

Holes in convex drawings*

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Abstract

The Erdős–Szekeres theorem states that, for every positive integer k , every sufficiently large point set in general position contains a subset of k points in convex position – a k -gon. In the same vein, Erdős later asked for the existence of k -holes which are k -gons with no additional points in their convex hulls. Today it is known that every sufficiently large point set in general position contains 6-holes, while there exist arbitrarily large point sets without 7-holes.

Harborth (1978) started the investigation of empty triangles in simple drawings of the complete graph. In a simple drawing, vertices are mapped to points in the plane and edges are drawn as simple curves connecting the corresponding endpoints such that any two edges intersect in at most one point, which is either a common vertex or a proper crossing. For the subclass of convex drawings, which in particular includes point sets, Arroyo et al. (2018) showed that quadratically many empty triangles exist.

In this article, we generalize the concept of k -holes to simple drawings of the complete graph K_n and investigate their existence. We provide arbitrarily large simple drawings without 4-holes, show that convex drawings contain quadratically many 4-holes, and generalize the Empty Hexagon theorem (Gerken 2006; Nicolás 2007) by proving the existence of 6-holes in sufficiently large convex drawings.

1 Introduction

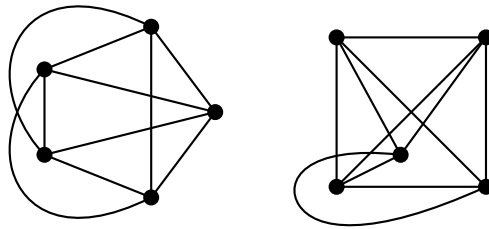
The study of holes in point sets was motivated by the Erdős–Szekeres theorem [12] and continues to be an active research branch. The classic theorem states that for every $k \in \mathbb{N}$ every sufficiently large point set in general position (i.e., no three points on a line) contains a subset of k points in convex position – a so called k -gon. A variation suggested by Erdős [11] is about the existence of holes. A k -hole H in a point set S is a k -gon with the property that there are no points of S in the interior of the convex hull of H .

In this article, we investigate holes in simple drawings of the complete graph K_n . Even though the notation of holes generalizes to simple drawings in a natural manner, we have to introduce some basic notation before we can talk about these structures and our results.

In a *simple drawing*, vertices are mapped to distinct points in the plane (or on the sphere) and edges are mapped to simple curves connecting the corresponding points such that two edges have at most one point in common which is either a common endpoint or a proper intersection. Furthermore we assume that no three edges cross in a common point. Simple drawings can be considered as a generalization of point sets because a set of n points in general position yields a *geometric drawing* of K_n where the vertices are the points and the edges are the straight-line segments connecting the vertices.

Moreover, we investigate the subclass of convex drawings introduced by Arroyo et al. [4]. To define convexity, we consider *triangles* which are subdrawings of K_3 induced by three

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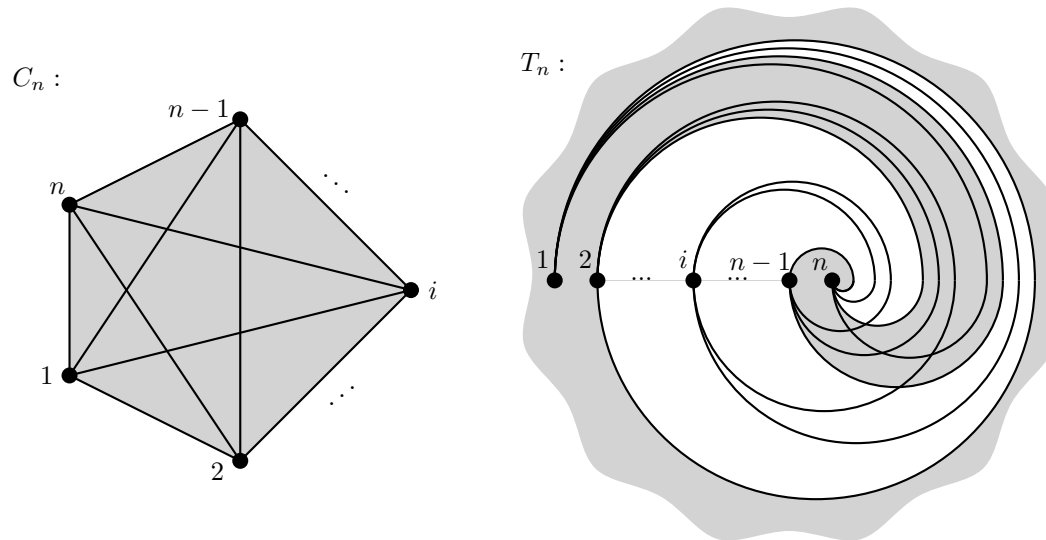
■ **Figure 1** Drawing of the two forbidden subconfigurations of convex drawings. Note that the right drawing is the T_5 .

