A Superlinear Lower Bound on the Number of 5-Holes*

Oswin Aichholzer[1], Martin Balko[2], Thomas Hackl[3], Jan Kynčl[4], Irene Parada[5], Manfred Scheucher[6], Pavel Valtr[7], and Birgit Vogtenhuber[8]

1 **Institute for Software Technology, Graz University of Technology, Austria**
`oaich@ist.tugraz.at`

2 **Dept. of Applied Mathematics and Institute for Theoretical Computer Science, Faculty of Mathematics and Physics, Charles University, Czech Republic; and Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary**
`balko@kam.mff.cuni.cz`

3 **Institute for Software Technology, Graz University of Technology, Austria**
`thackl@ist.tugraz.at`

4 **Dept. of Applied Mathematics and Institute for Theoretical Computer Science, Faculty of Mathematics and Physics, Charles University, Czech Republic**
`kyncl@kam.mff.cuni.cz`

5 **Institute for Software Technology, Graz University of Technology, Austria**
`iparada@ist.tugraz.at`

6 **Institute for Software Technology, Graz University of Technology, Austria; and Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary**
`mscheuch@ist.tugraz.at`

7 **Dept. of Applied Mathematics and Institute for Theoretical Computer Science, Faculty of Mathematics and Physics, Charles University, Czech Republic; and Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary**
`valtr@kam.mff.cuni.cz`

8 **Institute for Software Technology, Graz University of Technology, Austria**
`bvogt@ist.tugraz.at`

Abstract

Let P be a finite set of points in the plane in *general position*, that is, no three points of P are on a common line. We say that a set H of five points from P is a 5-*hole in* P if H is the vertex set of a convex 5-gon containing no other points of P. For a positive integer n, let $h_5(n)$ be the minimum number of 5-holes among all sets of n points in the plane in general position.

Despite many efforts in the last 30 years, the best known asymptotic lower and upper bounds for $h_5(n)$ have been of order $\Omega(n)$ and $O(n^2)$, respectively. We show that $h_5(n) = \Omega(n \log^{4/5} n)$, obtaining the first superlinear lower bound on $h_5(n)$.

* The research for this article was partially carried out in the course of the bilateral research project "Erdős–Szekeres type questions for point sets" between Graz and Prague, supported by the OEAD project CZ 18/2015 and project no. 7AMB15A T023 of the Ministry of Education of the Czech Republic. Aichholzer, Scheucher, and Vogtenhuber were partially supported by the ESF EUROCORES programme EuroGIGA – CRP ComPoSe, Austrian Science Fund (FWF): I648-N18. Parada was supported by the Austrian Science Fund (FWF): W1230. Balko and Valtr were partially supported by the grant GAUK 690214. Balko, Kynčl, and Valtr were partially supported by the project CE-ITI no. P202/12/G061 of the Czech Science Foundation (GAČR). Hackl and Scheucher were partially supported by the Austrian Science Fund (FWF): P23629-N18. Balko, Scheucher, and Valtr were partially supported by the ERC Advanced Research Grant no. 267165 (DISCONV).

© Oswin Aichholzer, Martin Balko, Thomas Hackl, Jan Kynčl, Irene Parada, Manfred Scheucher, Pavel Valtr, and Birgit Vogtenhuber;
licensed under Creative Commons License CC-BY

33rd International Symposium on Computational Geometry (SoCG 2017).
Editors: Boris Aronov and Matthew J. Katz; Article No. 8; pp. 8:1–8:16
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

The following structural result, which might be of independent interest, is a crucial step in the proof of this lower bound. If a finite set P of points in the plane in general position is partitioned by a line ℓ into two subsets, each of size at least 5 and not in convex position, then ℓ intersects the convex hull of some 5-hole in P. The proof of this result is computer-assisted.

1998 ACM Subject Classification G.2.1 Combinatorics

Keywords and phrases Erdős–Szekeres type problem, k-hole, empty k-gon, empty pentagon, planar point set

Digital Object Identifier 10.4230/LIPIcs.SoCG.2017.8

1 Introduction

We say that a set of points in the plane is in *general position* if it contains no three points on a common line. A point set is in *convex position* if it is the vertex set of a convex polygon. In 1935, Erdős and Szekeres [16] proved the following theorem, which is a classical result both in combinatorial geometry and Ramsey theory.

▶ **Theorem** ([16], The Erdős–Szekeres Theorem). *For every integer $k \geq 3$, there is a smallest integer $n = n(k)$ such that every set of at least n points in general position in the plane contains k points in convex position.*

The Erdős–Szekeres Theorem motivated a lot of further research, including numerous modifications and extensions of the theorem. Here we mention only results closely related to the main topic of our paper.

Let P be a finite set of points in general position in the plane. We say that a set H of k points from P is a *k-hole in P* if H is the vertex set of a convex k-gon containing no other points of P. In the 1970s, Erdős [15] asked whether, for every positive integer k, there is a k-hole in every sufficiently large finite point set in general position in the plane. Harborth [21] proved that there is a 5-hole in every set of 10 points in general position in the plane and gave a construction of 9 points in general position with no 5-hole. After unsuccessful attempts of researchers to answer Erdős' question affirmatively for any fixed integer $k \geq 6$, Horton [22] constructed, for every positive integer n, a set of n points in general position in the plane with no 7-hole. His construction was later generalized to so-called *Horton sets* and *squared Horton sets* [29] and to higher dimensions [30]. The question whether there is a 6-hole in every sufficiently large finite planar point set remained open until 2007 when Gerken [19] and Nicolás [23] independently gave an affirmative answer.

For positive integers n and k, let $h_k(n)$ be the minimum number of k-holes in a set of n points in general position in the plane. Due to Horton's construction, $h_k(n) = 0$ for every n and every $k \geq 7$. Asymptotically tight estimates for the functions $h_3(n)$ and $h_4(n)$ are known. The best known lower bounds are due to Aichholzer et al. [5] who showed that $h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7}$ and $h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - o(n)$. The best known upper bounds $h_3(n) \leq 1.6196n^2 + o(n^2)$ and $h_4(n) \leq 1.9397n^2 + o(n^2)$ are due to Bárány and Valtr [12].

For $h_5(n)$ and $h_6(n)$, no matching bounds are known. So far, the best known asymptotic upper bounds on $h_5(n)$ and $h_6(n)$ were obtained by Bárány and Valtr [12] and give $h_5(n) \leq 1.0207n^2 + o(n^2)$ and $h_6(n) \leq 0.2006n^2 + o(n^2)$. For the lower bound on $h_6(n)$, Valtr [31] showed $h_6(n) \geq n/229 - 4$.

In this paper we give a new lower bound on $h_5(n)$. It is widely conjectured that $h_5(n)$ grows quadratically in n, but to this date only lower bounds on $h_5(n)$ that are linear in

n have been known. As noted by Bárány and Füredi [10], a linear lower bound of $\lfloor n/10 \rfloor$ follows directly from Harborth's result [21]. Bárány and Károlyi [11] improved this bound to $h_5(n) \geq n/6 - O(1)$. In 1987, Dehnhardt [14] showed $h_5(11) = 2$ and $h_5(12) = 3$, obtaining $h_5(n) \geq 3\lfloor n/12 \rfloor$. However, his result remained unknown to the scientific community until recently. García [18] then presented a proof of the lower bound $h_5(n) \geq 3\lfloor \frac{n-4}{8} \rfloor$ and a slightly better estimate $h_5(n) \geq \lceil 3/7(n-11) \rceil$ was shown by Aichholzer, Hackl, and Vogtenhuber [6]. Quite recently, Valtr [31] obtained $h_5(n) \geq n/2 - O(1)$. This was strengthened by Aichholzer et al. [5] to $h_5(n) \geq 3n/4 - o(n)$. All improvements on the multiplicative constant were achieved by utilizing the values of $h_5(10)$, $h_5(11)$, and $h_5(12)$. In the bachelor's thesis of Scheucher [26] the exact values $h_5(13) = 3$, $h_5(14) = 6$, and $h_5(15) = 9$ were determined and $h_5(16) \in \{10, 11\}$ was shown. During the preparation of this paper, we further determined the value $h_5(16) = 11$; see our webpage [25]. The values $h_5(n)$ for $n \leq 16$ can be used to obtain further improvements on the multiplicative constant. By revising the proofs of [5, Lemma 1] and [5, Theorem 3], one can obtain $h_5(n) \geq n - 10$ and $h_5(n) \geq 3n/2 - o(n)$, respectively. We also note that it was shown in [24] that if $h_3(n) \geq (1+\epsilon)n^2 - o(n^2)$, then $h_5(n) = \Omega(n^2)$.

