the end of trading at time $t = T_1$.

Proof of Proposition 8.A.1. Using the notation from Theorem 8.1, the combined trading speed of the seller and all competitors amounts to

$$\sum_{i=0}^{n} \dot{X}_i(t) = \sum_{i=0}^{n} (a e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t} + b_i e^{\frac{\gamma}{\lambda}t}) = (n+1) a e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t}.$$

The change in combined asset position at time t is therefore:

$$\begin{aligned}\sum_{i=0}^{n}(X_i(t)-X_i(0)) &= \sum_{i=0}^{n}\int_0^t \dot{X}_i(s)ds = \int_0^t \sum_{i=0}^{n}\dot{X}_i(s)ds \\ &= \int_0^t (n+1)ae^{-\frac{n}{n+2}\frac{\gamma}{\lambda}s}ds = (n+1)\frac{n+2}{n}\frac{\lambda}{\gamma}a\left(1-e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t}\right).\end{aligned}$$

Now, we can compute the expected market price:

$$\begin{aligned}\bar{P}(t) &= \tilde{P}_0 + \gamma\sum_{i=0}^{n}(X_i(t)-X_i(0)) + \lambda\sum_{i=0}^{n}\dot{X}_i(t) \\ &= \tilde{P}_0 + \gamma(n+1)\frac{n+2}{n}\frac{\lambda}{\gamma}a(1-e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t}) + \lambda(n+1)ae^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t} \\ &= \tilde{P}_0 + \lambda\frac{n+1}{n}(n+2-2e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t})a \qquad (8.8) \\ &= \tilde{P}_0 + \lambda\frac{n+1}{n}(n+2-2e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t})\frac{n}{n+2}\frac{\gamma}{\lambda}\left(1-e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}\right)^{-1}\frac{-X_0}{n+1} \\ &= \tilde{P}_0 - \gamma X_0\frac{1}{1-e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}} + \gamma X_0\frac{2}{n+2}\frac{e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t}}{1-e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}} \qquad (8.9)\end{aligned}$$

Only the last term in the expression above is time dependent; its influence decreases with increasing n. In the limit, we obtain that the expected market price $\bar{P}(t)$ is constant:

$$\lim_{n\to\infty}\bar{P}(t) \equiv \tilde{P}_0 - \gamma X_0\frac{1}{1-e^{-\frac{\gamma}{\lambda}T_1}}.$$

□

Proposition 8.A.2. Without any competitors (i.e., nobody is aware of the seller's intentions), the price overshoots by $\lambda X_0/T_1$. If competitors are present, the price overshooting is reduced to

$$\frac{n}{n+2}\gamma X_0\frac{e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}}{1-e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}},$$

which is a decreasing function of the number n of competitors.

Proof of Proposition 8.A.2. Without any competitors, the optimal strategy for the seller is to liquidate her asset position linearly: $X_0(t) = (T_1 - t)X_0/T_1$. The market price thus drops to

$$\bar{P}(T_1-) = \tilde{P}_0 - \gamma X_0 - \lambda X_0/T_1.$$

and price overshooting amounts to $\lambda X_0/T_1$.

From Equation (8.8), we know the structure of $\bar{P}(t)$ when competitors are present and deduce that the market price decreases to

$$\bar{P}(T_1) = \tilde{P}_0 - \gamma X_0\frac{1}{1-e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}} + \gamma X_0\frac{2}{n+2}\frac{e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}}{1-e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}}.$$

Thus, the price overshoots with magnitude

$$\bar{P}(T_1) - \bar{P}(T_1-) = \frac{n}{n+2}\gamma X_0 \frac{e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}}{1-e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}}.$$

The monotonicity follows directly. □

8.B Proofs for propositions on the two stage model

The proofs of the theorems, propositions and corollaries presented in this chapter are given in order of appearance in the main body of text. In order to keep the proofs compact, they sometimes use results that are independently proven later in this appendix.

Proof of Theorem 8.2. The actual computations are lengthy; we will therefore only sketch the approach.

Let us first discuss the case $n = 1$, i.e., the seller is facing only one competitor. By computations similar to the ones in Proposition 8.A.1, we can express the expected market price $\bar{P}(t)$ as a *linear* function of the seller's asset position X_0 and the competitors asset position $X_1(T_1) = Z_1$ at the end of stage 1. Furthermore, by Theorem 8.1 the competitor's trading speed $\dot{X}_1(t)$ is linear in X_0 and Z_1. Therefore we can then calculate the return for the competitor in the two stages as *quadratic* functions of X_0 and Z_1:

$$Return_{Competitor} = Return_{Stage1}(X_0, Z_1) + Return_{Stage2}(X_0, Z_1)$$

Now, we can determine the optimal Z_1 by maximizing the quadratic function $Return_{Competitor}$, i.e., by determining the root of its derivative, which is a linear function in X_0.

Let us turn to the case $n \geq 2$, i.e., the seller is facing at least two competitors. We assume that $n-1$ competitors acquire optimal asset positions $X_i(T_1) = Y_i$ for $1 \leq i \leq n-1$ and solve for the optimal asset position $X_n(T_1) = Z_n$ for the last competitor. Similar to the case $n = 1$ discussed above, we can calculate the return for the last competitor as a *quadratic* function of $X_0 + \sum_{i=1}^{n-1} Y_i$ and Z_n:

$$Return_{Competitor_n} = Return_{Stage1}(X_0 + \sum_{i=1}^{n-1} Y_i, Z_n) + Return_{Stage2}(X_0 + \sum_{i=1}^{n-1} Y_i, Z_n)$$

We can again determine the optimal Z_n by maximizing $Return_{Competitor_n}$ and obtain a *linear* function of $X_0 + \sum_{i=1}^{n-1} Y_i$:

$$Z_n^{optimal} = f(X_0 + \sum_{i=1}^{n-1} Y_i)$$

Similarly we obtain the linear equations

$$Z_j^{optimal} = f(X_0 + \sum_{i=1, i\neq j}^{n} Y_i)$$

for all $1 \le j \le n$. Since we assumed that $(Y_1, \dots, Y_n)$ was optimal in the first place, we know that the optimal $Z_j^{optimal}$ has to be equal to Y_j; we therefore obtain

$$Y_j = f(X_0 + \sum_{i=1, i\neq j}^{n} Y_i) \tag{8.10}$$

for all $1 \le j \le n$. The set of linear equations (8.10) constitutes a symmetric, non-singular linear problem of n equations in n variables. Its unique solution therefore has to fulfill $Y_1 = \cdots = Y_n$ and these Y_i are a linear function of X_0. By computing this linear function precisely, we obtain the functional form

$$F\left(\frac{\gamma T_1}{\lambda}, \frac{T_2}{T_1}, n\right) = -\frac{A_2 n^2 + A_1 n + A_0}{B_3 n^3 + B_2 n^2 + B_1 n + B_0}$$

with parameters

$$\begin{aligned} A_0 = 2\Bigg(& - e^{\frac{\gamma(-T_1+(2+n)T_2)}{(1+n)\lambda}} - e^{\frac{\gamma(n(3+2n)T_1+(2+n)T_2)}{\left(2+3n+n^2\right)\lambda}} + \\ & e^{\frac{\gamma\left(\left(2+2n+n^2\right)T_1+n(2+n)T_2\right)}{\left(2+3n+n^2\right)\lambda}} + e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)^2T_2\right)}{\left(2+3n+n^2\right)\lambda}} + \\ & e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} - e^{\frac{\gamma\left(-nT_1+\left(2+5n+2n^2\right)T_2\right)}{\left(2+3n+n^2\right)\lambda}} + e^{\frac{n\gamma T_1+\gamma T_2}{\lambda+n\lambda}} - \\ & e^{\frac{\gamma T_1+n\gamma T_2}{\lambda+n\lambda}}\Bigg) \end{aligned}$$

$$\begin{aligned} A_1 = {} & 3e^{\frac{(2+n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - 3e^{\frac{(1+2n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - 3e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} + \\ & 3e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} - 2e^{\frac{\gamma(-T_1+(2+n)T_2)}{(1+n)\lambda}} - e^{\frac{n\gamma(-T_1+(2+n)T_2)}{\left(2+3n+n^2\right)\lambda}} + \\ & e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)T_2\right)}{\left(2+3n+n^2\right)\lambda}} - e^{\frac{\gamma\left(-(4+3n)T_1+(2+n)^2T_2\right)}{\left(2+3n+n^2\right)\lambda}} + \\ & 2e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} + e^{\frac{\gamma\left(-\left(2+4n+n^2\right)T_1+\left(2+5n+2n^2\right)T_2\right)}{\left(2+3n+n^2\right)\lambda}} + \\ & 2e^{\frac{n\gamma T_1+\gamma T_2}{\lambda+n\lambda}} - 2e^{\frac{\gamma T_1+n\gamma T_2}{\lambda+n\lambda}} \end{aligned}$$

$$\begin{aligned} A_2 = {} & e^{\frac{(2+n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - e^{\frac{(1+2n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} + \\ & e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} - e^{\frac{n\gamma(-T_1+(2+n)T_2)}{\left(2+3n+n^2\right)\lambda}} + e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)T_2\right)}{\left(2+3n+n^2\right)\lambda}} - \\ & e^{\frac{\gamma\left(-(4+3n)T_1+(2+n)^2T_2\right)}{\left(2+3n+n^2\right)\lambda}} + e^{\frac{\gamma\left(-\left(2+4n+n^2\right)T_1+\left(2+5n+2n^2\right)T_2\right)}{\left(2+3n+n^2\right)\lambda}} \end{aligned}$$

and

$$B_0 = -2\Bigg(2e^{\frac{(1+2n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} - e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} - e^{\frac{\gamma(-T_1+(2+n)T_2)}{(1+n)\lambda}} + e^{\frac{n\gamma(-T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}} - 2e^{\frac{\gamma(n(3+2n)T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}} + e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)T_2\right)}{(1+n)(2+n)\lambda}} + e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)^2T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} + e^{\frac{\gamma\left(-nT_1+\left(2+5n+2n^2\right)T_2\right)}{(1+n)(2+n)\lambda}} - 2e^{\frac{\gamma\left(-\left(2+4n+n^2\right)T_1+\left(2+5n+2n^2\right)T_2\right)}{(1+n)(2+n)\lambda}} + 2e^{\frac{n\gamma T_1+\gamma T_2}{\lambda+n\lambda}}\Bigg)$$

$$B_1 = 2e^{\frac{(2+n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} - e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} - e^{\frac{\gamma(-T_1+(2+n)T_2)}{(1+n)\lambda}} + e^{\frac{n\gamma(-T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}} + e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)T_2\right)}{(1+n)(2+n)\lambda}} - 2e^{\frac{\gamma\left(\left(2+2n+n^2\right)T_1+n(2+n)T_2\right)}{(1+n)(2+n)\lambda}} - 2e^{\frac{\gamma\left(-(4+3n)T_1+(2+n)^2T_2\right)}{(1+n)(2+n)\lambda}} + e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)^2T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} + e^{\frac{\gamma\left(-nT_1+\left(2+5n+2n^2\right)T_2\right)}{(1+n)(2+n)\lambda}} + 2e^{\frac{\gamma T_1+n\gamma T_2}{\lambda+n\lambda}}$$

$$B_2 = 2\Bigg(e^{\frac{(2+n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - 2e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} + e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} + e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma\left(\left(2+2n+n^2\right)T_1+n(2+n)T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)^2T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} + 2e^{\frac{\gamma\left(-nT_1+\left(2+5n+2n^2\right)T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma\left(-\left(2+4n+n^2\right)T_1+\left(2+5n+2n^2\right)T_2\right)}{(1+n)(2+n)\lambda}} + e^{\frac{n\gamma T_1+\gamma T_2}{\lambda+n\lambda}}\Bigg)$$

$$B_3 = -e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} + e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} + e^{\frac{\gamma(-T_1+(2+n)T_2)}{(1+n)\lambda}} - e^{\frac{n\gamma(-T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}} + e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)^2T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} + e^{\frac{\gamma\left(-nT_1+\left(2+5n+2n^2\right)T_2\right)}{(1+n)(2+n)\lambda}}$$

Note that the denominator $B_3n^3 + B_2n^2 + B_1n + B_0$ of the general expression

$$X_i(T_1) = -\frac{A_2n^2 + A_1n + A_0}{B_3n^3 + B_2n^2 + B_1n + B_0}X_0$$

is 0 in the case $n = 1$; however, the general expression as a whole converges for $n \to 1$ against the optimal value of $X_1(T_1)$ for $n = 1$ as given in Equation (8.3).

In the following proofs, we will need the limits $\lim_{n\to\infty} A_i$ and $\lim_{n\to\infty} B_i$. All of these limits exist and can be established by direct calculations. We obtain:

$$\begin{aligned}
\lim_{n\to\infty} A_0 &= 2e^{\frac{\gamma T_1}{\lambda}}\Big(-1+e^{\frac{\gamma T_1}{\lambda}}\Big)\Big(-1+e^{\frac{\gamma(T_2-T_1)}{\lambda}}\Big)^2\\
\lim_{n\to\infty} A_1 &= -3\Big(-1+e^{\frac{\gamma T_1}{\lambda}}\Big)\Big(-1+e^{\frac{\gamma(T_2-T_1)}{\lambda}}\Big)^2\\
\lim_{n\to\infty} A_2 &= -\Big(-1+e^{\frac{\gamma T_1}{\lambda}}\Big)\Big(-1+e^{\frac{\gamma(T_2-T_1)}{\lambda}}\Big)^2\\
\lim_{n\to\infty} B_0 &= -2\Big(-1+e^{\frac{\gamma T_1}{\lambda}}\Big)\Big(-1+e^{\frac{\gamma(T_2-T_1)}{\lambda}}\Big)\Big(-1+2e^{\frac{\gamma T_1}{\lambda}}-2e^{\frac{\gamma(T_2-T_1)}{\lambda}}+e^{\frac{\gamma T_2}{\lambda}}\Big)\\
\lim_{n\to\infty} B_1 &= \Big(-1+e^{\frac{\gamma T_1}{\lambda}}\Big)\Big(-1+e^{\frac{\gamma(T_2-T_1)}{\lambda}}\Big)\Big(-1+e^{\frac{\gamma T_2}{\lambda}}\Big)\\
\lim_{n\to\infty} B_2 &= 4\Big(-1+e^{\frac{\gamma T_1}{\lambda}}\Big)\Big(-1+e^{\frac{\gamma(T_2-T_1)}{\lambda}}\Big)\Big(-1+e^{\frac{\gamma T_2}{\lambda}}\Big)\\
\lim_{n\to\infty} B_3 &= \Big(-1+e^{\frac{\gamma T_1}{\lambda}}\Big)\Big(-1+e^{\frac{\gamma(T_2-T_1)}{\lambda}}\Big)\Big(-1+e^{\frac{\gamma T_2}{\lambda}}\Big)
\end{aligned}$$

□

Proof of Proposition 8.3. We apply Theorem 8.2 and obtain:

$$\lim_{n\to\infty}\sum_{i=1}^{n} X_i(T_1) = -\frac{\lim_{n\to\infty} A_2}{\lim_{n\to\infty} B_3}X_0.$$

From the proof of Theorem 8.2, we know the values of the limits of A_2 and B_3 and the desired result follows. □

Proof of Corollary 8.4. Using Proposition 8.3 and L'Hospitale's rule, we calculate

$$\lim_{\lambda\to\infty}\lim_{n\to\infty}\sum_{i=1}^{n} X_i(T_1) = \lim_{\lambda\to\infty}\frac{e^{\frac{\gamma(T_2-T_1)}{\lambda}}-1}{e^{\frac{\gamma T_2}{\lambda}}-1} = \frac{T_2-T_1}{T_2}.$$

□

Proof of Corollary 8.5. We observe that by Theorem 8.2 all derivatives of $X_i(T_1)$ converge locally uniformly. Hence, we have

$$\lim_{n\to\infty}\frac{d}{d\gamma}X_i(T_1) = \frac{d}{d\gamma}\lim_{n\to\infty}X_i(T_1),$$

and by computing the derivatives of $\lim_{n\to\infty} X_i(T_1)$ using Proposition 8.3 we obtain the first two relations of the corollary.

Similar to the proof of Theorem 8.6, it can be shown that for large n, $X_i(T_1)$ is increasing in n. This shows the last of the three relations stated in the corollary. □

Proof of Theorem 8.6. Using Theorems 8.1 and 8.2 and Propositions 8.A.1 and 8.8, we can calculate the return for the seller in a straightforward way and obtain:

$$\begin{aligned} R_0 &= X_0\left(\tilde{P}_0 - \gamma X_0 \frac{A_7n^7 + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0}{B_7n^7 + B_6n^6 + B_5n^5 + B_4n^4 + B_3n^3 + B_2n^2 + B_1n + B_0}\right) \\ &=: X_0\left(\tilde{P}_0 - \gamma X_0 G\left(\frac{\gamma T_1}{\lambda}, \frac{T_2}{T_1}, n\right)\right) \end{aligned} \tag{8.11}$$

The coefficients A_i and B_i are functions of $\frac{\gamma T_1}{\lambda}$, $\frac{T_2}{T_1}$ and n. They are of a similar structure as the coefficients derived in the proof of Theorem 8.2, but even more complex. The coefficients A_6, A_7, B_6 and B_7 are presented in Appendix A as examples; the other coefficients are omitted for brevity.

The coefficients A_i and B_i converge for $n \to \infty$; furthermore, their derivatives $\frac{dA_i}{dn}$ and $\frac{dB_i}{dn}$ converge to 0 as $n \to \infty$. We compute

$$\lim_{n\to\infty} R_0 = \lim_{n\to\infty} \mathbb{E}(\text{Return for the seller}) = X_0\left(\tilde{P}_0 - \gamma X_0 \frac{\lim_{n\to\infty} A_7}{\lim_{n\to\infty} B_7}\right).$$

Inserting A_7 and B_7 and computing the limit gives the desired limit.

