

A superlinear lower bound on the number of 5-holes*

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Abstract

Let P be a finite set of points in the plane in *general position*, that is, no three points of P are on a common line. We say that a set H of five points from P is a *5-hole in P* if H is the vertex set of a convex 5-gon containing no other points of P . For a positive integer n , let $h_5(n)$ be the minimum number of 5-holes among all sets of n points in the plane in general position.

Despite many efforts in the last 30 years, the best known asymptotic lower and upper bounds for $h_5(n)$ have been of order $\Omega(n)$ and $O(n^2)$, respectively. We show that $h_5(n) = \Omega(n \log^{4/5} n)$, obtaining the first superlinear lower bound on $h_5(n)$.

The following structural result, which might be of independent interest, is a crucial step in the proof of this lower bound. If a finite set P of points in the plane in general position is partitioned by a line ℓ into two subsets, each of size at least 5 and not in convex position, then ℓ intersects the convex hull of some 5-hole in P . The proof of this result is computer-assisted.

1 Introduction

We say that a set of points in the plane is in *general position* if it contains no three points on a common

line. A point set is in *convex position* if it is the vertex set of a convex polygon. Let P be a finite set of points in general position in the plane. We say that a set H of k points from P is a *k -hole in P* if H is the vertex set of a convex polygon containing no other points of P .

In the 1970s, Erdős [6] asked whether, for every positive integer k , there is a k -hole in every sufficiently large finite point set in general position in the plane. Harborth [8] proved that there is a 5-hole in every set of 10 points in general position in the plane and gave a construction of 9 points in general position with no 5-hole. After unsuccessful attempts of researchers to answer Erdős' question affirmatively for any fixed integer $k \geq 6$, Horton [9] constructed, for every positive integer n , a set of n points in general position in the plane with no 7-hole. The question whether there is a 6-hole in every sufficiently large finite planar point set remained open until 2007 when Gerken [7] and Nicolás [10] independently gave an affirmative answer.

For positive integers n and k , let $h_k(n)$ be the minimum number of k -holes in a set of n points in general position in the plane. Due to Horton's construction [9], $h_k(n) = 0$ for every n and every $k \geq 7$. The functions $h_3(n)$ and $h_4(n)$ are both known to be asymptotically quadratic [2, 4]. For $h_5(n)$ and $h_6(n)$, the best known asymptotic bounds are $\Omega(n)$ and $O(n^2)$ [4, 7, 8, 10]. See, e.g., [2] for more details.

As our main result, we give the first superlinear lower bound on $h_5(n)$. This solves an open problem, which was explicitly stated, for example, in a book by Brass, Moser, and Pach [5, Chapter 8.4, Problem 5] and in the survey [1].

Theorem 1 *There is a fixed constant $c > 0$ such that for every integer $n \geq 10$ we have $h_5(n) \geq cn \log^{4/5} n$.*

Let P be a finite set of points in the plane in general position and let ℓ be a line that contains no point of P and that partitions P into two non-empty subsets A and B . We then say that $P = A \cup B$ is *ℓ -divided*.

The following result, which might be of independent interest, is a crucial step in the proof of Theorem 1.

Theorem 2 *Let $P = A \cup B$ be an ℓ -divided set with $|A|, |B| \geq 5$ and with neither A nor B in convex position. Then there is an ℓ -divided 5-hole in P .*

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The proof of Theorem 2 is computer-assisted. We reduce the result to several statements about point sets of size at most 11 and then verify each of these statements by an exhaustive computer search. To verify the computer-aided proofs we have implemented two independent programs, which, in addition, are based on different abstractions of point sets. Some of the tools that we use originate from a bachelor's thesis of Scheucher [12].

In the rest of the paper, we assume that every point set P is planar, finite, and in general position. We also assume, without loss of generality, that all points in P have distinct x -coordinates. We use $\text{conv}(P)$ to denote the convex hull of P and $\partial \text{conv}(P)$ to denote the boundary of the convex hull of P .