As our main result, we give the first superlinear lower bound on $h_5(n)$. This solves an open problem, which was explicitely stated, for example, in a book by Brass, Moser, and Pach [13, Chapter 8.4, Problem 5] and in the survey [2].

▶ **Theorem 1.** *There is an absolute constant $c > 0$ such that for every integer $n \geq 10$ we have $h_5(n) \geq cn \log^{4/5} n$.*

Let P be a finite set of points in the plane in general position and let ℓ be a line that contains no point of P. We say that P is *ℓ-divided* if there is at least one point of P in each of the two halfplanes determined by ℓ. For an ℓ-divided set P, we use $P = A \cup B$ to denote the fact that ℓ partitions P into the subsets A and B.

The following result, which might be of independent interest, is a crucial step in the proof of Theorem 1.

▶ **Theorem 2.** *Let $P = A \cup B$ be an ℓ-divided set with $|A|, |B| \geq 5$ and with neither A nor B in convex position. Then there is an ℓ-divided 5-hole in P.*

The proof of Theorem 2 is computer-assisted. We reduce the result to several statements about point sets of size at most 11 and then verify each of these statements by an exhaustive computer search. To verify the computer-aided proofs we have implemented two independent programs, which, in addition, are based on different abstractions of point sets; see Subsection 4.2. Some of our tools originate from the bachelor's theses of Scheucher [26, 27].

In the rest of the paper, we assume that every point set P is planar, finite, and in general position. We also assume, without loss of generality, that all points in P have distinct x-coordinates. We use $\operatorname{conv}(P)$ to denote the convex hull of P and $\partial \operatorname{conv}(P)$ to denote the boundary of the convex hull of P.

A subset Q of P that satisfies $P \cap \operatorname{conv}(Q) = Q$ is called an *island of P*. Note that every k-hole in an island Q of P is also a k-hole in P. For any subset R of the plane, if R contains no point of P, then we say that R is *empty of points of P*.

In Section 2 we derive quite easily Theorem 1 from Theorem 2. Then, in Section 3, we give some preliminaries for the proof of Theorem 2, which is presented in Section 4.

2 Proof of Theorem 1

We apply Theorem 2 to obtain a superlinear lower bound on the number of 5-holes in a given set of n points. Without loss of generality, we assume that $n = 2^t$ for some integer $t \geq 5^5$.

We prove by induction on $t \geq 5^5$ that the number of 5-holes in an arbitrary set P of $n = 2^t$ points is at least $f(t) := c \cdot 2^t t^{4/5} = c \cdot n \log_2^{4/5} n$ for some absolute constant $c > 0$. For $t = 5^5$, we have $n > 10$ and, by the result of Harborth [21], there is at least one 5-hole in P. If c is sufficiently small, then $f(t) = c \cdot n \log_2^{4/5} n \leq 1$ and we have at least $f(t)$ 5-holes in P, which constitutes our base case.

For the inductive step we assume that $t > 5^5$. We first partition P with a line ℓ into two sets A and B of size $n/2$ each. Then we further partition A and B into smaller sets using the following well-known lemma, which is, for example, implied by a result of Steiger and Zhao [28, Theorem 1].

▶ **Lemma 3** ([28]). *Let $P' = A' \cup B'$ be an ℓ-divided set and let r be a positive integer such that $r \leq |A'|, |B'|$. Then there is a line that is disjoint from P' and that determines an open halfplane h with $|A' \cap h| = r = |B' \cap h|$.*

We set $r := \lfloor \log_2^{1/5} n \rfloor$, $s := \lfloor n/(2r) \rfloor$, and apply Lemma 3 iteratively in the following way to partition P into islands $P_1, \dots, P_{s+1}$ of P so that the sizes of $P_i \cap A$ and $P_i \cap B$ are exactly r for every $i \in \{1, \dots, s\}$. Let $P'_0 := P$. For every $i = 1, \dots, s$, we consider a line that is disjoint from P'_{i-1} and that determines an open halfplane h with $|P'_{i-1} \cap A \cap h| = r = |P'_{i-1} \cap B \cap h|$. Such a line exists by Lemma 3 applied to the ℓ-divided set P'_{i-1}. We then set $P_i := P'_{i-1} \cap h$, $P'_i := P'_{i-1} \setminus P_i$, and continue with $i + 1$. Finally, we set $P_{s+1} := P'_s$.

For every $i \in \{1, \dots, s\}$, if one of the sets $P_i \cap A$ and $P_i \cap B$ is in convex position, then there are at least $\binom{r}{5}$ 5-holes in P_i and, since P_i is an island of P, we have at least $\binom{r}{5}$ 5-holes in P. If this is the case for at least $s/2$ islands P_i, then, given that $s = \lfloor n/(2r) \rfloor$ and thus $s/2 \geq \lfloor n/(4r) \rfloor$, we obtain at least $\lfloor n/(4r) \rfloor \binom{r}{5} \geq c \cdot n \log_2^{4/5} n$ 5-holes in P for a sufficiently small $c > 0$.

We thus further assume that for more than $s/2$ islands P_i, neither of the sets $P_i \cap A$ nor $P_i \cap B$ is in convex position. Since $r = \lfloor \log_2^{1/5} n \rfloor \geq 5$, Theorem 2 implies that there is an ℓ-divided 5-hole in each such P_i. Thus there is an ℓ-divided 5-hole in P_i for more than $s/2$ islands P_i. Since each P_i is an island of P and since $s = \lfloor n/(2r) \rfloor$, we have more than $s/2 \geq \lfloor n/(4r) \rfloor$ ℓ-divided 5-holes in P. As $|A| = |B| = n/2 = 2^{t-1}$, there are at least $f(t-1)$ 5-holes in A and at least $f(t-1)$ 5-holes in B by the inductive assumption. Since A and B are separated by the line ℓ, we have at least

$$2f(t-1) + n/(4r) = 2c(n/2) \log_2^{4/5} (n/2) + n/(4r) \geq cn(t-1)^{4/5} + n/(4t^{1/5})$$

5-holes in P. The right side of the above expression is at least $f(t) = cnt^{4/5}$, because the inequality $cn(t-1)^{4/5} + n/(4t^{1/5}) \geq cnt^{4/5}$ is equivalent to the inequality $(t-1)^{4/5} t^{1/5} + 1/(4c) \geq t$, which is true if c is sufficiently small, as $(t-1)^{4/5} t^{1/5} \geq t - 1$. This completes the proof of Theorem 1.

3 Preliminaries

Before proceeding with the proof of Theorem 2, we first introduce some notation and definitions, and state some immediate observations.

Let a, b, c be three distinct points in the plane. We denote the line segment spanned by a and b as ab, the ray starting at a and going through b as $\overrightarrow{ab}$, and the line through a and b

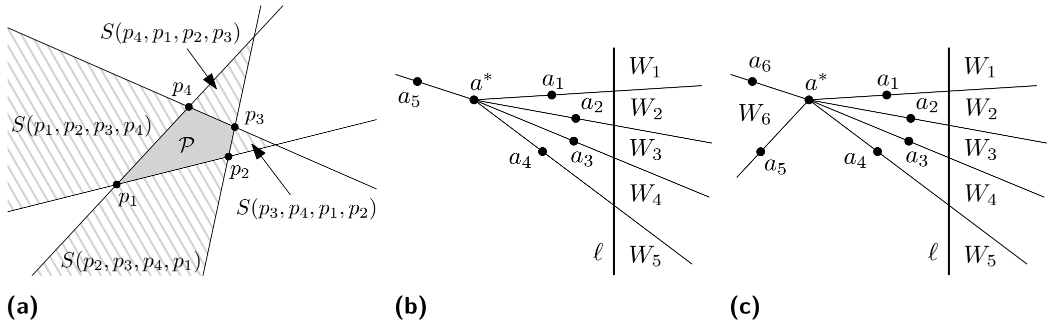

Figure 1 (a) An example of sectors. (b) An example of a^*-wedges with $t = |A| - 1$. (c) An example of a^*-wedges with $t < |A| - 1$.

directed from a to b as $\overline{ab}$. We say c is to the *left* (*right*) of $\overline{ab}$ if the triple (a, b, c) traced in this order is oriented counterclockwise (clockwise). Note that c is to the left of $\overline{ab}$ if and only if c is to the right of $\overline{ba}$, and that the triples (a, b, c), (b, c, a), and (c, a, b) have the same orientation. We say a point set S is to the *left* (*right*) of $\overline{ab}$ if every point of S is to the left (right) of $\overline{ab}$.

