On the dynamics of sup-norm nonexpansive maps

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Abstract

We present several results for the periods of periodic points of sup-norm nonexpansive maps. In particular, we show that the period of each periodic point of a sup-norm nonexpansive map $f\colon M \to M$, where $M \subset \mathbb{R}^n$, is at most $\max_k 2^k\binom{n}{k}$. This upper bound is smaller than 3^n and improves the previously known bounds. Further we consider a special class of sup-norm nonexpansive maps, namely topical functions. For topical functions $f\colon \mathbb{R}^n \to \mathbb{R}^n$ Gunawardena and Sparrow have conjectured that the optimal upper bound for the periods of periodic points is $\binom{n}{\lfloor n/2 \rfloor}$. We give a proof of this conjecture. To obtain the results we use combinatorial and geometric arguments. In particular, we analyse the cardinality of anti-chains in certain partially ordered sets.

1 Introduction

In this paper we are concerned with the dynamics of sup-norm nonexpansive maps $f\colon M \to M$, where $M \subset \mathbb{R}^n$. A characteristic property of the iterative behaviour of sup-norm nonexpansive maps is that every bounded orbit converges to a periodic orbit. Moreover there exists an upper bound for the possible periods of periodic points of sup-norm nonexpansive maps that only depends on the dimension of the ambient space. It is therefore an interesting problem to determine the optimal upper bound for the periods of periodic points of sup-norm nonexpansive maps. This problem has been considered in [4, 14, 15, 16, 18, 25], and Nussbaum [18] has conjectured that 2^n is the optimal upper bound. At present however the conjecture is proved only for $n \leq 3$ (see [15]).

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The main goal of the paper is to present two results concerning the periodic points of sup-norm nonexpansive maps. As a first result we show that the period of each periodic point of a sup-norm nonexpansive map $f\colon M \to M$, where $M \subset \mathbb{R}^n$, is at most $\max_k 2^k\binom{n}{k}$. This upper bound is smaller than 3^n and is a considerable improvement of the previously best known estimate $n!2^n$ by Martus [16].

It turns out that the ideas for proving our estimate can be used to analyse the periods of periodic points of topical functions. Topical functions are maps $f\colon \mathbb{R}^n \to \mathbb{R}^n$ that are order-preserving and additively homogeneous. They are sup-norm nonexpansive by a result of Crandall and Tartar [7], and they appear in a variety of applications, such as, the analysis of discrete event systems [3, 5, 9, 10], optimal control theory [2], and nonlinear Perron-Frobenius theory [8, 19, 20]. As a second result we prove a conjecture of Gunawardena and Sparrow (see [9, §4.3] and [10, §2.1.3]), which asserts that the optimal upper bound for the periods of periodic points of topical functions $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is $\binom{n}{\lfloor n/2 \rfloor}$.

To obtain the results we analyse transitive actions of abelian groups of sup-norm isometries. As explained in the next section this is a more general setting than is required. In fact, in order to obtain an optimal upper bound for the periods of periodic points of sup-norm nonexpansive maps, one could consider transitive actions of cyclic groups of sup-norm isometries. But, as yet, we do not know how to (or if it is necessary to) exploit this additional assumption.

The dynamics of nonexpansive maps has been studied for various norms on $\mathbb{R}^n$. In particular it is known that if the unit ball of the norm is a polyhedron, then every bounded orbit of a nonexpansive map converges to a periodic orbit (see [18] or [26]). This is an immediate consequence of the result for the sup-norm, because any finite dimensional polyhedrally normed space can be isometrically embedded into $\mathbb{R}^n$ with the sup-norm, as long as n is sufficiently large. Another interesting polyhedral norm on $\mathbb{R}^n$, besides the sup-norm, is the ℓ_1-norm. There exist many detailed results for the possible periods of periodic points of ℓ_1-norm nonexpansive maps and we refer the reader to [1, 12, 13, 17, 21, 22, 23, 24] for further information.

In addition to the introduction the paper contains four sections. In Section 2 we collect several definitions and results for sup-norm nonexpansive maps. Furthermore we formulate a theorem from which it follows that the period of each periodic point of a sup-norm nonexpansive map $f\colon M \to M$, where $M \subset \mathbb{R}^n$, is at most $\max_k 2^k\binom{n}{k}$. Subsequently, we prove this more general theorem in Section 3. Section 4 contains several definitions and results concerning topical functions. In addition we state a theorem that implies that the period of each periodic point of a topical function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is at most $\binom{n}{\lfloor n/2 \rfloor}$. Section 5 is devoted to a proof of this theorem. We conclude Section 5 with several remarks. In particular, we explain a consequence of our results for the dynamics of certain cone maps.