To prove that $\lim_{n\to\infty} R_0$ is increasing for large n, we compute the derivative of the seller's return R_0 with respect to n as

$$\frac{d}{dn}R_0 = -\gamma X_0 \frac{\text{Numerator}}{(B_7n^7 + B_6n^6 + B_5n^5 + B_4n^4 + B_3n^3 + B_2n^2 + B_1n + B_0)^2}$$

with

$$\begin{aligned} \text{Numerator} = &\Bigg(7A_7B_7n + 7A_7B_6 + 6A_6B_7 + \frac{dA_7}{dn}B_7n^2 + \frac{dA_7}{dn}B_6n \\ &+ \frac{dA_7}{dn}B_5 + \frac{dA_6}{dn}B_7n + \frac{dA_6}{dn}B_6 + \frac{dA_5}{dn}B_7\Bigg)n^{12} \\ &- \Bigg(7B_7A_7n + 7B_7A_6 + 6B_6A_7 + \frac{dB_7}{dn}A_7n^2 + \frac{dB_7}{dn}A_6n \\ &+ \frac{dB_7}{dn}A_5 + \frac{dB_6}{dn}A_7n + \frac{dB_6}{dn}A_6 + \frac{dB_5}{dn}A_7\Bigg)n^{12} + o(n^{11}). \end{aligned}$$

For large n, we can omit the $o(n^{11})$ term; furthermore, we know that all derivatives converge to 0 as $n \to \infty$. We therefore obtain for large n:

$$\begin{aligned}\text{Numerator} \approx \Bigg(&\left(\frac{dA_7}{dn}B_7 - \frac{dB_7}{dn}A_7\right) n^2 \\ &+ \left(\frac{dA_7}{dn}B_6 + \frac{dA_6}{dn}B_7 - \frac{dB_7}{dn}A_6 n - \frac{dB_6}{dn}A_7\right) n \\ &+ A_7B_6 - B_7A_6 \Bigg) n^{12}\end{aligned}$$

Inserting the expressions for A_i and B_i, we obtain

$$\lim_{n\to\infty} \left(\frac{dA_7}{dn}B_7 - \frac{dB_7}{dn}A_7\right) n^2 = 0$$

$$\lim_{n\to\infty} \left(\frac{dA_7}{dn}B_6 + \frac{dA_6}{dn}B_7 - \frac{dB_7}{dn}A_6 n - \frac{dB_6}{dn}A_7\right) n = 0$$

$$\lim_{n\to\infty} (A_7B_6 - B_7A_6) = -e^{\frac{\gamma T_1}{\lambda}} \left(e^{\frac{\gamma T_1}{\lambda}} - 1\right)^7 \left(e^{\frac{\gamma (T_2 - T_1)}{\lambda}} - 1\right)^5 \left(e^{\frac{\gamma T_2}{\lambda}} - 1\right)^3 < 0.$$

The derivative of the seller's return has the opposite sign of the $Numerator$ and is thus positive for large values of n.

To prove that the seller's return is decreasing in $\gamma T_1/\lambda$ and increasing in T_2/T_1 for large n, we proceed similar to the proof of Corollary 8.5, observe that the derivatives of R_0 converge locally uniformly for $n \to \infty$ and obtain the desired relations by inspection of the limit $\lim_{n\to\infty} R_0$. □

Proof of Corollary 8.7. The condition

$$\frac{1}{2} + \frac{\lambda}{\gamma T_1} > \frac{1}{1 - e^{-\frac{\gamma}{\lambda} T_2}}$$

is obtained by direct comparison of the returns of sunshine and stealth trading given in Equations (8.4) and (8.5). Equation (8.7) can be derived by passing to the limit $T_2 \to \infty$. □

Proof of Proposition 8.8. First, we note that by arguments similar to the proof of Proposition 8.A.1 (in particular Formula (8.9)), the price during stage 1 ($t \in [0, T_1)$) is

$$\begin{aligned}\bar{P}(t) = \tilde{P}_0 - \gamma &\left(X_0 - \sum_{i=1}^n X_i(T_1)\right) \frac{1}{1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}} \\ + \gamma &\left(X_0 - \sum_{i=1}^n X_i(T_1)\right) \frac{2}{n+2} \frac{e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t}}{1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}}\end{aligned} \tag{8.12}$$

and the price during stage 2 ($t \in [T_1, T_2]$) is

$$\begin{aligned}\bar{P}(t) = \tilde{P}_0 &- \gamma \left(X_0 - \sum_{i=1}^{n} X_i(T_1) \right) - \gamma \left(\sum_{i=1}^{n} X_i(T_1) \right) \frac{1}{1 - e^{-\frac{n-1}{n+1}\frac{\gamma}{\lambda}(T_2 - T_1)}} \\ &+ \gamma \left(\sum_{i=1}^{n} X_i(T_1) \right) \frac{2}{n+1} \frac{e^{-\frac{n-1}{n+1}\frac{\gamma}{\lambda}(t - T_1)}}{1 - e^{-\frac{n-1}{n+1}\frac{\gamma}{\lambda}(T_2 - T_1)}}. \end{aligned} \tag{8.13}$$

Again, the time-dependent terms vanish as n increases. For the first stage, we obtain the limit

$$\begin{aligned}\lim_{n\to\infty} \bar{P}(t) = \tilde{P}_0 &- \gamma \left(X_0 - \lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1) \right) \frac{1}{1 - e^{-\lim_{n\to\infty} \frac{n}{n+2}\frac{\gamma}{\lambda}T_1}} \\ &+ \gamma \left(X_0 - \lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1) \right) \left(\lim_{n\to\infty} \frac{2}{n+2} \right) \frac{e^{-\lim_{n\to\infty} \frac{n}{n+2}\frac{\gamma}{\lambda}t}}{1 - e^{-\lim_{n\to\infty} \frac{n}{n+2}\frac{\gamma}{\lambda}T_1}} \\ = \tilde{P}_0 &- \gamma \left(X_0 - \frac{e^{\frac{\gamma(T_2 - T_1)}{\lambda}} - 1}{e^{\frac{\gamma T_2}{\lambda}} - 1} X_0 \right) \frac{1}{1 - e^{-\frac{\gamma}{\lambda}T_1}} \\ = \tilde{P}_0 &- \gamma X_0 \frac{e^{\frac{\gamma T_2}{\lambda}}}{e^{\frac{\gamma T_2}{\lambda}} - 1} \end{aligned}$$

For the second stage, we compute

$$\begin{aligned}\lim_{n\to\infty} \bar{P}(t) = \tilde{P}_0 &- \gamma \left(X_0 - \lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1) \right) \\ &- \gamma \left(\lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1) \right) \frac{1}{1 - e^{-\lim_{n\to\infty} \frac{n-1}{n+1}\frac{\gamma}{\lambda}(T_2 - T_1)}} \\ &+ \gamma \left(\lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1) \right) \lim_{n\to\infty} \frac{2}{n+1} \frac{e^{-\lim_{n\to\infty} \frac{n-1}{n+1}\frac{\gamma}{\lambda}(t - T_1)}}{1 - e^{-\lim_{n\to\infty} \frac{n-1}{n+1}\frac{\gamma}{\lambda}(T_2 - T_1)}} \\ = \tilde{P}_0 &- \gamma \left(X_0 - \frac{e^{\frac{\gamma(T_2 - T_1)}{\lambda}} - 1}{e^{\frac{\gamma T_2}{\lambda}} - 1} X_0 \right) - \gamma \left(\frac{e^{\frac{\gamma(T_2 - T_1)}{\lambda}} - 1}{e^{\frac{\gamma T_2}{\lambda}} - 1} X_0 \right) \frac{1}{1 - e^{-\frac{\gamma}{\lambda}(T_2 - T_1)}} \\ = \tilde{P}_0 &- \gamma X_0 \frac{e^{\frac{\gamma T_2}{\lambda}}}{e^{\frac{\gamma T_2}{\lambda}} - 1}. \end{aligned}$$

□

Proof of Proposition 8.9. By Formulas (8.12) and (8.13), it is easy to see that within each stage the price $\bar{P}(t)$ moves monotonously. Therefore, the only four possible times at which the minimum price can be achieved are T_0, T_1-, T_1 and T_2. It is straightforward to calculate the prices for these four points in time using Theorem 8.2 and Formulas (8.12) and (8.13), to show that $\bar{P}(T_1-)$ is the minimum of these four values and that it is lower than $\bar{P}(T_2+)$. Furthermore, it is direct to show that $\bar{P}(T_1-)$ is an increasing function of the number of competitors n.

The different effect of competitors on price overshooting in plastic and elastic markets is shown by the examples in Section 8.6. $\square$

Chapter 9

A cautionary note on multiple players in limit order book models

9.1 Introduction

In the previous chapter, we chose a multiple player liquidity model and then derived the optimal trading strategies for all players. By comparing optimal strategies in different liquidity models, we can identify the features of the liquidity model that are required to produce reasonable results in multiple player situations. In this chapter, we consider a multiple player extension of the limit order book model of Obizhaeva and Wang (2006). In this model, the price after a large trade does not recover instantaneously, but with a finite speed ("resilience").

We derive the optimal discrete-time trading strategies for two players in a one stage framework, i.e., a framework in which both players are trading during the same time span. As we observed in the previous chapter, this one stage framework is the fundamental building block of multiple stage setups. We consider both open-loop and closed-loop strategies and find that they exhibit the same qualitative properties in equilibrium. Surprisingly, if the discretization time step is small, the optimal strategy is to quickly buy and sell large asset positions. Such a strategy provides a hedge against market manipulations by the other player. The low cost of round trip trades in this market model encourages such oscillatory trading by two effects: First, the hedge trading strategy is relatively cheap, since it consists mainly of round trip trades. Second, market manipulation by round trip trades is also cheap and therefore a large and realistic threat for traders that do not pursue a hedge trading strategy. Both of these effects result in an incentive to pursue the hedge strategy.

By no means do we believe that the strategies derived in this chapter describe trading behavior in practice, nor do we recommend them to traders as normatively "rational" strategies. On the contrary, we believe that our results show that round trip costs are an important part of market models for multiple players. In reality, round trip costs are significant and grow with the size of the round trip (see Loeb (1983)). Such round trip costs can be modeled for example by including a completely temporary price impact component as in the market model considered in Chapter 8, or by modeling a trading dependent spread, e.g., by modeling the transient impacts on both sides of the limit order book independently.

In this chapter, we first introduce the market model (Section 9.2) and then solve for the unique open-loop equilibrium (Section 9.3) and closed-loop equilibrium (Section 9.4). In Section 9.5, we compare the findings of this chapter with those of Chapter 8 and suggest adjustments to the model considered in this chapter. Supplementary material to this chapter is contained in Appendix B of this dissertation. Section B.1 presents an example recursion function for the dynamic programming solution obtained in Section 9.4, and Section B.2

illustrates the stability of the findings in this chapter with additional numerical examples. The Mathematica source code used for generation of the figures in this chapter and in Appendix B is given in Section B.3.

9.2 Market model with finite resilience and no spread

In this section, we present a multiple player extension of the limit order book market model of Obizhaeva and Wang (2006). We limit our discussion to trading in discrete time; a generalization of the model to continuous time is straightforward. Similar to our approach in Section 8.2, we assume that the market consists of a risk-free asset without interest and a risky asset. In this market, two strategic players and a number of noise traders are trading. We denote the asset positions of the two strategic players at time t_n by X_n and Y_n and assume that they are both aware of their initial asset positions X_0 and Y_0 and their target asset positions $X_{N+1} = Y_{N+1} = 0$. The (sell) orders[1] $\dot{X}_n := -(X_{n+1} - X_n)$ and $\dot{Y}_n := -(Y_{n+1} - Y_n)$ at time t_n are executed at a market price that depends on previous trades through a linear permanent impact and a transient impact $D_n := D(t_n)$:

$$P(t_n) := P_n := \tilde{P}(t_n) - \gamma(X_0 - X_n + Y_0 - Y_n) - D_n - \frac{\gamma + \lambda}{2}(\dot{X}_n + \dot{Y}_n).$$

We already know the first two terms on the right hand side from the market model introduced in Section 3.2: they are the market price in absence of large traders modeled as a martingale $\tilde{P}$, and the permanent impact of previous trades. The third term reflects the transient price impact of previous trades, while the last term accounts for the average permanent and transient price impact of the current trade. We assume that the transient impact is changed in a linear way by each trade and decreases exponentially between trades:

$$D_{n+1} := (D_n + \lambda(\dot{X}_n + \dot{Y}_n))e^{-\rho(t_{n+1} - t_n)}.$$

The parameter $\rho \geq 0$ quantifies the resilience, i.e., the speed of price recovery. The structural difference between this market model and the market model we considered in Chapter 8 is that the transient impact D_n depends on the entire trade history, whereas the temporary impact in Chapter 8 only depends on the current trade. It is assumed that the market is in equilibrium at the beginning of trading, i.e., that $D_0 = 0$. This market model was first introduced by Obizhaeva and Wang (2006) as a limit order book model of liquidity: the transient price impact represents the limit orders that were consumed by previous market orders of the large trader, and the price resilience reflects the arrival of new limit orders in the order book. Alternatively, this model can be economically motivated as a market maker model with utility indifference pricing, where the market maker can continuously offload her inventory after large trades. We assume that at all points in time, there is only one market price, i.e., there is no spread. Again following our approach in Section 8.2, we assume that both players are risk neutral.

[1] In line with previous notation, we use the convention $\dot{X} > 0$ for sell orders.

9.3 Optimal open-loop strategies

First, we determine the optimal open-loop strategies for two players in this market, i.e., strategies that do not depend on the previous trades of the other player. Therefore, we assume in this section that both players choose deterministic strategies at the beginning of trading, i.e., strategies that only depend on time. As discussed in Section 7.3, we require that these strategies do not depend on P_n or $\tilde{P}_n$.

Theorem 9.1. *Let $\lambda, \rho, \tau > 0$. In the unique open-loop Nash equilibrium, both players follow a trading strategy that is linear in X_0 and Y_0:*

$$\dot{X}_n = \Phi_n X_0 + \Psi_n Y_0 \tag{9.1}$$

$$\dot{Y}_n = \Phi_n Y_0 + \Psi_n X_0 \tag{9.2}$$

with constants Φ_n and Ψ_n that only depend on γ and λ and are independent of X_0 and Y_0.

Proof. When player 1 follows the strategy $(X_n)_{0\le n\le N}$ and player 2 follows the strategy $(Y_n)_{0\le n\le N}$, then the expected cash proceeds for player 1 are given by

$$\begin{aligned} R_X &:= \mathbb{E}\left[\sum_{n=0}^{N} \dot{X}_n P_n\right] \\ &= X_0\tilde{P}_0 + \sum_{n=0}^{N} \dot{X}_n \Bigg(-\gamma\left(\sum_{i=0}^{n-1}\dot{X}_i + \sum_{i=0}^{n-1}\dot{Y}_i\right) \\ &\qquad - \Big(\sum_{i=0}^{n-1} \lambda(\dot{X}_i + \dot{Y}_i)e^{-\rho(t_n - t_i)}\Big) - \frac{\gamma+\lambda}{2}(\dot{X}_n + \dot{Y}_n)\Bigg). \end{aligned}$$

Similarly we define R_Y as the cash proceeds for player 2. Under the optimal strategy for player 1, the combined effect of reducing $\dot{X}_i$ and increasing $\dot{X}_0$ by the same amount either leaves R_X unchanged or decreases it; the same holds for the opposite change of the strategy, i.e., an increase in $\dot{X}_i$ and a reduction of $\dot{X}_0$. Therefore the optimal open-loop strategy for player 1 has to satisfy

$$\frac{dR_X}{d\dot{X}_i} = \frac{dR_X}{d\dot{X}_0}$$

for all $i = 1, \ldots, N$. Analogous equations need to hold for player 2. The optimal strategies for player 1 and player 2 therefore need to fulfill the linear equations

$$\begin{aligned}
\frac{dR_X}{d\dot{X}_1} &= \frac{dR_X}{d\dot{X}_0} \\
&\dots \\
\frac{dR_X}{d\dot{X}_N} &= \frac{dR_X}{d\dot{X}_0} \\
\frac{dR_Y}{d\dot{Y}_1} &= \frac{dR_Y}{d\dot{Y}_0} \\
&\dots \\
\frac{dR_Y}{d\dot{Y}_N} &= \frac{dR_Y}{d\dot{Y}_0} \\
\sum_{n=0}^{N} \dot{X}_n &= X_0 \\
\sum_{n=0}^{N} \dot{Y}_n &= Y_0.
\end{aligned}$$

This set of $2N+2$ linear equations in $2N+2$ variables $\dot{X}_0, \dots, \dot{X}_N, \dot{Y}_0, \dots, \dot{Y}_N$ is non-degenerate and linear in X_0 and Y_0. This establishes the existence and uniqueness of the open-loop equilibrium as well as the functional form of the optimal trading strategies given in the statement of the theorem. □

The proof of the previous theorem provides a simple method to compute the equilibrium trading strategies. In this chapter, we illustrate these for the predatory trading case of $X_0 > 0$ and $Y_0 = 0$; numerical examples for the case $Y_0 \neq 0$ are given in Appendix B.2. Figure 9.1 shows the optimal trading trajectories for the predatory trading situation in an elastic market (see Table 9.1 for parameter values). The qualitative properties of the strategies are very similar to the strategies in the one stage model discussed in Section 8.3: while player 1 sells her asset position, player 2 pursues predatory trading by first selling in parallel and then covering her short position later on at a lower price.

Unfortunately, the intuitive properties of Figure 9.1 disappear when the transient impact parameter λ is reduced (Figure 9.2), the resilience parameter ρ is reduced (Figure 9.3), or the time interval τ between trades is reduced (Figure 9.4). In all of these cases, oscillations appear in the optimal strategies. Most disturbingly, the change of the number of discrete time points $N+1$ from an odd number to an even number can dramatically change the optimal trading strategies (Figure 9.5) and lead to first trades that increase the asset positions of both players. In order to analyze the source of these counterintuitive results, we first turn to the case of trading at an odd number $N+1$ of points in time and the limiting case $\lambda = 0$, $\rho = 0$ or $\tau = 0$. For this simple setting, we can derive explicit formulas for $\dot{X}$ and $\dot{Y}$.

