On Disjoint Holes in Point Sets*†

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Abstract

Given a set of points $S \subseteq \mathbb{R}^2$, a subset $X \subseteq S$, $|X| = k$, is called *k-gon* if all points of X lie on the boundary of the convex hull $\mathrm{conv}(X)$, and *k-hole* if, in addition, no point of $S \setminus X$ lies in $\mathrm{conv}(X)$. We use computer assistance to show that every set of 17 points in general position admits two *disjoint* 5-holes, that is, holes with disjoint respective convex hulls. This answers a question of Hosono and Urabe (2001).

In a recent article, Hosono and Urabe (2018) present new results on interior-disjoint holes – a variant, which also has been investigated in the last two decades. Using our program, we show that every set of 15 points contains two interior-disjoint 5-holes. Moreover, our program can also be used to verify that every set of 17 points contains a 6-gon within significantly smaller computation time than the original program by Szekeres and Peters (2006).

1 Introduction

A set of points in the Euclidean plane $S \subseteq \mathbb{R}^2$ is *in general position* if no three points lie on a common line. Throughout this paper all point sets are considered to be in general position. A subset $X \subseteq S$ of size $|X| = k$ is a *k-gon* if all points of X lie on the boundary of the convex hull of X. A classical result from the 1930s by Erdős and Szekeres asserts that, for fixed $k \in \mathbb{N}$, every sufficiently large point set contains a k-gon [12, 25]. They also constructed point sets of size 2^{k-2} with no k-gon. Recently, Suk [31] significantly improved the upper bound by showing that every set of $2^{k+o(k)}$ points contains a k-gon. However, the precise minimum number $g(k)$ of points needed to guarantee the existence of a k-gon is still unknown for $k \geq 7$ (cf. [32]).

In the 1970s, Erdős [11] asked whether every sufficiently large point set contains a *k-hole*, that is, a k-gon with no other points of S lying inside its convex hull. Harborth [17] showed that every set of 10 points contains a 5-hole and Horton [18] introduced a construction of large point sets without 7-holes. The question, whether 6-holes exist in sufficiently large point sets, remained open until 2007, when Nicolas [26] and Gerken [15] independently showed that point sets with large k-gons also contain a 6-hole (see also [33]). The currently best bound is by Koshelev [23], who showed that every set of 463 points contains a 6-hole. However, the largest set without 6-holes currently known has 29 points and was found by Overmars [27].

In 2001, Hosono and Urabe [19] started the investigation of disjoint holes, where two holes X_1, X_2 of a given point set S are said to be *disjoint* if their respective convex hulls are disjoint (that is, $\mathrm{conv}(X_1) \cap \mathrm{conv}(X_2) = \emptyset$). This led to the following question: What is the smallest number $h(k_1, \ldots, k_l)$ such that every set of $h(k_1, \ldots, k_l)$ points determines a k_i-hole for every $i = 1, \ldots, l$, such that the holes are pairwise disjoint [21]?

* The full version of this paper is available online [30].

† Research was supported by the DFG Grant FE 340/12-1. We thank Stefan Felsner, Linda Kleist, Felix Schröder, and Martin Balko for fruitful discussions and helpful comments. Thanks to Adrian Dumitrescu for pointing out the variant of interior-disjoint holes.

35th European Workshop on Computational Geometry, Utrecht, The Netherlands, March 18–20, 2019.
This is an extended abstract of a presentation given at EuroCG'19. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

In Sections 2 and 3, we summarize the current state of the art for two- and three-parametetric values and we present some new results that were obtained using computer-assistance. The basic idea behind our computer-assisted proofs is to encode point sets and disjoint holes only using triple orientations (see Section 4), and then to use a SAT solver to disprove the existence of sets with certain properties (see Section 5).