Let $P = A \cup B$ be an ℓ-divided set. In the rest of the paper, we assume without loss of generality that ℓ is vertical and directed upwards, A is to the left of ℓ, and B is to the right of ℓ.

Sectors of polygons

For an integer $k \geq 3$, let $\mathcal{P}$ be a convex polygon with vertices $p_1, p_2, \ldots, p_k$ traced counterclockwise in this order. We denote by $S(p_1, p_2, \ldots, p_k)$ the open convex region to the left of each of the three lines $\overline{p_1p_2}$, $\overline{p_1p_k}$, and $\overline{p_{k-1}p_k}$. We call $S(p_1, p_2, \ldots, p_k)$ a *sector* of $\mathcal{P}$. Note that every convex k-gon defines exactly k sectors. Figure 1(a) gives an illustration.

We use $\triangle(p_1, p_2, p_3)$ to denote the closed triangle with vertices p_1, p_2, p_3. We also use $\square(p_1, p_2, p_3, p_4)$ to denote the closed quadrilateral with vertices p_1, p_2, p_3, p_4 traced in the counterclockwise order along the boundary.

The following simple observation summarizes some properties of sectors of polygons.

▶ **Observation 4.** *Let $P = A \cup B$ be an ℓ-divided set with no ℓ-divided 5-hole in P. Then the following conditions are satisfied.*
(i) *Every sector of an ℓ-divided 4-hole in P is empty of points of P.*
(ii) *If S is a sector of a 4-hole in A and S is empty of points of A, then S is empty of points of B.*

ℓ-critical sets and islands

An ℓ-divided set $C = A \cup B$ is called *ℓ-critical* if it fulfills the following two conditions.
(i) Neither A nor B is in convex position.
(ii) For every extremal point x of C, one of the sets $(C \setminus \{x\}) \cap A$ and $(C \setminus \{x\}) \cap B$ is in convex position.

Note that every ℓ-critical set $C = A \cup B$ contains at least four points in each of A and B. If $P = A \cup B$ is an ℓ-divided set with neither A nor B in convex position, then there exists an ℓ-critical island of P. This can be seen by iteratively removing extremal points so that none of the parts is in convex position after the removal.

a-wedges and a^*-wedges

Let $P = A \cup B$ be an ℓ-divided set. For a point a in A, the rays $\overrightarrow{aa'}$ for all $a' \in A \setminus \{a\}$ partition the plane into $|A| - 1$ regions. We call the closures of those regions *a-wedges* and label them as $W_1^{(a)}, \ldots, W_{|A|-1}^{(a)}$ in the clockwise order around a, where $W_1^{(a)}$ is the topmost a-wedge that intersects ℓ. Let $t^{(a)}$ be the number of a-wedges that intersect ℓ. Note that $W_1^{(a)}, \ldots, W_{t^{(a)}}^{(a)}$ are the a-wedges that intersect ℓ sorted in top-to-bottom order on ℓ. Also note that all a-wedges are convex if a is an inner point of A, and that there exists exactly one non-convex a-wedge otherwise. The indices of the a-wedges are considered modulo $|A| - 1$. In particular, $W_0^{(a)} = W_{|A|-1}^{(a)}$ and $W_{|A|}^{(a)} = W_1^{(a)}$.

If A is not in convex position, we denote the rightmost inner point of A as a^* and write $t := t^{(a^*)}$ and $W_k := W_k^{(a^*)}$ for $k = 1, \ldots, |A| - 1$. Recall that a^* is unique, since all points have distinct x-coordinates. Figures 1(b) and 1(c) give an illustration. We set $w_k := |B \cap W_k|$ and label the points of A so that W_k is bounded by the rays $\overrightarrow{a^*a_{k-1}}$ and $\overrightarrow{a^*a_k}$ for $k = 1, \ldots, |A| - 1$. Again, the indices are considered modulo $|A| - 1$. In particular, $a_0 = a_{|A|-1}$ and $a_{|A|} = a_1$.

▶ **Observation 5.** *Let $P = A \cup B$ be an ℓ-divided set with A not in convex position. Then the points $a_1, \ldots, a_{t-1}$ lie to the right of a^* and the points $a_t, \ldots, a_{|A|-1}$ lie to the left of a^*.*

4 Proof of Theorem 2

First, we give a high-level overview of the main ideas of the proof of Theorem 2. We proceed by contradiction and we suppose that there is no ℓ-divided 5-hole in a given ℓ-divided set $P = A \cup B$ with $|A|, |B| \geq 5$ and with neither A nor B in convex position. If $|A|, |B| = 5$, then the statement follows from the result of Harborth [21]. Thus we assume that $|A| \geq 6$ or $|B| \geq 6$. We reduce P to an island Q of P by iteratively removing points from the convex hull until one of the two parts $Q \cap A$ and $Q \cap B$ contains exactly five points or Q is ℓ-critical with $|Q \cap A|, |Q \cap B| \geq 6$. If $|Q \cap A| = 5$ and $|Q \cap B| \geq 6$ or vice versa, then we reduce Q to an island of Q with eleven points and, using a computer-aided result (Lemma 12), we show that there is an ℓ-divided 5-hole in that island and hence in P. If Q is ℓ-critical with $|Q \cap A|, |Q \cap B| \geq 6$, then we show that $|A \cap \partial\,\mathrm{conv}(Q)|, |B \cap \partial\,\mathrm{conv}(Q)| \leq 2$ and that, if $|A \cap \partial\,\mathrm{conv}(Q)| = 2$, then a^* is the single interior point of $Q \cap A$ and similarly for B (Lemma 17). Without loss of generality, we assume that $|A \cap \partial\,\mathrm{conv}(Q)| = 2$ and thus a^* is the single interior point of $Q \cap A$. Using this assumption, we prove that $|Q \cap B| < |Q \cap A|$ (Proposition 19). By exchanging the roles of $Q \cap A$ and $Q \cap B$, we obtain $|Q \cap A| \leq |Q \cap B|$ (Proposition 22), which gives a contradiction.

To bound $|Q \cap B|$, we use three results about the sizes of the parameters $w_1, \ldots, w_t$ for the ℓ-divided set Q, that is, about the numbers of points of $Q \cap B$ in the a^*-wedges $W_1, \ldots, W_t$ of Q. We show that if we have $w_i = 2 = w_j$ for some $1 \leq i < j \leq t$, then $w_k = 0$ for some k with $i < k < j$ (Lemma 10). Further, for any three or four consecutive a^*-wedges whose union is convex and contains at least four points of $Q \cap B$, each of those a^*-wedges contains at most two such points (Lemma 16). Finally, we show that $w_1, \ldots, w_t \leq 3$ (Lemma 18). The proofs of Lemmas 16 and 18 rely on some results about small ℓ-divided sets with computer-aided proofs (Lemmas 13, 14, and 15). Altogether, this is sufficient to show that $|Q \cap B| < |Q \cap A|$.

We now start the proof of Theorem 2 by showing that if there is an ℓ-divided 5-hole in the intersection of P with a union of consecutive a^*-wedges, then there is an ℓ-divided 5-hole in P.

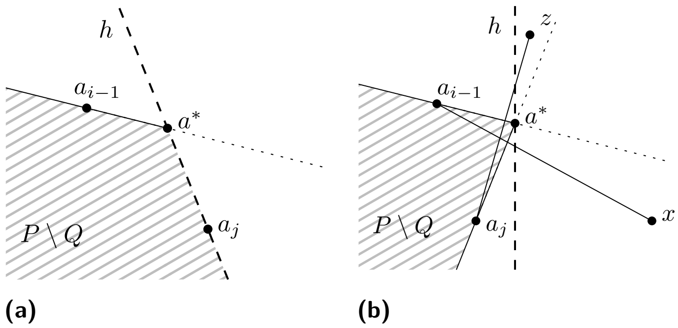

Figure 2 Illustration of the proof of Lemma 6. (a) The point a_j is to the right of a^*. (b) The point a_j is to the left of a^*. (c) The hole H properly intersects the ray $\overrightarrow{a^*a_j}$. The boundary of the convex hull of H is drawn red and the convex hull of H' is drawn blue.