Proposition 9.2. *Let $N+1$ be odd and let $\lambda = 0$, $\rho = 0$ or $\tau = 0$. Then the unique*

Parameter	Fig. 9.1, 9.6a	Fig. 9.2, 9.6b	Fig. 9.3, 9.6c	Fig. 9.4, 9.6d	Fig. 9.5, 9.7
Player 1's asset position X_0	1				
Player 2's asset position Y_0	0				
Duration T of trading time	1				
Permanent impact sensitivity γ	1				
Transient impact sensitivity λ	10	*0.1*	10	10	*0.1*
Resilience parameter ρ	200	200	*10*	200	*10*
Time interval between trades τ	1/100	1/100	1/100	*1/1000*	*var.*
Number of time steps N	100	100	100	*1000*	*var.*

Table 9.1: Parameter values used for numerical computation in Sections 9.3 and 9.4.

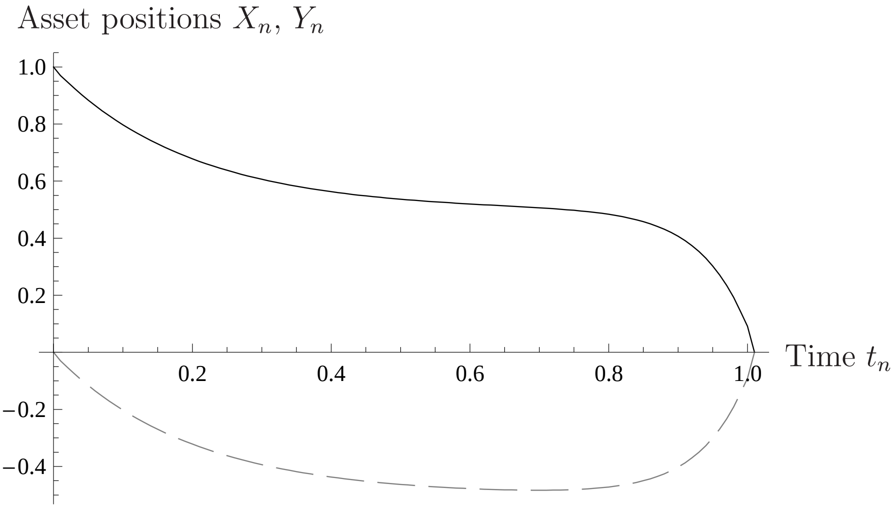


Figure 9.1: Asset positions $X(t_n) = X_n$ and $Y(t_n) = Y_n$ in the open-loop equilibrium over time. The solid line represents player 1 (the seller), the dashed line player 2 (the competitor). Base case; see Table 9.1 for parameter values.

open-loop Nash equilibrium is given by

$$X_n = X_0 \textit{ for even } n \qquad X_n = 0 \textit{ for odd } n$$
$$Y_n = Y_0 \textit{ for even } n \qquad Y_n = 0 \textit{ for odd } n.$$

Proof. First, observe that when $\lambda = 0$, $\rho = 0$ or $\tau = 0$, then the trading game for player 1 and player 2 is a constant sum game, i.e., player 2 can only gain at player 1's expense. Now assume that $X_n \neq X_{n+2}$. Then the strategy $Y_i = 0$ for all $i \neq 0, n+1$ and $Y_{n+1} = Y$ yields expected cash proceeds for player 2 of

$$R_Y = Y(\gamma + \lambda)(X_{n+2} - X_n)/2 + Y_0(\tilde{P}_0 - (\gamma + \lambda)(Y_0 + \dot{X}_0)/2).$$

By choosing a very large respectively small Y, player 2 can therefore achieve arbitrarily large gains and impose arbitrarily large losses on player 1. Hence, any strategy for player 1 with $X_n \neq X_{n+2}$ cannot be optimal for player 1. By the same argument, $Y_n = Y_{n+2}$ needs to hold for any optimal strategy for player 2. This shows that the strategies suggested in

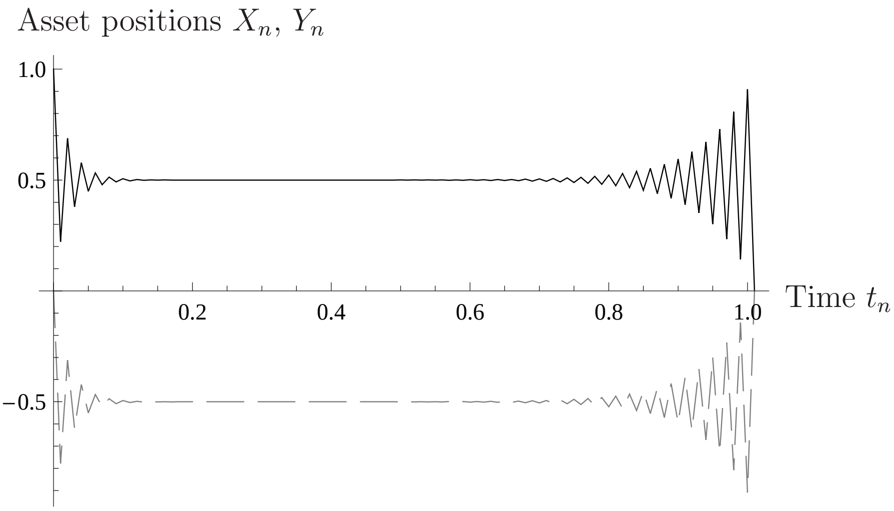


Figure 9.2: Asset positions $X(t_n) = X_n$ and $Y(t_n) = Y_n$ in the open-loop equilibrium over time. The solid line represents player 1 (the seller), the dashed line player 2 (the competitor). Reduced transient impact parameter λ; see Table 9.1 for parameter values.

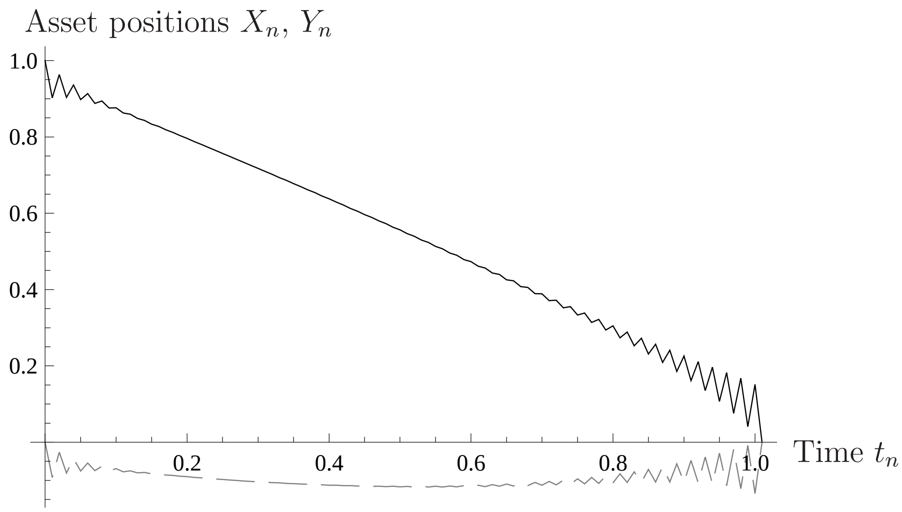


Figure 9.3: Asset positions $X(t_n) = X_n$ and $Y(t_n) = Y_n$ in the open-loop equilibrium over time. The solid line represents player 1 (the seller), the dashed line player 2 (the competitor). Reduced resilience parameter ρ; see Table 9.1 for parameter values.

the proposition represent the only candidate for an open-loop equilibrium. To see that they indeed establish an equilibrium, observe that when player 1 follows the suggested strategy, the expected cash proceeds of player 1 and player 2 are independent of player 2's trading strategy. The same holds for player 2's strategy. $\square$

For small λ, ρ or τ, the oscillations act as a kind of hedge; they provide a cushion against market manipulations of the other player. Such a hedge is only possible for an odd number $N+1$ of trading time points; for an even number, $X_n = X_{n+2}$ cannot hold for all n since $X_0 \neq X_{N+1} = 0$. Hence, no Nash equilibrium exists for even $N+1$ and $\lambda = 0$, $\rho = 0$ or $\tau = 0$. If $\lambda, \rho, \tau > 0$, then an equilibrium exists, and for the special case of trading at two points in time, we can state closed form expressions for the optimal trading strategy.

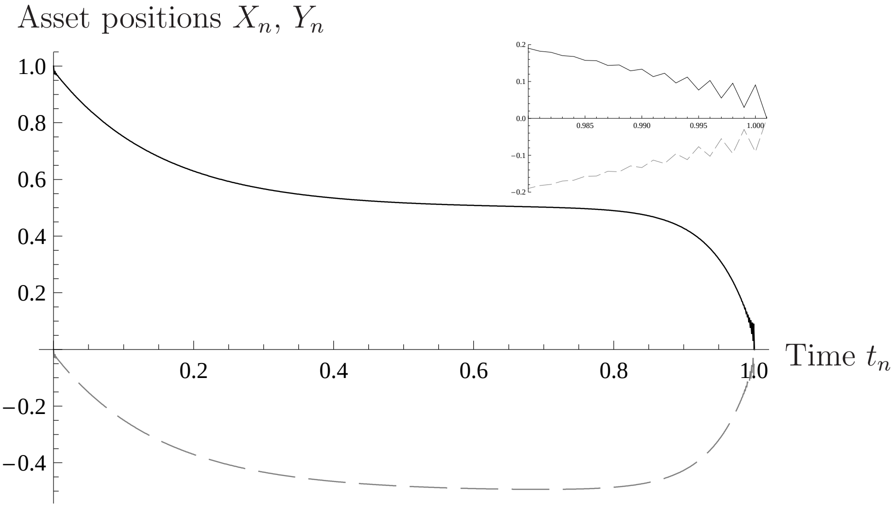


Figure 9.4: Asset positions $X(t_n) = X_n$ and $Y(t_n) = Y_n$ in the open-loop equilibrium over time; the inset shows a closeup for the final time period $t \in [0.98, 1.0]$. The solid line represents player 1 (the seller), the dashed line player 2 (the competitor). Reduced discretization time step τ; see Table 9.1 for parameter values.

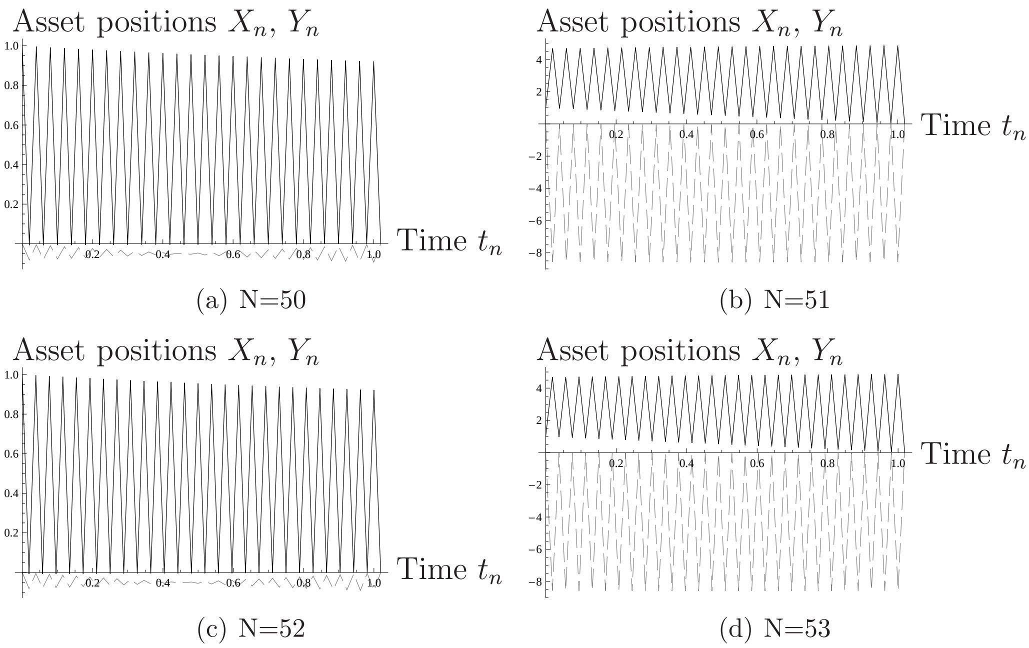


Figure 9.5: Asset positions $X(t_n) = X_n$ and $Y(t_n) = Y_n$ in the open-loop equilibrium over time. The solid line represents player 1 (the seller), the dashed line player 2 (the competitor). Reduced transient impact parameter λ and resilience ρ; see Table 9.1 for parameter values. Four different numbers of simulation time steps are compared: $N = 50, 51, 52, 53$. Note the different scales in the subfigures.

Proposition 9.3. *Consider the case of trading at two points in time t_0 and t_1 with $\tau := t_1 - t_0$. Let $\lambda > 0$, $\rho > 0$ and $\tau > 0$. If player 1 needs to sell X_0 shares and player 2 does not want to change her overall asset position ($Y_0 = 0$), then the open-loop equilibrium is given* *by*

$$\begin{aligned}
\dot{X}_0 &= \left(\frac{2}{3} - \frac{(\gamma+\lambda)e^{\rho\tau}}{6\lambda(e^{\rho\tau}-1)}\right) X_0 \\
\dot{X}_1 &= \left(\frac{1}{3} + \frac{(\gamma+\lambda)e^{\rho\tau}}{6\lambda(e^{\rho\tau}-1)}\right) X_0 \\
\dot{Y}_0 &= \left(\frac{\gamma e^{\rho\tau}+\lambda}{3\lambda(e^{\rho\tau}-1)}\right) X_0 \\
&= -\dot{Y}_1.
\end{aligned}$$

The expected cash proceeds are

$$\begin{aligned}
\mathbb{E}\left[\dot{X}_0 P_0 + \dot{X}_1 P_1\right] &= X_0 \tilde{P}_0 - \left(\frac{2\gamma + (e^{-\rho\tau}+1)\lambda}{9} + \frac{5(\gamma+\lambda)^2}{36\lambda(1-e^{-\rho\tau})}\right) X_0^2 \\
\mathbb{E}\left[\dot{Y}_0 P_0 + \dot{Y}_1 P_1\right] &= \frac{(\gamma+e^{-\rho\tau}\lambda)^2}{9\lambda(1-e^{-\rho\tau})} X_0^2.
\end{aligned}$$

Proof. This follows by a straightforward application of the computation method for the optimal strategies given in the proof of Theorem 9.1. □

By the formulas of the preceding proposition, we see that if the transient impact parameter λ, the resilience parameter ρ and the time between the two trades τ is large, then player 1 is selling a part of her asset position at the first time point and the remainder at the second time point. As the three parameters λ, ρ and τ decrease, player 1's first trade $\dot{X}_1$ changes sign and she *increases* her asset position at the first time point. This behavior can best be explained by first looking at player 2's situation. Player 2 knows that player 1 sells X_0 shares and that player 1 cannot completely hedge against market manipulation. By front running or preying on player 1's sell order, player 2 can thus make a profit. She will therefore sell at time t_0 and cover her position at time t_1. The size of her orders is driven by the trade-off between increased gains of predation and increased cost of trading due to her own transient price impact. The smaller λ, ρ and τ, i.e., the smaller this effect of the transient impact, the larger her trades. For player 1, such large trades of player 2 offer a profitable opportunity to provide liquidity; the more shares player 2 is selling at time t_0, the more shares player 1 will be willing to buy in spite of her own selling intentions. The same logic extends to larger even numbers $N+1$ of trading time points: player 2 exploits the inability of player 1 to hedge by large sell orders at the first point in time; player 1 counteracts this by providing liquidity and buying a large number of shares. Thereafter, both players follow an oscillating strategy that hedges them against market manipulation by the other player.

9.4 Optimal closed-loop strategies

In the previous section, we determined the open-loop Nash equilibrium and observed surprising oscillations. We now turn to closed-loop strategies, i.e. strategies, that at time t_n take into account the trades of the other player at times $t_0, \ldots, t_{n-1}$. As discussed in Section 7.4, the execution of such strategies requires market transparency. For the limiting case of $\lambda = 0$, $\rho = 0$ or $\tau = 0$, we note that the oscillations of the optimal open-loop strategies for odd $N+1$ in Proposition 9.2 disappear in the closed-loop setting: both players can stop trading at time t_1 when they realize that both of them have a zero asset position. They only need to execute their oscillatory hedge strategy if they find out that the other player deviated from her optimal strategy and manipulated the market price.

We now turn to general parameter values of $\lambda, \rho, \tau > 0$. In order to account for the feedback effect of trades at time t_n on future trades of the other player at times $t_{n+1}, \ldots, t_N$, we follow a dynamic programming approach. We define the "cost function" $J_n(X, Y, D)$ as the difference between $\tilde{P}_0 X$ (the mark-to-market value of the asset position X before the beginning of the liquidation) and the expected cash proceeds from optimally selling X shares between time t_n and t_N in a market where

- another player optimally liquidates Y shares in the same time frame
- the transient impact at the beginning of trading is D
- $X_0 - X + Y_0 - Y$ shares were sold between t_0 and t_n, and this sale permanently impacted the market price.

The following theorem provides a recursive description of the optimal trading strategy and the cost function $J_n(X, Y, D)$.

Theorem 9.4. *In the unique closed-loop equilibrium, both players follow a trading strategy that is linear in X_n, Y_n and D_n:*

$$\dot{X}_n = \Phi_n X_n + \Psi_n Y_n + \Xi_n D_n \tag{9.3}$$

$$\dot{Y}_n = \Phi_n Y_n + \Psi_n X_n + \Xi_n D_n. \tag{9.4}$$

The cost function is quadratic in X_n, Y_n and D_n:

$$J_n(X_n, Y_n, D_n) = \gamma(X_0+Y_0)X_n + \kappa_n X_n^2 + \mu_n X_n D_n + \nu_n D_n^2 + \phi_n Y_n^2 + \psi_n Y_n D_n + \xi_n X_n Y_n. \tag{9.5}$$

All of the parameters Φ_n, Ψ_n, Ξ_n, κ_n, μ_n, ν_n, ϕ_n, ψ_n and ξ_n at time t_n are real numbers that can be recursively computed from the values of the parameters at time t_{n+1} and are independent of X_n, Y_n and D_n.