2 Preliminaries

Let $\mathbb{R}^n$ be equipped with the sup-norm, given by $\|x\|_\infty = \max_i |x_i|$ for $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$. A map $f\colon M \to M$, where $M \subset \mathbb{R}^n$, is called *sup-norm nonexpansive* if $\|f(x) - f(y)\|_\infty \leq \|x - y\|_\infty$ for all $x, y \in M$. It is said to be a *sup-norm isometry* if equality holds for all $x, y \in M$. A group Γ of sup-norm isometries $g\colon X \to X$, where $X \subset \mathbb{R}^n$, is said to *act transitively on* X if for each $x, y \in X$ there exists $g \in \Gamma$ such that $g(x) = y$.

A point $x \in M$ is called a *periodic point* of $f\colon M \to M$ if there exists an integer $p \geq 1$ such that $f^p(x) = x$. The minimal such $p \geq 1$ is called the *period* of x under f. Transitive actions of groups of isometries are intimately connected with periodic points of nonexpansive maps, as we shall explain now. Let $x \in M$ be a periodic point of $f\colon M \to M$ with period p and let X be the orbit of x under f; so, $X = \{f^k(x)\colon 0 \leq k < p\}$. If f is sup-norm nonexpansive, then the restriction of f to X is a sup-norm isometry that maps X onto itself. Moreover, if we put $g = f_{|X}$, then $\{g^k\colon X \to X \mid 0 \leq k < p\}$ is a cyclic group of sup-norm isometries that acts transitively on X. Thus to prove that the period of each periodic point of a sup-norm nonexpansive map $f\colon M \to M$, where $M \subset \mathbb{R}^n$ is at most $\max_k 2^k\binom{n}{k}$, it suffices to show that each finite set X in $\mathbb{R}^n$, with a transitive abelian group of sup-norm isometries, has at most $\max_k 2^k\binom{n}{k}$ elements. In the paper we take this approach. In fact, to estimate the periods of periodic points of sup-norm nonexpansive maps, we prove the following theorem in Section 3.

Theorem 2.1. *If X is a finite set in $\mathbb{R}^n$ on which an abelian group of sup-norm isometries acts transitively, then $|X| \leq \max_k 2^k\binom{n}{k}$.*

Let us first discuss some preliminary results. We would like to mention that several of these results can be found in [15, Section 2]. For the reader's convenience we prove them here once more.

Actions of abelian groups of isometries enjoy the following useful property (compare [15, Lemma 2.3]).

Lemma 2.2. *If X is a set in $\mathbb{R}^n$ on which an abelian group Γ of sup-norm isometries acts transitively, then for each $f \in \Gamma$ there exists a constant $d(f)$ such that $\|x - f(x)\|_\infty = d(f)$ for all $x \in X$;*

Proof. Since Γ is an abelian group of sup-norm isometries we have that

$$\|x - f(x)\|_\infty = \|g(x) - g(f(x))\|_\infty = \|g(x) - f(g(x))\|_\infty \quad \text{for all } g \in \Gamma.$$

As Γ acts transitively on X, it follows that $\|x - f(x)\|_\infty$ is the same for all $x \in X$, which completes the proof. $\square$

A sequence of points $x^1, x^2, \ldots, x^m$ in $\mathbb{R}^n$ is called an *additive chain* if

$$\|x^1 - x^m\|_\infty = \sum_{k=1}^{m-1} \|x^k - x^{k+1}\|_\infty.$$

We say that it has *length* m if it consists of m distinct points. Lemma 2.2 has the following corollary for additive chains (compare [15, Proposition 2.1]).

Corollary 2.3. *Suppose that X is a finite set in $\mathbb{R}^n$ on which an abelian group Γ of sup-norm isometries acts transitively. If $x^1, x^2, \ldots, x^m$ is an additive chain in X of length $m \geq 2$, and for $1 \leq k < m$ the map $f_k \in \Gamma$ maps x^k to x^{k+1}, then for each permutation π on $\{1, \ldots, m-1\}$ and for each $x \in X$ the sequence*

$$x,\ f_{\pi(1)}(x),\ \ldots,\ f_{\pi(m-1)} \circ \cdots \circ f_{\pi(1)}(x)$$

is an additive chain of length m.