A subset Q of P that satisfies $P \cap \text{conv}(Q) = Q$ is called an *island* of P . Note that every k -hole in an island of P is also a k -hole in P .

2 Proof of Theorem 1

We now show how to apply Theorem 2 to obtain a superlinear lower bound on the number of 5-holes in a given set of n points. Without loss of generality, we assume that $n = 2^t$ for some integer $t \geq 5^5$.

We prove by induction on $t \geq 5^5$ that the number of 5-holes in an arbitrary set P of $n = 2^t$ points is at least $f(t) := c \cdot 2^{t4/5} = c \cdot n \log_2^{4/5} n$ for some absolute constant $c > 0$. For $t = 5^5$, we have $n > 10$ and, by the result of Harborth [8], there is at least one 5-hole in P . If c is sufficiently small, then $f(t) = c \cdot n \log_2^{4/5} n \leq 1$ and we have at least $f(t)$ 5-holes in P , which constitutes our base case.

For the inductive step we assume that $t > 5^5$. We first partition P with a line ℓ into two sets A and B of size $n/2$ each. Then we further partition A and B into smaller sets using the following well-known lemma, which is, for example, implied by a result of Steiger and Zhao [13, Theorem 1].

Lemma 3 ([13]) *Let $P' = A' \cup B'$ be an ℓ -divided set and let r be a positive integer such that $r \leq |A'|, |B'|$. Then there is a line disjoint from P' that determines an open halfplane h with $|A' \cap h| = r = |B' \cap h|$.*

We set $r := \lfloor \log_2^{1/5} n \rfloor$, $s := \lfloor n/(2r) \rfloor$, and apply Lemma 3 iteratively in the following way to partition P into islands P_1, \dots, P_{s+1} of P so that the sizes of $P_i \cap A$ and $P_i \cap B$ are exactly r for every $i \in \{1, \dots, s\}$. Let $P'_0 := P$. For every $i = 1, \dots, s$, we consider a line that is disjoint from P'_{i-1} and that determines an open halfplane h with $|P'_{i-1} \cap A \cap h| = r = |P'_{i-1} \cap B \cap h|$. Such a line exists by Lemma 3 applied to the ℓ -divided set P'_{i-1} . We then set $P_i := P'_{i-1} \cap h$, $P'_i := P'_{i-1} \setminus P_i$, and continue with $i + 1$. Finally, we set $P_{s+1} := P'_s$.

For every $i \in \{1, \dots, s\}$, if one of the sets $P_i \cap A$ and $P_i \cap B$ is in convex position, then there are at least

$\binom{r}{5}$ 5-holes in P_i and, since P_i is an island of P , we have at least $\binom{r}{5}$ 5-holes in P . If this is the case for at least $s/2$ islands P_i , then, given that $s = \lfloor n/(2r) \rfloor$ and thus $s/2 \geq \lfloor n/(4r) \rfloor$, we obtain at least $\lfloor n/(4r) \rfloor \binom{r}{5} \geq c \cdot n \log_2^{4/5} n$ 5-holes in P for a sufficiently small $c > 0$.

We thus further assume that for more than $s/2$ islands P_i , neither of the sets $P_i \cap A$ nor $P_i \cap B$ is in convex position. Since $r = \lfloor \log_2^{1/5} n \rfloor \geq 5$, Theorem 2 implies that there is an ℓ -divided 5-hole in each such P_i . Thus there is an ℓ -divided 5-hole in P_i for more than $s/2$ islands P_i . Since each P_i is an island of P and since $s = \lfloor n/(2r) \rfloor$, we have more than $s/2 \geq \lfloor n/(4r) \rfloor$ ℓ -divided 5-holes in P . As $|A| = |B| = n/2 = 2^{t-1}$, there are at least $f(t-1)$ 5-holes in A and at least $f(t-1)$ 5-holes in B by the inductive assumption. Since A and B are separated by ℓ , we have at least

$$\begin{aligned} 2f(t-1) + n/(4r) &= 2c(n/2) \log_2^{4/5} (n/2) + n/(4r) \\ &\geq cn(t-1)^{4/5} + n/(4t^{1/5}) \end{aligned}$$