As a representative example, the recursion equation for κ_n is presented in Appendix B.1. It is interesting to note that the evolution of D_n (reflecting trading of both players with the general market) depends only on $X_0 + Y_0$. From the evolution of market prices, it is hence not distinguishable whether two players each liquidate a medium sized asset position or whether one player liquidates a large asset position and is preyed on by another player.

Proof. We show by backwards induction that J, $\dot{X}$ and $\dot{Y}$ can be expressed as in Equations (9.3), (9.4) and (9.5), and we prove that explicit recursion formulas for the parameters Φ_n, Ψ_n, Ξ_n, κ_n, μ_n, ν_n, ϕ_n, ψ_n and ξ_n exist.

For the base case $n = N$, Equations (9.3), (9.4) and (9.5) hold with parameters

$$\begin{aligned}
\Phi_N &= 1\\
\Psi_N &= 0\\
\Xi_N &= 0\\
\kappa_N &= (\lambda-\gamma)/2\\
\mu_N &= 1\\
\nu_n &= \phi_N = \psi_N = 0\\
\xi_N &= (\lambda-\gamma)/2.
\end{aligned}$$

Since both players have no choice but to liquidate their remaining asset positions at time t_N, the equilibrium is unique.

We now prove the inductive step. Assuming that selling $\dot{Y}_n$ shares at time t_n is optimal for player 2, we can determine player 1's minimal cost by the following dynamic programming equation:

$$\begin{aligned}
J_n(X_n, Y_n, D_n) &= \min_{\dot{X}_n}\Bigg\{\left[\gamma(X_0 - X_n + Y_0 - Y_n) + D_n + \frac{\gamma+\lambda}{2}(\dot{X}_n + \dot{Y}_n)\right]\dot{X}_n\\
&\quad + J_{n+1}\left(X_n - \dot{X}_n, Y_n - \dot{Y}_n, (D_n + \lambda(\dot{X}_n + \dot{Y}_n))e^{-\rho(t_{n+1}-t_n)}\right)\Bigg\}\\
&=: \min_{\dot{X}_n} f_n(\dot{X}_n)
\end{aligned}$$

By the inductive hypothesis, Equation (9.5) holds for $n+1$, and we have

$$\begin{aligned}
f_n(\dot{X}_n) = &\left[\gamma(X_0 - X_n + Y_0 - Y_n) + D_n + \frac{\gamma+\lambda}{2}(\dot{X}_n + \dot{Y}_n)\right]\dot{X}_n\\
&+ \gamma(X_0 + Y_0)(X_n - \dot{X}_n)\\
&+ \kappa_{n+1}(X_n - \dot{X}_n)^2\\
&+ \mu_{n+1}(X_n - \dot{X}_n)(D_n + \lambda(\dot{X}_n + \dot{Y}_n))e^{-\rho(t_{n+1}-t_n)}\\
&+ \nu_{n+1}((D_n + \lambda(\dot{X}_n + \dot{Y}_n))e^{-\rho((t_{n+1}-t_n)})^2\\
&+ \phi_{n+1}(Y_n - \dot{Y}_n)^2\\
&+ \psi_{n+1}(Y_n - \dot{Y}_n)(D_n + \lambda(\dot{X}_n + \dot{Y}_n))e^{-\rho(t_{n+1}-t_n)}\\
&+ \xi_{n+1}(X_n - \dot{X}_n)(Y_n - \dot{Y}_n).
\end{aligned} \tag{9.6}$$

The function f_n therefore is a quadratic function of $\dot{X}_n$. We can find its minimum by setting its derivative to zero and obtain that in equilibrium

$$-\Theta_n \dot{X}_n = a_n D_n + b_n X_n + c_n Y_n + e_n \dot{Y}_n \tag{9.7}$$

with

$$\begin{aligned}
a_n &= 1 - \mu_n e^{-\rho(t_{n+1}-t_n)} + 2\nu_n \lambda e^{-2\rho(t_{n+1}-t_n)} \\
b_n &= -\gamma - 2\kappa_n + \mu_n \lambda e^{-\rho(t_{n+1}-t_n)} \\
c_n &= -\gamma + \psi_n \lambda e^{-\rho(t_{n+1}-t_n)} - \xi_n \\
e_n &= (\gamma+\lambda)/2 - \mu_n \lambda e^{-\rho(t_{n+1}-t_n)} + 2\nu_n \lambda^2 e^{-2\rho(t_{n+1}-t_n)} - \psi_n \lambda e^{-\rho(t_{n+1}-t_n)} + \xi_n \\
\Theta_n &= \gamma + \lambda + 2\kappa_n - 2\mu_n \lambda e^{-\rho(t_{n+1}-t_n)} + 2\nu_n \lambda^2 e^{-2\rho(t_{n+1}-t_n)}.
\end{aligned}$$

By symmetry, we know that if player 2 is following an optimal strategy, we must have

$$-\Theta_n \dot{Y}_n = a_n D_n + b_n Y_n + c_n X_n + e_n \dot{X}_n. \tag{9.8}$$

We can now solve this interdependency and obtain the unique solution

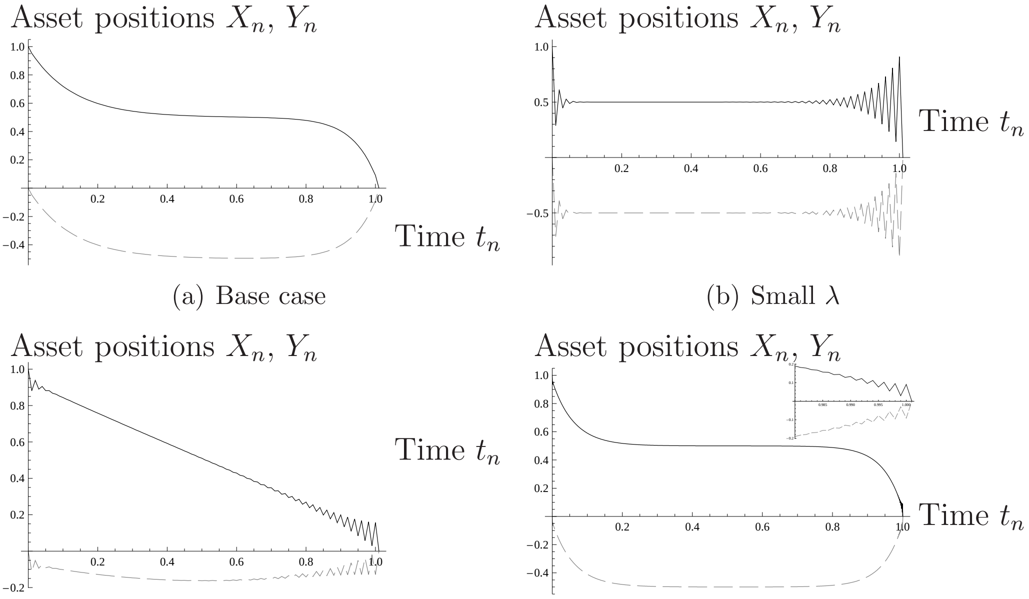
$$\dot{X}_n = \frac{1}{\Theta_n^2 - e_n^2}\left[D_n a_n(e_n - \Theta_n) + X_n(c_n e_n - \Theta_n b_n) + Y_n(b_n e_n - \Theta_n c_n)\right] \tag{9.9}$$

$$\dot{Y}_n = \frac{1}{\Theta_n^2 - e_n^2}\left[D_n a_n(e_n - \Theta_n) + Y_n(c_n e_n - \Theta_n b_n) + X_n(b_n e_n - \Theta_n c_n)\right]. \tag{9.10}$$

This establishes Equations (9.3) and (9.4) for n. By plugging the previous two equations into Equation (9.6), we obtain $J_n(X_n, Y_n, D_n)$ and find that it is of the form of Equation (9.5). The recursion formulas for the parameters κ_n, μ_n, ν_n, ϕ_n, ψ_n and ξ_n follow explicitly, but are complex. An example is presented in Appendix B.1.

In the derivation of Equations (9.9) and (9.10), we implicitly assumed that $\Theta_n \neq \pm e_n$. By way of contradiction, assume that $\Theta_n = \pm e_n$. If $X_n = Y_n = D_n = 0$, then this implies that the optimal trade $\dot{X}_n$ for player 1 is $\dot{X}_n = \pm\dot{Y}_n$ by Equation (9.7), irrespective of player 2's choice of $\dot{Y}_n$. Trading $\pm\dot{Y}_n$ however cannot be optimal for player 1 in general, since it brings her in a symmetric position to player 2 and thus does not promise her any profit, while there clearly exists a trade $\dot{X}_n$ that promises a profit to her. □

The recursive description of the optimal trading strategies given in Theorem 9.4 can be implemented to analyze the properties of optimal trading in different market environments. The Mathematica source code is provided in Appendix B.3. Figures 9.6 and 9.7 show the optimal closed-loop strategies for the same parameters as Figures 9.1 to 9.5. In all cases, the same qualitative behavior can be observed for the optimal closed-loop strategies as for the optimal open-loop strategies. The breakdown of oscillation that we discussed for odd $N+1$ and $\lambda = 0$, $\rho = 0$ or $\tau = 0$ at the beginning of this section does not generalize to positive values of λ, ρ and τ. To gain an intuitive understanding, assume that λ, ρ and τ are small and consider the time point t_{N-1}. Because Proposition 9.3 also holds in the closed-loop setting, we see that having a large position X_{N-1} leads to large trades at time t_{N-1} and large costs. Therefore both players will try hard to achieve $X_{N-1} \approx 0 = X_{N+1}$ respectively $Y_{N-1} \approx 0 = Y_{N+1}$. Now we realize that the situation at t_{N-3} is similar to the situation at t_{N-1}: Within two time steps, both players need to liquidate (almost) all of their asset positions X_{N-3} respectively Y_{N-3} and no hedge trading strategy is available. Therefore even small asset positions X_{N-3} and Y_{N-3} will lead to large trades $\dot{X}_{N-3}$ and $\dot{Y}_{N-3}$ and large costs. Hence both players will try to have small asset positions at time t_{N-3},



(a) Base case (b) Small λ

(c) Small ρ (d) Small τ

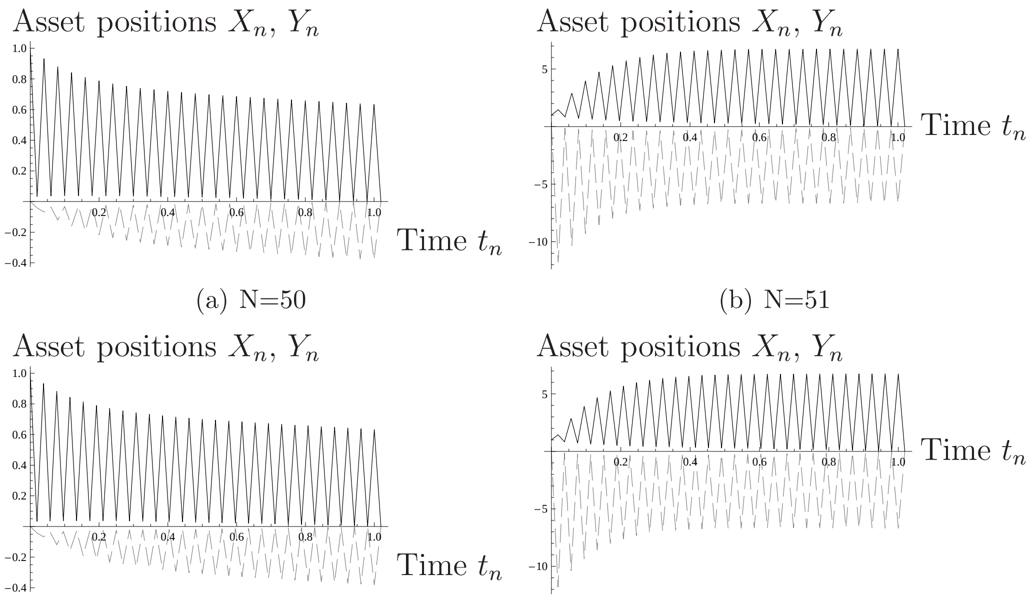
Figure 9.6: Asset positions $X(t_n) = X_n$ and $Y(t_n) = Y_n$ in the closed-loop equilibrium over time; the inset in Subfigure (d) shows a closeup for the final time period $t \in [0.98, 1.0]$. The solid line represents player 1 (the seller), the dashed line player 2 (the competitor). See Table 9.1 for parameter values.

and we see that the situation at time t_{N-5} is similar to t_{N-3}. The persistence of oscillations in the closed-loop framework can therefore be understood by the player's need to have small asset positions at times $t_{N-1}, t_{N-3}, \ldots$ and the inflation of the positions at the intermediary time points $t_N, t_{N-2}, \ldots$.

9.5 Comparison with Chapter 8 and potential adjustments

In this chapter, we determined the open-loop and closed-loop equilibria for two players trading in a market with finite resilience and no spread. We found that for small values of λ, ρ and τ, the optimal trading strategies oscillate: both players perform large round trip trades. From a mathematical point of view, the main issue is that this effect occurs for small τ. For a reasonable multiple player model, one can expect the optimal trading strategies to converge when the discretization time step τ is reduced further and further. For the model considered in this chapter, we found numerically that such a convergence does not hold, and we supported this observation by intuitive explanations and an analytical treatment of limiting cases (Propositions 9.2 and 9.3). The source of the oscillations is that the cost of round trip trading (selling at time t_n and buying back at time t_{n+1}) goes to zero as the discretization time step τ goes to zero. This leads to an ever decreasing cost of market manipulation and hedging and thus a shift in focus from interacting with the market (liquidating the initial asset position) to interacting with the other strategic player (profiting from her trading intentions respectively hedging against market manipulations by the other player).

The market model described in Section 8.2 builds on the discrete-time precursors of Almgren and Chriss (1999) and Almgren and Chriss (2001). In these discrete-time models,



(a) N=50 (b) N=51 (c) N=52 (d) N=53

Figure 9.7: Asset positions $X(t_n) = X_n$ and $Y(t_n) = Y_n$ in the closed-loop equilibrium over time. The solid line represents player 1 (the seller), the dashed line player 2 (the competitor). Reduced transient impact parameter λ and resilience ρ; see Table 9.1 for parameter values. Four different numbers of simulation time steps are compared: $N = 50, 51, 52, 53$. Note the different scales in the subfigures.

the discretization time step τ has the exact opposite effect: when τ is reduced, the cost of a round trip trade increases. This effect appears to be connected to the existence of optimal trading strategies in multiple player settings in continuous time.

In order to account for the significant cost of round trip trades in reality, we suggest two adjustments to the market model of Section 9.2. First, a purely temporary price impact component can be added to the model. The resulting model reflects three different price impacts: a permanent price impact, a transient price impact that decays over time, and a temporary price impact that instantaneously vanishes. Practitioners find that including all three price impacts improves pre-trade analysis accuracy (see Simmonds (2007)). Schulz (2007) analyzes the optimal trading strategies for a single player in such a model. For our multiple player setting, we expect that the optimal strategies are similar to the ones stated in Section 8.3, i.e., that the qualitative features of Figure 9.1 carry over to general values of λ, ρ and τ.

As a second modeling approach, both sides of the limit order book can be modeled explicitly. A large sell order widens the spread; after the trade, the spread narrows again until it is increased by the next order. In such a model, the cost of a round trip trade grows when the time between the two orders is reduced, because the spread has less time to narrow. Based on the favorable results of Figure 9.1, we expect that the oscillations of Figures 9.2, 9.3 and 9.4 disappear and optimal trading behavior similar to the results of Chapter 8 emerges. Unfortunately, explicitly modeling both sides of the limit order book significantly complicates the already complex computations. We therefore leave this task for future research.

