On 4-Crossing-Families in Point Sets and an Asymptotic Upper Bound*

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Abstract

A k-crossing family in a point set S in general position is a set of k segments spanned by points of S such that all k segments mutually cross. In this short note we present two statements on crossing families which are based on sets of small cardinality: (1) Any set of at least 15 points contains a crossing family of size 4. (2) There are sets of n points which do not contain a crossing family of size larger than $4\lceil \frac{n}{20} \rceil < \frac{n}{5} + 4$. Both results improve the previously best known bounds.

1 Introduction

Let S be a set of n points in the Euclidean plane *in general position*, that is, no three points in S are collinear. A *segment* of S is a line segment with its two endpoints (which we will also call vertices) being points of S.

▶ **Definition 1.1.** A *k-crossing family* in a point set S is a set of k segments spanned by points of S such that all k segments mutually cross in their interior.

For a point set S, let $\mathrm{cf}(S)$ be the maximum size of a crossing family in S, and let $\mathrm{cf}(n)$ be the minimum of $\mathrm{cf}(S)$ over all point sets S of cardinality n in general position.

It is easy to see that $\mathrm{cf}(n)$ is a monotone function. From the fact that the complete graph with 5 vertices is not planar it follows that any set of $n \geq 5$ points has a crossing family of size at least 2. In [1] it is shown that every set with $n \geq 10$ points admits a crossing family of size 3. The result is based on analyzing the set of all order types of size 10. The bound on n is tight, that is, there exist 12 order types of 9 points which do have a maximal crossing family of size 2. One such set is shown in Figure 1.

Aronov et al. [3] proved in 1994 the existence of crossing families of size $\sqrt{n/12}$ for every set of n points, which until recently was the best general lower bound. Their proof relies on mutually avoiding sets, for which the given bound is asymptotically tight.

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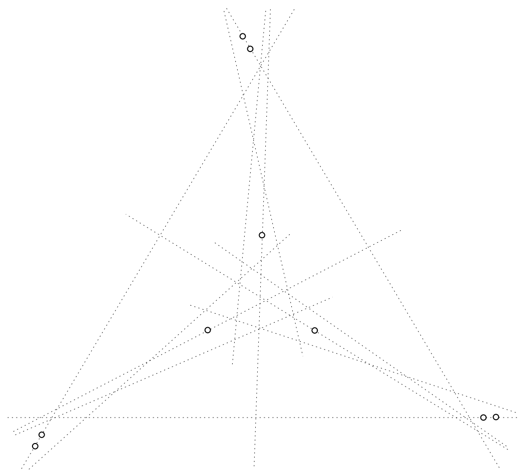


Figure 1 A set S of 9 points which does not contain a 3-crossing family, that is, $\mathrm{cf}(S) = 2$.

Only in 2019 Pach, Rubin, and Tardos [9] showed in a breakthrough result that any set of n points in general position in the plane contains a crossing family of size $n^{1-o(1)}$. This almost shows the generally accepted conjecture that $\mathrm{cf}(n)$ should be $\Theta(n)$.

The conjecture is also supported by the currently best upper bounds. Recently, Evans and Saeedi [6] showed that $\mathrm{cf}(n) \leq 5\lceil \frac{n}{24} \rceil$. We will improve that bound to $\mathrm{cf}(n) \leq 4\lceil \frac{n}{20} \rceil$ in Section 3. In the next section we start by showing that $\mathrm{cf}(15) = 4$.

2 Sets of 15 points always contain a 4-crossing family

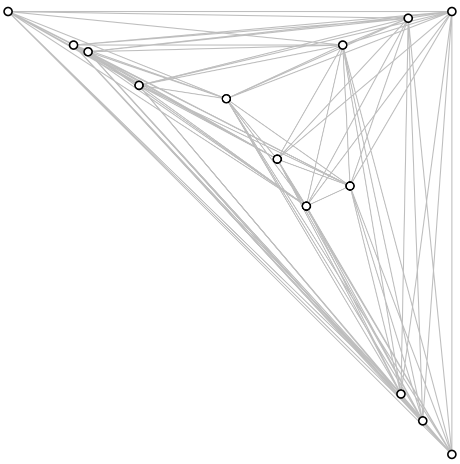
From the mentioned result $\mathrm{cf}(10) = 3$ [1] we know that also any set of 11 points contains a 3-crossing family, and no such set can contain a crossing family of size more than 5 (as at least $2k$ points are needed for a k-crossing family). The following table shows how the maximal size of a crossing family is distributed among all combinatorially different point sets of size 11 and was computed with the help of the database of all order types of size 11 obtained in [2].

