Shooting Stars in Simple Drawings of $K_{m,n}$*

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Abstract

In this work we study the existence of plane spanning trees in simple drawings of the complete bipartite graph $K_{m,n}$. We show that every simple drawing of $K_{2,n}$ and $K_{3,n}$, $n \geq 1$, as well as every outer drawing of $K_{m,n}$ for any $m, n \geq 1$, contains plane spanning trees. Moreover, for all these cases we show the existence of special plane spanning trees, which we call shooting stars. Shooting stars are spanning trees that contain the star of a vertex, i.e., all its incident edges.

1 Introduction

In a drawing of a graph in the Euclidean plane the vertices are drawn as distinct points and the edges are drawn as continuous arcs connecting its two end points. Depending on which properties of the graph are to be considered, there might be additional requirements on how the graph is drawn. Typically the drawing of an edge has to be non-self-crossing and must not pass through any point representing a vertex other than its two end points. In addition, in a *simple drawing* of a graph any pair of edges crosses at most once, either in their interior or at a common end point, no tangencies are allowed and no three edges pass through a single crossing. These drawings are also called *good drawings* [1, 3] or *(simple) topological graphs* [5, 6].

The probably most restricted version of drawings are straight-line drawings, also called *geometric graphs*, where an edge is drawn as straight-line segment connecting its two end points. Thus, the placement of the vertices in the plane entirely determines the full drawing.

Both classes of drawings are of special interest if we want to minimize the number of crossings in a drawing of a given graph. If such a drawing does not contain any crossing at all then it is called *plane*. In this work we are interested in *plane spanning subdrawings* of a given drawing, that is, drawings without crossings that contain all the vertices of the given drawing and a subset of its edges.

The existence of plane subdrawings of simple drawings of the complete graph K_n has received quite a lot of attention. For example, Ruiz-Vargas [9] showed that every simple drawing of K_n contains $\Omega(n^{1/2-\epsilon})$ pairwise disjoint edges for any $\epsilon > 0$, by this improving over many previous bounds [6, 7, 10]. Fulek and Ruiz-Vargas [4] proved that given a simple drawing of K_n, a plane cycle C in the drawing, and any vertex v that is not part of C, at least two edges connecting v to C do not intersect C. Hence every simple drawing of K_n contains a plane sub-drawing with at least $2n - 3$ edges. Rafla [8] conjectured that every simple drawing of K_n contains a plane Hamiltonian cycle, a statement that is known to be

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true for several classes of simple drawings (e.g., 2-page book drawings, monotone drawings, cylindrical drawings), but is still open in the general case. Pach et al. [6] proved that every simple drawing of K_n contains a plane drawing of any fixed tree with at most $c \log^{1/6} n$ vertices.

In this paper we concentrate on the existence of plane spanning trees in simple drawings. Obviously, any simple (or straight-line) drawing of the complete graph K_n contains some plane spanning trees: choose any vertex v and all the edges incident to v. As these edges cannot cross we obtain a plane spanning star.

For the complete bipartite graph $K_{m,n}$, the situation is less obvious. As a warm up exercise, let us first consider straight-line drawings of $K_{m,n}$. Let V_1 and V_2 be the sides of the bipartition of the vertex set of $K_{m,n}$. Chose any vertex $v_0 \in V_1$ and draw the star consisting of all edges v_0v with $v \in V_2$. This star induces a partition of the plane into wedges centered at v_0, where one wedge might have an opening angle larger than π. Draw a virtual angular bisector within each wedge and connect the vertices of V_1 which lie in each half of a wedge to the corresponding end point $v \neq v_0$ of the star edge. This results in a plane spanning tree with root v_0 and height 2 that includes all the edges incident to v_0. In the following, we call such a (not necessarily rectilinear) plane spanning tree a *shooting star (rooted at v_0)*.