Appendix A

Supplementary material for Chapter 8

A.1 Explicit function of the seller's cash position after liquidation

The coefficients A_i and B_i in Formula (8.11) are complex. As an illustration of their structure, we explicitly state A_7 and B_7 as examples of "simple" coefficients, and A_6 and B_6 as more complex examples.

$$A_7 = \Big(-1+e^{\frac{\gamma T_1}{\lambda}}\Big)\Big(-1+e^{\frac{n\gamma T_1}{2\lambda+n\lambda}}\Big)\Big(e^{\frac{\gamma((1+n)T_1+(2+n)T_2)}{(1+n)\lambda}}-e^{\frac{\gamma(2(1+n)^2T_1+(2+n)^2T_2)}{(1+n)(2+n)\lambda}}-$$
$$e^{\frac{\gamma((1+n)T_1+(1+2n)T_2)}{(1+n)\lambda}}+e^{\gamma(\frac{2(1+n)T_1}{(2+n)\lambda}+\frac{T_2+2nT_2}{\lambda+n\lambda})}\Big)\Big(-e^{\frac{\gamma T_2}{\lambda+n\lambda}}+e^{\frac{n\gamma T_2}{\lambda+n\lambda}}-e^{\frac{\gamma((1+n)T_1+nT_2)}{(1+n)\lambda}}+$$
$$e^{\frac{\gamma((1+n)T_1+(2+n)T_2)}{(1+n)\lambda}}-e^{\frac{n\gamma((1+n)T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}}-e^{\frac{\gamma(2(1+n)^2T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}}+e^{\frac{\gamma(2(1+n)^2T_1+n(2+n)T_2)}{(1+n)(2+n)\lambda}}-$$
$$e^{\frac{\gamma(2(1+n)^2T_1+(2+n)^2T_2)}{(1+n)(2+n)\lambda}}-e^{\frac{\gamma((1+n)T_1+(1+2n)T_2)}{(1+n)\lambda}}+e^{\frac{2(1+n)\gamma T_1}{(2+n)\lambda}+\frac{\gamma T_2}{\lambda+n\lambda}}+e^{\frac{n\gamma T_1}{2\lambda+n\lambda}+\frac{\gamma T_2}{\lambda+n\lambda}}+$$
$$e^{\frac{\gamma T_1}{\lambda}+\frac{n\gamma T_2}{\lambda+n\lambda}}-e^{\frac{2(1+n)\gamma T_1}{(2+n)\lambda}+\frac{n\gamma T_2}{\lambda+n\lambda}}+e^{\gamma(\frac{2(1+n)T_1}{(2+n)\lambda}+\frac{T_2+2nT_2}{\lambda+n\lambda})}\Big)$$

$$B_7 = (-1+e^{\frac{\gamma T_1}{\lambda}})(-1+e^{\frac{n\gamma T_1}{2\lambda+n\lambda}})(e^{\frac{\gamma T_2}{\lambda+n\lambda}}-e^{\frac{n\gamma T_2}{\lambda+n\lambda}}+e^{\frac{\gamma((1+n)T_1+nT_2)}{(1+n)\lambda}}-$$
$$e^{\frac{\gamma((1+n)T_1+(2+n)T_2)}{(1+n)\lambda}}+e^{\frac{n\gamma((1+n)T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}}+e^{\frac{\gamma(2(1+n)^2T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}}-e^{\frac{\gamma(2(1+n)^2T_1+n(2+n)T_2)}{(1+n)(2+n)\lambda}}+$$
$$e^{\frac{\gamma(2(1+n)^2T_1+(2+n)^2T_2)}{(1+n)(2+n)\lambda}}+e^{\frac{\gamma((1+n)T_1+(1+2n)T_2)}{(1+n)\lambda}}-e^{\frac{2(1+n)\gamma T_1}{(2+n)\lambda}+\frac{\gamma T_2}{\lambda+n\lambda}}-e^{\frac{n\gamma T_1}{2\lambda+n\lambda}+\frac{\gamma T_2}{\lambda+n\lambda}}-$$
$$e^{\frac{\gamma T_1}{\lambda}+\frac{n\gamma T_2}{\lambda+n\lambda}}+e^{\frac{2(1+n)\gamma T_1}{(2+n)\lambda}+\frac{n\gamma T_2}{\lambda+n\lambda}}-e^{\gamma(\frac{2(1+n)T_1}{(2+n)\lambda}+\frac{T_2+2nT_2}{\lambda+n\lambda})})^2$$

$$A_6 = -16e^{\frac{2\gamma(T_1+T_2)}{\lambda}}+8e^{\frac{3\gamma(T_1+T_2)}{\lambda}}+2e^{\frac{\gamma(2T_1+T_2)}{\lambda}}+12e^{\frac{\gamma(T_1+2T_2)}{\lambda}}-4e^{\frac{\gamma(T_1+3T_2)}{\lambda}}-$$
$$2e^{\frac{\gamma(2T_1+3T_2)}{\lambda}}-2e^{\frac{2\gamma((1+n)T_1+nT_2)}{(1+n)\lambda}}+e^{\frac{\gamma((1+n)T_1+2nT_2)}{(1+n)\lambda}}+e^{\frac{\gamma(3(1+n)T_1+2nT_2)}{(1+n)\lambda}}+2e^{\frac{3\gamma(nT_1+(2+n)T_2)}{(2+n)\lambda}}-$$
$$2e^{\frac{2\gamma((1+n)T_1+(2+n)T_2)}{(1+n)\lambda}}-32e^{\frac{2\gamma((1+n)T_1+(2+n)T_2)}{(2+n)\lambda}}+4e^{\frac{2(1+2n)\gamma((1+n)T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}}-32e^{\frac{2\gamma(2(1+n)T_1+(2+n)T_2)}{(2+n)\lambda}}-$$
$$14e^{\frac{3\gamma(2(1+n)T_1+(2+n)T_2)}{(2+n)\lambda}}+4e^{\frac{2\gamma(3(1+n)T_1+(2+n)T_2)}{(2+n)\lambda}}+6e^{\frac{\gamma(4(1+n)T_1+(2+n)T_2)}{(2+n)\lambda}}+3e^{\frac{2\gamma((1+n)^2T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}}+$$
$$2e^{\frac{2\gamma(2(1+n)^2T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}}-e^{\frac{2\gamma(3(1+n)^2T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}}-8e^{\frac{2\gamma((1+2n)T_1+(2+n)T_2)}{(2+n)\lambda}}+4e^{\frac{2\gamma((3+2n)T_1+(2+n)T_2)}{(2+n)\lambda}}-$$
$$6e^{\frac{\gamma((4+3n)T_1+(2+n)T_2)}{(2+n)\lambda}}-2e^{\frac{\gamma((4+5n)T_1+(2+n)T_2)}{(2+n)\lambda}}+e^{\frac{2\gamma((1+3n+2n^2)T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}}+3e^{\frac{\gamma((1+n)T_1+2(2+n)T_2)}{(1+n)\lambda}}-$$
$$2e^{\frac{\gamma(3(1+n)T_1+2(2+n)T_2)}{(1+n)\lambda}}+28e^{\frac{\gamma((2+3n)T_1+2(2+n)T_2)}{(2+n)\lambda}}+40e^{\frac{\gamma((4+3n)T_1+2(2+n)T_2)}{(2+n)\lambda}}+8e^{\frac{\gamma((4+5n)T_1+2(2+n)T_2)}{(2+n)\lambda}}-$$
$$8e^{\frac{\gamma((6+5n)T_1+2(2+n)T_2)}{(2+n)\lambda}}-3e^{\frac{\gamma((2+5n+3n^2)T_1+2(2+n)T_2)}{(1+n)(2+n)\lambda}}+2e^{\frac{\gamma(nT_1+3(2+n)T_2)}{(2+n)\lambda}}-4e^{\frac{\gamma(2nT_1+3(2+n)T_2)}{(2+n)\lambda}}+$$
$$2e^{\frac{\gamma(2(1+n)T_1+3(2+n)T_2)}{(2+n)\lambda}}-34e^{\frac{\gamma(4(1+n)T_1+3(2+n)T_2)}{(2+n)\lambda}}+8e^{\frac{\gamma((2+3n)T_1+3(2+n)T_2)}{(2+n)\lambda}}+20e^{\frac{\gamma((4+3n)T_1+3(2+n)T_2)}{(2+n)\lambda}}-$$
$$6e^{\frac{\gamma((2+4n)T_1+3(2+n)T_2)}{(2+n)\lambda}}-30e^{\frac{\gamma((6+4n)T_1+3(2+n)T_2)}{(2+n)\lambda}}+16e^{\frac{\gamma((4+5n)T_1+3(2+n)T_2)}{(2+n)\lambda}}+36e^{\frac{\gamma((6+5n)T_1+3(2+n)T_2)}{(2+n)\lambda}}-$$
$$3e^{\frac{2\gamma((1+n)^2T_1+n(2+n)T_2)}{(1+n)(2+n)\lambda}}-6e^{\frac{2\gamma(2(1+n)^2T_1+n(2+n)T_2)}{(1+n)(2+n)\lambda}}-e^{\frac{2\gamma(3(1+n)^2T_1+n(2+n)T_2)}{(1+n)(2+n)\lambda}}-e^{\frac{2\gamma((1+3n+2n^2)T_1+n(2+n)T_2)}{(1+n)(2+n)\lambda}}+$$
$$3e^{\frac{\gamma((2+5n+3n^2)T_1+2n(2+n)T_2)}{(1+n)(2+n)\lambda}}+6e^{\frac{\gamma((4+7n+3n^2)T_1+2n(2+n)T_2)}{(1+n)(2+n)\lambda}}+2e^{\frac{\gamma((4+9n+5n^2)T_1+2n(2+n)T_2)}{(1+n)(2+n)\lambda}}+$$
$$2e^{\frac{2\gamma(n(1+n)T_1+(2+n)^2T_2)}{(1+n)(2+n)\lambda}}-4e^{\frac{2\gamma((1+n)^2T_1+(2+n)^2T_2)}{(1+n)(2+n)\lambda}}+8e^{\frac{2\gamma(2(1+n)^2T_1+(2+n)^2T_2)}{(1+n)(2+n)\lambda}}+5e^{\frac{2\gamma(3(1+n)^2T_1+(2+n)^2T_2)}{(1+n)(2+n)\lambda}}+$$
$$2e^{\frac{2\gamma((1+3n+2n^2)T_1+(2+n)^2T_2)}{(1+n)(2+n)\lambda}}+9e^{\frac{2\gamma((3+5n+2n^2)T_1+(2+n)^2T_2)}{(1+n)(2+n)\lambda}}-e^{\frac{\gamma(n(1+n)T_1+2(2+n)^2T_2)}{(1+n)(2+n)\lambda}}-$$

$$
\begin{gathered}
e^{\frac{\gamma(3n(1+n)T_1+2(2+n)^2T_2)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma((2+5n+3n^2)T_1+2(2+n)^2T_2)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma((4+7n+3n^2)T_1+2(2+n)^2T_2)}{(1+n)(2+n)\lambda}} - \\
5e^{\frac{\gamma((4+9n+5n^2)T_1+2(2+n)^2T_2)}{(1+n)(2+n)\lambda}} - 12e^{\frac{\gamma((6+11n+5n^2)T_1+2(2+n)^2T_2)}{(1+n)(2+n)\lambda}} - 6e^{\frac{\gamma((1+n)T_1+(3+n)T_2)}{(1+n)\lambda}} + \\
10e^{\frac{\gamma(2(1+n)T_1+(3+n)T_2)}{(1+n)\lambda}} - 2e^{\frac{\gamma(3(1+n)T_1+(3+n)T_2)}{(1+n)\lambda}} + 4e^{\frac{2\gamma((1+n)T_1+(1+2n)T_2)}{(1+n)\lambda}} + e^{\frac{\gamma((1+n)T_1+2(1+2n)T_2)}{(1+n)\lambda}} - \\
6e^{\frac{\gamma(3(1+n)T_1+2(1+2n)T_2)}{(1+n)\lambda}} - 6e^{\frac{\gamma((1+n)T_1+(1+3n)T_2)}{(1+n)\lambda}} + 6e^{\frac{\gamma(2(1+n)T_1+(1+3n)T_2)}{(1+n)\lambda}} + 2e^{\frac{\gamma(3(1+n)T_1+(1+3n)T_2)}{(1+n)\lambda}} + \\
16e^{\frac{\gamma(2(1+n)^2T_1+(6+5n+n^2)T_2)}{(1+n)(2+n)\lambda}} + 22e^{\frac{\gamma(4(1+n)^2T_1+(6+5n+n^2)T_2)}{(1+n)(2+n)\lambda}} + 4e^{\frac{\gamma(2(3+5n+2n^2)T_1+(6+5n+n^2)T_2)}{(1+n)(2+n)\lambda}} - \\
14e^{\frac{\gamma((2+5n+3n^2)T_1+(6+5n+n^2)T_2)}{(1+n)(2+n)\lambda}} - 26e^{\frac{\gamma((4+7n+3n^2)T_1+(6+5n+n^2)T_2)}{(1+n)(2+n)\lambda}} + 4e^{\frac{\gamma((2+6n+4n^2)T_1+(6+5n+n^2)T_2)}{(1+n)(2+n)\lambda}} - \\
6e^{\frac{\gamma((4+9n+5n^2)T_1+(6+5n+n^2)T_2)}{(1+n)(2+n)\lambda}} - 2e^{\frac{\gamma((6+11n+5n^2)T_1+(6+5n+n^2)T_2)}{(1+n)(2+n)\lambda}} + 2e^{\frac{2\gamma(n(1+n)T_1+(2+5n+2n^2)T_2)}{(1+n)(2+n)\lambda}} + \\
2e^{\frac{2\gamma((1+n)^2T_1+(2+5n+2n^2)T_2)}{(1+n)(2+n)\lambda}} + 26e^{\frac{2\gamma(2(1+n)^2T_1+(2+5n+2n^2)T_2)}{(1+n)(2+n)\lambda}} + 9e^{\frac{2\gamma(3(1+n)^2T_1+(2+5n+2n^2)T_2)}{(1+n)(2+n)\lambda}} + \\
21e^{\frac{2\gamma((3+5n+2n^2)T_1+(2+5n+2n^2)T_2)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma(n(1+n)T_1+2(2+5n+2n^2)T_2)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma(3n(1+n)T_1+2(2+5n+2n^2)T_2)}{(1+n)(2+n)\lambda}} - \\
7e^{\frac{\gamma((2+5n+3n^2)T_1+2(2+5n+2n^2)T_2)}{(1+n)(2+n)\lambda}} - 19e^{\frac{\gamma((4+7n+3n^2)T_1+2(2+5n+2n^2)T_2)}{(1+n)(2+n)\lambda}} - 11e^{\frac{\gamma((4+9n+5n^2)T_1+2(2+5n+2n^2)T_2)}{(1+n)(2+n)\lambda}} - \\
24e^{\frac{\gamma((6+11n+5n^2)T_1+2(2+5n+2n^2)T_2)}{(1+n)(2+n)\lambda}} + 16e^{\frac{\gamma(2(1+n)^2T_1+(2+7n+3n^2)T_2)}{(1+n)(2+n)\lambda}} + 10e^{\frac{\gamma(4(1+n)^2T_1+(2+7n+3n^2)T_2)}{(1+n)(2+n)\lambda}} - \\
4e^{\frac{\gamma(6(1+n)^2T_1+(2+7n+3n^2)T_2)}{(1+n)(2+n)\lambda}} - 8e^{\frac{\gamma(2(3+5n+2n^2)T_1+(2+7n+3n^2)T_2)}{(1+n)(2+n)\lambda}} - 14e^{\frac{\gamma((2+5n+3n^2)T_1+(2+7n+3n^2)T_2)}{(1+n)(2+n)\lambda}} - \\
14e^{\frac{\gamma((4+7n+3n^2)T_1+(2+7n+3n^2)T_2)}{(1+n)(2+n)\lambda}} + 4e^{\frac{\gamma((2+6n+4n^2)T_1+(2+7n+3n^2)T_2)}{(1+n)(2+n)\lambda}} - 2e^{\frac{\gamma((4+9n+5n^2)T_1+(2+7n+3n^2)T_2)}{(1+n)(2+n)\lambda}} + \\
10e^{\frac{\gamma((6+11n+5n^2)T_1+(2+7n+3n^2)T_2)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma T_1}{\lambda}+\frac{2\gamma T_2}{\lambda+n\lambda}} - 2e^{\frac{4(1+n)\gamma T_1}{(2+n)\lambda}+\frac{2\gamma T_2}{\lambda+n\lambda}} + e^{\frac{6(1+n)\gamma T_1}{(2+n)\lambda}+\frac{2\gamma T_2}{\lambda+n\lambda}} - \\
e^{\frac{3\gamma T_1}{\lambda}+\frac{2n\gamma T_2}{\lambda+n\lambda}} + e^{\frac{6(1+n)\gamma T_1}{(2+n)\lambda}+\frac{2n\gamma T_2}{\lambda+n\lambda}}
\end{gathered}
$$