k	number of order types
3	63 978 178
4	1 783 117 647
5	487 417 082
total	2 334 512 907

Table 1 Number of realizable order types of size 11 containing at most a k-crossing family.

To obtain the largest point set containing no crossing family of size 4, we made a complete abstract order type extension from $n = 11$ to $n = 15$. The database of all realizable order types of cardinality 11 contains 2 334 512 907 sets [2], of which 63 978 178 (about 2.7 %) contain no 4-crossing family; see Table 1. Since adding points to an existing set can never decrease the size of the maximal crossing family, we need to consider only those sets.

The approach of extending order types in an abstract way as described in [2] has the advantage that there is no need to realize the obtained sets, as we actually are interested in



coordinates:

61	66
61	0
57	5
54	9
47	40
41	37
37	44
30	53
46	61
18	55
55	65
11	60
9	61
0	66

Figure 2 A set S of 14 points which does not contain a 4-crossing family, that is, $\mathrm{cf}(S) = 3$.

the smallest cardinality where no such sets exist. This avoids dealing with the notoriously hard problem of realizing abstract order types, which is know to be $\exists\mathbb{R}$-hard [8]. The extension is done iteratively by adding one more (abstract) point in each step. Afterwards each obtained abstract order type is checked for the maximum size of a crossing family, and if this is larger than 4 the abstract order type is discarded.

After the first three rounds we obtained 2 727 858 abstract order types of cardinality 14 which do not contain a 4-crossing family. All these sets were extended by one further point (in the abstract setting), but all resulting sets contained a 4-crossing family. Thus, the largest possible set with no 4-crossing family has size at most 14. The whole process of abstract extension took about 100 hours, computed in parallel on 40 standard CPUs.

To show that the obtained bound is best possible it is sufficient to realize at least one generated abstract order type of size 14, and such an example is given in Figure 2. Together with the set of 20 points with no 5-crossing family depicted in Figure 5, we conclude the following.

▶ **Theorem 2.1.** *Every set of at least* 15 *points in the plane in general position contains a* 4*-crossing family. Moreover, we have*

$$\mathrm{cf}(n) = 3 \textit{ for } 10 \le n \le 14 \quad \textit{and} \quad \mathrm{cf}(n) = 4 \textit{ for } 15 \le n \le 20.$$

Besides the above described computer proof, we have also developed a SAT framework which allowed us to verify $\mathrm{cf}(15) > 3$ within less than 2 CPU days using the SAT solver CaDiCaL [5]. We have also used this framework to verify $\mathrm{cf}(10) > 2$ [1]. The python program creating the instance is available on our supplemental website [11].

The idea behind the SAT model is very similar as in [12]: We assume towards a contradiction that $\mathrm{cf}(15) \le 3$, that is, there is a set of 15 points with no 4-crossing family. We have Boolean variables X_{abc} to indicate whether three points a, b, c are positively or negatively oriented. As outlined in [12], these variable have to fulfill the signotope axioms [7, 4]. Based on the variables for triple orientations, we then assign auxiliary variables $Y_{ab,cd}$ to indicate whether the two segments ab, cd cross. Finally we assert that for any set of four segments with pairwise distinct endpoints, at least one pair of them does not cross. As the SAT solver CaDiCaL terminates with "unsatisfiable", no such point set exists, and hence $\mathrm{cf}(15) > 3$.

3 Small sets and an upper bound

The following theorem was already implicitly used by Aronov et al. [3, Section 6] in their discussion, where they stated the upper bound $\mathrm{cf}(n) \le \frac{n}{4}$. It can also be found as Lemma 3 in [6]. Since in [3] no proof is given and the proof from [6] appears to be incomplete (see Figure 3), we include a full proof here.