▶ **Lemma 6.** *Let $P = A \cup B$ be an ℓ-divided set with A not in convex position. For integers i, j with $1 \le i \le j \le t$, let $W := \bigcup_{k=i}^{j} W_k$ and $Q := P \cap W$. If there is an ℓ-divided 5-hole in Q, then there is an ℓ-divided 5-hole in P.*

Proof. If W is convex then Q is an island of P and the statement immediately follows. Hence we assume that W is not convex. The region W is bounded by the rays $\overrightarrow{a^*a_{i-1}}$ and $\overrightarrow{a^*a_j}$ and all points of $P \setminus Q$ lie in the convex region $\mathbb{R}^2 \setminus W$; see Figure 2.

Since W is non-convex and every a^*-wedge contained in W intersects ℓ, at least one of the points a_{i-1} and a_j lies to the left of a^*. Moreover, the points $a_i, \dots, a_{j-1}$ are to the right of a^* by Observation 5. Without loss of generality, we assume that a_{i-1} is to the left of a^*.

Let H be an ℓ-divided 5-hole in Q. If a_j is to the left of a^*, then we let h be the closed halfplane determined by the vertical line through a^* such that a_{i-1} and a_j lie in h. Otherwise, if a_j is to the right of a^*, then we let h be the closed halfplane determined by the line $\overline{a^*a_j}$ such that a_{i-1} lies in h. In either case, $h \cap A \cap Q = \{a^*, a_{i-1}, a_j\}$.

We say that H *properly intersects* a ray r if there are points $p, q \in H$ such that the interior of the segment pq intersects r. Now we show that if H properly intersects the ray $\overrightarrow{a^*a_j}$, then H contains a_{i-1}. Assume there are points $p, q \in H$ such that pq properly intersects $r := \overrightarrow{a^*a_j}$. Since r lies in h and neither of p and q lies in r, at least one of the points p and q lies in $h \setminus r$. Without loss of generality, we assume $p \in h \setminus r$. From $h \cap A \cap Q = \{a^*, a_{i-1}, a_j\}$ we have $p = a_{i-1}$. By symmetry, if H properly intersects the ray $\overrightarrow{a^*a_{i-1}}$, then H contains a_j.

Suppose for contradiction that H properly intersects both rays $\overrightarrow{a^*a_{i-1}}$ and $\overrightarrow{a^*a_j}$. Then H contains the points a_{i-1}, a_j, x, y, z for some points $x, y, z \in Q$, where $a_{i-1}x$ intersects $\overrightarrow{a^*a_j}$, and a_jz intersects $\overrightarrow{a^*a_{i-1}}$. Observe that z is to the left of $\overline{a_{i-1}a^*}$ and that x is to the right of $\overline{a_ja^*}$. If a_j lies to the right of a^*, then z is to the left of a^*, and thus z is in A; see Figure 2(a). However, this is impossible as z also lies in h. Hence, a_j lies to the left of a^*; see Figure 2(b). As x and z are both to the right of a^*, the point a^* is inside the convex quadrilateral $\square(a_{i-1}, a_j, x, z)$. This contradicts the assumption that H is a 5-hole in Q.

So assume that H properly intersects exactly one of the rays $\overrightarrow{a^*a_{i-1}}$ and $\overrightarrow{a^*a_j}$, say $\overrightarrow{a^*a_j}$; see Figure 2(c). In this case, H contains a_{i-1}. The interior of the triangle $\triangle(a^*, a_{i-1}, a_j)$ is empty of points of Q, since the triangle is contained in h. Moreover, $\text{conv}(H)$ cannot intersect the line that determines h both strictly above and strictly below a^*. Thus, all remaining points of $H \setminus \{a_{i-1}\}$ lie to the right of $\overline{a_{i-1}a^*}$ and to the right of $\overline{a_ja^*}$. If H is empty of points of $P \setminus Q$, we are done. Otherwise, we let $H' := (H \setminus \{a_{i-1}\}) \cup \{p'\}$ where $p' \in P \setminus Q$ is a point inside $\triangle(a^*, a_{i-1}, a_j)$ closest to $\overline{a_ja^*}$. Note that the point p' might not be unique. By construction, H' is an ℓ-divided 5-hole in P. An analogous argument shows that there is an ℓ-divided 5-hole in P if H properly intersects $\overrightarrow{a^* a_{i-1}}$.

Finally, if H does not properly intersect any of the rays $\overrightarrow{a^* a_{i-1}}$ and $\overrightarrow{a^* a_j}$, then conv(H) contains no point of $P \setminus Q$ in its interior, and hence H is an ℓ-divided 5-hole in P. ◀

4.1 Sequences of a^*-wedges with at most two points of B

In this subsection we consider an ℓ-divided set $P = A \cup B$ with A not in convex position. We consider the union W of consecutive a^*-wedges, each containing at most two points of B, and derive an upper bound on the number of points of B that lie in W if there is no ℓ-divided 5-hole in $P \cap W$; see Corollary 11.

▶ **Observation 7.** *Let $P = A \cup B$ be an ℓ-divided set with A not in convex position. Let W_k be an a^*-wedge with $w_k \geq 1$ and $1 \leq k \leq t$ and let b be the leftmost point in $W_k \cap B$. Then the points a^*, a_{k-1}, b, and a_k form an ℓ-divided 4-hole in P.*

From Observation 4(i) and Observation 7 we obtain the following result.

▶ **Observation 8.** *Let $P = A \cup B$ be an ℓ-divided set with A not in convex position and with no ℓ-divided 5-hole in P. Let W_k be an a^*-wedge with $w_k \geq 2$ and $1 \leq k \leq t$ and let b be the leftmost point in $W_k \cap B$. For every point b' in $(W_k \cap B) \setminus \{b\}$, the line $\overline{bb'}$ intersects the segment $a_{k-1}a_k$. Consequently, b is inside $\triangle(a_{k-1}, a_k, b')$, to the left of $\overline{a_k b'}$, and to the right of $\overline{a_{k-1} b'}$.*

The following lemma states that there is an ℓ-divided 5-hole in P if two consecutive a^*-wedges both contain exactly two points of B. Its proof can be found in the full version of the paper [4].

▶ **Lemma 9.** *Let $P = A \cup B$ be an ℓ-divided set with A not in convex position and with $|A|, |B| \geq 5$. Let W_i and W_{i+1} be consecutive a^*-wedges with $w_i = 2 = w_{i+1}$ and $1 \leq i < t$. Then there is an ℓ-divided 5-hole in P.*

Next we show that if there is a sequence of consecutive a^*-wedges where the first and the last a^*-wedge both contain two points of B and every a^*-wedge in between them contains exactly one point of B, then there is an ℓ-divided 5-hole in P.

▶ **Lemma 10.** *Let $P = A \cup B$ be an ℓ-divided set with A not in convex position and with $|A| \geq 5$ and $|B| \geq 6$. Let $W_i, \ldots, W_j$ be consecutive a^*-wedges with $1 \leq i < j \leq t$, $w_i = 2 = w_j$, and $w_k = 1$ for every k with $i < k < j$. Then there is an ℓ-divided 5-hole in P.*

The proof of Lemma 10 can be found in the full version of the paper [4]. We now use Lemma 10 to show the following upper bound on the total number of points of B in a sequence $W_i, \ldots, W_j$ of consecutive a^*-wedges with $w_i, \ldots, w_j \leq 2$.

▶ **Corollary 11.** *Let $P = A \cup B$ be an ℓ-divided set with no ℓ-divided 5-hole, with A not in convex position, and with $|A| \geq 5$ and $|B| \geq 6$. For $1 \leq i \leq j \leq t$, let $W_i, \ldots, W_j$ be consecutive a^*-wedges with $w_k \leq 2$ for every k with $i \leq k \leq j$. Then $\sum_{k=i}^{j} w_k \leq j - i + 2$.*

Proof. Let n_0, n_1, and n_2 be the number of a^*-wedges from $W_i, \ldots, W_j$ with 0, 1, and 2 points of B, respectively. Due to Lemma 10, we can assume that between any two a^*-wedges from $W_i, \ldots, W_j$ with two points of B each, there is an a^*-wedge with no point of B. Thus $n_2 \leq n_0 + 1$. Since $n_0 + n_1 + n_2 = j - i + 1$, we have $\sum_{k=i}^{j} w_k = 0n_0 + 1n_1 + 2n_2 = (j - i + 1) + (n_2 - n_0) \leq j - i + 2$. ◀

4.2 Computer-assisted results

We now provide lemmas that are key ingredients in the proof of Theorem 2. All these lemmas have computer-aided proofs. Each result was verified by two independent implementations, which are also based on different abstractions of point sets; see below for details.