Proof. Put $f = f_{m-1} \circ \cdots \circ f_2 \circ f_1$ and observe that as Γ is abelian $f = f_{\pi(m-1)} \circ \cdots \circ f_{\pi(2)} \circ f_{\pi(1)}$. For simplicity we write $z^k = f_{\pi(k-1)} \circ \cdots \circ f_{\pi(1)}(x)$ for $2 \leq k \leq m$ and $z^1 = x$. Using Lemma 2.2(i) we obtain the equalities:

$$\begin{aligned} \|z^1 - z^m\|_\infty = d(f) = \|x^1 - x^m\|_\infty &= \sum_{k=1}^{m-1} \|x^{k+1} - x^k\|_\infty \\ = \sum_{k=1}^{m-1} d(f_k) = \sum_{k=1}^{m-1} d(f_{\pi(k)}) &= \sum_{k=1}^{m-1} \|z^{k+1} - z^k\|_\infty. \end{aligned}$$

Thus, $z^1, z^2, \ldots, z^m$ is an additive chain of length m, and this proves the corollary. $\square$

A useful idea in this exposition is the notion of an extreme pair. A pair of distinct points $\{x, y\}$ in X is said to be an *extreme pair in X* if there exists no $z \in X$ such that z, x, y or x, y, z is an additive chain of length 3. Extreme pairs have the following property.

Lemma 2.4. *Suppose that X is a finite set in $\mathbb{R}^n$ on which an abelian group Γ of sup-norm isometries acts transitively. Then $\{x, y\}$ is an extreme pair in X if and only if there exists no $z \in X$ such that x, y, z is an additive chain of length 3.*

Proof. It is clear from the definition that if $\{x, y\}$ is an extreme pair in X, then there exists no $z \in X$ such that x, y, z is an additive chain of length 3. To prove the other implication we assume that $\{x, y\}$ is not an extreme pair in X. Then there exists $z \in X$ such that z, x, y or x, y, z is an additive chain of length 3. If x, y, z is an additive chain of length 3, we are done. On the other hand, if z, x, y is an additive chain of length 3, then we can use Corollary 2.3 to see that $x, y, f(y)$ is an additive chain of length 3, where $f \in \Gamma$ is such that $f(z) = x$. This proves the lemma. $\square$

Further we need to define a partial ordering $\preceq$ on $\{0,1,2\}^n$ by

$$a \preceq b \text{ if } a_i = b_i \text{ for all } i \text{ for which } b_i \in \{0,1\}.$$

It is easy to verify that $\preceq$ is reflexive, anti-symmetric, and transitive. A subset $\mathcal{A}$ of the partially ordered set $(\{0,1,2\}^n, \preceq)$ is called an *anti-chain* if there exist no distinct $a, b \in \mathcal{A}$ with $a \preceq b$. Using a standard method in extremal set theory, called the LYM technique [6, Chapter 7], we obtain the following upper bound for the cardinality of anti-chains in $(\{0,1,2\}^n, \preceq)$.

Proposition 2.5. *If $\mathcal{A}$ is an anti-chain in $(\{0,1,2\}^n, \preceq)$, then*

$$|\mathcal{A}| \leq \max_k 2^k \binom{n}{k}.$$

Proof. Consider chains $a^0 \preceq a^1 \preceq \ldots \preceq a^n$ in $(\{0,1,2\}^n, \preceq)$ consisting of $n+1$ distinct elements. There are $2^n n!$ such chains. Indeed, to obtain such a chain one has to select a point a^0 such that a^0 has no coordinates equal to 2. There are 2^n possibilities. Subsequently one changes one by one the coordinates of a^0 to 2. There are $n!$ ways to do this. Thus, in total we have $2^n n!$ such chains.

Now consider an element a in $\mathcal{A}$ with k coordinates equal to 2, and we ask ourselves how many of the chains $a^0 \preceq a^1 \preceq \ldots \preceq a^n$ contain a. Clearly there are $2^k k!$ chains $a^0 \preceq \ldots \preceq a^{k-1} \preceq a$, with $k+1$ distinct elements and each of these can be extended in $(n-k)!$ ways. Thus there are $2^k k!(n-k)!$ such chains.