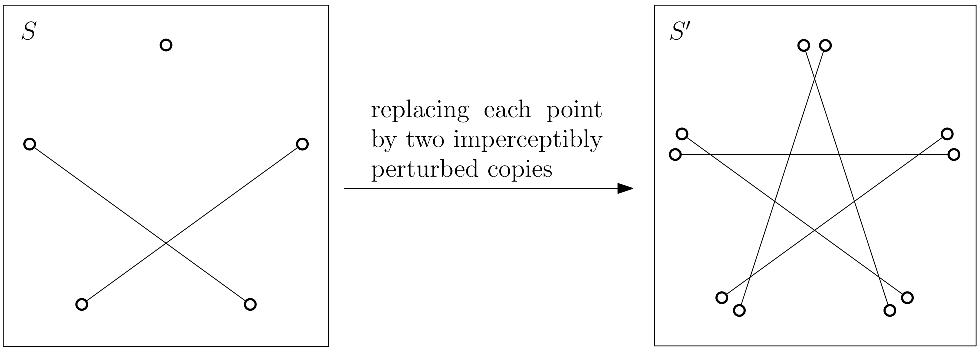


Figure 3 Five points in convex position have at most a 2-crossing family (left). Replacing each point by two imperceptibly perturbed copies with no prescribed structure as in [6] may result in a set of 10 points with a crossing family of size larger than 4 (right). The process needs to avoid generating odd cycles of pairwise crossing edges; see the proof of Theorem 3.1 for details.

▶ **Theorem 3.1.** *Let $S \subset \mathbb{R}^2$ be a set of n points in general position with $\mathrm{cf}(S) = k$. Then for any $N \ge n$ there exists a set $S' \subset \mathbb{R}^2$ of N points in general position with $\mathrm{cf}(S') \le k\lceil \frac{N}{n} \rceil$.*

For our proof of Theorem 3.1, we will use a simple property of geometric thrackles. A *geometric thrackle* is a geometric graph such that each pair of edges (drawn as line segments) either meets at a common vertex or crosses properly. Woodall proved that a graph can be drawn as a geometric thrackle if and only if it is a a subgraph of a graph obtained by attaching leaves (vertices of degree 1) to the vertices of an odd cycle [13, Theorem 2]. We will need only the following weaker characterization.

▶ **Lemma 3.2.** *A geometric thrackle T contains no even cycles.*

A strenghtening to monotone thrackles was proved by Pach and Sterling [10]. Since the proof for geometric thrackles is substantially simpler, we include it here to keep this note self-contained.

Proof of Lemma 3.2. Assume there exists an even cycle $C = p_0, p_1, \ldots, p_n$ for some even $n \ge 4$ with $p_0 = p_n$ and $\overline{p_{i-1}p_i} \in T$ for $i = 1, \ldots, n$. We set $p_i = p_j$ for $i = j \pmod{n}$. Consider a line segment $\ell = \overline{p_i p_{i+1}}$ in C. The supporting line through ℓ divides the plane into two half-planes. Since T is a thrackle, the previous segment $\overline{p_{i-1}p_i}$ and the next segment $\overline{p_{i+1}p_{i+2}}$ cross, and thus p_{i-1} and p_{i+2} lie in the same half-plane. Moreover, since all segments in the path $P = p_{i+2}, p_{i+3}, \ldots, p_{i-1}$ cross the segment $\overline{p_i p_{i+1}}$, P is an alternating path with respect to the side of the half-plane, and hence P has even length $|P|$, that is, an even number of edges. Since the cycle C has length $|C| = |P| + 3$ this is a contradiction, because C was assumed to have even length. ◀

Proof of Theorem 3.1. Let S be a set of n points in general position in the plane and let $m = \lceil \frac{N}{n} \rceil$. Without loss of generality we may assume that $N = mn$; in case $N < mn$ we may later remove some points from the constructed point set. We can also assume that all points of S have distinct x- and y-coordinates; otherwise we slightly rotate S. Our aim is to construct a set S' of mn points by creating m *copies* of each point from S, such that the following two properties hold:

(S1) a line segment between two copies of $p \in S$ only intersects line segments incident to another copy of p,

(S2) all copies of a point are almost on a horizontal line; that is, if $p \in S$ is above (below) $q \in S$ then any line through two different copies of p is above (below) any copy of q.

