On Area-Universal Graphs: How to win a Mug

June 22, 2017

We answered the following prize question posed at EuroCG[1].: "Is the graph in Figure 1 area-universal?"

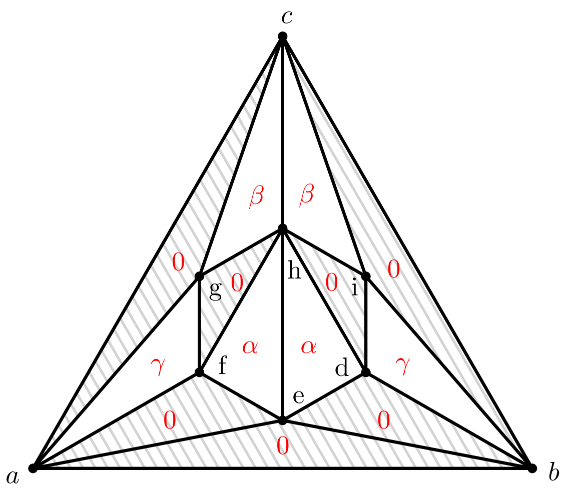


Figure 1: The graph

Theorem 1. *The graph shown in Figure 1 is not area-universal.*

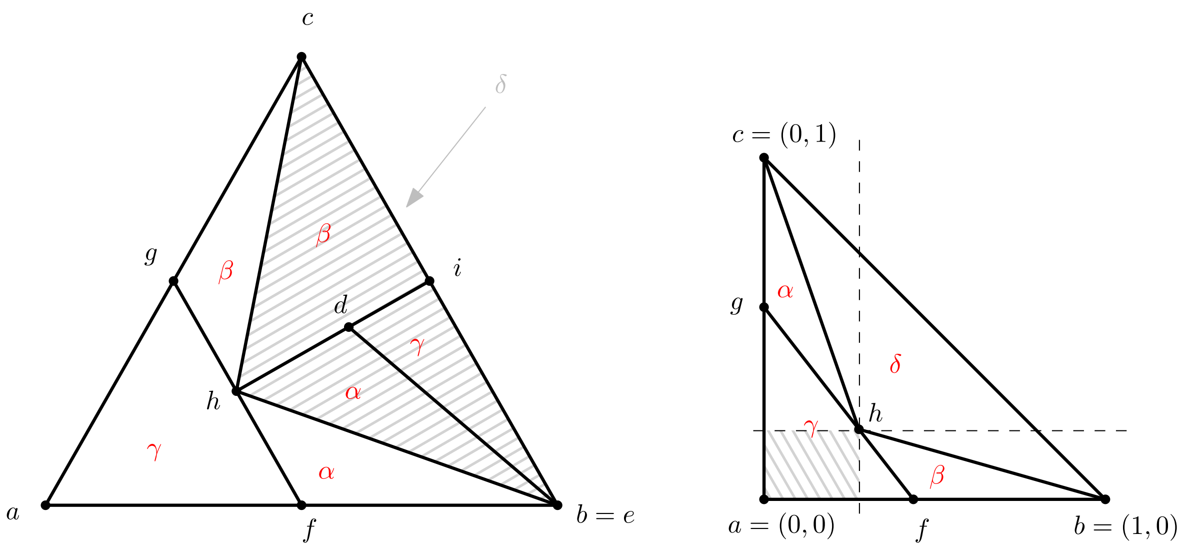
Proof. We show that the face-assignment as shown in Figure 1 cannot be realized for certain values of $\alpha, \beta, \gamma > 0$. Suppose for a contradiction that there exists a realization. Since the triangle $\triangle(a, b, e)$ has area 0, the three points a, b, e are collinear. Moreover, since a and b lie on the outer face and since $a \neq b$ (otherwise the drawing would not have any area), the vertex e lies on the (closed) segment ab. Note that e might coincide with a or b. Similarly, g lies on the segment ac and i lies on the segment bc. Since $\beta, \gamma > 0$, the points a, b, c, g, i are all distinct. We distinguish the following two cases:

Suppose for a contradiction, that e does not coincide with a or b. A similar argument as above shows that d lies on the line $\overline{be}$, and f lies on the line $\overline{ae}$, and thus both lie on the (closed) segment ab. Since $\alpha, \gamma > 0$, the points a, d, e, f, b are all distinct. Similarly, since $\alpha, \beta, \gamma > 0$, the three points h, i, d are distinct. Since $\triangle(i, d, h)$ has area 0, the three points d, i, h are collinear. Moreover, since d, i both lie on the boundary of the convex hull, h lies on the (open) segment id. Similarly, h lies on the (open) segment gf. However, this is not possible as the path $c - h - e$ separates those two segments - a contradiction to the assumption that e does not coincide with a or b.

Without loss of generality, we assume that e coincides with b, i.e., $b = e$. By the same arguments as above, f lies on the segment ab. Recall that g lies on the segment ac. As $\alpha, \gamma > 0$, f lies in the (open) segment ab. Similarly, g lies in the (open) segment ac and i lies in the (open) segment bc. Figure 2a illustrates the combinatorics.

In the following we consider the placement $a = (0, 0)$, $b = (1, 0)$, $c = (0, 1)$ and we consider the union of the four triangle inside $\triangle(b, c, h)$ with area $\delta := \alpha + \beta + \gamma$, as illustrated in Figure 2b. Note that afb, agc, and fhg are collinear triples due to this union.

[1] http://csconferences.mah.se/eurocg2017/openproblems.pdf



(a) (b)

Figure 2: The graph

Claim 1.1. *Not all assignments of $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha+\beta+\gamma+\delta = \frac{1}{2}$ are realizable for the configuration from Figure 2b.*

To prove this claim, observe that the axis-parallel rectangle $\Box$ through a and h has area $h_x \cdot h_y = (1-\lambda)f_x \cdot \lambda g_y$, while writing $h = (1-\lambda)f + \lambda g$ for some $\lambda \in (0,1)$. It is easy to see that this expression would be maximized for $\lambda = \frac{1}{2}$ and thus the area of $\Box$ is at most half of the area of the triangle $\triangle(a, f, g)$. Now note that $h_x > 2\alpha$ since $\triangle(c, g, h)$ has area α and since $0 < g_y$. Similarly, $h_y > 2\alpha$. Altogether, we obtain that the area of the triangle $\triangle(a, f, g)$ is strictly greater than $2 \cdot 2\alpha \cdot 2\beta$, or equivalently, $\gamma > 8\alpha\beta$. This finishes the proof of the claim.

As a direct consequence of the claim, the assignment $\alpha = \beta = \frac{1}{10}, \gamma = \frac{1}{20}, \delta = \frac{1}{4}$ (note that $\alpha + \beta + \gamma = \frac{1}{4} = \delta$) to the graph from Figure 2a is not realizable because $\gamma \not> 8\alpha\beta$. Consequently, the original graph from Figure 1 is also not realizable with this assignment. This finishes the proof of the theorem. $\Box$