▶ **Lemma 12.** *Let $P = A \cup B$ be an ℓ-divided set with $|A| = 5$, $|B| = 6$, and with A not in convex position. Then there is an ℓ-divided 5-hole in P.*

▶ **Lemma 13.** *Let $P = A \cup B$ be an ℓ-divided set with no ℓ-divided 5-hole in P, $|A| = 5$, $4 \leq |B| \leq 6$, and with A in convex position. Then for every point a of A, every convex a-wedge contains at most two points of B.*

▶ **Lemma 14.** *Let $P = A \cup B$ be an ℓ-divided set with no ℓ-divided 5-hole in P, $|A| = 6$, and $|B| = 5$. Then for each point a of A, every convex a-wedge contains at most two points of B.*

▶ **Lemma 15.** *Let $P = A \cup B$ be an ℓ-divided set with no ℓ-divided 5-hole in P, $5 \leq |A| \leq 6$, $|B| = 4$, and with A in convex position. Then for every point a of A, if the non-convex a-wedge is empty of points of B, every a-wedge contains at most two points of B.*

We remark that all the assumptions in the statements of Lemmas 12 to 15 are necessary; see the full version of the paper [4]. To prove these lemmas, we employ an exhaustive computer search through all combinatorially different sets of $|P| \leq 11$ points in the plane. Since none of these statements depends on the actual coordinates of the points but only on the relative positions of the points, we distinguish point sets only by orientations of triples of points as proposed by Goodman and Pollack [20]. That is, we check all possible equivalence classes of point sets in the plane with respect to their triple-orientations, which are known as *order types*.

We wrote two independent programs to verify Lemmas 12 to 15. Both programs are available online [25, 8].

The first implementation is based on programs from the two bachelor's theses of Scheucher [26, 27]. For our verification purposes we reduced the framework from there to a very compact implementation [25]. The program uses the order type database [3, 7], which stores all order types realizable as point sets of size up to 11. The order types realizable as sets of ten points are available online [1] and the ones realizable as sets of eleven points need about 96 GB and are available upon request from Aichholzer. The running time of each of the programs in this implementation does not exceed two hours on a standard computer.

The second implementation [8] neither uses the order type database nor the program used to generate the database. Instead it relies on the description of point sets by so-called *signature functions* [9, 17]. In this description, points are sorted according to their x-coordinates and every unordered triple of points is represented by a sign from $\{-, +\}$, where the sign is $-$ if the triple traced in the order by increasing x-coordinates is oriented clockwise and the sign is $+$ otherwise. Every 4-tuple of points is then represented by four signs of its triples, which are ordered lexicographically. There are only eight 4-tuples of signs that we can obtain (out of 16 possible ones); see [9, Theorem 3.2] or [17, Theorem 7] for details. In our algorithm, we generate all possible signature functions using a simple depth-first search algorithm and verify the conditions from our lemmas for every signature. The running time of each of the programs in this implementation may take up to a few hundreds of hours.

4.3 Applications of the computer-assisted results

Here we present some applications of the computer-assisted results from Section 4.2.

▶ **Lemma 16.** *Let $P = A \cup B$ be an ℓ-divided set with no ℓ-divided 5-hole in P, with $|A| \geq 6$, and with A not in convex position. Then the following two conditions are satisfied.*

(i) *Let W_i, W_{i+1}, W_{i+2} be three consecutive a^*-wedges whose union is convex and contains at least four points of B. Then $w_i, w_{i+1}, w_{i+2} \leq 2$.*

(ii) *Let $W_i, W_{i+1}, W_{i+2}, W_{i+3}$ be four consecutive a^*-wedges whose union is convex and contains at least four points of B. Then $w_i, w_{i+1}, w_{i+2}, w_{i+3} \leq 2$.*

Proof. To show part (i), let $W := W_i \cup W_{i+1} \cup W_{i+2}$, $A' := A \cap W$, $B' := B \cap W$, and $P' := A' \cup B'$. Since W is convex, P' is an island of P and thus there is no ℓ-divided 5-hole in P'. Note that $|A'| = 5$ and A' is in convex position. If $|B'| \leq 5$, then every convex a^*-wedge in P' contains at most two points of B' by Lemma 13 applied to P'. So assume that $|B'| \geq 6$. We remove points from P' from the right to obtain $P'' = A' \cup B''$, where B'' contains exactly six points of B'. Note that there is no ℓ-divided 5-hole in P'', since P'' is an island of P'. By Lemma 13, each a^*-wedge in P'' contains exactly two points of B''. Let $\tilde{B}$ be the set of points of B that are to the left of the rightmost point of B'', including this point, and let $\tilde{P} := A \cup \tilde{B}$. Note that $B'' \subseteq \tilde{B}$. Since $|B''| = 6$ and since $W \cap \tilde{B} = B''$, each of the a^*-wedges W_i, W_{i+1}, W_{i+2} contains exactly two points of $\tilde{B}$. The a^*-wedges W_i, W_{i+1}, and W_{i+2} are also a^*-wedges in $\tilde{P}$. Thus, Lemma 9 applied to $\tilde{P}$ and W_i, W_{i+1} then gives us an ℓ-divided 5-hole in $\tilde{P}$. From the choice of $\tilde{P}$, we then have an ℓ-divided 5-hole in P, a contradiction.

To show part (ii), let $W := W_i \cup W_{i+1} \cup W_{i+2} \cup W_{i+3}$, $A' := A \cap W$, $B' := B \cap W$, and $P' := A' \cup B'$. Since W is convex, P' is an island of P and thus there is no ℓ-divided 5-hole in P'. Note that $|A'| = 6$ and A' is in convex position. If $|B'| = 4$, then the statement follows from Lemma 15 applied to P' since a^* is an extremal point of P'. If $|B'| = 5$, then the statement follows from Lemma 14 applied to P' and thus we can assume $|B'| \geq 6$. Suppose for contradiction that $w_j \geq 3$ for some $i \leq j \leq i+3$. We remove points from P from the right to obtain P'' so that $B'' := P'' \cap B$ contains exactly six points of $W \cap B$. By applying part (i) for P'' and $W_i \cup W_{i+1} \cup W_{i+2}$ and $W_{i+1} \cup W_{i+2} \cup W_{i+3}$, we obtain that $|B'' \cap W_i|, |B'' \cap W_{i+3}| = 3$ and $|B'' \cap W_{i+1}|, |B'' \cap W_{i+2}| = 0$. Let b be the rightmost point from $P'' \cap W$. By Lemma 14 applied to $W \cap (P'' \setminus \{b\})$, there are at most two points of $B'' \setminus \{b\}$ in every a^*-wedge in $W \cap (P'' \setminus \{b\})$. This contradicts the fact that either $|(B'' \cap W_i) \setminus \{b\}| = 3$ or $|(B'' \cap W_{i+3}) \setminus \{b\}| = 3$. ◀

4.4 Extremal points of ℓ-critical sets

Recall the definition of ℓ-critical sets: An ℓ-divided point set $C = A \cup B$ is called ℓ-critical if neither $C \cap A$ nor $C \cap B$ is in convex position and if for every extremal point x of C, one of the sets $(C \setminus \{x\}) \cap A$ and $(C \setminus \{x\}) \cap B$ is in convex position.

In this section, we consider an ℓ-critical set $C = A \cup B$ with $|A|, |B| \geq 5$. We first show that C has at most two extremal points in A and at most two extremal points in B. Later, under the assumption that there is no ℓ-divided 5-hole in C, we show that $|B| \leq |A| - 1$ if A contains two extremal points of C (Section 4.4.1) and that $|B| \leq |A|$ if B contains two extremal points of C (Section 4.4.2).