To finish the argument let m_k be the number of elements in $\mathcal{A}$ with k coordinates equal to 2. Obviously $|\mathcal{A}| = \sum_{k=0}^n m_k$. As $\mathcal{A}$ is an anti-chain, each of the chains $a^0 \preceq a^1 \preceq \ldots \preceq a^n$ contains at most one element of $\mathcal{A}$. Therefore

$$\sum_{k=0}^n m_k 2^k k!(n-k)! \leq 2^n n!,$$

so that

$$\sum_{k=0}^n m_k 2^{k-n} \binom{n}{k}^{-1} \leq 1.$$

From this we deduce that

$$|\mathcal{A}| = \sum_{k=0}^n m_k \leq \max_k 2^{n-k} \binom{n}{k} = \max_k 2^k \binom{n}{k}.$$

$\square$

The upper bound in Proposition 2.5 is sharp. Indeed, the collection of points in $(\{0,1,2\}^n, \preceq)$ with $n-k$ coordinates equal to 2 is an anti-chain that contains $2^k \binom{n}{n-k} = 2^k \binom{n}{k}$ elements. From this remark it follows directly that

$\max_k 2^k \binom{n}{k} < 3^n$. In fact, one can use Stirling's formula and some analysis to show that

$$\max_k 2^k \binom{n}{k} = \gamma(n)3^n/\sqrt{n}, \quad \text{where } 0 < \delta_1 \leq \gamma(n) \leq \delta_2 < \infty \text{ for all } n.$$

3 Proof of the first main result

Proof of Theorem 2.1. Let X be a finite set in $\mathbb{R}^n$ on which an abelian group Γ of sup-norm isometries acts transitively. Define $c\colon X \to \{0,1,2\}^n$ by

$$c(x)_i = \begin{cases} 0 & \text{if there exists } y \in X \text{ with } \{x,y\} \text{ an extreme pair in } X \\ & \text{and } \|x-y\|_\infty = y_i - x_i, \\ 1 & \text{if there exists } y \in X \text{ with } \{x,y\} \text{ an extreme pair in } X \\ & \text{and } \|x-y\|_\infty = x_i - y_i, \\ 2 & \text{otherwise,} \end{cases}$$

for all $1 \leq i \leq n$ and $x \in X$.

We remark that $c(x)$ is well-defined, because $c(x)_i = 0$ and $c(x)_i = 1$ implies that there exist $y, z \in X$ such that $\{x,y\}$ and $\{x,z\}$ are extreme pairs in X with $\|x-y\|_\infty = y_i - x_i$ and $\|x-z\|_\infty = x_i - z_i$. This implies that

$$\|y-z\|_\infty \leq \|y-x\|_\infty + \|x-z\|_\infty = y_i - x_i + x_i - z_i = y_i - z_i \leq \|y-z\|_\infty,$$

so that z, x, y is an additive chain of length 3. This however contradicts the fact that $\{x,y\}$ is an extreme pair in X.

By Proposition 2.5 it suffices to show that $c(y) \not\preceq c(x)$ for all $x \neq y$ in X. To see this let $\mathcal{F}$ be the collection of all additive chains $z^1, z^2, \ldots, z^r$ in X such that $z^1 = x$ and $z^2 = y$. Now consider an additive chain $z^1, z^2, \ldots, z^m$ of maximal length m in $\mathcal{F}$. We claim that $\{z^1, z^m\}$ is an extreme pair in X. Indeed, there exists no $u \in X$ such that z^1, z^m, u is an additive chain of length 3, as m is maximal. Therefore it follows from Lemma 2.4 that $\{z^1, z^m\}$ is an extreme pair in X.

We observe that there exists $1 \leq i \leq n$ such that either $\|z^1 - z^m\|_\infty = z^m_i - z^1_i$ or $\|z^1 - z^m\|_\infty = z^1_i - z^m_i$. In the first case $c(x)_i = 0$ by definition, and we now show that $c(y)_i \neq 0$. If $c(y)_i = 0$, then there exists $u \in X$ such that $\{y, u\}$ is an extreme pair in X with $\|y-u\|_\infty = u_i - y_i$. As $z^1, z^2, \ldots, z^m$ is an additive chain and $\|z^1 - z^m\|_\infty = z^m_i - z^1_i$, we have that

$$\|z^1 - z^m\|_\infty = z^m_i - z^1_i = \sum_{k=1}^{m-1} z^{k+1}_i - z^k_i \leq \sum_{k=1}^{m-1} \|z^{k+1} - z^k\|_\infty = \|z^1 - z^m\|_\infty.$$

This implies that $\|z^{k+1} - z^k\|_\infty = z^{k+1}_i - z^k_i$ for all $1 \leq k < m$, so that $\|x-y\|_\infty = y_i - x_i$. Hence x, y, u is an additive chain of length 3, which contradicts the fact that $\{y, u\}$ is an extreme pair in X. Thus, we conclude that $c(y)_i \neq 0$ and hence $c(y) \not\leq c(x)$.