vertices. Since the three edges of a triangle do not cross, the triangle separates the plane (resp. the sphere) into two connected components. The closure of each of the components is called a *side* of the triangle. A side S is *convex*, if for every two vertices in S , the connecting edge is fully contained in S . A simple drawing is *convex* if every triangle has a convex side. Furthermore, a convex drawing is *f-convex* if there is a marking point f in the plane such that for all triangles the side not containing f is convex. A *pseudolinear drawing* is a simple drawing in the plane such that all edges can be extended to pseudolines such that two have at most one point in common. A *pseudoline* is a simple curve partitioning the plane in two unbounded components. As shown by Arroyo et al. [3], a simple drawing of K_n is convex if and only if the two non-convex drawings of the K_5 (see Figure 1) do not appear as a subdrawing. Furthermore, they showed that a simple drawing of K_n is pseudolinear if and only if it is *f-convex* and the marking point f is in the unbounded cell. For more information about the convexity hierarchy, we refer the reader to [3, 4, 9].

Next, we introduce the notions of k -gons in simple drawings of the complete graph. A k -gon C_k is a subdrawing isomorphic to the geometric drawing of k points in convex position, see Figure 2(left). Two simple drawings are *isomorphic* if there exists a bijection on the vertex sets such that the same pairs of edges cross. Note that isomorphism is independent of the choice of the outer cell. Thus, in terms of crossings, a k -gon C_k is a (sub)drawing with vertices v_1, \dots, v_k such that $\{v_i, v_\ell\}$ crosses $\{v_j, v_m\}$ for $i < j < \ell < m$. In contrast to the geometric setting where every sufficiently large geometric drawing contains a k -gon, simple drawings of complete graphs do not necessarily contain k -gons [16]. For example, the perfect twisted drawing T_n depicted in Figure 2(right) does not contain any 5-gon. In terms of crossings, T_n can be characterized as a drawing of K_n with vertices v_1, \dots, v_n such that $\{v_i, v_j\}$ crosses $\{v_\ell, v_m\}$ for $i < j < \ell < m$. However, a theorem by Pach, Solymosi and Tóth [21] states that every sufficiently large drawing of K_n contains a k -gon or a T_k . The currently best known bound is due to Suk and Zeng [23] who showed that every simple drawing of K_n with $n > 2^{9 \cdot \log_2(a) \log_2(b) a^2 b^2}$ contains a C_a or a T_b . Since convex drawings do not contain T_5 as a subdrawing, every convex drawing of K_n with n sufficiently large contains a k -gon.

To eventually define k -holes for general k , let us first consider the special case of 3-holes, which are also known as empty triangles. A triangle is *empty* if one of its two sides does not contain any vertices in its interior. For general simple drawings Harborth [16] proved that there are at least two empty triangles and conjectured that the minimum among all simple drawings on n vertices is $2n - 4$, which is obtained by T_n . García et al. [13] recently showed that the conjecture holds for a class containing the perfect twisted drawings, the so called *generalized twisted drawing*. However, the conjecture remains open in general. The best known lower bound is by Aichholzer et al. [2], who proved that there are at least n empty triangles.

In the geometric setting, the number of empty triangles behaves quite differently: every point set has a quadratic number of empty triangles, and this bound is asymptotically optimal [6]. Moreover, determining the minimum number remains a challenging problem [10, Chapter 8.4]. For the current bounds, see [1]. The class of convex drawings behaves similarly to the geometric setting: the minimum number of empty triangles is asymptotically quadratic as shown by Arroyo et al. [3].



■ **Figure 2** A drawing of C_n (left) and T_n (right) for $n \geq 4$.

In this article, we go beyond empty triangles and investigate the existence of k -holes in simple drawings for $k \geq 4$. In the subdrawing induced by a k -gon with $k \geq 4$, all triangles have exactly one empty side which is the convex side. We define the *convex side* of a k -gon as the union of all convex sides of its triangles and call a vertex which lies in the interior of its convex side an *interior vertex*. The convex side of C_n in Figure 2 is highlighted grey. Note that the triangles of a k -gon for $k \geq 4$ have exactly one convex side, which is the one not containing the other vertices of the k -gon and hence the convex side of a k -gon is well-defined. A k -hole is a k -gon which has no interior vertices. For example the vertices $1, 2, n-1, n$ form a 4-hole in T_n which is highlighted grey in Figure 2. For a k -gon C_k in a convex drawing, Arroyo et al. [4] showed that the edges from an interior vertex to a vertex of C_k and edges between two interior vertices are fully contained in the convex side of C_k .