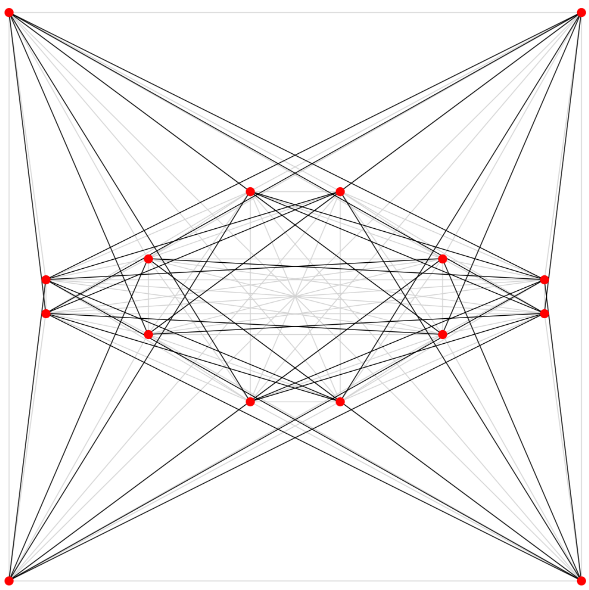
In the Final Remarks (Section 6) we outline how our SAT model can be adapted to tackle related questions on point sets. In particular, the program can be used to show that every set of 15 points contains two interior-disjoint 5-holes, and to prove $g(6) = 17$ with significantly smaller computation time than the original program from Szekeres and Peters [32].

2 Two Disjoint Holes

For two parameters, the value $h(k_1, k_2)$ has been determined for all $k_1, k_2 \leq 5$ except for $h(5,5)$ [19, 20, 21, 5]. Table 1 summarizes the currently best bounds for two-parametric values. Concerning the value $h(5,5)$, the best bounds are $17 \leq h(5,5) \leq 19$. The lower bound $h(5,5) \geq 17$ is witnessed by the set of 16 points with no two disjoint 5-holes (taken from Hosono and Urabe [21]), which is depicted Figure 1, and the upper bound $h(5,5) \leq 19$ was shown by Bhattacharya and Das [6] by an elaborate case distinction.

	2	3	4	5
2	4	5	6	10
3		6	7	10
4			9	12
5				17*

Table 1 Values of $h(k_1, k_2)$ [19, 20, 21, 5]. The entry marked with star (*) is new.



0	0
0	270
140	0
140	270
9	127
9	143
131	127
131	143
34	117
34	153
106	117
106	153
59	85
59	185
81	85
81	185

Figure 1 A set of 16 points with no two disjoint 5-holes. This point set and the one by Hosono and Urabe [21, Figure 3] are of the same order type (see Section 4.1 for the definition of order type).

As our main result of this paper, we determine the precise value of $h(5,5)$. The proof is based on a SAT model which we later describe in Section 5.

▶ **Theorem 2.1** (Computer-assisted). *Every set of 17 points contains two disjoint 5-holes, hence* $h(5,5) = 17$.

We remark that the computations for verifying Theorem 2.1 take about two hours on a single 3GHz CPU using a modern SAT solver such as glucose [3] or picosat [7]. Moreover, we have verified the output of glucose and picosat with the proof checking tool DRAT-trim [34].

3 Three Disjoint Holes

For three parameters, most values $h(k_1, k_2, k_3)$ for $k_1, k_2, k_3 \leq 4$ and also the values $h(2,3,5) = 11$ and $h(3,3,5) = 12$ are known [21, 35]. Tables 2 and 3 summarize the currently best known bounds for three-parametric values.

	2	3	4
2	8	9	11
3		10	12
4			14

Table 2 Values of $h(k_1, k_2, 4)$ [21, 35].

	2	3	4	5
2	10	11	11..14	17*
3		12	13..14	17..19*
4			15..17	17..23*
5				22*..27*

Table 3 Bounds for $h(k_1, k_2, 5)$ [21, 35].

We now use Theorem 2.1 to derive new bounds on the value $h(k,5,5)$ for $k = 2,3,4,5$.