Similarly, if $\|z^1 - z^m\|_\infty = z_i^1 - z_i^m$, then by definition $c(x)_i = 1$, but $c(y)_i \neq 1$. Indeed, if $c(y)_i = 1$, then there exists $v \in X$ such that $\{y, v\}$ is an extreme pair in X and $\|y - v\|_\infty = y_i - v_i$. Again, as $z^1, z^2, \ldots, z^m$ is an additive chain and $\|z^1 - z^m\|_\infty = z_i^1 - z_i^m$, it follows that $\|x - y\|_\infty = x_i - y_i$. This implies that x, y, v is an additive chain of length 3, which contradicts the fact that $\{y, v\}$ is an extreme pair. Thus, $c(y)_i \neq 1$ in this case and hence $c(y) \not\leq c(x)$. This completes the proof of the theorem. □

4 Topical functions

Before we give the definition of a topical function we introduce some notation. On $\mathbb{R}^n$ we let $\leq$ be a partial ordering, given by $x \leq y$ if $x_i \leq y_i$ for all $1 \leq i \leq n$. Two elements x and y in $\mathbb{R}^n$ are said to be *comparable* if $x \leq y$ or $y \leq x$. A map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is called *order-preserving* if for each $x, y \in \mathbb{R}^n$ with $x \leq y$ we have that $f(x) \leq f(y)$. It is said to *additively homogeneous* if $f(x + \lambda\mathbf{1}) = f(x) + \lambda\mathbf{1}$ for all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. (Here $\mathbf{1}$ denotes the vector with all coordinates unity.) A *topical function* is a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ that is order-preserving and additively homogeneous.

Crandall and Tartar [7] have proved that every topical functions is sup-norm nonexpansive. One can deduce this result from a useful observation of Gunawardena and Keane [11]. To formulate the observation we first need to introduce the function $t\colon \mathbb{R}^n \to \mathbb{R}$ given by, $t(x) = \max_i x_i$ for all $x \in \mathbb{R}^n$.

Proposition 4.1. *A map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a topical function if and only if*

$$t(f(x) - f(y)) \leq t(x - y) \quad \textit{for all } x, y \in \mathbb{R}^n.$$

Proof. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a topical function and $x, y \in \mathbb{R}^n$. Obviously $x \leq y + t(x - y)\mathbf{1}$, so that $f(x) \leq f(y) + t(x - y)\mathbf{1}$. This implies that $t(f(x) - f(y)) \leq t(x - y)$. To see the other implication assume that $x \leq y$. Then $t(x - y) \leq 0$, so that $t(f(x) - f(y)) \leq 0$ and hence $f(x) \leq f(y)$. To show that f is additively homogeneous put $y = x + \lambda\mathbf{1}$. Clearly $t(y - x) = \lambda$ and $t(x - y) = -\lambda$, which gives $t(f(y) - f(x)) \leq \lambda$ and $t(f(x) - f(y)) \leq -\lambda$. Thus $\lambda\mathbf{1} \leq f(y) - f(x) \leq \lambda\mathbf{1}$ and this completes the proof. □

As $\|x\|_\infty = \max\{t(x), t(-x)\}$, Proposition 4.1 immediately implies that every topical function is sup-norm nonexpansive.

The function t is not a norm; it is for instance not symmetric. Nevertheless we say that $f\colon M \to M$, where $M \subset \mathbb{R}^n$, is *t-nonexpansive* if $t(f(x) - f(y)) \leq t(x - y)$ for all $x, y \in M$. Further we call $f\colon M \to M$ a *t-isometry* if equality holds for all $x, y \in M$. We observe that if $x \in \mathbb{R}^n$ is a periodic point of a topical function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ with period p, then the orbit of x, given by $X = \{f^k(x)\colon 0 \leq k < p\}$, contains no comparable elements, as

f is order-preserving. Moreover it follows from Proposition 4.1 that on X a cyclic group of t-isometries acts transitively.

It has been conjectured by Gunawardena and Sparrow (see [9, §4.3]) that the optimal upper bound for the periods of periodic points of topical functions $f\colon\mathbb{R}^n \to \mathbb{R}^n$ is $\binom{n}{\lfloor n/2\rfloor}$. To see that the upper bound cannot be sharper they constructed the following example (compare [10, §2.1.3]). Put $m = \binom{n}{\lfloor n/2\rfloor}$ and let U be the collection of m points $u \in \{0,1\}^n$ with $\sum_i u_i = \lfloor n/2\rfloor$. Label the points in U by $u^1, u^2, \ldots, u^m$ and put $u^{m+1} = u^1$. Now define a map $f\colon\mathbb{R}^n \to \mathbb{R}^n$ by

$$f(x)_i = \max_{k:u_i^{k+1}=1} \big(\min_{j:u_j^k=1} \{x_j\} \big) \quad \text{for } 1 \le i \le n \text{ and } x \in \mathbb{R}^n.$$

It is easy to verify that f is a topical function and that $f(u^k) = u^{k+1}$ for all $1 \le k \le m$.