5-holes in P . The right side of the above expression is at least $f(t) = cnt^{4/5}$, because the inequality $cn(t-1)^{4/5} + n/(4t^{1/5}) \geq cnt^{4/5}$ is equivalent to the inequality $(t-1)^{4/5}t^{1/5} + 1/(4c) \geq t$, which is true if c is sufficiently small, as $(t-1)^{4/5}t^{1/5} \geq t-1$. This completes the proof of Theorem 1.

3 Preliminaries

Before proceeding with the proof of Theorem 2, we first introduce some notation and definitions, and state some immediate observations.

Let a, b be two points in the plane. We denote the ray starting at a and going through b as \overrightarrow{ab} and the line through a and b directed from a to b as \overleftrightarrow{ab} .

Let $P = A \cup B$ be an ℓ -divided set. In the rest of the paper, we assume without loss of generality that ℓ is vertical and directed upwards, A is to the left of ℓ , and B is to the right of ℓ .

ℓ -critical sets An ℓ -divided set $C = A \cup B$ is *ℓ -critical* if it fulfills the following two conditions.

- (i) Neither A nor B is in convex position.
- (ii) For every extremal point x of C , either $(C \setminus \{x\}) \cap A$ or $(C \setminus \{x\}) \cap B$ is in convex position.

a -wedges and a^* -wedges Let $P = A \cup B$ be an ℓ -divided set. For a point a in A , the rays $\overrightarrow{aa'}$ for all $a' \in A \setminus \{a\}$ partition the plane into $|A| - 1$ regions. We call the closures of those regions *a -wedges* and label them as $W_1^{(a)}, \dots, W_{|A|-1}^{(a)}$ in clockwise order around a , where $W_1^{(a)}$ is the topmost a -wedge that intersects ℓ . Let $t^{(a)}$ be the number of a -wedges that intersect ℓ . Note that $W_1^{(a)}, \dots, W_{t^{(a)}}^{(a)}$ are the a -wedges that intersect ℓ sorted in top-to-bottom order on ℓ . Also note

that all a -wedges are convex if a is an inner point of A , and that there exists exactly one non-convex a -wedge otherwise.

If A is not in convex position, we denote the right-most inner point of A as a^* and write $t := t^{(a^*)}$ and $W_k := W_k^{(a^*)}$ for $k = 1, \dots, |A| - 1$. Recall that a^* is unique, since all points have distinct x -coordinates. We set $w_k := |B \cap W_k|$ and label the points of A so that W_k is bounded by the rays $\overrightarrow{a^*a_{k-1}}$ and $\overrightarrow{a^*a_k}$ for $k = 1, \dots, |A| - 1$. Figure 1 gives an illustration.

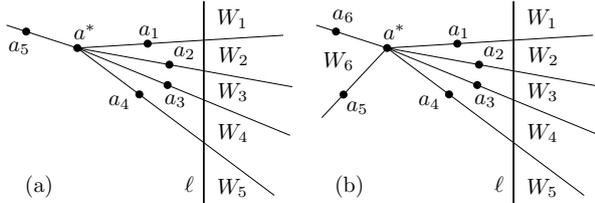


Figure 1: (a) An example of a^* -wedges with $t = |A| - 1$. (b) An example of a^* -wedges with $t < |A| - 1$.

4 Proof of Theorem 2

In the rest of the paper, we state results that we use to prove Theorem 2 and we then present the proof of this theorem. Due to lack of space, we omit the proofs of almost all auxiliary results.

4.1 a^* -wedges with at most two points of B

We first consider an ℓ -divided set $P = A \cup B$ with A not in convex position. We show that, if there is a sequence of consecutive a^* -wedges where the first and the last a^* -wedge both contain two points of B and every a^* -wedge in between them contains exactly one point of B , then there is an ℓ -divided 5-hole in P .