▶ **Lemma 17.** *Let $C = A \cup B$ be an ℓ-critical set. Then the following statements are true.*

(i) *If $|A| \geq 5$, then $|A \cap \partial \operatorname{conv}(C)| \leq 2$.*

(ii) *If $A \cap \partial \operatorname{conv}(C) = \{a, a'\}$, then a^* is the single interior point in A and every point of $A \setminus \{a, a'\}$ lies in the convex region spanned by the lines $\overline{a^*a}$ and $\overline{a^*a'}$ that does not have any of a and a' on its boundary.*

(iii) *If $A \cap \partial\,\mathrm{conv}(C) = \{a, a'\}$, then the a^*-wedge that contains a and a' contains no point of B.*

By symmetry, analogous statements hold for B.

The proof of Lemma 17 can be found in the full version of the paper [4].

We remark that the assumption $|A| \geq 5$ in part (i) of Lemma 17 is necessary. In fact, arbitrarily large ℓ-critical sets with only four points in A and with three points of A on $\partial\,\mathrm{conv}(C)$ exist, and analogously for B.

▶ **Lemma 18.** *Let $C = A \cup B$ be an ℓ-critical set with no ℓ-divided 5-hole in C and with $|A| \geq 6$. Then $w_i \leq 3$ for every $1 < i < t$. Moreover, if $|A \cap \partial\,\mathrm{conv}(C)| = 2$, then $w_1, w_t \leq 3$.*

Proof. Recall that, since C is ℓ-critical, we have $|B| \geq 4$. Let i be an integer with $1 \leq i \leq t$. We assume that there is a point a in $A \cap \partial\,\mathrm{conv}(C)$, which lies outside of W_i, as otherwise there is nothing to prove for W_i (either $|A \cap \partial\,\mathrm{conv}(C)| = 1$ and $i \in \{1, t\}$ or $|A \cap \partial\,\mathrm{conv}(C)| = 2$ and, by Lemma 17(iii), $W_i \cap B = \emptyset$). We consider $C' := C \setminus \{a\}$. Since C is an ℓ-critical set, $A' := C' \cap A$ is in convex position. Thus, there is a non-convex a^*-wedge W' of C'. Since W' is non-convex, all other a^*-wedges of C' are convex. Moreover, since W' is the union of the two a^*-wedges of C that contain a, all other a^*-wedges of C' are also a^*-wedges of C. Let W be the union of all a^*-wedges of C that are not contained in W'. Note that W is convex and contains at least $|A| - 3 \geq 3$ a^*-wedges of C. Since $|A| \geq 6$, the statement follows from Lemma 16(i). ◀

4.4.1 Two extremal points of C in A

▶ **Proposition 19.** *Let $C = A \cup B$ be an ℓ-critical set with no ℓ-divided 5-hole in C, with $|A|, |B| \geq 6$, and with $|A \cap \partial\,\mathrm{conv}(C)| = 2$. Then $|B| \leq |A| - 1$.*

Proof. Since $|A \cap \partial\,\mathrm{conv}(C)| = 2$, Lemma 18 implies that $w_i \leq 3$ for every $1 \leq i \leq t$. Let a and a' be the two points in $A \cap \partial\,\mathrm{conv}(C)$. By Lemma 17(ii), all points of $A \setminus \{a, a'\}$ lie in the convex region R spanned by the lines $\overline{a^*a}$ and $\overline{a^*a'}$ that does not have any of a and a' on its boundary. That is, without loss of generality, $a = a_{h-1}$ and $a' = a_h$ for some $1 \leq h \leq |A| - 1$ and, by Lemma 17(iii), we have $w_h = 0$. Since all points of $A \setminus \{a, a'\}$ lie in the convex region R, the regions $W := \mathrm{cl}(\mathbb{R}^2 \setminus (W_{h-1} \cup W_h))$ and $W' := \mathrm{cl}(\mathbb{R}^2 \setminus (W_h \cup W_{h+1}))$ are convex. Here $\mathrm{cl}(X)$ denotes the closure of a set $X \subseteq \mathbb{R}^2$. Recall that the indices of the a^*-wedges are considered modulo $|A| - 1$ and that $\mathbb{R}^2$ is the union of all a^*-wedges.

First, suppose for contradiction that $|A| = 6$ and $|B| \geq 6$. There are exactly five a^*-wedges $W_1, \ldots, W_5$, and only four of them can contain points of B, since $w_h = 0$. We apply Lemma 16(i) to W and to W' and obtain that either $w_i \leq 2$ for every $1 \leq i \leq t$ or $w_{h-1}, w_{h+1} = 3$ and $w_i = 0$ for every $i \notin \{h-1, h+1\}$. In the first case, Corollary 11 implies that $|B| \leq 5$ and in the latter case Lemma 14 applied to $P \setminus \{b\}$, where b is the rightmost point of B, gives $|B| \leq 5$, a contradiction. Hence, we assume $|A| \geq 7$.

▶ **Claim 20.** *For $1 \leq k \leq t - 3$, if one of the four consecutive a^*-wedges W_k, W_{k+1}, W_{k+2}, or W_{k+3} contains 3 points of B, then $w_k + w_{k+1} + w_{k+2} + w_{k+3} = 3$.*

There are $|A| - 1 \geq 6$ a^*-wedges and, in particular, W and W' are both unions of at least four a^*-wedges. For every W_i with $w_i = 3$ and $1 \leq i \leq t$, the a^*-wedge W_i is either contained in W or in W'. Thus we can find four consecutive a^*-wedges $W_k, W_{k+1}, W_{k+2}, W_{k+3}$ whose union is convex and contains W_i. Lemma 16(ii) implies that each of $W_k, W_{k+1}, W_{k+2}, W_{k+3}$ except of W_i is empty of points of B. This finishes the proof of Claim 20.

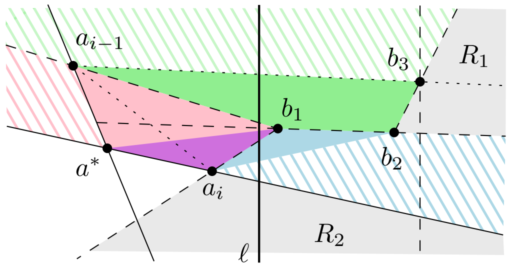

Figure 3 An illustration of the proof of Proposition 22.

▶ **Claim 21.** *For all integers i and j with $1 \le i < j \le t$, we have $\sum_{k=i}^{j} w_k \le j - i + 2$.*

Let $S := (w_i, \ldots, w_j)$ and let S' be the subsequence of S obtained by removing every 1-entry from S. If S contains only 1-entries, the statement clearly follows. Thus we can assume that S' is non-empty. Recall that S' contains only 0-, 2-, and 3-entries, since $w_i \le 3$ for all $1 \le i \le t$. Due to Claim 20, there are at least three consecutive 0-entries between every pair of nonzero entries of S' that contains a 3-entry. Together with Lemma 10, this implies that there is at least one 0-entry between every pair of 2-entries in S'.

By applying the following iterative procedure, we show that $\sum_{s \in S'} s \le |S'| + 1$. While there are at least two nonzero entries in S', we remove the first nonzero entry s from S'. If $s = 2$, then we also remove the 0-entry from S' that succeeds s in S. If $s = 3$, then we also remove the two consecutive 0-entries from S' that succeed s in S'. The procedure stops when there is at most one nonzero element s' in the remaining subsequence S'' of S'. If $s' = 3$, then S'' contains at least one 0-entry and thus S'' contains at least $s' - 1$ elements. Since the number of removed elements equals the sum of the removed elements in every step of the procedure, we have $\sum_{s \in S'} s \le |S'| + 1$. This implies

$$\sum_{k=i}^{j} w_k = \sum_{s \in S} s = |S| - |S'| + \sum_{s \in S'} s \le |S| - |S'| + |S'| + 1 = j - i + 2$$

and finishes the proof of Claim 21.