▶ **Corollary 3.1.** *We have*

$$h(2,5,5) = 17, \quad 17 \leq h(3,5,5) \leq 19, \quad 17 \leq h(4,5,5) \leq 23, \quad \textit{and} \quad 22 \leq h(5,5,5) \leq 27.$$

Proof. To show $h(2,5,5) \leq 17$, observe that, due to Theorem 2.1, every set of 17 points contains two disjoint 5 holes that are separated by a line ℓ. By the pigeonhole principle there are at least 9 points on one of the two sides of such a separating line ℓ. Again, using a SAT instance similar to the one for Theorem 2.1, one can easily verify that every set of 9 points with a 5-hole also contains a 2-hole which is disjoint from the 5-hole. We remark that also the order type database of 9 points can be used to verify this statement.

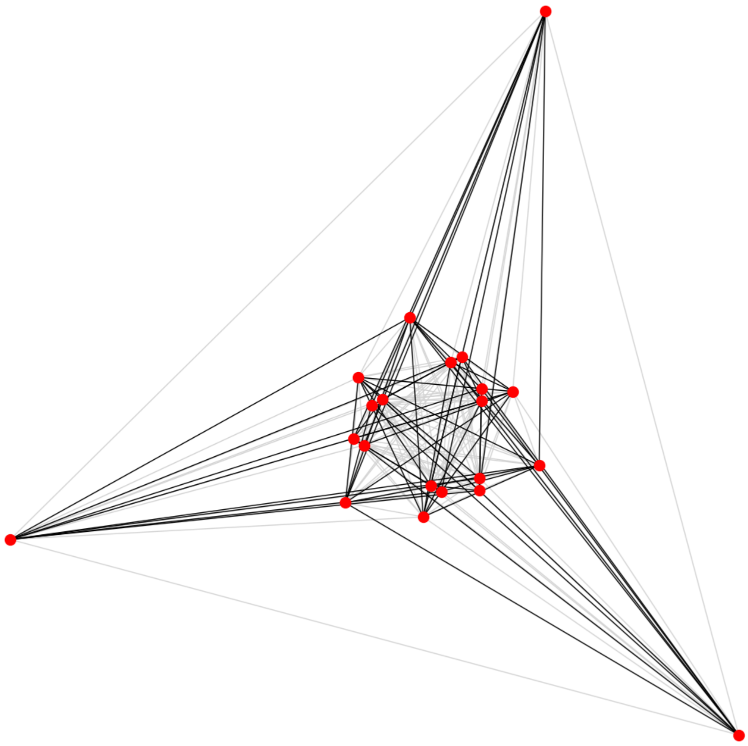
Similarly we show $h(3,5,5) \leq 19$: Every set of 19 points contains two disjoint 5-holes that are separated by a line ℓ. Now there are at least 10 points on one side of ℓ, and since $h(3,5) = 10$, there is a 3-hole and a 5-hole that are disjoint on that particular side.

An analogous argument shows $h(4,5,5) \leq 2 \cdot h(4,5) - 1 = 23$.

The point set from Figure 2 shows $h(5,5,5) > 21$, while $h(5,5,5) \leq h(5)+h(5,5) = 27$. ◀

4 Encoding with Triple Orientations

We describe how point sets and disjoint holes can be encoded only using triple orientations. This combinatorial description allows us to get rid of the actual point coordinates and to only consider a discrete parameter-space. This is essential for our SAT model of the problem.



0	161014
437034	595949
326347	343801
284425	294548
368806	311583
359850	306967
303825	276373
295136	271265
384946	285229
410465	282863
385025	275150
280383	244110
288858	238662
432159	221931
383508	211334
343366	205440
352134	200469
273710	191231
383027	201270
337326	179552
595182	0

Figure 2 A set of 21 points with no three disjoint 5-holes.