In order to prove the conjecture of Gunawardena and Sparrow we show the following theorem in Section 5.

Theorem 4.2. *If X is a finite set of incomparable elements in $\mathbb{R}^n$ on which an abelian group of t-isometries acts transitively, then $|X| \le \binom{n}{\lfloor n/2\rfloor}$.*

To establish this theorem we apply similar ideas as the ones used in the proof of Theorem 2.1. In particular, we use the notion of a t-additive chain. A sequence of points $x^1, x^2, \ldots, x^m$ in $\mathbb{R}^n$ is called a *t-additive chain* if

$$t(x^1 - x^m) = \sum_{k=1}^{m-1} t(x^k - x^{k+1}).$$

The *length* of a t-additive chain is the number of distinct points in it. Note that if $a^1, a^2, \ldots, a^m$ is a t-additive chain, then $a^m, a^{m-1}, \ldots, a^1$ need not be a t-additive chain, because $t(x) \ne t(-x)$ in general.

We now collect several preliminary results for the proof of Theorem 4.2, beginning with two simple observations concerning t-additive chains.

Lemma 4.3. *If $t(x-y) = x_i - y_i$ and $t(y-z) = y_i - z_i$, then $t(x-z) = x_i - z_i$ and x, y, z is a t-additive chain.*

Proof. Remark that

$$x_i - z_i \le t(x-z) \le t(x-y) + t(y-z) = x_i - y_i + y_i - z_i = x_i - z_i,$$

so that $t(x-z) = x_i - z_i$ and $t(x-z) = t(x-y) + t(y-z)$. $\square$

Lemma 4.4. *If $x^1, x^2, \ldots, x^m$ is a t-additive chain and $t(x^1 - x^m) = x_i^1 - x_i^m$, then $t(x^k - x^{k+1}) = x_i^k - x_i^{k+1}$ for all $1 \le k < m$.*

Proof. We have that

$$t(x^1 - x^m) = \sum_{k=1}^{m-1} t(x^k - x^{k+1}) \geq \sum_{k=1}^{m-1} x_i^k - x_i^{k+1} = x_i^1 - x_i^m = t(x^1 - x^m).$$

This implies that $t(x^k - x^{k+1}) = x_i^k - x_i^{k+1}$ for all $1 \leq k < m$. □

Completely analogous to Lemma 2.2 one can show that transitive actions of abelian groups of t-isometries satisfy the following property.

Lemma 4.5. *If X is a set in $\mathbb{R}^n$ on which an abelian group Γ of t-isometries acts transitively, then for each $f \in \Gamma$ there exists a constant $d(f)$ such that $t(x - f(x)) = d(f)$ for all $x \in X$.*

In the same fashion as Corollary 2.3 we deduce the following consequence.

Corollary 4.6. *Let X be a set in $\mathbb{R}^n$ on which an abelian group Γ of t-isometries acts transitively. If x, y, z is a t-additive chain in X and $f \in \Gamma$ is such that $f(x) = y$, then $y, z, f(z)$ is a t-additive chain in X.*

Proof. Let $g \in \Gamma$ be such that $g(y) = z$. Using Lemma 4.5 and the fact that Γ is abelian we find that

$$\begin{aligned} t(y - f(z)) &= t(y - g(f(y))) = t(x - g(f(x))) \\ &= t(x - y) + t(y - z) = t(z - f(z)) + t(y - z). \end{aligned}$$

□

Next we introduce the notion of t-extreme pair. An ordered pair (x, y) in $X \times X$, with $x \neq y$, is called *t-extreme in X* if there exists no $z \in X$ such that x, y, z or z, x, y is a t-additive chain of length 3. Note that (x, y) is a t-extreme pair need not imply that (y, x) is a t-extreme pair. Therefore we consider ordered pairs instead of unordered pairs. By using Corollary 4.6 we can prove the following lemma for t-extreme pairs. This lemma resembles Lemma 2.4 and the proofs are quite similar.