In the geometric setting it is known that for $k \leq 6$ every sufficiently large point set contains a k -hole [15, 14, 19] and that there are arbitrarily large point sets without 7-holes [17]. Since the latter applies to simple drawings, the remaining questions in simple drawings are about the existence of 4-, 5- and 6-holes.

Our Results: For $n \geq 5$ we present a non-convex simple drawing of K_n without 4-holes (Section 2). Furthermore, we show that – as in the geometric setting – the number of 4-holes in convex drawings of K_n is at least $\Omega(n^2)$ (Theorem 3.1), generalizing a result by Bárány and Füredi [6], and that every sufficiently large convex drawing contains a 5-hole and a 6-hole (Theorem 3.2), generalizing the Empty Hexagon theorem by Gerken [14] and Nicolás [19]. In order to show the latter, we prove that if a subdrawing of a convex drawing is induced by a minimal k -gon with $k \geq 5$ together with its interior vertices, then it is f -convex

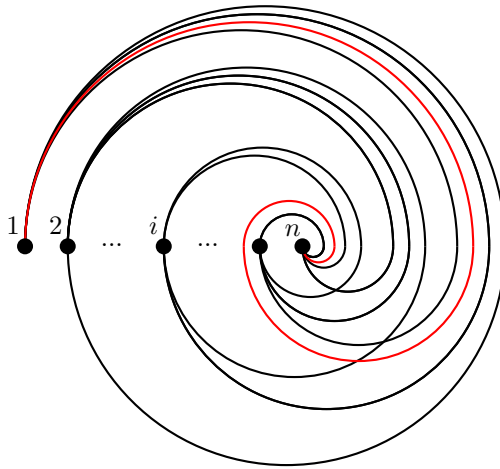
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(Theorem 3.3). This result might be of independent interest as it allows to transfer results from the straight-line, pseudolinear, and f -convex setting to convex drawings.

2 Holes in simple drawings

The perfect twisted drawing T_n depicted in Figure 2(right) has exactly $2n - 4$ empty triangles, which are spanned by the vertices $\{1, 2, i\}$ for $3 \leq i \leq n$ and $\{i, n - 1, n\}$ for $1 \leq i \leq n - 2$ [16]. For $n \geq 4$, T_n has exactly one 4-hole, which is spanned by $\{1, 2, n - 1, n\}$.

For $n \geq 5$, let \widetilde{T}_n denote the non-convex drawing of K_n that is obtained by starting with the drawing of T_n and rerouting the edge $\{1, n\}$ as illustrated in Figure 3. More precisely, while in T_n the edge $\{1, n\}$ crosses every edge $\{i, j\}$ with $2 \leq i < j \leq n - 1$, in \widetilde{T}_n it only crosses the edges $\{i, j\}$ with $2 \leq i < j \leq n - 2$. Recall that the pairs of crossing edges determine isomorphism class.



■ **Figure 3** An illustration of the drawing \widetilde{T}_n without 4-holes. The edge $\{1, n\}$ is highlighted red.

► **Proposition 2.1.** For $n \geq 5$ the drawing \widetilde{T}_n does not contain a 4-hole.

Proof. Rerouting the edge $\{1, n\}$ only affects the emptiness of triangles incident to both vertices 1 and n . In particular it only affects the vertex $n - 1$ which changes the side of every triangles incident to $\{1, n\}$. In \widetilde{T}_n , the only empty triangles incident to $\{1, n\}$ are $\{1, n - 1, n\}$ and $\{1, n - 2, n\}$. Note that the triangle $\{1, 2, n\}$ is not empty in \widetilde{T}_n . Hence, the empty triangles in \widetilde{T}_n are $\{1, 2, i\}$ for $3 \leq i \leq n - 1$, and $\{i, n - 1, n\}$ for $1 \leq i \leq n - 2$, and $\{1, n - 2, n\}$. Since no four vertices span four empty triangles, \widetilde{T}_n does not contain a 4-hole. ◀

3 Holes in convex drawings

In this section, we show that convex drawings of the complete graph behave similarly to geometric point sets when it comes to the existence of holes.

► **Theorem 3.1.** Every convex drawing of K_n contains at least $\Omega(n^2)$ 4-holes.

The proof generalizes the idea of Bárány and Füredi [6] and is deferred to the full version. The bound is asymptotically best possible as there are point sets (squared Horton sets [7]

and random point sets [5]) which have only quadratically many 3-holes, 4-holes, 5-holes, and 6-holes.