4.1 Triple Orientations

Given a set of points $S = \{s_1, \dots, s_n\}$ with $s_i = (x_i, y_i)$, we say that the triple (a, b, c) is *positively (negatively) oriented* if

$$\chi_{abc} := \operatorname{sgn}\det \begin{pmatrix} 1 & 1 & 1 \\ x_a & x_b & x_c \\ y_a & y_b & y_c \end{pmatrix} \in \{-1, 0, +1\}$$

is positive (negative). Note that $\chi_{abc} = 0$ indicates collinear points, in particular, $\chi_{aaa} = \chi_{aab} = \chi_{aba} = \chi_{baa} = 0$. It is easy to see, that convexity is a combinatorial rather than a geometric property since k-gons can be described only by the relative position of the points: If the points $s_1, \dots, s_k$ are the vertices of a convex polygon (ordered along the boundary), then, for every $i = 1, \dots, k$, the cyclic order of the other points around s_i is $s_{i+1}, s_{i+2}, \dots, s_{i-1}$ (indices modulo k). Similarly, one can also describe containment (and thus k-holes) only using relative positions: A point s_0 lies inside a convex polygon if the cyclic order around s_0 is precisely the order of the vertices along the boundary of the polygon.

To observe that the disjointness of two point sets can be described solely using triple orientations, suppose that a line ℓ separates point sets A and B. Then, for example by rotating ℓ, we can find another line ℓ' that contains a point $a \in A$ and a point $b \in B$ and separates $A \setminus \{a\}$ and $B \setminus \{b\}$. In particular, we have $\chi_{aba'} \leq 0$ for all $a' \in A$ and $\chi_{abb'} \geq 0$ for all $b' \in B$, or the other way round. Altogether, the existence of disjoint holes can be described solely using triple orientations.

Even though, for fixed $n \in \mathbb{N}$, there are uncountable possibilities to choose n points from the Euclidean plane, there are only finitely many equivalence classes of point sets when point sets inducing the same orientation triples are considered equal. As introduced by Goodman and Pollack [16], these equivalence classes are called *order types*.

4.2 An Abstraction of Point Sets

Consider a point set $S = \{s_1, \ldots, s_n\}$ where $s_1, \ldots, s_n$ have increasing x-coordinates. Using the *unit paraboloid duality transformation*, which maps point $s = (a, b)$ to line $s^* : y = 2ax - b$, we obtain the arrangement of dual lines $S^* = \{s_1^*, \ldots, s_n^*\}$, where the dual lines $s_1^*, \ldots, s_n^*$ have increasing slopes. By the increasing x-coordinates and the properties of the unit paraboloid duality (see e.g. [24, Chapter 1.3]), the following three statements are equivalent:

(i) The points s_i, s_j, s_k are positively oriented.
(ii) The point s_k lies above the line $\overline{s_i s_j}$.
(iii) The intersection-point of the two lines s_i^* and s_j^* lies above the line s_k^*.

Due to Felsner and Weil [14] (see also [4]), for every 4-tuple s_i, s_j, s_k, s_l with $i < j < k < l$ the sequence

$$\chi_{ijk},\ \chi_{ijl},\ \chi_{ikl},\ \chi_{jkl}$$

(index-triples are in lexicographic order) changes its sign at most once. These conditions are the *signotope axioms*. It is worth to note that the signotope axioms are necessary conditions but not sufficient for point sets. There exist χ-configurations which fulfill the conditions above – so-called *abstract point sets*, *abstract order types*, *abstract oriented matroids (of rank 3)*, or *signotopes* – that are **not** induced by any point set, and in fact, deciding whether an abstract point set has a realizing point set is known to be $\exists\mathbb{R}$-complete (see e.g. [13]).

4.3 Increasing Coordinates and Cyclic Order

In the following, we see why we can assume, without loss of generality, that in every point set $S = \{s_1, \ldots, s_n\}$ the following three conditions hold:

- the points $s_1, \ldots, s_n$ have increasing x-coordinates,
- in particular, s_1 is an extremal point, and
- the points $s_2, \ldots, s_n$ are sorted around s_1.