For simple drawings, we are not aware of a similarly easy construction. Actually, it is still an open problem whether every simple drawing of $K_{m,n}$ contains a plane spanning tree. In this paper we solve that problem for the cases of $K_{2,n}$ and $K_{3,n}$ (see Section 2), as well as for *outer drawings* of $K_{m,n}$ (see Section 3, where also a definition of these drawings can be found). In all those cases, we show the existence of a shooting star rooted at any of the vertices of one side of the bipartition (the smaller one in case of $K_{2,n}$ and $K_{3,n}$ and the one lying on the outer boundary in the case of outer drawings).

2 Plane Spanning Trees in Simple Drawings of $K_{2,n}$ and $K_{3,n}$

In this section we prove that every simple drawing of $K_{2,n}$ and $K_{3,n}$ contains plane spanning trees of a certain structure. In order to do so, we introduce some notions and provide some auxiliary results.

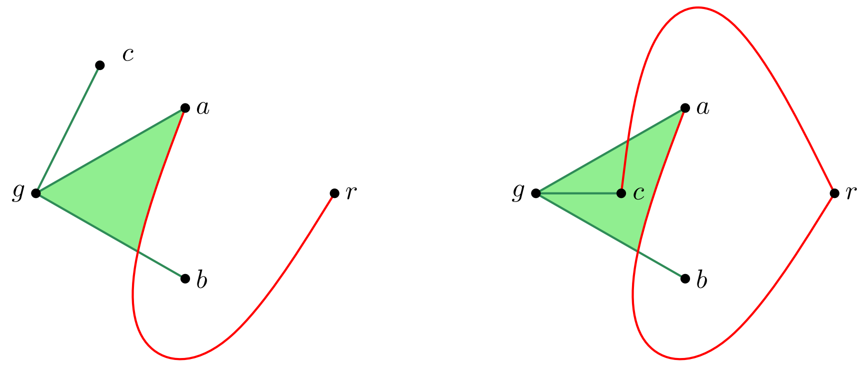
For a given simple drawing of K_n with vertex set V and two fixed vertices $g \neq r \in V$, we define a relation $\rightarrow_{gr}$ on the remaining vertices $V \setminus \{g, r\}$, where $a \rightarrow_{gr} b$ if and only if the arc ra properly crosses gb. In the following, we simply write $a \rightarrow b$ if the two vertices g and r are clear from the context.

▶ **Lemma 2.1.** *The relation $\rightarrow$ is asymmetric and acyclic, that is, there are no vertices $v_1, v_2, \ldots, v_k$ ($k \in \mathbb{N}$) with $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k \rightarrow v_1$.*

Proof. We give a proof by induction on k.
Induction basis: The case $k = 1$ is trivial. The case $k = 2$ follows from the fact that there is at most one proper crossing in every 4-tuple in a simple drawing – if ra crosses gb then rb cannot cross ga. For the case $k = 3$ assume there are three vertices a, b, c with $a \rightarrow b \rightarrow c \rightarrow a$. Let $\triangle$ denote the area bounded by the edges ga, gb, ra and not containing the vertex r, as illustrated in Figure 1. We distinguish the following two cases:

Case 1: $c \notin \triangle$. Since $c \rightarrow a$ holds, the arc rc crosses ga, and therefore the boundary of $\triangle$. Since $r \notin \triangle$ and since rc cannot cross ra, rc must also cross gb. Thus we have $c \rightarrow b$, which is a contradiction to $b \rightarrow c$.

Case 2: $c \in \triangle$. Since $a \rightarrow b$, the arc rb cannot cross ga. Moreover, since rb can neither cross ra nor gb, it is therefore completely outside of $\triangle$. Since gc is completely contained in $\triangle$, rb and gc cannot cross, and therefore, $b \not\rightarrow c$. Contradiction.



(a) Case 1 **(b)** Case 2

Figure 1 An illustration of the two cases of base case $k = 3$ from Lemma 2.1. The area $\triangle$ is colored light green.

Since c can neither be inside nor outside $\triangle$, the statement is proven.