If W_h does not intersect ℓ, that is, $t < h \le |A| - 1$, then the statement follows from Claim 21 applied with $i = 1$ and $j = t$. Otherwise, we have $h = 1$ or $h = t$ and we apply Claim 21 with $(i, j) = (2, t)$ or $(i, j) = (1, t - 1)$, respectively. Since $t \le |A| - 1$ and $w_h = 0$, this gives us $|B| \le |A| - 1$. ◀

4.4.2 Two extremal points of C in B

▶ **Proposition 22.** *Let $C = A \cup B$ be an ℓ-critical set with no ℓ-divided 5-hole in C, with $|A|, |B| \ge 6$, and with $|B \cap \partial\,\mathrm{conv}(C)| = 2$. Then $|B| \le |A|$.*

Proof. If $w_k \le 2$ for all $1 \le k \le t$, then the statement follows from Corollary 11, since $|B| = \sum_{k=1}^{t} w_k \le t + 1 \le |A|$. Therefore we assume that there is an a^*-wedge W_i that contains at least three points of B. Let b_1, b_2, and b_3 be the three leftmost points in $W_i \cap B$ from left to right. Without loss of generality, we assume that b_3 is to the left of $\overline{b_1 b_2}$. Otherwise we can consider a vertical reflection of P. Figure 3 gives an illustration.

Let R_1 be the region that lies to the left of $\overline{b_1 b_2}$ and to the right of $\overline{b_2 b_3}$ and let R_2 be the region that lies to the right of $\overline{a_i b_1}$ and to the right of $\overline{a^* a_i}$. Let $B' := B \setminus \{b_1, b_2, b_3\}$.

▶ **Claim 23.** *Every point of B' lies in $R_1 \cup R_2$.*

We first show that every point of B' that lies to the left of $\overline{b_1b_2}$ lies in R_1. Then we show that every point of B' that lies to the right of $\overline{b_1b_2}$ lies in R_2.

By Observation 8, both lines $\overline{b_1b_2}$ and $\overline{b_1b_3}$ intersect the segment $a_{i-1}a_i$. Since the segment $a_{i-1}b_1$ intersects ℓ and since b_1 is the leftmost point of $W_i \cap B$, all points of B' that lie to the left of $\overline{b_1b_2}$ lie to the left of $\overline{a_{i-1}b_1}$. The four points a_{i-1}, b_1, b_2, b_3 form an ℓ-divided 4-hole in P, since a_{i-1} is the leftmost and b_3 is the rightmost point of a_{i-1}, b_1, b_2, b_3 and both a_{i-1} and b_3 lie to the left of $\overline{b_1b_2}$. By Observation 4(i), the sector $S(a_{i-1}, b_1, b_2, b_3)$ is empty of points of P (green shaded area in Figure 3). Altogether, all points of B' that lie to the left of $\overline{b_1b_2}$ are to the right of $\overline{b_2b_3}$ and thus lie in R_1.

Since the segment a_ib_1 intersects ℓ and since b_1 is the leftmost point of $W_i \cap B$, all points of B' that lie to the right of $\overline{b_1b_2}$ lie to the right of $\overline{a_ib_1}$. By Observation 4(i), the sector $S(b_1, b_2, b_3, a_{i-1})$ is empty of points of P. Combining this with the fact that a^* is to the right of $\overline{a_{i-1}b_3}$, we see that a^* lies to the right of $\overline{b_1b_2}$. Since b_1 and b_2 both lie to the left of $\overline{a^*a_i}$ and since a^* and a_i both lie to the right of $\overline{b_1b_2}$, the points b_2, b_1, a^*, a_i form an ℓ-divided 4-hole in P. By Observation 4(i), the sector $S(b_2, b_1, a^*, a_i)$ (blue shaded area in Figure 3) is empty of points of P. Altogether, all points of B' that lie to the right of $\overline{b_1b_2}$ are to the right of $\overline{a^*a_i}$ and to the right of $\overline{a_ib_1}$ and thus lie in R_2. This finishes the proof of Claim 23.

▶ **Claim 24.** *If b_4 is a point from $B' \setminus R_1$, then b_2 lies inside the triangle $\triangle(b_3, b_1, b_4)$.*

By Claim 23, b_4 lies in R_2 and thus to the right of $\overline{a_ib_1}$ and to the right of $\overline{a^*a_i}$. We recall that b_4 lies to the right of $\overline{b_1b_2}$.

We distinguish two cases. First, we assume that the points b_2, b_3, b_1, a_i are in convex position. Then b_2, b_3, b_1, a_i form an ℓ-divided 4-hole in P and, by Observation 4(i), the sector $S(b_2, b_3, b_1, a_i)$ is empty of points from P. Thus b_4 lies to the right of $\overline{b_2b_3}$ and the statement follows.

Second, we assume that the points b_2, b_3, b_1, a_i are not in convex position. Due to Observation 8, b_2 and b_3 both lie to the right of $\overline{a_ib_1}$. Moreover, since b_3 is the rightmost of those four points, b_2 lies inside the triangle $\triangle(b_3, b_1, a_i)$. In particular, a_i lies to the right of $\overline{b_2b_3}$. Therefore, since b_2 and b_3 are to the left of $\overline{a^*a_i}$, the line $\overline{b_2b_3}$ intersects ℓ in a point p above $\ell \cap \overline{a^*a_i}$. Let q be the point $\ell \cap \overline{b_1b_2}$. Note that q is to the left of $\overline{a^*a_i}$. The point b_4 is to the right of $\overline{b_2b_3}$, as otherwise b_4 lies in $\triangle(p, q, b_2)$, which is impossible because the points p, q, b_2 are in W_i while b_4 is not. Altogether, b_2 is inside $\triangle(b_3, b_1, b_4)$ and this finishes the proof of Claim 24.

▶ **Claim 25.** *Either every point of B' is to the right of b_3 or b_3 is the rightmost point of B.*

By Observation 4(i), the sector $S(b_3, a_{i-1}, b_1, b_2)$ is empty of points of P and thus all points of $B' \cap R_1$ lie to the left of $\overline{a_{i-1}b_3}$ and, in particular, to the right of b_3.

Suppose for contradiction that the claim is not true. That is, there is a point $b_4 \in B'$ that is the rightmost point in B and there is a point $b_5 \in B'$ that is to the left of b_3. Note that b_4 is an extremal point of C. By Claim 23 and by the fact that all points of $B' \cap R_1$ lie to the right of b_3, b_5 lies in $R_2 \setminus R_1$. By Claim 24, b_2 lies in the triangle $\triangle(b_1, b_5, b_3)$, and thus $B \setminus \{b_4\}$ is not in convex position. This contradicts the assumption that C is an ℓ-critical island. This finishes the proof of Claim 25.

▶ **Claim 26.** *The point b_3 is the third leftmost point of B. In particular, W_i is the only a^*-wedge with at least three points of B.*

Suppose for contradiction that b_3 is not the third leftmost point of B. Then by Claim 25, b_3 is the rightmost point of B and therefore an extremal point of B. This implies that $B' \subseteq R_2 \setminus R_1$, since all points of $B' \cap R_1$ lie to the right of b_3. By Claim 24, each point of B'

then forms a non-convex quadrilateral together with b_1, b_2, and b_3. Since neither b_1 nor b_2 are extremal points of C and since $|B \cap \partial\,\mathrm{conv}(C)| = 2$, there is a point $b_4 \in B$ that is an extremal point of C. Since $|B| \geq 5$, the set $C \setminus \{b_4\}$ has none of its parts separated by ℓ in convex position, which contradicts the assumption that C is an ℓ-critical set. Since W_i is an arbitrary a^*-wedge with $w_i \geq 3$, Claim 26 follows.

▶ **Claim 27.** *Let W be a union of four consecutive a^*-wedges that contains W_i. Then $|W \cap B| \leq 4$.*

Suppose for contradiction that $|W \cap B| \geq 5$. Let $C' := C \cap W$. Note that $|C' \cap A| = 6$ and that a^*, a_{i-1}, a_i lie in C'. By Lemma 6, there is no ℓ-divided 5-hole in C'. We obtain C'' by removing points from C' from the right until $|C'' \cap B| = 5$. Since C'' is an island of C', there is no ℓ-divided 5-hole in C''. From Claim 26 we know that b_1, b_2, b_3 are the three leftmost points in C and thus lie in C''. We apply Lemma 14 to C'' and, since b_1, b_2, b_3 lie in a convex a^*-wedge of C'', we obtain a contradiction. This finishes the proof of Claim 27.

We now complete the proof of Proposition 22. First, we assume that $1 \leq i \leq 4$. Let $W := W_1 \cup W_2 \cup W_3 \cup W_4$. By Claim 27, $|W \cap B| \leq 4$. Claim 26 implies that $w_k \leq 2$ for every k with $5 \leq k \leq t$. By Corollary 11, we have

$$|B| = \sum_{k=1}^{4} w_k + \sum_{k=5}^{t} w_k \leq 4 + (t-3) = t + 1 \leq |A|.$$

The case $t - 3 \leq i \leq t$ follows by symmetry.