Lemma 4 *Let $P = A \cup B$ be an ℓ -divided set with A not in convex position and with $|A| \geq 5$ and $|B| \geq 6$. Let W_i, \dots, W_j be consecutive a^* -wedges with $1 \leq i < j \leq t$, $w_i = 2 = w_j$, and $w_k = 1$ for every k with $i < k < j$. Then there is an ℓ -divided 5-hole in P .*

The proof of this lemma is carried out by a rather elaborate case distinction, which we omit here.

4.2 Computer-assisted results

We now provide lemmas that are key ingredients in the proof of Theorem 2. All these lemmas have computer-aided proofs. Each result was verified by two independent implementations, which are also based on different abstractions of point sets. In particular, to prove these lemmas, we employ an exhaustive computer search through all combinatorially different sets of $|P| \leq 11$ points in the plane. Both programs and detailed information are available online [3, 11].

Lemma 5 *Let $P = A \cup B$ be an ℓ -divided set with $|A| = 5$, $|B| = 6$, and with A not in convex position. Then there is an ℓ -divided 5-hole in P .*

Lemma 6 *Let $P = A \cup B$ be an ℓ -divided set with no ℓ -divided 5-hole in P , $|A| = 5$, $4 \leq |B| \leq 6$, and with A in convex position. Then for every point a of A , every convex a -wedge contains at most two points of B .*

Lemma 7 *Let $P = A \cup B$ be an ℓ -divided set with no ℓ -divided 5-hole in P , $|A| = 6$, and $|B| = 5$. Then for each point a of A , every convex a -wedge contains at most two points of B .*

Lemma 8 *Let $P = A \cup B$ be an ℓ -divided set with no ℓ -divided 5-hole in P , $5 \leq |A| \leq 6$, $|B| = 4$, and with A in convex position. Then for every point a of A , if the non-convex a -wedge contains no point of B , every a -wedge contains at most two points of B .*

4.3 Applications of the computer-assisted results

As a first application of the computer-assisted results we prove the following statement, which restricts the number of points of B in a^* -wedges. Its proof uses Lemmas 6, 7, and 8 and also Lemma 4.

Lemma 9 *Let $P = A \cup B$ be an ℓ -divided set with no ℓ -divided 5-hole in P , with $|A| \geq 6$, and with A not in convex position. Then the following two conditions are satisfied.*

- (i) *Let W_i, W_{i+1}, W_{i+2} be three consecutive a^* -wedges whose union is convex and contains at least four points of B . Then $w_i, w_{i+1}, w_{i+2} \leq 2$.*
- (ii) *Let $W_i, W_{i+1}, W_{i+2}, W_{i+3}$ be four consecutive a^* -wedges whose union is convex and contains at least four points of B . Then $w_i, w_{i+1}, w_{i+2}, w_{i+3} \leq 2$.*

4.4 Extremal points of ℓ -critical sets

The following statement, whose relatively easy proof is omitted in this abstract, says that every ℓ -critical set has at most two extremal points on each side of ℓ .

Lemma 10 *Let $C = A \cup B$ be an ℓ -critical set with $|A \cap C| \geq 5$. Then $|A \cap \partial \text{conv}(C)| \leq 2$. By symmetry, an analogous statement holds for B .*

Now we use Lemma 9 to restrict the parameters w_i . Then we state the last auxiliary result used in the proof of Theorem 2.