Lemma 4.7. *If X is a finite set of incomparable elements in $\mathbb{R}^n$ on which an abelian group Γ of t-isometries acts transitively, then (x, y) in $X \times X$ with $x \neq y$ is t-extreme in X if and only if there exists no $z \in X$ such that x, y, z is a t-additive chain of length 3.*

Proof. It follows from the definition that if (x, y) is t-extreme, then there exists no $z \in X$ such that x, y, z is a t-additive chain of length 3. For the other implication we assume that (x, y) is not t-extreme. Hence there exists $z \in X$ such that either x, y, z or z, x, y is a t-additive chain of length 3. In the first case we are done. In the second case it follows from Corollary 4.6 that $x, y, f(y)$ is a t-additive chain, where $f \in \Gamma$ is such that $f(z) = x$. To see that it has length 3 we remark that $t(u-v)>0$ for all $u \neq v$ in X, as no two elements in X are comparable. Thus, $t(y-f(y)) = t(z-x) > 0$, because $x \neq z$, and $t(x-f(y)) = t(z-y) > 0$, as $y \neq z$. This completes the proof. $\square$

5 Proof of the second main result

By using the same type of arguments as in the proof of Theorem 2.1 we now prove Theorem 4.2. An important ingredient in the proof is a classic result in combinatorics, namely Sperner's Theorem, which asserts the following (compare [6, Chapter 7]).

Theorem 5.1 (Sperner). *If $\mathcal{A}$ is a collection of subsets of $\{1,\ldots,n\}$ such that there are no distinct $A, B \in \mathcal{A}$ with $A \subset B$, then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.*

More precisely, we use the following equivalent formulation: if $\mathcal{A}$ is an anti-chain in the partially ordered set $(\{0,1\}^n, \leq)$, then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Proof of Theorem 4.2. Let X be a finite set of incomparable points in $\mathbb{R}^n$ on which an abelian group Γ of t-isometries acts transitively. Define a map $c\colon X \to \{0,1\}^n$ by

$$c(x)_i = \begin{cases} 1 & \text{if there exists } y \in X \text{ such that } (x,y) \text{ is } t\text{-extreme in } X \\ & \text{and } t(x-y) = x_i - y_i, \\ 0 & \text{otherwise,} \end{cases}$$

for $1 \leq i \leq n$ and $x \in X$. By Sperner's Theorem it suffices to show that $c(x)$ and $c(y)$ are incomparable for all $x \neq y$ in X. So, suppose that $x \neq y$ in X. Let $z^1, z^2, \ldots, z^m$ be a t-additive chain in X consisting of m distinct elements, where $z^1 = x$, $z^2 = y$, and m is maximal. We claim that (z^1, z^m) is t-extreme in X. Indeed, if we suppose by way of contradiction that (z^1, z^m) is not t-extreme, then by Lemma 4.7 there exists $u \in X$ such that z^1, z^m, u is a t-additive chain of length 3. This implies that $z^1, z^2, \ldots, z^m, u$ is a t-additive chain. To see that it consists of $m+1$ distinct elements we use the fact that X contains no comparable elements. If $u = z^k$, then $k \neq m$ and moreover

$$t(z^1-u) = t(z^1-z^k)+t(z^k-z^m)+t(z^m-u) = t(z^1-u)+t(u-z^m)+t(z^m-u).$$

This implies that $t(u-z^m) + t(z^m-u) = 0$. But this contradicts the fact that $t(u-z^m) > 0$ and $t(z^m-u) > 0$, as u and z^m are not comparable. Thus, $z^1, z^2, \ldots, z^m, u$ is a t-additive chain with $m+1$ distinct points, which contradicts the maximality of m. Therefore (z^1, z^m) is t-extreme in X.

Clearly there exists $1 \leq i \leq n$ such that $t(z^1 - z^m) = z^1_i - z^m_i$, and hence the definition of $c\colon X \to \{0,1\}^n$ gives $c(x)_i = 1$. We claim that

$c(y)_i \neq 1$. Indeed, if $c(y)_i = 1$ there exists $v \in X$ such that (y, v) is t-extreme and $t(y - v) = y_i - v_i$. As $t(z^1 - z^m) = z_i^1 - z_i^m$ it follows from Lemma 4.4 that $t(x - y) = x_i - y_i$. Using Lemma 4.3 we find that x, y, v is a t-additive chain. This additive chain has length 3, because $x = v$ implies $y_i - x_i = y_i - v_i = t(y - v) = t(y - x) > 0$, which contradicts the fact that $x_i - y_i = t(x - y) > 0$. But x, y, v cannot be a t-additive chain of length 3, as (y, v) is t-extreme. Thus, $c(y)_i \neq 1$ and hence $c(y)_i = 0$. By interchanging the roles of x and y we also find j such that $c(y)_j = 1$ and $c(x)_j = 0$. Therefore $c(x)$ and $c(y)$ are not comparable and we are done. □