Furthermore, we investigate larger holes. We show that every sufficiently large convex drawing contains a 6-hole (and hence a 5-hole).

► **Theorem 3.2.** *Every convex drawing of K_n with n sufficiently large contains a 6-hole.*

For the proof below we use the existence of a k -gon in sufficiently large simple drawings [21, 23]. Even though the existence of 6-holes directly implies the existence of 5-holes, when adapting the proof to 5-holes one can obtain a better bound on the required number of vertices.

An important part of the proof is that the subdrawing induced by a minimal k -gon together with its interior vertices is f -convex, which then can be transformed into a pseudolinear drawing. A k -gon is *minimal* if its convex side does not contain the convex side of another k -gon.

► **Theorem 3.3.** *Let C_k be a minimal k -gon with $n \geq k \geq 5$ in a convex drawing of K_n . Then the subdrawing induced by the vertices on the convex side of C_k is f -convex.*

Since the proof for the existence of 6-holes in point sets [14] also applies to the setting of pseudolinear drawings [22], we can now use Theorem 3.3 to derive Theorem 3.2. Similarly, the text-book proof for the existence of 5-holes in every 6-gon of a point set (see e.g. Section 3.2 in [18]) applies to pseudolinear drawings as it only uses triple orientations. However, proving the existence of 6-holes via 9-gons¹ is far more technical. Hence we refer the interested reader to [22] for a computer-assisted proof and [24] for a simplified proof of the Empty Hexagon theorem with worse bounds.

Proof of Theorem 3.2. By the result of Suk and Zeng [23] every convex drawing of K_n with $n > 2^{225 \log_2(5) \cdot k^2 \log_2(k)}$ contains a k -gon. In order to find a 6-hole, we apply this result for $k = 9$. (To find a 5-hole, we can use $k = 6$.) Consider a minimal k -gon. By Theorem 3.3, the subdrawing induced by the vertices from the convex side of the k -gon is f -convex. Since the existence of holes is invariant under the choice of the outer cell, we can choose the cell containing f as the unbounded cell to make the subdrawing pseudolinear. Next we apply the results concerning the existence of a 6-hole (resp. 5-hole) in pseudolinear drawings and conclude that the subdrawing induced by the k -gon and the interior vertices contains a 6-hole (resp. 5-hole). This 6-hole (resp. 5-hole) in the subdrawing does not contain vertices of the original drawing of K_n since those vertices would be interior vertices of the k -gon. Therefore it is also a 6-hole (resp. 5-hole) in the original drawing. This completes the argument. ◀

4 Discussion

We have shown that every convex drawing of K_n with $n \geq 5$ contain a quadratic number of 4-holes and that sufficiently large drawings contain 5- and 6-holes, while 7-holes do not exist in general. However, it remains to determine the precise values of $h^{\text{conv}}(5)$ and $h^{\text{conv}}(6)$, where $h^{\text{conv}}(k)$ (resp. $h^{\text{geom}}(k)$) denotes the smallest integer such that every convex (resp. geometric) drawing of size $n \geq h^{\text{conv}}(k)$ contains a k -hole. In the geometric setting it is known that $h^{\text{geom}}(5) = 10$ [15] and $30 \leq h^{\text{geom}}(6) \leq 1717$ [14, 20]. In this article we showed

¹ Gerken [14] showed that every 9-gon in a point set yields a 6-hole and Nicolás [19] showed that a 25-gon yields a 6-hole. Both articles involve very long case distinctions.

$h^{\text{conv}}(k) \leq 2^{225 \cdot k^2 \cdot \log_2 5 \cdot \log_2 k} + 1$ for $k = 5$ and $k = 6$ (Theorem 3.2). Moreover, we used the SAT framework from [8] to find configurations for $n \leq 10$ and $n = 12$ without 5-holes and to prove that every drawing for $n = 11, 13, 14, 15, 16$ contains a 5-hole. Based on our computational data, we conjecture that $h^{\text{conv}}(5) = 13$. Note that unlike in the geometric setting, the existence of 5-holes in convex drawings is not monotone as the existence of a 5-hole in all convex drawings of K_{11} does not imply the existence for $n = 12$.

It would be interesting to obtain better bounds on the size of a largest k -gon and on the size of a largest f -convex subdrawing in a convex drawing of K_n . The currently best estimate for a k -gon is by Suk and Zeng [23], which yields $\Omega((\log n)^{1/2-o(1)})$, and combining this with Theorem 3.3 yields an f -convex drawing of the same size.

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