Lemma 11 *Let $C = A \cup B$ be an ℓ -critical set with no ℓ -divided 5-hole in C and with $|A| \geq 6$. Then $w_i \leq 3$ for every $1 < i < t$. Moreover, if $|A \cap \partial \text{conv}(C)| = 2$, then also $w_1, w_t \leq 3$.*

Proof. Recall that, since C is ℓ -critical, we have $|B| \geq 4$. Let i be an integer with $1 \leq i \leq t$. We assume that there is a point a in $A \cap \partial \text{conv}(C)$, which lies outside of W_i , as otherwise there is nothing to prove for W_i (either $|A \cap \partial \text{conv}(C)| = 1$ and $i \in \{1, t\}$ or $|A \cap \partial \text{conv}(C)| = 2$ and $W_i \cap B = \emptyset$). We consider $C' := C \setminus \{a\}$. Since C is an ℓ -critical set, $A' := C' \cap A$ is in convex position. Thus, there is a non-convex a^* -wedge W' of C' . Since W' is non-convex, all other a^* -wedges of C' are convex. Moreover, since W' is the union of the two a^* -wedges of C that contain a , all other a^* -wedges of C' are also a^* -wedges of C . Let W be the union of all a^* -wedges of C that are not contained in W' . Note that W is convex and contains at least $|A| - 3 \geq 3$ a^* -wedges of C . Since $|A| \geq 6$, the lemma follows from Lemma 9(i). \square

Proposition 12 *Let $C = A \cup B$ be an ℓ -critical set with no ℓ -divided 5-hole in C and with $|A|, |B| \geq 6$. Then the following two conditions are satisfied.*

- (i) *If $|A \cap \partial \text{conv}(C)| = 2$ then $|B| \leq |A| - 1$.*
- (ii) *If $|B \cap \partial \text{conv}(C)| = 2$ then $|B| \leq |A|$.*

In the proof of this statement, we use the restrictions from Lemmas 4 and 9 that bound the number of points of B in a^* -wedges to derive the desired bound $|B| \leq |A|$. In the proof of part (i) we also apply Lemma 11 to show strict inequality. The proof of Proposition 12 is quite involved and we omit it here.

4.5 Finalizing the proof of Theorem 2

We are now ready to prove Theorem 2. Namely, we show that for every ℓ -divided set $P = A \cup B$ with $|A|, |B| \geq 5$ and with neither A nor B in convex position there is an ℓ -divided 5-hole in P .

Suppose for the sake of contradiction that there is no ℓ -divided 5-hole in P . We know by the result of Harborth [8] that every set P of ten points contains a 5-hole in P . In the case $|A|, |B| = 5$, the statement then follows from the assumption that neither of A and B is in convex position.

So assume that at least one of the sets A and B has at least six points. We obtain an island Q of P by iteratively removing extremal points so that neither part is in convex position after the removal and until one of the following conditions holds.

- (i) One of the parts $Q \cap A$ and $Q \cap B$ has five points.
- (ii) Q is an ℓ -critical island of P with $|Q \cap A| \geq 6$ and $|Q \cap B| \geq 6$.

In case (i), we have $|Q \cap A| = 5$ or $|Q \cap B| = 5$. If $|Q \cap A| = 5$ and $|Q \cap B| \geq 6$, then we let Q' be the union of $Q \cap A$ with the six leftmost points of $Q \cap B$. Since $Q \cap A$ is not in convex position, Lemma 5 implies that there is an ℓ -divided 5-hole in Q' , which

is also an ℓ -divided 5-hole in Q , since Q' is an island of Q . However, this is impossible as then there is an ℓ -divided 5-hole in P because Q is an island of P . We proceed analogously if $|Q \cap A| \geq 6$ and $|Q \cap B| = 5$.

In case (ii), we have $|Q \cap A|, |Q \cap B| \geq 6$. There is no ℓ -divided 5-hole in Q , since Q is an island of P . By Lemma 10, we can assume without loss of generality that $|A \cap \partial \text{conv}(Q)| = 2$. Then it follows from Proposition 12(i) that $|Q \cap B| < |Q \cap A|$. By exchanging the roles of $Q \cap A$ and $Q \cap B$ and by applying Proposition 12(ii), we obtain that $|Q \cap A| \leq |Q \cap B|$, a contradiction. This finishes the proof of Theorem 2.

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