The result for the periods of periodic points of topical functions has consequences for the dynamics of certain cone maps. Let $\mathbb{R}^n_+$ be the standard positive cone in $\mathbb{R}^n$, so, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } 1 \leq i \leq n\}$ and let $\mathrm{int}(\mathbb{R}^n_+)$ denote its interior. We observe that there exists a homeomorphism $L: \mathrm{int}(\mathbb{R}^n_+) \to \mathbb{R}^n$ given by, $L(x)_i = \log(x_i)$ for $1 \leq i \leq n$ and $x \in \mathrm{int}(\mathbb{R}^n_+)$. The inverse $E: \mathbb{R}^n \to \mathrm{int}(\mathbb{R}^n_+)$ of L is of course given by, $E(x)_i = e^{x_i}$ for $1 \leq i \leq n$ and $x \in \mathbb{R}^n$. As both $z \to \log(z)$ and $z \to e^z$ are monotone functions, we have that if $g: \mathrm{int}(\mathbb{R}^n_+) \to \mathrm{int}(\mathbb{R}^n_+)$ is order-preserving, then $f: \mathbb{R}^n \to \mathbb{R}^n$ given by, $f(x) = L(g(E(x)))$ for $x \in \mathbb{R}^n$, is also order-preserving. Moreover, it is easy to see that if $g: \mathrm{int}(\mathbb{R}^n_+) \to \mathrm{int}(\mathbb{R}^n_+)$ is homogeneous, i.e., $g(\lambda x) = \lambda g(x)$ for all $x \in \mathrm{int}(\mathbb{R}^n_+)$ and $\lambda > 0$, then f is additively homogeneous. Conversely, one can show that if $f: \mathbb{R}^n \to \mathbb{R}^n$ is a topical function, then $g: \mathrm{int}(\mathbb{R}^n_+) \to \mathrm{int}(\mathbb{R}^n_+)$, given by $g(x) = E(f(L(x)))$ for all $x \in \mathrm{int}(\mathbb{R}^n_+)$, is order-preserving and homogeneous. Therefore order-preserving homogeneous maps $g: \mathrm{int}(\mathbb{R}^n_+) \to \mathrm{int}(\mathbb{R}^n_+)$ have the same dynamical behaviour as topical functions $f: \mathbb{R}^n \to \mathbb{R}^n$. Consequently, the following assertion is true.

Theorem 5.2. *The optimal upper bound for the periods of periodic points of maps $g: \mathrm{int}(\mathbb{R}^n_+) \to \mathrm{int}(\mathbb{R}^n_+)$ that are order-preserving and homogeneous is $\binom{n}{\lfloor n/2 \rfloor}$.*

We conclude the paper by indicating a connection between the two main results. To every sup-norm nonexpansive map $f: M \to M$, where $M \subset \mathbb{R}^n$, a t-nonexpansive map $g: M' \to M'$, where $M' \subset \mathbb{R}^{2n}$, can be associated such that g has the same dynamical behaviour as f. Indeed, if we let $\tau: \mathbb{R}^n \to \mathbb{R}^{2n}$ be given by $\tau(x) = (x, -x)$ for all $x \in \mathbb{R}^n$, then

$$t(\tau(x) - \tau(y)) = \max_{1 \leq i \leq 2n} \{\tau(x)_i - \tau(y)_i\} = \max\{t(x - y), t(y - x)\} = \|x - y\|_\infty.$$

This implies that if f is sup-norm nonexpansive, then $g = \tau \circ f \circ \tau^{-1}$ is t-nonexpansive. In particular, if Γ is an abelian group of sup-norm isometries $f: X \to X$ that acts transitively on X and $Y = \tau(X)$, then Y contains no comparable elements and $\Gamma' = \{\tau \circ f \circ \tau^{-1}: Y \to Y \mid f \in \Gamma\}$ is an abelian group of t-isometries that acts transitively on Y. By applying Theorem 4.2 we immediately deduce that each finite set X in $\mathbb{R}^n$, on which an abelian group of sup-norm isometries acts transitively, has at most $\binom{2n}{n}$ elements. From Stirling's formula it follows that $\binom{2n}{n} \sim 4^n/\sqrt{\pi n}$, which is a worse upper bound than the one in Theorem 2.1.

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