When modeling a computer program, one can use these constraints (which do not affect the output of the program) to restrict the search space and to possibly get a speedup. This idea, however, is not new and was already used for the generation of the *order type database*, which provides a complete list of all order types of up to 11 points [24, 1, 2].

▶ **Lemma 4.1.** *Let $S = \{s_1, \ldots, s_n\}$ be a point set where s_1 is extremal and $s_2, \ldots, s_n$ are sorted around s_1. Then there is a point set $\tilde{S} = \{\tilde{s_1}, \ldots, \tilde{s_n}\}$ of the same order type as S (in particular, $\tilde{s_2}, \ldots, \tilde{s_n}$ are sorted around $\tilde{s_1}$) such that $\tilde{s_1}, \ldots, \tilde{s_n}$ have increasing x-coordinates.*

Proof. We can assume $s_1 = (0, 0)$ and $x_i, y_i > 0$ for $i \geq 2$ – otherwise we can apply an affine-linear transformation. Moreover, x_i/y_i is increasing for $i \geq 2$ since $s_2, \ldots, s_n$ are sorted around s_1. Since S is in general position, there is an $\varepsilon > 0$ such that S and $S' := \{(0, \varepsilon)\} \cup \{s_2, \ldots, s_n\}$ are of the same order type. We apply the projective transformation $(x, y) \mapsto (x/y, -1/y)$ to S' to obtain $\tilde{S}$. By the multilinearity of the determinant, we obtain

$$\det\begin{pmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{pmatrix} = y_i \cdot y_j \cdot y_k \cdot \det\begin{pmatrix} 1 & 1 & 1 \\ x_i/y_i & x_j/y_j & x_k/y_k \\ -1/y_i & -1/y_j & -1/y_k \end{pmatrix}.$$

Since the points in S' have positive y-coordinates, S' and $\tilde{S}$ have the same triple orientations. Moreover, as $\tilde{x}_i = x'_i/y'_i$ is increasing for $i \geq 1$, the set $\tilde{S}$ fulfills all desired properties. ◀

5 SAT Model

The basic idea to prove Theorem 2.1 is to assume – towards a contradiction – that a point set $S = \{s_1, \ldots, s_{17}\}$ with no two disjoint 5-holes exists. We formulate a SAT instance, where boolean variables indicate whether triples are positively or negatively oriented and clauses encode the necessary conditions described in Section 4. To be precise, we also have auxiliary variables, e.g., to indicate whether 4 points are in convex position and whether 3 points form a 3-hole. A detailled description of our SAT model can be found in the full version [30] and the source code of our python program is available online on our supplemental website [29].

Using a SAT solver we verify that the SAT instance has no solution and conclude that the point set S does not exist. This contradiction then completes the proof of Theorem 2.1.

It is folklore that satisfiability is NP-hard in general, thus it is challenging for SAT solvers to terminate in reasonable time for certain inputs of SAT instances. We now highlight the two crucial parts of our SAT model, which are indeed necessary for reasonable computation times: First, due to Lemma 4.1, we can assume without loss of generality that the points are sorted from left to right and also around the first point s_1. Second, we teach the solver that every set of 10 points gives a 5-hole, that is, $h(5) = 10$ [17]. By dropping either of these two constraints (which only give additional information to the solver and do not affect the solution space), none of the tested SAT solvers terminated within days.

6 Final Remarks

Interior-disjoint Holes: Two holes X_1, X_2 are called *interior-disjoint* if their respective convex hulls are interior-disjoint [10, 28, 9, 8, 22]. In a recent article, Hosono and Urabe [22] summarized the current status and presented some new results. By slightly adapting the SAT model from Section 5, we managed to show that every set of 15 points contains two interior-disjoint 5-holes; this further improves their result [22, Theorem 3].

Classical Erdős–Szekeres: The computation time for the computer assisted proof by Szekeres and Peters [32] for $g(6) = 17$ was about 1500 hours. By slightly adapting the model from Section 5 we have been able to confirm $g(6) = 17$ using glucose and DRAT-trim with about one hour of computation time.

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