Induction step: Suppose – towards a contradiction – that there exist $v_1, \ldots, v_k$ with $k \geq 4$ and $v_1 \to v_2 \to \ldots \to v_k \to v_1$. We write $a = v_1$, $b = v_2$, $w = v_{k-1}$, and $z = v_k$. Let $\triangle$ denote the area bounded by the edges ga,gb, and ra that does not contain the vertex r. We distinguish the following two cases:

Case 1: $z \notin \triangle$. We continue analogously to Case 1 of base case $k = 2$. Since $z \to a$ holds, rz crosses ga, and therefore the boundary of $\triangle$. Since $r \notin \triangle$ and since rz cannot cross ra, rz must also cross gb. Thus we have $z \to b$.

Case 2: $z \in \triangle$. Since $w \to z$ holds, rw crosses gz at some point inside $\triangle$. Since $r \notin \triangle$ and since rw cannot cross ra, it must cross ga or gb (or both). Thus we have $w \to a$ or $w \to b$.

In both cases, we can find $v'_1, \ldots, v'_l$ for some $l < k$ with $v'_1 \to \ldots \to v'_l \to v'_1$, which is a contradiction. This completes the proof of the lemma. ◀

▶ **Theorem 2.2.** *Let D be a simple drawing of the complete bipartite graph $K_{2,n}$ with sides of the bipartition $\{g, r\}$ and P. Then, for every $k \in \{0, \ldots, n\}$, D contains a plane spanning tree with k edges incident to g and $n - k + 1$ edges incident to r.*

Proof. According to Lemma 2.1, we can find a labeling $v_1, \ldots, v_n$ of the vertices in P such that $v_i \to_{gr} v_j$ only holds if $i < j$. Let S_1 be the star with center g and children $\{v_1, \ldots, v_k\}$ and let S_2 be the star with center r and children $\{v_k, \ldots, v_n\}$. By definition of relation $\to_{gr}$, the edges of S_1 and S_2 do not cross, and hence we have a plane spanning tree. ◀

▶ **Corollary 2.3.** *Let D be a simple drawing of the complete bipartite graph $K_{2,n}$ with sides of the bipartition $\{g, r\}$ and P. Then for each $c \in \{g, r\}$, D contains a shooting star rooted at c.*

Proof. Consider again the proof of Theorem 2.2. With the according labeling of P, no edge rv_i can cross the edge gv_1. Hence, the plane spanning tree consisting of all the edges incident to r together with the edge gv_1 gives the desired shooting star rooted at r. Similarly, the tree with all edges incident to g and the edge rv_n is a shooting star rooted at g. ◀

We also have an analogous result, showing shooting stars also exist for simple drawings of $K_{3,n}$. Due to lack of space, we only state the theorem. Its proof is deferred to the foll version of this paper.

▶ **Theorem 2.4.** *Let D be a simple drawing of the complete bipartite graph $K_{3,n}$ with sides of the bipartition $\{g, r, b\}$ and P. Then for each $c \in \{g, r, b\}$, D contains a shooting star rooted at c.*

3 Shooting Stars in Outer Drawings of $K_{m,n}$

In this section we will study the problem of finding plane spanning trees in a special kind of simple drawings of bipartite graphs, namely outer drawings. The concept of outer drawings was recently introduced in [2]. They are defined as follows:

▶ **Definition 3.1.** A simple drawing of a $K_{m,n}$ in which all the m vertices of one side of the bipartition lie on the outer boundary of the drawing is called *outer drawing*.

We denote by P the side of the bipartition whose vertices must lie on the outer boundary of the drawing. The other side of the bipartition is denoted by S. Note that points of S may also lie on the outer boundary but don't have to.

We now proceed to prove that there is a plane spanning tree in every outer drawing of a $K_{m,n}$ with $m, n \in \mathbb{N}$.