Second, we assume that $5 \leq i \leq t-4$. Let $W := W_{i-3} \cup W_{i-2} \cup W_{i-1} \cup W_i$. Note that W is convex, since $2 \leq i-3$ and $i < t$. By Lemma 16(ii), we have $w_{i-3} + w_{i-2} + w_{i-1} + w_i \leq 3$ and $w_i + w_{i+1} + w_{i+2} + w_{i+3} \leq 3$. By Claim 26, $w_k \leq 2$ for all k with $1 \leq k \leq i-4$. Thus, by Corollary 11, $\sum_{k=1}^{i-4} w_k \leq i-3$. Similarly, we have $\sum_{k=i+4}^{t} w_k \leq t-i-2$. Altogether, we obtain that

$$|B| = \sum_{k=1}^{i-4} w_k + \sum_{k=i-3}^{i-1} w_k + w_i + \sum_{k=i+1}^{i+3} w_k + \sum_{k=i+4}^{t} w_k \leq (i-3)+3+(t-i-2) = t-2 \leq |A|-3.$$

◀

4.5 Finalizing the proof of Theorem 2

We are now ready to prove Theorem 2. Namely, we show that for every ℓ-divided set $P = A \cup B$ with $|A|, |B| \geq 5$ and with neither A nor B in convex position there is an ℓ-divided 5-hole in P.

Suppose for the sake of contradiction that there is no ℓ-divided 5-hole in P. By the result of Harborth [21], every set P of ten points contains a 5-hole in P. In the case $|A|, |B| = 5$, the statement then follows from the assumption that neither of A and B is in convex position.

So assume that at least one of the sets A and B has at least six points. We obtain an island Q of P by iteratively removing extremal points so that neither part is in convex position after the removal and until one of the following conditions holds.

(i) One of the parts $Q \cap A$ and $Q \cap B$ has only five points.
(ii) Q is an ℓ-critical island of P with $|Q \cap A|, |Q \cap B| \geq 6$.

In case (i), we have $|Q \cap A| = 5$ or $|Q \cap B| = 5$. If $|Q \cap A| = 5$ and $|Q \cap B| \geq 6$, then we let Q' be the union of $Q \cap A$ with the six leftmost points of $Q \cap B$. Since $Q \cap A$ is not in convex position, Lemma 12 implies that there is an ℓ-divided 5-hole in Q', which is also an ℓ-divided 5-hole in Q, since Q' is an island of Q. However, this is impossible as then there is an ℓ-divided 5-hole in P because Q is an island of P. If $|Q \cap A| \geq 6$ and $|Q \cap B| = 5$, then we proceed analogously.

In case (ii), we have $|Q \cap A|, |Q \cap B| \geq 6$. There is no ℓ-divided 5-hole in Q, since Q is an island of P. By Lemma 17(i), we can assume without loss of generality that $|A \cap \partial \operatorname{conv}(Q)| = 2$. Then it follows from Proposition 19 that $|Q \cap B| < |Q \cap A|$. By exchanging the roles of $Q \cap A$ and $Q \cap B$ and by applying Proposition 22, we obtain that $|Q \cap A| \leq |Q \cap B|$, a contradiction. This completes the proof of Theorem 2.

References

1. O. Aichholzer. Enumerating order types for small point sets with applications. `http://www.ist.tugraz.at/aichholzer/research/rp/triangulations/ordertypes/`.
2. O. Aichholzer. [Empty] [colored] k-gons. Recent results on some Erdős–Szekeres type problems. In *Proceedings of XIII Encuentros de Geometría Computacional*, pages 43–52, Zaragoza, Spain, 2009.
3. O. Aichholzer, F. Aurenhammer, and H. Krasser. Enumerating order types for small point sets with applications. *Order*, 19(3):265–281, 2002.
4. O. Aichholzer, M. Balko, T. Hackl, J. Kynčl, I. Parada, M. Scheucher, P. Valtr, and B. Vogtenhuber. A superlinear lower bound on the number of 5-holes. `http://arXiv.org/abs/1703.05253`, 2017.
5. O. Aichholzer, R. Fabila-Monroy, T. Hackl, C. Huemer, A. Pilz, and B. Vogtenhuber. Lower bounds for the number of small convex k-holes. *Computational Geometry: Theory and Applications*, 47(5):605–613, 2014.
6. O. Aichholzer, T. Hackl, and B. Vogtenhuber. On 5-gons and 5-holes. *Lecture Notes in Computer Science*, 7579:1–13, 2012.
7. O. Aichholzer and H. Krasser. Abstract order type extension and new results on the rectilinear crossing number. *Computational Geometry: Theory and Applications*, 36(1):2–15, 2007.
8. M. Balko. `http://kam.mff.cuni.cz/~balko/superlinear5Holes`.
9. M. Balko, R. Fulek, and J. Kynčl. Crossing numbers and combinatorial characterization of monotone drawings of K_n. *Discrete & Computational Geometry*, 53(1):107–143, 2015.
10. I. Bárány and Z. Füredi. Empty simplices in Euclidean space. *Canadian Mathematical Bulletin*, 30(4):436–445, 1987.
11. I. Bárány and Gy. Károlyi. Problems and results around the Erdős–Szekeres convex polygon theorem. In Akiyama, Kano, and Urabe, editors, *Discrete and Computational Geometry*, volume 2098 of *Lecture Notes in Computer Science*, pages 91–105. Springer, 2001.
12. I. Bárány and P. Valtr. Planar point sets with a small number of empty convex polygons. *Studia Scientiarum Mathematicarum Hungarica*, 41(2):243–266, 2004.
13. P. Brass, W. Moser, and J. Pach. *Research Problems in Discrete Geometry*. Springer, 2005.
14. K. Dehnhardt. *Leere konvexe Vielecke in ebenen Punktmengen*. PhD thesis, TU Braunschweig, Germany, 1987. In German.
15. P. Erdős. Some more problems on elementary geometry. *Australian Mathematical Society Gazette*, 5(2):52–54, 1978.
16. P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Mathematica*, 2:463–470, 1935.
17. S. Felsner and H. Weil. Sweeps, arrangements and signotopes. *Discrete Applied Mathematics*, 109(1–2):67–94, 2001.
18. A. García. A note on the number of empty triangles. *Lecture Notes in Computer Science*, 7579:249–257, 2012.

19 T. Gerken. Empty convex hexagons in planar point sets. *Discrete & Computational Geometry*, 39(1–3):239–272, 2008.

20 J. E. Goodman and R. Pollack. Multidimensional sorting. *SIAM Journal on Computing*, 12(3):484–507, 1983.

21 H. Harborth. Konvexe Fünfecke in ebenen Punktmengen. *Elemente der Mathematik*, 33:116–118, 1978. In German.

22 J. D. Horton. Sets with no empty convex 7-gons. *Canadian Mathematical Bulletin*, 26(4):482–484, 1983.

23 C. M. Nicolás. The empty hexagon theorem. *Discrete & Computational Geometry*, 38(2):389–397, 2007.

24 R. Pinchasi, R. Radoičić, and M. Sharir. On empty convex polygons in a planar point set. *Journal of Combinatorial Theory, Series A*, 113(3):385–419, 2006.

25 M. Scheucher. `http://www.ist.tugraz.at/scheucher/5holes`.

26 M. Scheucher. Counting convex 5-holes, Bachelor's thesis, 2013. In German.

27 M. Scheucher. On order types, projective classes, and realizations, Bachelor's thesis, 2014.

28 W. Steiger and J. Zhao. Generalized ham-sandwich cuts. *Discrete & Computational Geometry*, 44(3):535–545, 2010.

29 P. Valtr. Convex independent sets and 7-holes in restricted planar point sets. *Discrete & Computational Geometry*, 7(2):135–152, 1992.

30 P. Valtr. Sets in $\mathbb{R}^d$ with no large empty convex subsets. *Discrete Mathematics*, 108(1):115–124, 1992.

31 P. Valtr. On empty pentagons and hexagons in planar point sets. In *Proceedings of Computing: The Eighteenth Australasian Theory Symposium (CATS 2012)*, pages 47–48, Melbourne, Australia, 2012.