To this end, we place the m copies of a point $p = (x, y)$ at $p_i = (x + i\varepsilon, y + (i\varepsilon)^2)$ for $i = 0, \dots, m-1$. For sufficiently small $\varepsilon > 0$, all points from S' have distinct x- and y-coordinates and the above conditions are fulfilled.

Let F' be a maximum crossing family in S'. We will show how to find a crossing family F in S of size at least $|F'|/m$. If $|F'| \leq m$, we are clearly done since any segment in S is a crossing family of size 1. Hence assume that F' contains more than m segments. Then no segment $f' \in F'$ can be incident to two copies of the same point of S due to property (S1). Thus every segment $f' \in F'$ connects (copies of) two distinct points of S.

Let F be the set of segments (without multiplicity) on S induced by F' and by contracting the m copies of p_i to p, for each $p \in S$. Formally, $F = \{\overline{pq} \mid \exists i, j : \overline{p_i q_j} \in F'\}$; Figure 4 gives an illustration. Let G_F be the geometric graph induced by F, which has the set of end points of F as (drawn) vertices and F as (drawn) edges. More formally, $G_F = (V, E, \phi, \psi)$ with $V = \{p \in S \mid \exists q : \overline{pq} \in F\}$, $E = \{\{p, q\} \mid \overline{pq} \in F\}$, $\phi : V \to \mathbb{R}^2, p \mapsto p$, and $\psi : \{p, q\} \mapsto \overline{pq}$ for any $\{p, q\} \in E$. By construction G_F is a geometric thrackle, and due to Lemma 3.2, G_F contains no even cycles. Observe that according to property (S2), the neighbors of a vertex p in G_F can either be all above p or all below p, as no segment of F connected from above to a copy p_i can cross a segment of F connected from below to a copy p_j. As a consequence, G_F is bipartite and thus contains no odd cycles. Hence, G_F is acyclic; equivalently, a forest.

As long as G_F contains vertices of degree larger than 1, we continue as follows: Let $p \in V(G_F)$ be a leaf incident to a vertex $q \in V(G_F)$ of degree larger than 1. We construct F'' by removing all segments in F' incident to copies of q, and by inserting segments connecting all copies of q to copies of p in the way such that all those segments cross; Figure 4 gives an illustration of this modification. By construction, $|F''| \geq |F'|$ holds and thus F'' is another maximal crossing family in S'. We can replace F' by F''.

We can iteratively repeat this process, and in every step the number of vertices of degree larger than 1 strictly decreases. Therefore we can assume that G_F contains no vertices of degree greater than 1. As a consequence, F is a (not necessarily maximal) crossing family in S with $|F| \geq |F'|/m$. Therefore, $\mathrm{cf}(S') \leq |F'| \leq m \cdot |F| \leq m \cdot \mathrm{cf}(S) = mk$. ◀

From the point set depicted in Figure 1, it follows by Theorem 3.1 that there are sets of n points with no crossing family larger than $2\lceil \frac{n}{9} \rceil$. Evans and Saeedi [6] constructed a set of 24 points with no crossing family of size 6 or more, which yields the upper bound $\mathrm{cf}(n) \leq 5\lceil \frac{n}{24} \rceil$ presented there.

k	1	2	3	4	5	6	7	8	9	10
Currently best example	4	9	14	20	25	29	34	40	44	50

Table 2 Largest known point sets S_k containing at most a k-crossing family, that is, with $\mathrm{cf}(S_k) = k$. For $k \leq 3$ the sets are best possible.