$$
\begin{gathered}
B_6 = (-1+e^{\frac{\gamma T_1}{\lambda}})(-1+e^{\frac{n\gamma T_1}{2\lambda+n\lambda}})(-14e^{\frac{\gamma T_2}{\lambda}} + 4e^{\frac{2\gamma T_2}{\lambda}} - 4e^{\frac{(3+n)\gamma T_2}{(1+n)\lambda}} + \\
9e^{\frac{2\gamma T_2}{\lambda+n\lambda}} + 5e^{\frac{2n\gamma T_2}{\lambda+n\lambda}} + 4e^{\frac{\gamma(T_1+T_2)}{\lambda}} - 4e^{\frac{2\gamma(T_1+T_2)}{\lambda}} + 20e^{\frac{\gamma(T_1+2T_2)}{\lambda}} - 4e^{\frac{\gamma(T_1+3T_2)}{\lambda}} - \\
6e^{\frac{\gamma(2T_1+3T_2)}{\lambda}} + 20e^{\frac{\gamma(nT_1+(2+n)T_2)}{(2+n)\lambda}} - 4e^{\frac{2\gamma(nT_1+(2+n)T_2)}{(2+n)\lambda}} - 6e^{\frac{\gamma(2nT_1+(2+n)T_2)}{(2+n)\lambda}} + e^{\frac{2\gamma((1+n)T_1+(2+n)T_2)}{(1+n)\lambda}} - \\
40e^{\frac{2\gamma((1+n)T_1+(2+n)T_2)}{(2+n)\lambda}} + e^{\frac{2n\gamma((1+n)T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}} + 4e^{\frac{2\gamma(2(1+n)T_1+(2+n)T_2)}{(2+n)\lambda}} + 4e^{\frac{2\gamma((1+n)^2T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}} - \\
9e^{\frac{2\gamma(2(1+n)^2T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}} - 4e^{\frac{\gamma((2+3n)T_1+(2+n)T_2)}{(2+n)\lambda}} + 4e^{\frac{\gamma((1+n)T_1+2(2+n)T_2)}{(1+n)\lambda}} - 6e^{\frac{n\gamma((1+n)T_1+2(2+n)T_2)}{(1+n)(2+n)\lambda}} + \\
20e^{\frac{\gamma((2+3n)T_1+2(2+n)T_2)}{(2+n)\lambda}} - 14e^{\frac{\gamma(4(1+n)T_1+3(2+n)T_2)}{(2+n)\lambda}} + 4e^{\frac{\gamma((2+3n)T_1+3(2+n)T_2)}{(2+n)\lambda}} + 20e^{\frac{\gamma((4+3n)T_1+3(2+n)T_2)}{(2+n)\lambda}} - \\
4e^{\frac{2\gamma((1+n)^2T_1+n(2+n)T_2)}{(1+n)(2+n)\lambda}} - 9e^{\frac{2\gamma(2(1+n)^2T_1+n(2+n)T_2)}{(1+n)(2+n)\lambda}} + 4e^{\frac{\gamma((2+5n+3n^2)T_1+2n(2+n)T_2)}{(1+n)(2+n)\lambda}} - \\
4e^{\frac{2\gamma((1+n)^2T_1+(2+n)^2T_2)}{(1+n)(2+n)\lambda}} + 5e^{\frac{2\gamma(2(1+n)^2T_1+(2+n)^2T_2)}{(1+n)(2+n)\lambda}} - 6e^{\frac{\gamma((4+7n+3n^2)T_1+2(2+n)^2T_2)}{(1+n)(2+n)\lambda}} - \\
10e^{\frac{\gamma((1+n)T_1+(3+n)T_2)}{(1+n)\lambda}} + 4e^{\frac{\gamma(2(1+n)T_1+(3+n)T_2)}{(1+n)\lambda}} + 5e^{\frac{2\gamma((1+n)T_1+(1+2n)T_2)}{(1+n)\lambda}} - 10e^{\frac{\gamma((1+n)T_1+(1+3n)T_2)}{(1+n)\lambda}} + \\
4e^{\frac{\gamma(n(1+n)T_1+(6+5n+n^2)T_2)}{(1+n)(2+n)\lambda}} + 20e^{\frac{\gamma(2(1+n)^2T_1+(6+5n+n^2)T_2)}{(1+n)(2+n)\lambda}} - 10e^{\frac{\gamma((2+5n+3n^2)T_1+(6+5n+n^2)T_2)}{(1+n)(2+n)\lambda}} - \\
4e^{\frac{\gamma((4+7n+3n^2)T_1+(6+5n+n^2)T_2)}{(1+n)(2+n)\lambda}} + 4e^{\frac{2\gamma((1+n)^2T_1+(2+5n+2n^2)T_2)}{(1+n)(2+n)\lambda}} + 9e^{\frac{2\gamma(2(1+n)^2T_1+(2+5n+2n^2)T_2)}{(1+n)(2+n)\lambda}} - \\
4e^{\frac{\gamma((2+5n+3n^2)T_1+2(2+5n+2n^2)T_2)}{(1+n)(2+n)\lambda}} - 14e^{\frac{\gamma((4+7n+3n^2)T_1+2(2+5n+2n^2)T_2)}{(1+n)(2+n)\lambda}} - 4e^{\frac{\gamma(n(1+n)T_1+(2+7n+3n^2)T_2)}{(1+n)(2+n)\lambda}} + \\
4e^{\frac{\gamma(2n(1+n)T_1+(2+7n+3n^2)T_2)}{(1+n)(2+n)\lambda}} + 20e^{\frac{\gamma(2(1+n)^2T_1+(2+7n+3n^2)T_2)}{(1+n)(2+n)\lambda}} - 4e^{\frac{\gamma(4(1+n)^2T_1+(2+7n+3n^2)T_2)}{(1+n)(2+n)\lambda}} - \\
10e^{\frac{\gamma((2+5n+3n^2)T_1+(2+7n+3n^2)T_2)}{(1+n)(2+n)\lambda}} + 4e^{\frac{\gamma((4+7n+3n^2)T_1+(2+7n+3n^2)T_2)}{(1+n)(2+n)\lambda}} + 5e^{2\gamma(\frac{nT_1}{2\lambda+n\lambda}+\frac{T_2}{\lambda+n\lambda})} - \\
4e^{\frac{\gamma T_1}{\lambda}+\frac{2\gamma T_2}{\lambda+n\lambda}} + 9e^{\frac{4(1+n)\gamma T_1}{(2+n)\lambda}+\frac{2\gamma T_2}{\lambda+n\lambda}} - 14e^{\frac{n\gamma T_1}{2\lambda+n\lambda}+\frac{2\gamma T_2}{\lambda+n\lambda}} + 9e^{\frac{4(1+n)\gamma T_1}{(2+n)\lambda}+\frac{2n\gamma T_2}{\lambda+n\lambda}})
\end{gathered}
$$

A.2 Mathematica source code

In this section, we provide the Mathematica source code used to generate the figures in Chapter 8. Two slightly different programs were used for the case of one competitor and the case of more than one competitor. We only provide the program for more than one competitor; the simpler version for only one competitor can be obtained by replacing the multiple player optimization in the second stage by a linear trading strategy for the single competitor and by adjusting the computations for the optimal asset position Z_0.

Stage 1 (all players active)

First, we set the parameters for the first stage according to Theorem 8.1. We divide the players in three groups: group X consists of the seller liquidating X_0 shares, group Y consists of $n-1$ competitors each buying Y_0 shares, and group Z consists of the remaining competitor buying Z_0 shares. Furthermore, we let Mathematica know that T_1 is a positive real number; this allows Mathematica to simplify the computations.

```
a₁ = n/(n + 2)γ/λ (1 − Exp [−(n)/(n + 2) ∗ γ/λT₁]) ^(−1)
      (−X₀ + (n − 1)Y₀ + Z₀)/ (n + 1);
b_X1 = γ/λ (Exp [γ/λT₁] − 1) ^(−1)
      (−X₀ − (−X₀ + (n − 1)Y₀ + Z₀)/ (n + 1)) ;
b_Y1 = γ/λ (Exp [γ/λT₁] − 1) ^(−1)
      (Y₀ − (−X₀ + (n − 1)Y₀ + Z₀)/ (n + 1)) ;
b_Z1 = γ/λ (Exp [γ/λT₁] − 1) ^(−1)
      (Z₀ − (−X₀ + (n − 1)Y₀ + Z₀)/ (n + 1)) ;
T₁ > 0;
```

Now we can define the optimal trading speeds x_1, y_1, z_1 in stage 1 for players in all three groups according to Theorem 8.1.

```
x₁[t_] = a₁Exp[−(n)/(n + 2)γ/λt] + b_X1Exp[γ/λt];
y₁[t_] = a₁Exp[−(n)/(n + 2)γ/λt] + b_Y1Exp[γ/λt];
z₁[t_] = a₁Exp[−(n)/(n + 2)γ/λt] + b_Z1Exp[γ/λt];
```

The asset positions X_1, Y_1, Z_1 are obtained by integration.

```
X₁[t_] = Integrate [x₁[s], {s, 0, t}] + X₀;
Y₁[t_] = Integrate [y₁[s], {s, 0, t}] ;
Z₁[t_] = Integrate [z₁[s], {s, 0, t}] ;
```

Now we can compute the price P_1 in stage 1.

```
P₁[t_] = Simplify[P₀ + γ(X₁[t] − X₀ + (n − 1)Y₁[t] + Z₁[t])
      +λ(x₁[t] + (n − 1)y₁[t] + z₁[t])];
```

The gains for players in all three groups follow.

```
Gain_X1 = −Simplify [Integrate [P₁[t]x₁[t], {t, 0, T₁}] , T₁ > 0] ;
Gain_Y1 = −Simplify [Integrate [P₁[t]y₁[t], {t, 0, T₁}] , T₁ > 0] ;
Gain_Z1 = −Simplify [Integrate [P₁[t]z₁[t], {t, 0, T₁}] , T₁ > 0] ;
```

Stage 2 (only competitors active)

Similar to the first stage, we compute the parameters, trading speeds, asset positions and gains for the players in all three groups. The seller does not trade, and the competitors in group Y respectively Z sell their asset positions Y_0 respectively Z_0. To keep the commands for the second stage similar to the commands for the first stage, we denote the length of the second stage by T_2 (in Chapter 8, T_2 denoted the end point of the second stage, i.e., the combined length of both the first and the second stage).

$$\begin{aligned}
&a_2 = (n-1)/(n+1) * \gamma/\lambda \\
&\quad (1 - \text{Exp}\,[-(n-1)/(n+1) * \gamma/\lambda T_2])\,\hat{}\,(-1) \\
&\quad ((-(n-1)Y_0 - Z_0)/\,(n))\,; \\
&b_{Y2} = \gamma/\lambda\,(\text{Exp}\,[\gamma/\lambda T_2] - 1)\,\hat{}\,(-1) \\
&\quad (-Y_0 - ((n-1)\,(-Y_0) - Z_0)/\,(n))\,; \\
&b_{Z2} = \gamma/\lambda\,(\text{Exp}\,[\gamma/\lambda T_2] - 1)\,\hat{}\,(-1) \\
&\quad (-Z_0 - ((n-1)\,(-Y_0) - Z_0)/\,(n))\,;
\end{aligned}$$

$$\begin{aligned}
&x_2[\text{t_}] = 0; \\
&y_2[\text{t_}] = a_2\text{Exp}[-(n-1)/(n+1)\gamma/\lambda t] + b_{Y2}\text{Exp}[\gamma/\lambda t]; \\
&z_2[\text{t_}] = a_2\text{Exp}[-(n-1)/(n+1)\gamma/\lambda t] + b_{Z2}\text{Exp}[\gamma/\lambda t];
\end{aligned}$$

$$\begin{aligned}
&X_2[\text{t_}] = X_1\,[T_1] + \text{Integrate}\,[x_2[s], \{s, 0, t\}]\,; \\
&Y_2[\text{t_}] = Y_1\,[T_1] + \text{Integrate}\,[y_2[s], \{s, 0, t\}]\,; \\
&Z_2[\text{t_}] = Z_1\,[T_1] + \text{Integrate}\,[z_2[s], \{s, 0, t\}]\,;
\end{aligned}$$

$$\begin{aligned}
&P_2[\text{t_}] = \text{Simplify}[P_0 + \gamma(X_2[t] - X_0 + (n-1)Y_2[t] + Z_2[t]) \\
&\quad +\lambda(x_2[t] + (n-1)y_2[t] + z_2[t])];
\end{aligned}$$

$$\begin{aligned}
&\text{Gain}_{X2} = -\text{Simplify}\,[\text{Integrate}\,[P_2[t] * x_2[t], \{t, 0, T_2\}]]\,; \\
&\text{Gain}_{Y2} = -\text{Simplify}\,[\text{Integrate}\,[P_2[t] * y_2[t], \{t, 0, T_2\}]]\,; \\
&\text{Gain}_{Z2} = -\text{Simplify}\,[\text{Integrate}\,[P_2[t] * z_2[t], \{t, 0, T_2\}]]\,;
\end{aligned}$$

Calculation of optimal Y_0

The overall gain for each player is the sum of the gains in the two stages.

$$\begin{aligned}
&\text{Gain}_X = \text{Simplify}\,[\text{Gain}_{X1} + \text{Gain}_{X2}]\,; \\
&\text{Gain}_Y = \text{Simplify}\,[\text{Gain}_{Y1} + \text{Gain}_{Y2}]\,; \\
&\text{Gain}_Z = \text{Simplify}\,[\text{Gain}_{Z1} + \text{Gain}_{Z2}]\,;
\end{aligned}$$

We compute the optimal asset position Z_0 for the single competitor by setting the derivative of $Gain_Z$ to zero. This gives an expression that depends on Y_0. In equilibrium, we must have $Y_0 = Z_0$, which allows us to solve for Y_0 and Z_0.

$$\begin{aligned}
&Y_0 = \text{Simplify}[Y_0/.\text{Solve}[Y_0 == Z_0/. \\
&\quad \text{Solve}[\text{Simplify}[D[\text{Gain}_Z, Z_0], T_2 > 0] == 0, Z_0][[1]], Y_0][[1]]]; \\
&Z_0 = Y_0;
\end{aligned}$$

Numerical example

To generate figures, we first set the parameters.

$$
\begin{aligned}
&T_1 = 1;\\
&T_2 = 1;\\
&X_0 = 1;\\
&\gamma = 3;\\
&\lambda = 1;\\
&P_0 = 10;
\end{aligned}
$$

Then, we concatenate the asset positions of stage 1 and stage 2.

$$
\begin{aligned}
&X[\mathrm{t\_}] = \mathrm{If}[t < T_1, X_1[t], X_2[t - T_1]];\\
&Y[\mathrm{t\_}] = \mathrm{If}[t < T_1, Y_1[t], Y_2[t - T_1]];
\end{aligned}
$$

Finally, we can plot the asset position evolution in equilibrium for varying numbers of competitors.

$$
\begin{aligned}
&\mathrm{Plot}[\{X[t]/.n \to 2, X[t]/.n \to 10, X[t]/.n \to 100,\\
&\quad n * Y[t]/.n \to 2, n * Y[t]/.n \to 10, n * Y[t]/.n \to 100\},\\
&\quad \{t, 0, T_1 + T_2\}, \mathrm{PlotStyle} \to \{\mathrm{GrayLevel}[0],\\
&\quad \mathrm{GrayLevel}[0.5], \mathrm{GrayLevel}[0.8],\\
&\quad \{\mathrm{GrayLevel}[0], \mathrm{Dashing}[\{0.05, 0.02\}]\},\\
&\quad \{\mathrm{GrayLevel}[0.5], \mathrm{Dashing}[\{0.05, 0.02\}]\},\\
&\quad \{\mathrm{GrayLevel}[0.8], \mathrm{Dashing}[\{0.05, 0.02\}]\}\}]
\end{aligned}
$$

Appendix B

Supplementary material for Chapter 9

B.1 Recursion function for the parameter κ_n

As explained in the proof of Theorem 9.4, the coefficients in Formula (9.5) for time t_{n-1} can be computed recursively from the coefficients for time t_n. The recursion equations are very complex. We therefore provide only the equation for κ_{n-1}; the recursion equations for the other parameters have a similar form and complexity.

$$\kappa_{n-1} = (4\lambda^2\nu_n(e^{\rho\tau}(2\gamma+2\kappa_n+\xi_n)-\lambda(\mu_n+\varphi))^2+(\kappa_n(e^{3\rho\tau}(\gamma^2+\lambda(8\kappa_n+3\lambda-2\xi_n)+2\gamma(2\lambda+\xi_n))-8\lambda^2\nu_n(-2\kappa_n\psi_n+\lambda(\mu_n+\psi_n))+4e^{\rho\tau}(2\lambda^3\nu_n-\kappa_n^2\psi_n^2+\kappa_n\lambda\psi_n(-2\mu_n+\psi_n)+\lambda^2(\mu_n^2+2\nu_n(\gamma+2\kappa_n-\xi_n)+\mu_n\psi_n))-2e^{2\rho\tau}(\gamma\lambda(2\mu_n+\psi_n)+\kappa_n(4\lambda\mu_n-2\lambda\psi_n-2\xi_n\psi_n)+\lambda(4\lambda\mu_n-2\mu_n\xi_n+\lambda\psi_n+2\xi_n\psi_n)))^2)/(-2\lambda\mu_n+e^{\rho\tau}(\gamma+4\kappa_n+\lambda-2\xi_n)+2\kappa_n\psi_n)^2+\frac{1}{-2\lambda\mu_n+e^{\rho\tau}(\gamma+4\kappa_n+\lambda-2\xi_n)+2\kappa_n\psi_n}2\lambda\mu_n(-e^{\rho\tau}(2\gamma+2\kappa_n+\xi_n)+\lambda(\mu_n+\psi_n))(-e^{3\rho\tau}(\gamma^2+\lambda(8\kappa_n+3\lambda-2\xi_n)+2\gamma(2\lambda+\xi_n))+8\lambda^2\nu_n(-2\kappa_n\psi_n+\lambda(\mu_n+\psi_n))-4e^{\rho\tau}(2\lambda^3\nu_n-\kappa_n^2\psi_n^2+\kappa_n\lambda\psi_n(-2\mu_n+\psi_n)+\lambda^2(\mu_n^2+2\nu_n(\gamma+2\kappa_n-\xi_n)+\mu_n\psi_n))+2e^{2\rho\tau}(\gamma\lambda(2\mu_n+\psi_n)+\kappa_n(4\lambda\mu_n-2\lambda\psi_n-2\xi_n\psi_n)+\lambda(4\lambda\mu_n-2\mu_n\xi_n+\lambda\psi_n+2\xi_n\psi_n)))-\frac{1}{-2\lambda\mu_n+e^{\rho\tau}(\gamma+4\kappa_n+\lambda-2\xi_n)+2\kappa_n\psi_n}2(8\gamma\lambda^2\nu_n+e^{2\rho\tau}(\gamma^2-\lambda(2\kappa_n+\xi_n)+\gamma(2\kappa_n+\lambda+\xi_n))+e^{\rho\tau}(\lambda^2(\mu_n+\psi_n)+\gamma(-5\lambda\mu_n-2\kappa_n\psi_n+\lambda\psi_n)))(e^{3\rho\tau}(\gamma^2+8\kappa_n^2+4\kappa_n\lambda+\gamma(8\kappa_n+\lambda-3\xi_n)-\xi_n(\lambda+2\xi_n))+4\lambda^3\nu_n(-\mu_n+\psi_n)+2e^{\rho\tau}\lambda(-\kappa_n\psi_n^2+\lambda(2\mu_n^2+4\kappa_n\nu_n-2\nu_n\xi_n-\mu_n\psi_n))+e^{2\rho\tau}(\kappa_n(-12\lambda\mu_n+2\xi_n\psi_n)+\gamma(2\kappa_n\psi_n+\lambda(-4\mu_n+\psi_n))+\lambda(\lambda(-2\mu_n+\psi_n)+2\xi_n(\mu_n+\psi_n))))-(2\xi_n(e^{3\rho\tau}(\gamma^2+\lambda(8\kappa_n+3\lambda-2\xi_n)+2\gamma(2\lambda+\xi_n))-8\lambda^2\nu_n(-2\kappa_n\psi_n+\lambda(\mu_n+\psi_n))+4e^{\rho\tau}(2\lambda^3\nu_n-\kappa_n^2\psi_n^2+\kappa_n\lambda\psi_n(-2\mu_n+\psi_n)+\lambda^2(\mu_n^2+2\nu_n(\gamma+2\kappa_n-\xi_n)+\mu_n\psi_n))-2e^{2\rho\tau}(\gamma\lambda(2\mu_n+\psi_n)+\kappa_n(4\lambda\mu_n-2\lambda\psi_n-2\xi_n\psi_n)+\lambda(4\lambda\mu_n-2\mu_n\xi_n+\lambda\psi_n+2\xi_n\psi_n)))(e^{3\rho\tau}(\gamma^2+\gamma(2\kappa_n+\lambda)+2\lambda(-\kappa_n+\xi_n))+4\lambda^3\nu_n(\mu_n-\psi_n)-2e^{\rho\tau}\lambda(\kappa_n\mu_n\psi_n+\lambda(\mu_n^2+4\kappa_n\nu_n-2\nu_n\xi_n-2\mu_n\psi_n))+e^{2\rho\tau}(4\kappa_n\lambda(\mu_n-\psi_n)+4\kappa_n^2\psi_n+\lambda(\lambda\mu_n-2\mu_n\xi_n-2\lambda\psi_n)+\gamma(2\kappa_n\psi_n-\lambda(\mu_n+2\psi_n)))))/(-2\lambda\mu_n+e^{\rho\tau}(\gamma+4\kappa_n+\lambda-2\xi_n)+2\kappa_n\psi_n)^2+(4\phi_n(e^{3\rho\tau}(\gamma^2+\gamma(2\kappa_n+\lambda)+2\lambda(-\kappa_n+\xi_n))+4\lambda^3\nu_n(\mu_n-\psi_n)-2e^{\rho\tau}\lambda(\kappa_n\mu_n\psi_n+\lambda(\mu_n^2+4\kappa_n\nu_n-2\nu_n\xi_n-2\mu_n\psi_n))+e^{2\rho\tau}(4\kappa_n\lambda(\mu_n-\psi_n)+4\kappa_n^2\psi_n+\lambda(\lambda\mu_n-2\mu_n\xi_n-2\lambda\psi_n)+\gamma(2\kappa_n\psi_n-\lambda(\mu_n+2\psi_n))))^2)/(-2\lambda\mu_n+e^{\rho\tau}(\gamma+4\kappa_n+\lambda-2\xi_n)+2\kappa_n\psi_n)^2-\frac{1}{-2\lambda\mu_n+e^{\rho\tau}(\gamma+4\kappa_n+\lambda-2\xi_n)+2\kappa_n\psi_n}4\lambda\psi_n(-e^{\rho\tau}(2\gamma+2\kappa_n+\xi_n)+\lambda(\mu_n+\psi_n))(-e^{3\rho\tau}(\gamma^2+\gamma(2\kappa_n+\lambda)+2\lambda(-\kappa_n+\xi_n))+4\lambda^3\nu_n(-\mu_n+\psi_n)+2e^{\rho\tau}\lambda(\kappa_n\mu_n\psi_n+\lambda(\mu_n^2+4\kappa_n\nu_n-2\nu_n\xi_n-2\mu_n\psi_n))+e^{2\rho\tau}(-4\kappa_n^2\psi_n+4\kappa_n\lambda(-\mu_n+\psi_n)+\lambda(-\lambda\mu_n+2\mu_n\xi_n+2\lambda\psi_n)+\gamma(-2\kappa_n\psi_n+\lambda(\mu_n+2\psi_n)))))/(8\lambda^2\nu_n+e^{2\rho\tau}(3\gamma+4\kappa_n+3\lambda+2\xi_n)-2e^{\rho\tau}(3\lambda\mu_n+\kappa_n\psi_n))^2$$