▶ **Theorem 3.2.** *Let D be an outer drawing of the complete bipartite graph $K_{m,n}$ with sides of the bipartition P and S where the vertices of P lie on the outer boundary. Let p be an arbitrary vertex in P. Then D contains a shooting star rooted at p.*

Proof. First, we label the vertices in P. We start in $p_1 := p$ and go clockwise along the outer boundary and denote the vertices of P by p_2 to p_m following the order in which they occur. Let T_1 be the sub graph that is induced by all edges incident to p_1. Notice that T_1 is a plane tree. We will add edges to T_1 until it becomes a spanning tree. We do so inductively by first adding an edge incident to p_2, then an edge incident to p_3 and so on until we add an edge incident to the vertex p_m. We denote by T_i the tree that we get by adding to T_{i-1} the selected edge incident to p_i for $2 \leq i \leq m$. We will show that it is possible to add edges such that T_i is always plane. After adding the last edge the statement then follows.

In the first step, for T_2, we need to find an edge that is incident to p_2 and does not cross any edge incident to p. We know from Theorem 2.2 that there is at least one such edge. We add that edge to T_1 and get a plane tree T_2. For T_i we need to find an edge that is incident to p_i and does not cross any of the edges of T_{i-1}. We denote by e_{i-1} the edge in T_{i-1} that is incident to p_{i-1} and by s_{i-1} the vertex in S that e_{i-1} is incident to. We also denote the edge that is incident to s_{i-1} and p by e_p. See Figure 2 for an illustration. The part of the boundary that goes from p clockwise until p_{i-1} together with the edges e_{i-1} and e_p encloses a region that we call R_1. The vertices p_2 to p_{i-1} all lie on that part of the boundary, because of the way we labeled them. We call the rest of the area inside the outer boundary R_2.

▶ Claim 1. All edges in T_{i-1} that are not incident to p lie completely inside R_1.

Proof. Since the boundary of R_1 consists of edges in T_{i-1} and the outer boundary, all edges of T_{i-1} that lie partly inside R_2 have to lie completely inside it. The edges in T_{i-1} that are not incident to p are incident with the vertices p_2 to p_{i-1}. As they have to lie on the part of the outer boundary that is also part of the boundary of R_1, the edges incident to these vertices have to lie partly inside R_1. Thus these edges have to lie completely inside R_1. ◀

Let us now consider the region R_2. The sub graph induced by p, p_i, and all vertices of S that lie in R_2 is a $K_{2,n'}$ with $n' \in \mathbb{N}$. By Theorem 2.2 there is an edge incident to p_i that does not cross any edges incident to p. This edge can neither cross the outer boundary nor

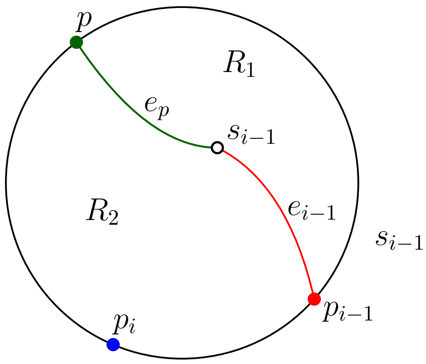


Figure 2 The edges e_p and e_{i-1} together with the outer boundary form two regions.

e_p and it can only cross e_{i-1} once. Since the edge has both end points in R_2, it follows that the edge has to lie completely in R_2. From Claim 1 it follows that it does not cross any of the edges of T_{i-1} that are not incident with p. As it doesn't cross any edges incident with p either, it follows that it doesn't cross any of the edges of T_{i-1}. Thus, we can add that edge and obtain a plane tree T_i. We continue to do so until we added an edge for every vertex in P. The plane spanning tree T_m is a shooting star. ◀

4 Conclusion

We have shown that particular cases of simple drawings of the complete bipartite graph $K_{m,n}$, namely all simple drawings of $K_{2,n}$ and $K_{3,n}$, $n \geq 1$, as well as all outer drawings of $K_{m,n}$ for any $n, m \geq 1$, contain plane spanning trees that are shooting stars. As we showed in the introduction, a similar result applies to straight-line drawings of $K_{m,n}$. In the full version of this paper we show that our results also extend to other classes of drawings of complete bipartite graphs. It is still an interesting open question whether every simple drawing of $K_{m,n}$ has a shooting star, or at least a plane spanning tree.

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