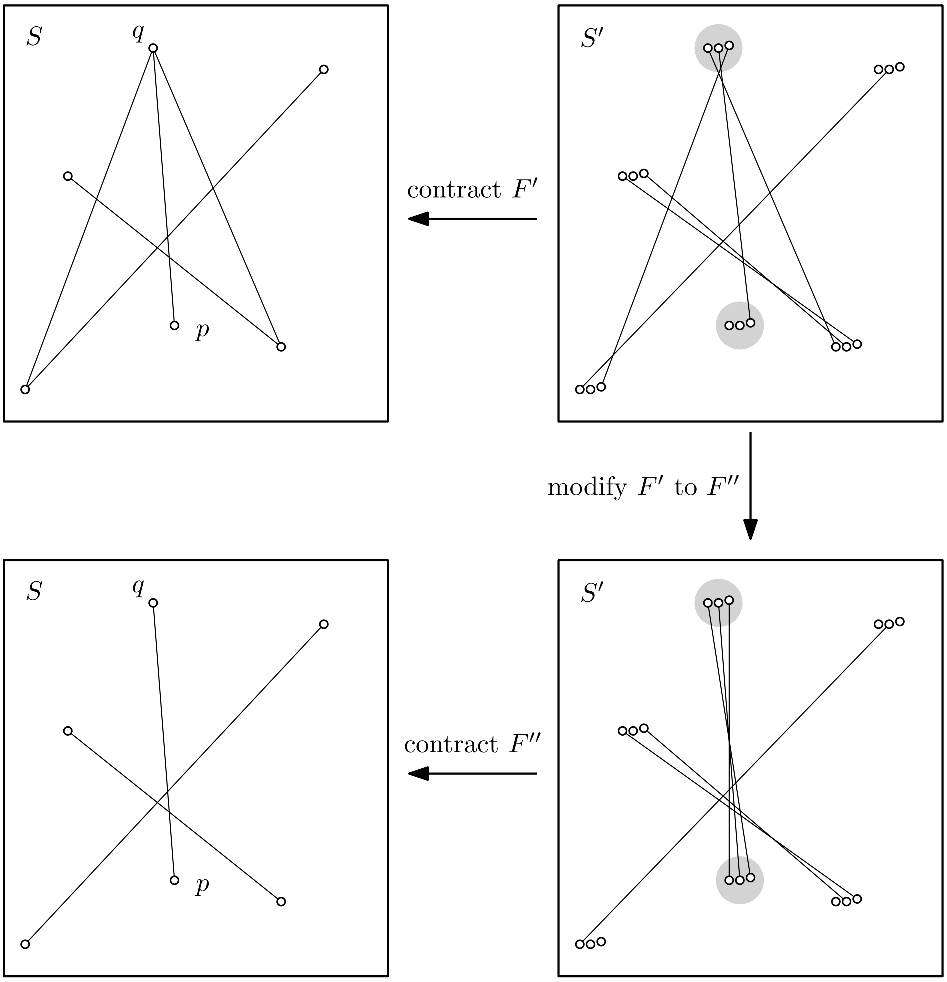


Figure 4 An illustration of the contracting process to attain F from F', and an illustration of the modification of F' in which all copies of p and q are "connected" to each other.

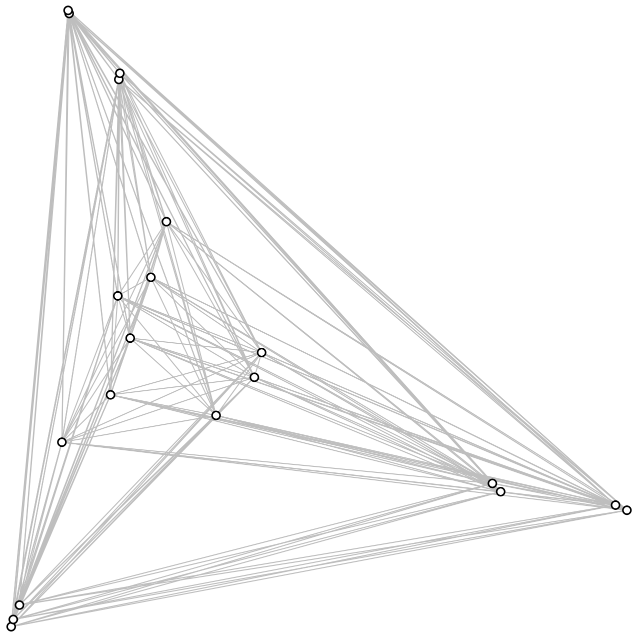
To further improve this bound we searched for sets with small crossing families. For $k \geq 5$ we partially extended several smaller sets (for example by doubling the number of points similarly to the process described in the proof of Theorem 3.1) and used heuristics such as simulated annealing to optimize them. Table 2 summarizes the sizes of the currently largest sets containing at most a k-crossing family for k up to 10.