B.2 Additional numerical examples

The numerical examples in this section shall illustrate the behavior of the optimal trading strategies for the case $Y_0 \neq 0$. We compare the evolution of the optimal asset positions over time for the three cases $Y_0 = X_0$, $Y_0 = 0$ and $Y_0 = -X_0$ in a market with large λ and ρ and in a market with small λ and ρ. The exact parameter values are given in Table B.1.

Parameter	Large λ, ρ	Small λ, ρ
Player 1's asset position X_0	1	
Duration T of trading time interval	1	
Permanent impact sensitivity γ	1	
Transient impact sensitivity λ	10	0.1
Resilience parameter ρ	200	10
Time interval between trades τ	1/50	
Number of time steps N	50	

Table B.1: Parameter values used for numerical computation in Section B.2.

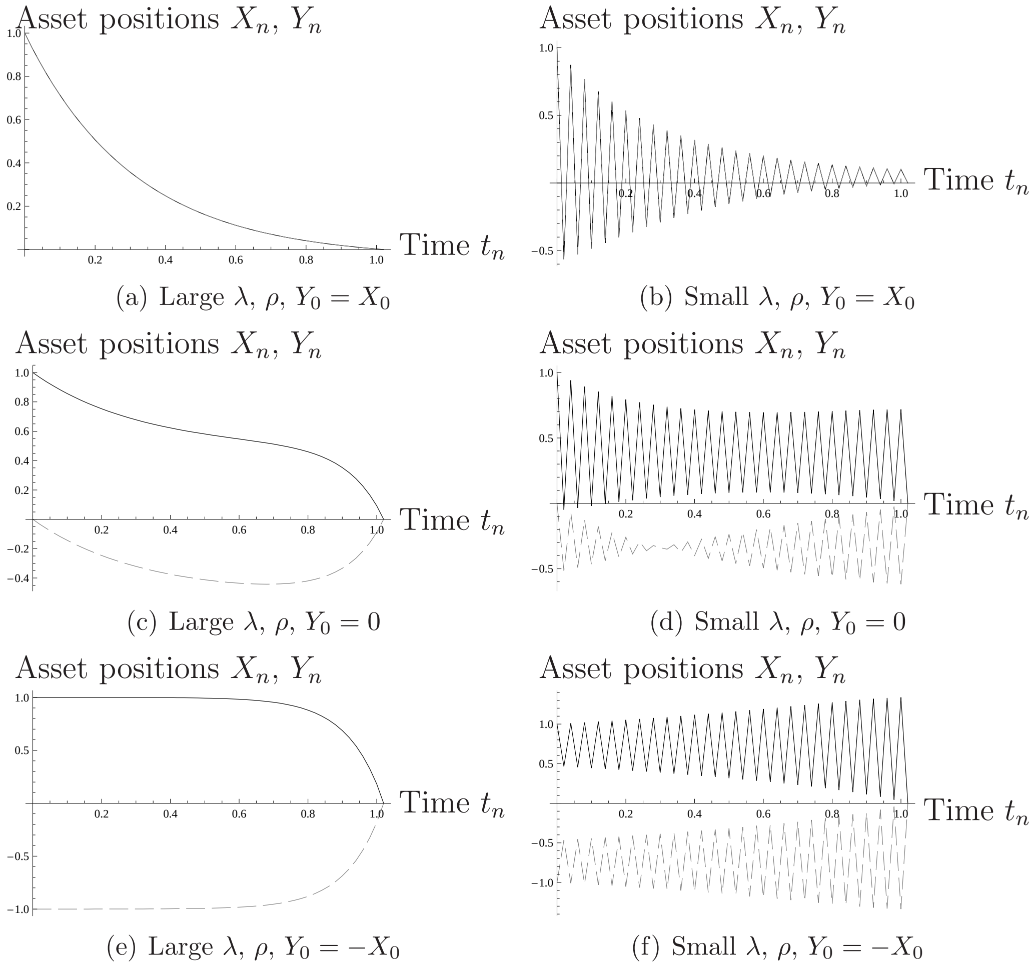
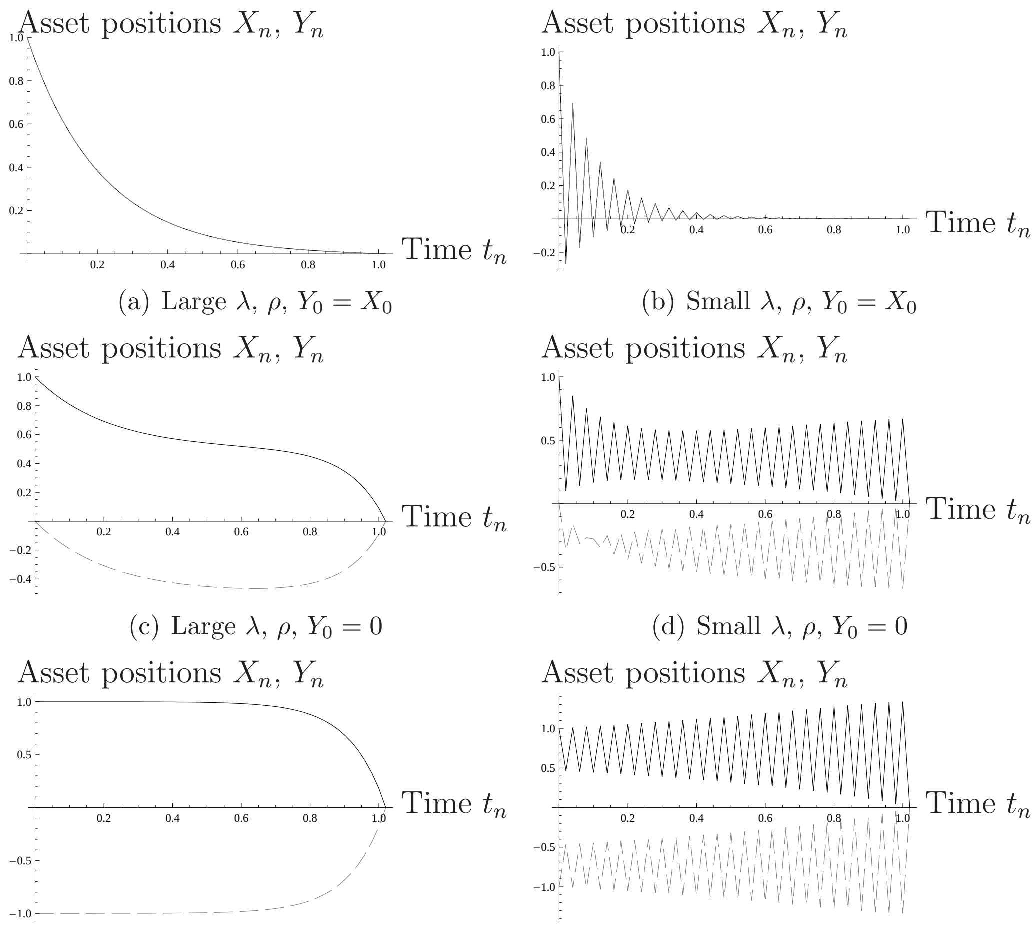


Figure B.1: Optimal asset positions $X(t_n) = X_n$ and $Y(t_n) = Y_n$ over time for the open-loop setting. The solid line represents player 1 (the seller), the dashed line player 2 (the competitor). Note the different scales in the subfigures.

Figures B.1 and B.2 graphically show the open-loop and closed-loop equilibria. We find that in line with our observation in Chapter 9, open-loop and closed-loop strategies are similar. For large λ and ρ, the results are qualitatively similar to the behavior derived in Chapter 8. For small λ and ρ, oscillations appear irrespective of the asset position Y_0 of the second player.



(a) Large λ, ρ, $Y_0 = X_0$ (b) Small λ, ρ, $Y_0 = X_0$

(c) Large λ, ρ, $Y_0 = 0$ (d) Small λ, ρ, $Y_0 = 0$

(e) Large λ, ρ, $Y_0 = -X_0$ (f) Small λ, ρ, $Y_0 = -X_0$

Figure B.2: Optimal asset positions $X(t_n) = X_n$ and $Y(t_n) = Y_n$ over time for the closed-loop setting. The solid line represents player 1 (the seller), the dashed line player 2 (the competitor). Note the different scales in the subfigures.

B.3 Mathematica source code for the closed-loop setting

In this section, we provide the Mathematica source code used to generate the *closed-loop* figures in Chapter 9 and in this appendix. The Mathematica computations for the *open-loop* figures are straightforward and therefore omitted.

Derivation of recursion equations

First, we define the parameters a, b, etc., such that $-\Theta\dot{X} = aD + bX + cY + e\dot{Y}$.

```
a = 1 − μExp[−ρτ] + 2νλExp[−2ρτ];
b = −γ − 2κ + μλExp[−ρτ];
c = −γ + ψλExp[−ρτ] − ξ;
e = (γ + λ)/2 − μλExp[−ρτ] + 2νλ^2Exp[−2ρτ]
      −ψκExp[−ρτ] + ξ;
Θ = γ + λ + 2κ − 2μλExp[−ρτ] + 2νλ^2Exp[−2ρτ];
```

Now, we transform these such that $xtilde := \dot{X} = a_1 D + b_1 X + c_1 Y$.

$$\begin{aligned}
&\mathrm{a1} = a(e - \Theta)/(\Theta\verb|^|2 - e\verb|^|2);\\
&\mathrm{b1} = (ce - \Theta b)/(\Theta\verb|^|2 - e\verb|^|2);\\
&\mathrm{c1} = (be - \Theta c)/(\Theta\verb|^|2 - e\verb|^|2);\\
&\mathrm{xtilde} = \mathrm{a1}D + \mathrm{b1}X + \mathrm{c1}Y;\\
&\mathrm{ytilde} = \mathrm{a1}D + \mathrm{b1}Y + \mathrm{c1}X;
\end{aligned}$$

Next, we define J to be the cost incurred from the next period onwards when player 1 still needs to sell X shares, player 2 needs to sell Y shares, and the market exhibits a transient impact of D. $J2$ is the cost expected now given sales of x respectively y at the current point in time.

$$\begin{aligned}
&J[\mathrm{X_, Y_, D_}] = \gamma(\mathrm{X0} + \mathrm{Y0})X + \kappa X\verb|^|2 + \mu XD + \nu D\verb|^|2\\
&\quad +\phi Y\verb|^|2 + \psi YD + \xi XY;\\
&\mathrm{J2}[\mathrm{x_, y_}] = (\gamma(\mathrm{X0} - X + \mathrm{Y0} - Y) + D + (x+y)(\gamma+\lambda)/2)x\\
&\quad +J[X - x, Y - y, (D + \lambda(x+y))\mathrm{Exp}[-\rho\tau]];
\end{aligned}$$

The optimal cost is therefore

$$\mathrm{Dummy} = \mathrm{J2}[\mathrm{xtilde}, \mathrm{ytilde}];$$

We separate this term into its different components as a polynomial of X, Y and D.

$$\begin{aligned}
&\mathrm{kappaG} = \mathrm{Simplify}[\mathrm{Coefficient}[\mathrm{Dummy}, X\verb|^|2]];\\
&\mathrm{muG} = \mathrm{Simplify}[\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Dummy}, X], D]];\\
&\mathrm{nuG} = \mathrm{Simplify}[\mathrm{Coefficient}[\mathrm{Dummy}, D\verb|^|2]];\\
&\mathrm{phiG} = \mathrm{Simplify}[\mathrm{Coefficient}[\mathrm{Dummy}, Y\verb|^|2]];\\
&\mathrm{psiG} = \mathrm{Simplify}[\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Dummy}, Y], D]];\\
&\mathrm{xiG} = \mathrm{Simplify}[\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Dummy}, X], Y]];
\end{aligned}$$

Definition of the market

$$\begin{aligned}
&\mathrm{AuctionNumber} = 100;\\
&T = 1;\\
&\tau = T/\mathrm{AuctionNumber};\\
&\gamma = 1.0;\\
&\lambda = 10.0;\\
&\rho = 200.0;
\end{aligned}$$

Recursion for parameter evolution over time

The parameters for the cost function at the last time step are given as a starting point for the recursion.

```
Clear[kappa, mu, nu, phi, psi, xi]
kappa[AuctionNumber] = (λ − γ)/2;
mu[AuctionNumber] = 1;
nu[AuctionNumber] = 0;
phi[AuctionNumber] = 0;
psi[AuctionNumber] = 0;
xi[AuctionNumber] = (λ − γ)/2;
```

Based on these, we can define the previous parameters recursively.

```
kappa[n_]:=kappa[n] = kappaG/.{κ → kappa[n + 1],
    μ → mu[n + 1], ν → nu[n + 1], ϕ → phi[n + 1],
    ψ → psi[n + 1], ξ → xi[n + 1]}
mu[n_]:=mu[n] = muG/.{Y → 0}/.{κ → kappa[n + 1],
    μ → mu[n + 1], ν → nu[n + 1], ϕ → phi[n + 1],
    ψ → psi[n + 1], ξ → xi[n + 1]}
nu[n_]:=nu[n] = nuG/.{κ → kappa[n + 1],
    μ → mu[n + 1], ν → nu[n + 1], ϕ → phi[n + 1],
    ψ → psi[n + 1], ξ → xi[n + 1]}
phi[n_]:=phi[n] = phiG/.{κ → kappa[n + 1],
    μ → mu[n + 1], ν → nu[n + 1], ϕ → phi[n + 1],
    ψ → psi[n + 1], ξ → xi[n + 1]}
psi[n_]:=psi[n] = psiG/.{κ → kappa[n + 1],
    μ → mu[n + 1], ν → nu[n + 1], ϕ → phi[n + 1],
    ψ → psi[n + 1], ξ → xi[n + 1]}
xi[n_]:=xi[n] = xiG/.{κ → kappa[n + 1],
    μ → mu[n + 1], ν → nu[n + 1], ϕ → phi[n + 1],
    ψ → psi[n + 1], ξ → xi[n + 1]}
```

In order to stay within Mathematica's recursion limits, we need to compute the parameters in 200-step-sizes.

```
For[counter = AuctionNumber − 200, counter > 0,
    counter = counter − 200, kappa[counter]]
```

Definition of the sales

```
X0 = 1;
Y0 = 0;
```

Recursion for sales trajectory evolution

With the parameters readily at hand, we can determine the exact values for the optimal trading strategies.

$$
\begin{aligned}
&\mathrm{Clear}[x, y, \mathrm{RemainingX}, \mathrm{RemainingY}, \mathrm{DAtTime}];\\
&x[\mathrm{n\_}]{:=}\\
&x[n] = \mathrm{xtilde}/.\{\kappa \to \mathrm{kappa}[n+1], \mu \to \mathrm{mu}[n+1],\\
&\quad \nu \to \mathrm{nu}[n+1], \phi \to \mathrm{phi}[n+1], \psi \to \mathrm{psi}[n+1],\\
&\quad \xi \to \mathrm{xi}[n+1], X \to \mathrm{RemainingX}[n],\\
&\quad Y \to \mathrm{RemainingY}[n], D \to \mathrm{DAtTime}[n]\}\\
&y[\mathrm{n\_}]{:=}\\
&y[n] = \mathrm{ytilde}/.\{\kappa \to \mathrm{kappa}[n+1], \mu \to \mathrm{mu}[n+1],\\
&\quad \nu \to \mathrm{nu}[n+1], \phi \to \mathrm{phi}[n+1], \psi \to \mathrm{psi}[n+1],\\
&\quad \xi \to \mathrm{xi}[n+1], X \to \mathrm{RemainingX}[n],\\
&\quad Y \to \mathrm{RemainingY}[n], D \to \mathrm{DAtTime}[n]\}\\
&x[\mathrm{AuctionNumber}]{:=}\mathrm{RemainingX}[\mathrm{AuctionNumber}];\\
&y[\mathrm{AuctionNumber}]{:=}\mathrm{RemainingY}[\mathrm{AuctionNumber}];
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{RemainingX}[0] = \mathrm{X0};\\
&\mathrm{RemainingX}[\mathrm{n\_}]{:=}\mathrm{RemainingX}[n]\\
&\quad = \mathrm{RemainingX}[n-1] - x[n-1];\\
&\mathrm{RemainingY}[0] = \mathrm{Y0};\\
&\mathrm{RemainingY}[\mathrm{n\_}]{:=}\mathrm{RemainingY}[n]\\
&\quad = \mathrm{RemainingY}[n-1] - y[n-1];\\
&\mathrm{DAtTime}[0] = 0;\\
&\mathrm{DAtTime}[\mathrm{n\_}]{:=}\mathrm{DAtTime}[n]\\
&\quad = (\mathrm{DAtTime}[n-1] + \lambda(x[n-1] + y[n-1]))\mathrm{Exp}[-\rho\tau];
\end{aligned}
$$