Using Theorem 3.1 together with the sets of 20 points containing no 5-crossing family (see Figure 5) we get the bound $\mathrm{cf}(n) \leq 4\lceil \frac{n}{20} \rceil$. Also we have a set of 25 points containing no 6-crossing family (see Figure 6), which implies $\mathrm{cf}(n) \leq 5\lceil \frac{n}{25} \rceil$ and therefore gives a slightly better upper bound for certain values of n.

▶ **Corollary 3.3.** *We have* $\mathrm{cf}(n) \leq \min\{4\lceil \frac{n}{20} \rceil, 5\lceil \frac{n}{25} \rceil\} < \frac{n}{5} + 4$.

4 Conclusion

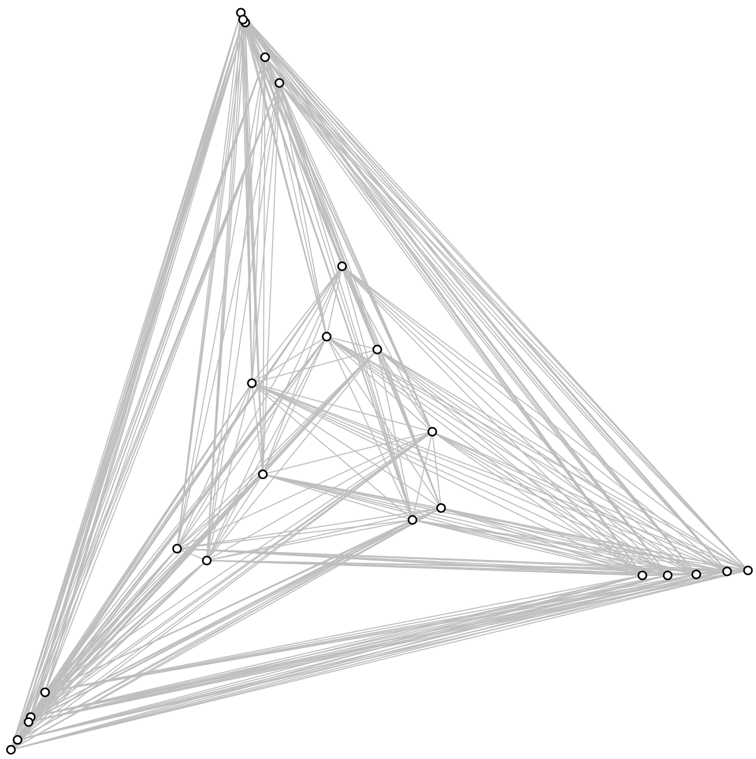
Based on exhaustive abstract extension of order types we showed that $\mathrm{cf}(15) = 4$. We conjecture that every set of 21 or more points contains a crossing family of size 5. We plan



coordinates:

595	113
0	0
2	7
8	21
49	179
473	131
465	139
96	225
198	205
115	280
235	242
103	321
242	266
584	118
135	339
150	393
104	531
105	537
56	595
55	598

Figure 5 A set S of 20 points with no 5-crossing family.



coordinates:

209	744
670	181
651	180
623	177
597	176
574	176
231	699
244	673
213	734
383	321
333	404
391	244
301	488
365	232
211	737
287	417
229	278
219	370
178	191
151	203
31	58
18	33
16	28
6	10
0	0

Figure 6 A set S of 25 points with no 6-crossing family.

to use our SAT framework to either settle the conjecture or improve the upper bound on the size of a crossing family to $4\lceil \frac{n}{21} \rceil$.

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