Output of derived sales trajectories

We can now plot the optimal trajectories.

$$
\begin{aligned}
&\mathrm{For}[\mathrm{xvector} = \{\}; \mathrm{counter} = 0, \mathrm{counter} \le \mathrm{AuctionNumber},\\
&\quad \mathrm{counter}{+}{+}, \mathrm{xvector} = \mathrm{Append}[\mathrm{xvector},\\
&\quad \{\mathrm{counter}/\mathrm{AuctionNumber}T, x[\mathrm{counter}]\}]]\\
&\mathrm{For}[\mathrm{yvector} = \{\}; \mathrm{counter} = 0, \mathrm{counter} \le \mathrm{AuctionNumber},\\
&\quad \mathrm{counter}{+}{+}, \mathrm{yvector} = \mathrm{Append}[\mathrm{yvector},\\
&\quad \{\mathrm{counter}/\mathrm{AuctionNumber}T, y[\mathrm{counter}]\}]]
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{ListPlot}[\{\mathrm{xvector}, \mathrm{yvector}\}, \mathrm{PlotJoined} \to \mathrm{True},\\
&\quad \mathrm{AxesOrigin} \to \{0, 0\}, \mathrm{PlotRange} \to \mathrm{All},\\
&\quad \mathrm{PlotStyle} \to \{\{\mathrm{GrayLevel}[0], \mathrm{Dashing}[\{\}]\},\\
&\quad \{\mathrm{GrayLevel}[0.5], \mathrm{Dashing}[\{0.05, 0.02\}]\}\}]
\end{aligned}
$$

```
For[RemainingXvector = {}; counter = 0,
    counter ≤ AuctionNumber, counter++,
    RemainingXvector = Append[RemainingXvector,
    {counter/AuctionNumber T, RemainingX[counter]}]]
RemainingXvector = Append[RemainingXvector,
    {T + 1/AuctionNumber, 0}];
For[RemainingYvector = {}; counter = 0,
    counter ≤ AuctionNumber, counter++,
    RemainingYvector = Append[RemainingYvector,
    {counter/AuctionNumber T, RemainingY[counter]}]]
RemainingYvector = Append[RemainingYvector,
    {T + 1/AuctionNumber, 0}];
```

```
ListPlot[{RemainingXvector, RemainingYvector},
    PlotJoined → True, PlotRange → All,
    PlotStyle → {{GrayLevel[0], Dashing[{}]},
    {GrayLevel[0.5], Dashing[{0.05, 0.02}]}}]
```

```
For[DAtTimevector = {}; counter = 0,
    counter ≤ AuctionNumber, counter++,
    DAtTimevector = Append[DAtTimevector,
    {counter/AuctionNumber T, DAtTime[counter]}]]
```

```
ListPlot[DAtTimevector, PlotJoined → True,
    AxesOrigin → {0, 0}, PlotRange → All,
    PlotStyle → {{GrayLevel[0], Dashing[{}]},
    {GrayLevel[0], Dashing[{0.05, 0.02}]}}]
```

Bibliography

Abramowitz, P. (2006). Tool of the trade. *Institutional Investor's Alpha Magazine 6*, 41–44.

Admati, A. R. and P. Pfleiderer (1991). Sunshine trading and financial market equilibrium. *Review of Financial Studies 4*(3), 443–81.

Alfonsi, A., A. Schied, and A. Schulz (2007a). Constrained portfolio liquidation in a limit order book model. *Working paper*.

Alfonsi, A., A. Schied, and A. Schulz (2007b). Optimal execution strategies in limit order books with general shape functions. *Working paper*.

Allen, F. and G. Gorton (1992, April). Stock price manipulation, market microstructure and asymmetric information. *European Economic Review 36*(2-3), 624–630.

Almgren, R. (2003, March). Optimal execution with nonlinear impact functions and trading-enhanced risk. *Applied Mathematical Finance 10*(1), 1–18.

Almgren, R. and N. Chriss (1999, December). Value under liquidation. *Risk 12*, 61–63.

Almgren, R. and N. Chriss (2001, Winter). Optimal execution of portfolio transactions. *Journal of Risk 3*(2), 5–39.

Almgren, R. and J. Lorenz (2007, Spring). Adaptive arrival price. *Algorithmic Trading III: Precision, Control, Execution*.

Almgren, R., C. Thum, E. Hauptmann, and H. Li (2005, July). Direct estimation of equity market impact. *Risk 18*(7), 57–62.

Attari, M., A. S. Mello, and M. E. Ruckes (2005, October). Arbitraging arbitrageurs. *Journal of Finance 60*(5), 2471–2511.

Back, K. (1992). Insider trading in continuous time. *Review of Financial Studies 5*(3), 387–409.

Bagehot, W. (1971, March/April). The only game in town. *Financial Analysts Journal 27*, 12–14.

Bangia, A., F. X. Diebold, T. Schuermann, and J. D. Stroughair (1998, December). Modeling liquidity risk, with implications for traditional market risk measurement and management. Center for Financial Institutions Working Papers 99-06, Wharton School Center for Financial Institutions, University of Pennsylvania.

Bank, P. and D. Baum (2004, January). Hedging and portfolio optimization in financial markets with a large trader. *Mathematical Finance 14*(1), 1–18.

Barclay, M. J. and J. B. Warner (1993, December). Stealth trading and volatility: Which trades move prices? *Journal of Financial Economics 34*(3), 281–305.

Beltrami, E. (1868). Sulla teoria delle linee geodetiche. *Rendiconti del Reale Istituto Lombardo (serie II) 1*, 708–718.

Benton, S. H. (1977). *The Hamilton-Jacobi equation: a global approach.* Mathematics in Science & Engineering. Academic Press Inc.,U.S.

Berkowitz, J. (2000, September). Incorporating liquidity risk into Value-at-Risk models. *Working paper*.

Bertsimas, D., P. Hummel, and A. Lo (1999, November/December). Optimal control of execution costs for portfolios. *Computing in Science and Engineering 1*(6), 40–53.

Bertsimas, D. and A. Lo (1998, April). Optimal control of execution costs. *Journal of Financial Markets 1*(1), 1–50.

Biais, B., L. Glosten, and C. Spatt (2005, May). Market microstructure: A survey of microfoundations, empirical results, and policy implications. *Journal of Financial Markets 8*(2), 217–264.

Biais, B., P. Hillion, and C. Spatt (1995, December). An empirical analysis of the limit order book and the order flow in the Paris Bourse. *Journal of Finance 50*(5), 1655–89.

Bolza, O. (1909). *Vorlesungen über Variationsrechnung.* Leipzig; Berlin: B.G. Teubner.

Bondarenko, O. (2001, June). Competing market makers, liquidity provision, and bid-ask spreads. *Journal of Financial Markets 4*(3), 269–308.

Bouchaud, J.-P., Y. Gefen, M. Potters, and M. Wyart (2004, April). Fluctuations and response in financial markets: The subtle nature of 'random' price changes. *Quantitative Finance 4*, 176–190.

Brunnermeier, M. K. and L. H. Pedersen (2005, August). Predatory trading. *Journal of Finance 60*(4), 1825–1863.

Butenko, S., A. Golodnikov, and S. Uryasev (2005, October). Optimal security liquidation algorithms. *Computational Optimization and Applications 32*(1-2), 9–27.

Butler, G. (2007, Spring). Liquidity aggregation: What institutional investors need to know. *Algorithmic Trading III: Precision, Control, Execution*.

Cai, F. (2003, February). Was there front running during the LTCM crisis? International Finance Discussion Papers 758, Board of Governors of the Federal Reserve System (U.S.).

Carlin, B. I., M. S. Lobo, and S. Viswanathan (2007, October). Episodic liquidity crises: Cooperative and predatory trading. *Journal of Finance 65*(5), 2235–2274.

Çetin, U., R. A. Jarrow, and P. Protter (2004, August). Liquidity risk and arbitrage pricing theory. *Finance and Stochastics 8*(3), 311–341.

Çetin, U. and L. C. G. Rogers (2007, January). Modelling liquidity effects in discrete time. *Mathematical Finance 17*(1), 15–29.

Cesari, L. (1983). *Optimization - Theory and Applications: Problems With Ordinary Differential Equations.* Number 17 in Applications of Mathematics. Springer.

Chakravarty, S. (2001, August). Stealth-trading: Which traders' trades move stock prices? *Journal of Financial Economics 61*(2), 289–307.

Chan, L. K. C. and J. Lakonishok (1995, September). The behavior of stock prices around institutional trades. *Journal of Finance 50*(4), 1147–74.

Chincarini, L. B. (2007, Winter). The Amaranth debacle: A failure of risk measures or a failure of risk management? *Journal of Alternative Investments 4*.

Chordia, T., R. Roll, and A. Subrahmanyam (2001, April). Market liquidity and trading activity. *Journal of Finance 56*(2), 501–530.

Coval, J. D. and E. Stafford (2007, November). Asset fire sales (and purchases) in equity markets. *Journal of Financial Economics 86*(2), 479–512.

Demsetz, H. (1968, February). The cost of transacting. *Quarterly Journal of Economics 82*(1), 33–53.

Dia, M. and S. Pouget (2006, June). Sunshine trading in an emerging stock market. Cahier de recherche 2006 Ű 174, IAE de Toulouse.

Dubil, R. (2002). Optimal liquidation of large security holdings in thin markets. In *Research in International Entrepreneurial Finance and Business Ventures at the Turn of the Third Millennium.*

Duffie, D. and A. Ziegler (2003, May/June). Liquidation risk. *Financial Analysts Journal 59*(3), 42–51.

Durrett, R. (1996). *Stochastic Calculus: A practical Introduction.* Probability and Stochastics Series. CRC-Press.

Easley, D. and M. O'Hara (1987, September). Price, trade size, and information in securities markets. *Journal of Financial Economics 19*(1), 69–90.

Engle, R. and R. Ferstenberg (2007, Winter). Execution risk. *Journal of Portfolio Management 33*(2), 34–44.

Fama, E. F. (1970, May). Efficient capital markets: a review of theory and empirical work. *Journal of Finance 25*(2), 383–417.

Finger, C. C. (2006, October). The lights are on. *RiskMetrics Group Research monthly*, 1–7.

Föllmer, H. and A. Schied (2004). *Stochastic Finance.* Gruyter.

Foster, F. D. and S. Viswanathan (1996, September). Strategic trading when agents forecast the forecasts of others. *Journal of Finance 51*(4), 1437–78.

Frey, R. (1997). Derivative asset analysis in models with level-dependent and stochastic volatility. *CWI Quarterly 10*(1), 1–34.

Frey, R. and P. Patie (2002). Risk management for derivatives in illiquid markets: a simulation-study. In *Advances in finance and stochastics*, pp. 137–159. Berlin: Springer.

Fudenberg, D. and D. Levine (1988, February). Open-loop and closed-loop equilibria in dynamic games with many players. *Journal of Economic Theory 44*(1), 1–18.

Glosten, L. R. and P. R. Milgrom (1985, March). Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. *Journal of Financial Economics 14*(1), 71–100.

Grossman, S. J. and M. H. Miller (1988, July). Liquidity and market structure. *Journal of Finance 43*(3), 617–37.

Gruber, U. M. (2004, September). *Convergence of Binomial Large Investor Models and General Correlated Random Walks*. Ph. D. thesis, Technische Universität Berlin.

Harris, L. E. (1997, December). Order exposure and parasitic traders. *Working paper*.

Harris, L. E. and E. Gurel (1986, September). Price and volume effects associated with changes in the s&p 500 list: New evidence for the existence of price pressures. *Journal of Finance 41*(4), 815–29.

He, H. and H. Mamaysky (2005, May). Dynamic trading policies with price impact. *Journal of Economic Dynamics and Control 29*(5), 891–930.

Hisata, Y. and Y. Yamai (2000, December). Research toward the practical application of liquidity risk evaluation methods. *Monetary and Economic Studies 18*(2), 83–127.

Holthausen, R. W., R. W. Leftwich, and D. Mayers (1987, December). The effect of large block transactions on security prices: A cross-sectional analysis. *Journal of Financial Economics 19*(2), 237–267.

Holthausen, R. W., R. W. Leftwich, and D. Mayers (1990, July). Large-block transactions, the speed of response, and temporary and permanent stock-price effects. *Journal of Financial Economics 26*(1), 71–95.

Huberman, G. and W. Stanzl (2004, July). Price manipulation and quasi-arbitrage. *Econometrica 72*(4), 1247–1275.

Huberman, G. and W. Stanzl (2005, June). Optimal liquidity trading. *Review of Finance 9*(2), 165–200.

Jarrow, R. A. and P. Protter (2007, October). Liquidity risk and option pricing theory. In J. Birge and V. Linetsky (Eds.), *Handbooks in operations research and management science: Financial Engineering*, Volume 15, Chapter 17. Elsevier.

Jorion, P. (2000, September). Risk management lessons from Long-Term Capital Management. *European Financial Management 6*(3), 277–300.

Kahneman, D. and A. Tversky (1979, March). Prospect theory: An analysis of decision under risk. *Econometrica 47*(2), 263–291.

Kempf, A. and O. Korn (1999, Febuary). Market depth and order size. *Journal of Financial Markets 2*(1), 29–48.

Kissell, R. and M. Glantz (2003). *Optimal Trading Strategies: Quantitative Approaches for Managing Market Impact and Trading Risk*. Mcgraw-Hill Professional.

Kissell, R. and R. Malamut (2005, February). Understanding the profit and loss distribution of trading algorithms. *Algorithmic Trading: Precision, Control, Execution*.

Kissell, R. and R. Malamut (2006, Winter). Algorithmic decision making framework. *Journal of Trading 1*, 12–21.

Konishi, H. and N. Makimoto (2001, Summer). Optimal slice of a block trade. *Journal of Risk 3*(4), 33–51.

Kraus, A. and H. R. Stoll (1972, June). Price impacts of block trading on the New York Stock Exchange. *Journal of Finance 27*(3), 569–588.

Kyle, A. S. (1985, November). Continuous auctions and insider trading. *Econometrica 53*(6), 1315–1336.

Ladyzhenskaya, O. A., V. A. Solonnikov, and N. N. Ural'ceva (1968). *Linear and Quasi-linear Equations of Parabolic Type*. Number 23 in Translations of Mathematical Monographs. American Mathematical Society.

Large, J. (2007, February). Measuring the resiliency of an electronic limit order book. *Journal of Financial Markets 10*(1), 1–25.

Leinweber, D. (2007, Febuary). Algo vs. algo. *Institutional Investor's Alpha Magazine 2*, 44–51.

Lillo, F., J. D. Farmer, and R. N. Mantegna (2003, January). Econophysics: Master curve for price-impact function. *Nature 421*, 129–130.

Loeb, T. (1983, May/June). Trading cost: The critical link between investment information and results. *Financial Analysts Journal 39*(3), 39–44.

Lowenstein, R. (2001). *When genius failed: The rise and fall of Long-Term Capital Management*. Fourth Estate.

Mönch, B. (2004, January). Optimal liquidation strategies. *EFMA 2004 Basel Meetings Paper*.

Obizhaeva, A. (2007, January). Information vs. liquidity: Evidence from portfolio transition trades. *Working paper*.

Obizhaeva, A. and J. Wang (2006, April). Optimal trading strategy and supply/demand dynamics. *Forthcoming in Journal of Financial Markets*.

O'Hara, M. (1998). *Market Microstructure Theory*. Blackwell Publishers.

Perold, A. F. (1998, Spring). The implementation shortfall: Paper versus reality. *Journal of Portfolio Management 14*(3), 4–9.

Potters, M. and J.-P. Bouchaud (2003, June). More statistical properties of order books and price impact. *Physica A 324*(1-2), 133–140.

Pratt, J. (1964, January/April). Risk aversion in the small and in the large. *Econometrica 32*(1/2), 122–136.

Rockafellar, R. T. (1970). *Convex Analysis*. Princeton Landmarks in Mathematics and Physics. Princeton University Press.

Rogers, L. C. G. and S. Singh (2007, August). The cost of illiquidity and its effects on hedging. *Working paper*.

Roux, A. (2007, October). Optimal portfolio liquidation with quadratic and non-quadratic risk measures. Master's thesis, Technical University Berlin.

Schack, J. (2004, November). The orders of battle. *Institutional Investor 11*, 77–84.

Schulz, A. (2007, July). Optimal execution in limit order book markets with call auctions. Master's thesis, Technical University Berlin.

Simmonds, M. (2007, February). The use of quantitative models in execution analytics and algorithmic trading. Presentation at the University Finance Seminar, Judge Business School, Cambridge University.

Submaranian, A. and R. A. Jarrow (2001, October). The liquidity discount. *Mathematical Finance 11*(4), 447–474.

Till, H. (2006). EDHEC comments on the Amaranth case: Early lessons from the debacle. *EDHEC Comments*.

Weber, P. and B. Rosenow (2005, August). Order book approach to price impact. *Quantitative Finance 5*(4), 357–364.